The Picard group of a $K3$ surface and its reduction modulo p

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Abstract

We present a method to compute the geometric Picard rank of a $K3$ surface over $\mathbb{Q}$. Contrary to a widely held belief, we show that it is possible to verify Picard rank 1 using reduction at a single prime.

1 Introduction

1.1. —— For complex, projective $K3$ surfaces, the Picard group is a highly interesting invariant. In general, it is isomorphic to $\mathbb{Z}^n$ for some $n = 1, \ldots, 20$. A generic $K3$ surface has Picard rank 1. Nevertheless, the first explicit examples of $K3$ surfaces over $\mathbb{Q}$ having geometric Picard rank 1 were constructed by R. van Luijk [vL] as late as 2004. Van Luijk's method is based on reduction modulo p. It works as follows.

1.2. Approach (van Luijk). —— Let S be a $K3$ surface over $\mathbb{Q}$.
i) At a place p of good reduction, the Picard group $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ of the surface injects into the Picard group $\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ of its reduction modulo p.
ii) On its part, $\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ injects into the second étale cohomology group $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$.
iii) Only roots of unity can arise as eigenvalues of the Frobenius Frob on the image of $\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ in $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$. The number of eigenvalues of this form, counted with multiplicities, is therefore an upper bound for the Picard rank of $S_{\overline{\mathbb{F}}_p}$. One may

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compute the eigenvalues of Frob by counting the points on S, defined over $\mathbb{F}_p$ and some finite extensions.

Doing this for one prime, one obtains an upper bound for $\mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$, which is always even. The Tate conjecture asserts that this bound is actually sharp. Therefore, the best that could happen is to find a prime p that yields an upper bound of 2 for the rank of $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$.

iv) In this case, the assumption that the surface has Picard rank 2 over $\overline{\mathbb{Q}}$ implies that the discriminants of both Picard groups, $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ and $\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$, belong to the same square class. Note here that reduction modulo p respects the intersection pairing.

v) To obtain a contradiction, one combines information from two primes. It may happen that one has a rank bound of 2 at both places but that different square classes arise for the discriminants. Then, these data are incompatible with Picard rank 2 over $\overline{\mathbb{Q}}$. Geometric Picard rank 1 is proven.

1.3. *The improvement.* —— The idea behind Approach 1.2 is to consider the specialization $\mathrm{sp}\colon\ \mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ as an injection of lattices. Then, the two possibilities $\mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) < \mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ and $\mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) = \mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ are distinguished. In the latter, the standard fact is used that $\mathrm{disc}\,\mathrm{Pic}(S_{\overline{\mathbb{Q}}})/\,\mathrm{disc}\,\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ is a perfect square.

We will show in this article that the assertion for the second case may be refined to $\mathrm{disc}\,\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) = \mathrm{disc}\,\mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$. More precisely, we shall prove that, at least for $p \neq 2$, the cokernel of $\mathrm{sp}\colon\ \mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \hookrightarrow \mathrm{Pic}(S_{\overline{\mathbb{F}}_p})$ is always torsion-free. This is true actually in a by far more general situation than just for $K3$ surfaces.

1.4. Theorem. —— *Let R be a discrete valuation ring with quotient field K of characteristic 0 and residue field k of characteristic $p > 0$. Further, let $\pi\colon X \to \mathrm{Spec}\,R$ be a morphism of schemes that is proper and smooth.*

Suppose that R is of ramification degree $e < p-1$ and that k is perfect. Then, the cokernel of the specialization homomorphism $\mathrm{sp}_{\overline{K}}\colon\ \mathrm{Pic}(X_{\overline{K}}) \to \mathrm{Pic}(X_{\overline{k}})$ is torsion-free.

1.5. Remarks. —— a) In the applications, we will have $R = \mathbb{Z}_{(p)} \subset \mathbb{Q}$. Then, the assumption simply means $p \neq 2$.

b) We will show this theorem in section 3. As an application, one may prove $\mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) = 1$ for a $K3$ surface S using its reduction at a single prime. This works as follows.

1.6. Approach. —— Let a $K3$ surface S over $\mathbb{Q}$ be given.

i) For a prime $p \neq 2$ of good reduction, perform steps i), ii) and iii) as in 1.2. Thereby, the hope is to prove $\mathrm{rk}\,\mathrm{Pic}(S_{\overline{\mathbb{F}}_p}) = 2$. Further, compute the discriminant giving two explicit generators.

Alternatively, to determine the discriminant, one might use the Artin-Tate formula [Mi]. In this case, $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_p}) = 2$ is shown only relative to the Tate conjecture. Observe, however, that a surface with $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_p}) = 1$, due to a failure of the Tate conjecture, would serve our purposes, as well.

ii) Assume $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 2$. Then, according to Theorem 1.4, every invertible sheaf on $S_{\overline{\mathbb{F}}_p}$ must lift to $S_{\overline{\mathbb{Q}}}$. Estimate the degree of a hypothetical effective divisor. Finally, use Gröbner bases to verify that such a divisor does not exist.

1.7. Example. —— Consider the $K3$ surface S over $\mathbb{Q}$, given by

$$w^2 = x^5y + x^4y^2 + 2x^3y^3 + x^2y^4 + xy^5 + 4y^6 + 2x^5z + 2x^4z^2 + 4x^3z^3 + 2xz^5 + 4z^6 \,.$$

Then, $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

Proof. For the reduction of S at the prime 5, one sees that the branch locus has a tritangent line given by $z - 2y = 0$. It meets the branch locus at $(1:0:0)$, $(1:3:1)$, and $(0:1:2)$.

The numbers of points on S over $\mathbb{F}_{5^d}$ are, in this order, 41, 751, 15 626, 392 251, 9 759 376, 244 134 376, 6 103 312 501, 152 589 156 251, 3 814 704 296 876, and 95 367 474 609 376. Thus, the traces of Frob on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_5}, \overline{\mathbb{Q}}_l)$ are 15, 125, 0, 1 625, $-6\,250$, $-6\,250$, $-203\,125$, 1 265 625, 7 031 250, and 42 968 750.

[EJ1, Algorithm 23] shows that the sign in the functional equation is positive. The characteristic polynomial of Frob is therefore completely determined. For its decomposition into prime polynomials, we find (after Tate twist to $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_5}, \overline{\mathbb{Q}}_l(1))$)

$$\begin{aligned}\frac{1}{5}(t-1)^2(5t^{20} - 5t^{19} - 5t^{18} + 10t^{17} - 2t^{16} - 3t^{15} + 4t^{14} - 2t^{13} - 2t^{12} + t^{11}\\ + 3t^{10} + t^9 - 2t^8 - 2t^7 + 4t^6 - 3t^5 - 2t^4 + 10t^3 - 5t^2 - 5t + 5)\,.\end{aligned}$$

This shows $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_5}) \leq 2$.

The irreducible components of the pull-back of the tritangent line are explicit generators for $\operatorname{Pic}(S_{\overline{\mathbb{F}}_5})$. Such a component l, being a projective line, has self-intersection number $l^2 = -2$. Further, $lh = 1$ for h the pull-back of a line. If we had $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 2$ then the invertible sheaf $\mathscr{O}(l)$ would lift to $S_{\overline{\mathbb{Q}}}$. We would have a divisor L on $S_{\overline{\mathbb{Q}}}$ such that $HL = 1$ and $L^2 = -2$. By [BPV, Proposition VIII.3.6.i], such a divisor is automatically effective.

The equation $HL = 1$ shows that L is obtained from a line on $\mathbf{P}^2$, the pull-back of which splits into two components. This is possible only for a line tritangent to the branch locus. [EJ1, Algorithm 8] shows, however, using Gröbner bases, that such a tritangent line does not exist. $\square$

2 The cokernel of the restriction map

2.1. Notation. —— i) Let R be a discrete valuation ring of unequal characteristic. We will write $K := \operatorname{Quot}(R)$ for its quotient field, $\mathfrak{p}$ for the maximal ideal,

$k := R/\mathfrak{p}$ for the residue field of characteristic p, and $\nu\colon K \twoheadrightarrow \mathbb{Z}$ for the normalized valuation. Let $e := \nu(p)$ denote the ramification degree of R.

ii) Let X be an R-scheme. Then, we will write $X_\mathfrak{p}$ for the special fiber and X_η for the generic fiber of X. For L an extension of K, we will denote by X_L the base extension of X_η to L. Analogously, for l an extension of k, we will write X_l for the base extension of $X_\mathfrak{p}$ to l. In the particular case that $l = \mathbb{F}_q$, the shortcut X_q shall be used for X_l.

2.2. Proposition. —— *Let $\pi\colon X \to \operatorname{Spec} R$ be a morphism of schemes that is proper and flat. Suppose that the special fiber $X_\mathfrak{p}$ is normal.*

If R is complete and satisfies the condition $e < p-1$ then the cokernel of the restriction homomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X_\mathfrak{p})$ is torsion-free.

Proof. This result was obtained by M. Raynaud in the course of his investigations on the Picard scheme [Ra2, Théorème 4.1.2.1)]. □

2.3. Remark. —— Assume, in addition, that the restriction homomorphism $H^1(X, \mathscr{O}_X) \to H^1(X_\mathfrak{p}, \mathscr{O}_{X_\mathfrak{p}})$ is surjective. Then, the assertion of Proposition 2.2 may be established using the following elementary argument, which is also due to M. Raynaud [Ra2, section 1].

Consider the functors T^i on the category of all finitely generated R-modules to finitely generated R-modules, given by $T^i(M) := H^i(X, \pi^*\widetilde{M})$. Here, $\widetilde{M}$ denotes the coherent sheaf associated with the R-module M. According to [EGA III, Proposition (7.7.10)], the functor T^1 is right exact. Hence, by [EGA III, Théorème (7.7.5.II)], T^2 is left exact. This, in turn, immediately implies that $H^2(X, \mathscr{O}_X)$ is torsion-free.

Further, the short exact sequence

$$0 \longrightarrow \mathscr{U}_1 \longrightarrow \mathscr{O}_X^* \longrightarrow \mathscr{O}_{X_\mathfrak{p}}^* \longrightarrow 0$$

shows that $\operatorname{coker}(\operatorname{Pic}(X) \to \operatorname{Pic}(X_\mathfrak{p}))$ injects into $H^2(X, \mathscr{U}_1)$. Finally, as $e < p-1$, the exponential map provides us with an isomorphism $\mathscr{O}_X \stackrel{\cdot p}{\to} p\mathscr{O}_X \stackrel{\exp}{\to} \mathscr{U}_1$.

2.4. Remarks. —— i) The additional assumption of 2.3 is fulfilled in our applications.

ii) For prime-to-p torsion, the assertion of Proposition 2.2 is true in a more general situation.

2.5. Proposition. —— *Let $\pi\colon X \to \operatorname{Spec} R$ be a proper morphism of schemes.*

If R is Henselian then the cokernel of the restriction homomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X_\mathfrak{p})$ has no prime-to-p torsion.

Proof. Let $l \neq p$ be a prime number. We will show that there is no l-torsion. For this, we observe at first that, according to a consequence of the theorem on proper base change [SGA 4, Exp. XII, Corollaire 5.5.iii)], the restriction morphism induces bijections $H^1_{\text{ét}}(X,\mu_l) \xrightarrow{\cong} H^1_{\text{ét}}(X_\mathfrak{p},\mu_l)$ and $H^2_{\text{ét}}(X,\mu_l) \xrightarrow{\cong} H^2_{\text{ét}}(X_\mathfrak{p},\mu_l)$.

Using the fact [SGA 6, Exp. X, diagramme (7.13.10)] that the restriction homomorphisms on the Picard groups and étale cohomology commute with the Chern maps, we see that restriction induces a surjection $\mathrm{Pic}(X)_l \twoheadrightarrow \mathrm{Pic}(X_\mathfrak{p})_l$ and an injection $\mathrm{Pic}(X)/l \hookrightarrow \mathrm{Pic}(X_\mathfrak{p})/l$.

Applied to the two commutative diagrams of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(X)_l \arrow[r] \arrow[d, two heads] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & P_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}(X_\mathfrak{p})_l \arrow[r] & \mathrm{Pic}(X_\mathfrak{p}) \arrow[r] & P_{X_\mathfrak{p}} \arrow[r] & 0\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & P_X \arrow[r, "\cdot l"] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X)/l \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & P_{X_\mathfrak{p}} \arrow[r, "\cdot l"] & \mathrm{Pic}(X_\mathfrak{p}) \arrow[r] & \mathrm{Pic}(X_\mathfrak{p})/l \arrow[r] & 0\,,
\end{tikzcd}$$

the snake lemma now shows that the induced homomorphism

$$\mathrm{coker}(\mathrm{Pic}(X) \to \mathrm{Pic}(X_\mathfrak{p})) \longrightarrow \mathrm{coker}(P_X \to P_{X_\mathfrak{p}})$$

is a bijection, while

$$\mathrm{coker}(P_X \to P_{X_\mathfrak{p}}) \xrightarrow{\cdot l} \mathrm{coker}(\mathrm{Pic}(X) \to \mathrm{Pic}(X_\mathfrak{p}))$$

is injective. Consequently, $\mathrm{coker}(\mathrm{Pic}(X) \to \mathrm{Pic}(X_\mathfrak{p}))$ has no l-torsion. □

3 The cokernel of the specialization map

3.1. —— In this section, we will continue to use the notation from 2.1. Let $\pi\colon X \to \operatorname{Spec} R$ be a morphism of schemes that is proper and smooth. We have the restriction homomorphisms

$$\mathrm{Pic}(X_\eta) \longleftarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_\mathfrak{p})\,.$$

As π is smooth, the arrow to the left is a bijection [SGA 6, Exp. X, App. 7.8]. Consequently, there is a natural homomorphism $\mathrm{sp}\colon \mathrm{Pic}(X_\eta) \to \mathrm{Pic}(X_\mathfrak{p})$, which is called the *specialization*.

3.2. Lemma. —— *Let $\pi\colon X \to \operatorname{Spec} R$ be a morphism of schemes that is proper and smooth.*

If R is complete and satisfies the condition $e < p-1$ then the cokernel of the specialization homomorphism $\mathrm{sp}\colon \mathrm{Pic}(X_\eta) \to \mathrm{Pic}(X_\mathfrak{p})$ is torsion-free.

Proof. The assertion follows directly from Proposition 2.2. □

3.3. —— Let K'/K be an extension field equipped with a discrete valuation extending that on K. Denote by R' the discrete valuation ring and by k' the residue field. The morphism $X \times_{\operatorname{Spec} R} \operatorname{Spec} R' \to \operatorname{Spec} R'$, obtained by base change, induces a specialization homomorphism $\mathrm{sp}_{K'}\colon \operatorname{Pic}(X_{K'}) \to \operatorname{Pic}(X_{k'})$.

There are the following two applications.

i) Suppose R to be complete. Then, for every finite extension K'/K, there is a unique [Se, Chap. II, §2, Proposition 3] discrete valuation extending the valuation on K. The direct limit of the homomorphisms $\mathrm{sp}_{K'}\colon \operatorname{Pic}(X_{K'}) \to \operatorname{Pic}(X_{k'})$ is a natural homomorphism $\mathrm{sp}_{\overline{K}}\colon \operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(X_{\overline{k}})$, again called the specialization.

ii) For general R, fix an embedding $\overline{K} \hookrightarrow \overline{\widehat{K}}$ of the algebraic closure of K into that of its completion. By functoriality, this induces a homomorphism $\operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(X_{\overline{\widehat{K}}})$. Composing with $\mathrm{sp}_{\overline{\widehat{K}}}$, constructed in i), one has a specialization homomorphism $\mathrm{sp}_{\overline{K}}\colon \operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(X_{\overline{k}})$.

3.4. Proposition. —— *Let $\pi\colon X \to \operatorname{Spec} R$ be a morphism of schemes that is proper and smooth.*

Suppose that R is complete, satisfies the condition $e < p-1$, and that k is perfect. Then, the cokernel of the specialization homomorphism $\mathrm{sp}_{\overline{K}}\colon \operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(X_{\overline{k}})$ is torsion-free.

Proof. By [Se, Chap. III, §5, Corollaire 1 du Théorème 3], K has a unique maximal unramified extension K^{nr}, which is actually the filtered direct limit of all finite unramified extensions K'/K.

An unramified extension does not change the ramification degree. Hence, according to Lemma 3.2, the homomorphisms $\mathrm{sp}_{K'}\colon \operatorname{Pic}(X_{K'}) \to \operatorname{Pic}(X_{k'})$ have torsion-free cokernels. As the filtered direct limit is an exact functor, the same is true for $\mathrm{sp}_{K^{\mathrm{nr}}}\colon \operatorname{Pic}(X_{K^{\mathrm{nr}}}) \to \operatorname{Pic}(X_{\overline{k}})$.

We claim that the specialization homomorphism $\mathrm{sp}_{\overline{K}}$ has the same image in $\operatorname{Pic}(X_{\overline{k}})$ as $\mathrm{sp}_{K^{\mathrm{nr}}}$. For this, let $\mathscr{L} \in \operatorname{Pic}(X_{\overline{K}})$. The inertia group $I := \operatorname{Gal}(\overline{K}/K^{\mathrm{nr}})$ sends $\mathscr{L}$ to a finite orbit $\{\mathscr{L}_1, \ldots, \mathscr{L}_m\}$. The specializations of $\mathscr{L}_1, \ldots, \mathscr{L}_m$ in $\operatorname{Pic}(X_{\overline{k}})$ are all the same. Therefore,

$$m\cdot\mathrm{sp}_{\overline{K}}(\mathscr{L}) = \mathrm{sp}_{\overline{K}}(\mathscr{L}^{\otimes m}) = \mathrm{sp}_{\overline{K}}(\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_m) = \mathrm{sp}_{K^{\mathrm{nr}}}(\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_m)\,,$$

since $\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_m$ is I-invariant. Hence, $m\cdot\mathrm{sp}_{\overline{K}}(\mathscr{L}) \in \operatorname{im} \mathrm{sp}_{K^{\mathrm{nr}}}$. As $\mathrm{sp}_{K^{\mathrm{nr}}}$ has a torsion-free cokernel, we see that $\mathrm{sp}_{\overline{K}}(\mathscr{L}) \in \operatorname{im} \mathrm{sp}_{K^{\mathrm{nr}}}$, too. $\square$

3.5. Remark. —— The argument above uses that $\operatorname{Pic}(X_L) = \operatorname{Pic}(X_K)^{\operatorname{Gal}(L/K)}$. This equality is certainly not correct, in general. It is true as soon as $Y(K) \neq \emptyset$ for every connected component Y of X.

As π is smooth, we indeed have $Y(K^{\mathrm{nr}}) \neq \emptyset$. To see this, let $s\colon \operatorname{Spec} l \to Y_k$ be a point defined over a finite extension. By [EGA IV, Proposition (17.5.3)], s may be lifted to a morphism $\operatorname{Spf} S \to Y$ for S the corresponding unramified extension of R. [EGA III, Théorème (5.4.1)] yields the desired point.

3.6. Theorem. —— *Let R be a discrete valuation ring with quotient field K of characteristic 0 and residue field k of characteristic $p > 0$. Further, let $\pi\colon X \to \operatorname{Spec} R$ be a morphism of schemes that is proper and smooth.*

Suppose that R is of ramification degree $e < p-1$ and that k is perfect. Then, the cokernel of the specialization homomorphism $\mathrm{sp}_{\overline{K}}\colon \operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(X_{\overline{k}})$ is torsion-free.

3.7. Corollary. —— *Let $p \neq 2$ be a prime number and X be a scheme proper and flat over $\mathbb{Z}$. Suppose that the special fiber X_p is non-singular.*

Then, the cokernel of the specialization homomorphism $\mathrm{sp}_{\overline{\mathbb{Q}}}\colon \operatorname{Pic}(X_{\overline{\mathbb{Q}}}) \to \operatorname{Pic}(X_{\overline{\mathbb{F}}_p})$ is torsion-free. $\square$

3.8. Remark. —— The technical condition on the ramification degree cannot be omitted. In fact, D. Maulik and B. Poonen [MP, Example 3.12] constructed counterexamples to the assertion of Theorem 3.6 in the situation that $e \geq p-1$.

3.9. Remarks (Elementary reductions**).** —— i) Let R' be a discrete valuation ring, finite and flat over R. Then, the assertion for $\mathrm{pr}_2\colon X \times_{\operatorname{Spec} R} \operatorname{Spec} R' \to \operatorname{Spec} R'$, obtained by base-change, implies that for π.

ii) In particular, we may suppose that $\pi\colon X \to \operatorname{Spec} R$ has a section.

iii) We may suppose that the fibers of π are geometrically connected.

Indeed, as $\pi\colon X \to \operatorname{Spec} R$ is proper and smooth, one has $\pi_*\mathscr{O}_X = \widetilde{S}$ for S a finite étale R-algebra [EGA III, Remarque (7.8.10.i)]. Hence, there exists a discrete valuation ring R', étale over R, such that $S \otimes_R R'$ is a direct product of finitely many copies of R'. This means that the connected components of $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ have geometrically connected fibers. Knowing the assertion for each component separately, the proof will be complete.

3.10. Proposition. —— *Let R be a discrete valuation ring of characteristic 0 and $\pi\colon X \to \operatorname{Spec} R$ a proper and smooth morphism of schemes. Suppose that π has a section and that the fibers of π are geometrically connected.*

Then, the specialization homomorphisms $\mathrm{sp}_{\overline{K}}\colon \operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(X_{\overline{k}})$ and $\mathrm{sp}_{\overline{\widehat{K}}}\colon \operatorname{Pic}(X_{\overline{\widehat{K}}}) \to \operatorname{Pic}(X_{\overline{k}})$ have the same image.

Proof. As $\mathrm{sp}_{\overline{K}}$ factors via $\mathrm{sp}_{\overline{\widehat{K}}}$, we clearly have $\operatorname{im} \mathrm{sp}_{\overline{K}} \subseteq \operatorname{im} \mathrm{sp}_{\overline{\widehat{K}}}$. We will show the reverse inclusion in several steps. Let an invertible sheaf $\mathscr{L} \in \operatorname{Pic}(X_{\overline{\widehat{K}}})$ be given. We have to construct an invertible sheaf $\mathscr{L}' \in \operatorname{Pic}(X_{\overline{K}})$ having the same specialization as $\mathscr{L}$.

First step. The Picard scheme.

Our assumptions on π imply that it is cohomologically flat in dimension zero [EGA III, Proposition (7.8.6)]. Hence, by [Ar1, Theorem 7.3], the Picard functor $\mathrm{Pic}_{X/R}$ is representable by an algebraic space $P := \mathbf{Pic}_{X/R}$ that is locally of finite type over R. According to [FGA, Exp. 236, Théorème 2.1.i)], P is separated. This is enough to ensure that P is actually a scheme [Ra1, Théorème (3.3.1)]. Further, every closed subset $Z \subseteq P$, being of finite type, is proper over R.

Second step. The representing morphism.

The invertible sheaf $\mathscr{L} \in \mathrm{Pic}(X_{\overline{\widehat{K}}})$ is defined over a finite extension L of $\widehat{K}$. Hence, it defines a morphism $i\colon \operatorname{Spec} L \to P$. As $\widehat{K}$ is complete, there is a unique prolongation to L of the discrete valuation on $\widehat{K}$. I.e., we have a discrete valuation ring $S \supseteq \widehat{R}$. There is a unique continuation $j\colon \operatorname{Spec} S \to P$ of i.

Third step. Artin approximation.

By Lemma 3.12, we have $S = \widehat{\underline{S}}$ for a discrete valuation ring $\underline{S}$, finite over R. Write $\underline{L}$ for the quotient field of $\underline{S}$. This is a finite extension of K.

We now recall that discrete valuation rings of characteristic zero are excellent [EGA IV, Scholie (7.8.3.iii)]. In particular, M. Artin's approximation results [Ar2] are applicable. According to [Ar2, Corollary (2.5)], there are an étale extension S' of $\underline{S}$ and a morphism $j'\colon \operatorname{Spec} S' \to P$ of schemes that coincides, up to extensions of the base field, with j on the special fiber.

Corresponding to j', there is some $\xi \in \mathrm{Pic}_{X/R}(\operatorname{Spec} S')$.

Fourth step. An invertible sheaf.

As the fibers of X are geometrically connected, we have $\pi_*\mathscr{O}_X = \mathscr{O}_{\operatorname{Spec} R}$. Further, since π has a section, one has [FGA, Exp.232, Proposition 2.1]

$$\mathrm{Pic}_{X/R}(T) = \mathrm{Pic}(X \times_{\operatorname{Spec} R} T)/\,\mathrm{Pic}(T)$$

for every R-scheme T. In particular,

$$\begin{aligned}\mathrm{Pic}_{X/R}(\operatorname{Spec} S') &= \mathrm{Pic}(X \times_{\operatorname{Spec} R} \operatorname{Spec} S')/\,\mathrm{Pic}(\operatorname{Spec} S') \\ &= \mathrm{Pic}(X \times_{\operatorname{Spec} R} \operatorname{Spec} S')\,.\end{aligned}$$

Hence, ξ defines an invertible sheaf on $X \times_{\operatorname{Spec} R} \operatorname{Spec} S'$. Let $\mathscr{L}' \in \mathrm{Pic}(X_{\underline{L}})$ be its restriction to the generic fiber. Then, by construction, $\mathscr{L}'$ has the same specialization as $\mathscr{L}$. The assertion follows. $\square$

3.11. Remark. — Suppose that $H^1(X, \mathscr{O}_X) = 0$. Then, Proposition 3.10 is significantly more elementary. In fact, the Picard scheme P_K is of dimension zero [FGA, Exp. 236, Proposition 2.10.iii)] in this case. Hence, every point on P_K is defined over $\overline{K}$. No approximation argument is necessary.

Actually, the assumption $H^1(X, \mathscr{O}_X) = 0$ is fulfilled in the examples, discussed in 1.7 and below in section 4.

3.12. Lemma. —— *Let R be a discrete valuation ring with quotient field K of characteristic zero and $L/\widehat{K}$ a finite field extension of its completion.*

Then, there exists a subfield $\underline{L} \subset L$, finite over K, such that $\widehat{\underline{L}} = L$.

Proof. Choose a primitive element x of L over $\widehat{K}$ and let $f \in \widehat{K}[X]$ be its minimal polynomial. Then, the assertion is an immediate consequence of [Se, Chapitre II, §2, Exercice 2]. $\square$

3.13. Proof of Theorem 3.6. —— Consider the completion $\widehat{R}$ of R and denote by $\widehat{K}$ the corresponding quotient field. The ramification degree of $\widehat{R}$ is the same as that of R. Therefore, Proposition 3.4 shows that the specialization homomorphism $\mathrm{sp}_{\overline{\widehat{K}}}\colon \mathrm{Pic}(X_{\overline{\widehat{K}}}) \to \mathrm{Pic}(X_{\overline{k}})$ has a torsion-free cokernel. Further, by Proposition 3.10, $\mathrm{sp}_{\overline{K}}$ has the same image in $\mathrm{Pic}(X_{\overline{k}})$ as $\mathrm{sp}_{\overline{\widehat{K}}}$. This implies the assertion. $\square$

4 The obstruction to first order deformations

The obstructions to lifting invertible sheaves were essential for the elementary proof of Proposition 2.2, as discussed in 2.3. In some cases, they can be made explicit.

4.1. Proposition. —— *Let S be a $K3$ surface of degree 2 over $\mathbb{Q}$, given explicitly by*

$$w^2 = f_6(x, y, z)$$

for $f_6 \in \mathbb{Z}[x, y, z]$ of degree 6. Suppose, for a prime $p \neq 2$ of good reduction, there is an $\mathbb{F}_p$-rational line "$\ell = 0$", tritangent to the ramification locus of S_p. Write l for an irreducible component of the pull-back of the tritangent.

One has $f_6 \equiv f_3^2 + \ell f_5 \pmod p$ for homogeneous forms $f_3, f_5 \in \mathbb{Z}[x, y, z]$. Put

$$G(x, y, z) := (f_6 - f_3^2 - \ell f_5)/p\,.$$

Then, $\mathscr{O}(l)$ lifts to S_{p^2} if and only if G vanishes in $\mathbb{F}_p[x, y, z]/(\ell, f_3, f_5)$.

Proof. Suppose that $\mathscr{O}(l)$ has a lift $\mathscr{L} \in \mathrm{Pic}(X_{p^2})$. Then, $\mathscr{L}/p\mathscr{L} \cong \mathscr{O}(l)$. Since multiplication by p induces an isomorphism $\mathscr{L}/p\mathscr{L} \cong p\mathscr{L}$, we automatically have a short exact sequence

$$0 \longrightarrow \mathscr{O}(l) \longrightarrow \mathscr{L} \longrightarrow \mathscr{O}(l) \longrightarrow 0\,.$$

As $H^1(X_p, \mathscr{O}(l)) = 0$, the restriction map $H^0(X_{p^2}, \mathscr{L}) \to H^0(X_p, \mathscr{O}(l))$ is a surjection. I.e., the divisor l on X_p necessarily lifts to an effective Cartier divisor on X_{p^2}.

This is possible only when the line defined by ℓ may be lifted to $\mathbf{P}^2_{p^2}$ in such a way that it is still a tritangent. On the other hand, if ℓ may be lifted to $\mathbf{P}^2_{p^2}$ such that it is still a tritangent then clearly $\mathscr{O}(l)$ lifts to X_{p^2}.

Explicitly, the condition means that f_6 is a square modulo p^2 and some lift of ℓ. Writing

$$f_6 \equiv (f_3 + pf_3')^2 + (\ell + p\ell')(f_5 + pf_5') \pmod{p^2},$$

one immediately sees that this is equivalent to the assertion that G vanishes in $\mathbb{F}_p[x, y, z]/(\ell, f_3, f_5)$. $\square$

4.2. Remark. —— There is another proof that consists of the determination of the cohomological obstruction to lifting $\mathscr{O}(l)$. I.e., of the image of $\mathscr{O}(l)$ under the connecting homomorphism $d\colon \operatorname{Pic}(X_p) \to H^2(X_p, \mathscr{O}_{X_p})$ that is induced by the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X_p} \longrightarrow \mathscr{O}^*_{X_{p^2}} \longrightarrow \mathscr{O}^*_{X_p} \longrightarrow 0.$$

The obstruction may easily be computed in Čech cohomology for a suitable affine open covering of X_{p^2}. Via the corresponding isomorphism $H^2(X_p, \mathscr{O}_{X_p}) \cong \mathbb{F}_p$, our result is indeed $((-G) \bmod (p, \ell, f_3, f_5))$. The necessary calculations are, however, rather lengthy and shall not be reproduced here.

4.3. —— In the examples below, we will use the obstruction in its explicit form, as given in Proposition 4.1. The methods for point counting, which we apply, are explained in some detail in [EJ1, EJ2, EJ4].

4.4. Example. —— Let S be a $K3$ surface over $\mathbb{Q}$ given by $w^2 = f_6(x, y, z)$. Suppose

$$\begin{aligned} f_6(x,y,z) \equiv x^6 + 2x^5z + 2x^4y^2 + 2x^4z^2 + 2x^3y^3 + 2x^3z^3 \\ + 2x^2y^4 + 2x^2y^3z + x^2z^4 + xy^3z^2 + 2xz^5 + y^6 \pmod 3. \end{aligned}$$

Assume further that the coefficient of y^2z^4 is not divisible by 9.

Then, $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

Proof. A direct calculation shows that, modulo 3, the right hand side is $f_3^2 + xf_5$ for $f_3 = 2x^3 + 2x^2z + xz^2 + 2y^3$ and $f_5 = 2x^3y^2 + x^2z^3 + 2xy^4 + 2z^5$. Thus, the branch locus of S_3 has a tritangent line given by $x = 0$.

The numbers of points over $\mathbb{F}_{3^d}$ are, in this order, 19, 127, 676, 6 751, 58 564, 532 414, 4 791 232, 43 038 703, 387 383 311, and 3 486 675 052. For the decomposition of the characteristic polynomial of the Frobenius on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l(1))$, we find

$$\begin{aligned} \frac{1}{3}(t-1)^2(3t^{20} - 3t^{19} - 3t^{18} + 8t^{17} - 3t^{16} - 4t^{15} + 6t^{14} - 4t^{13} + 2t^{12} + 4t^{11} \\ - 7t^{10} + 4t^9 + 2t^8 - 4t^7 + 6t^6 - 4t^5 - 3t^4 + 8t^3 - 3t^2 - 3t + 3). \end{aligned}$$

This shows $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_3}) \leq 2$.

Let l be an irreducible component of the pull-back of the tritangent line. We have to show that the obstruction to lifting $\mathscr{O}(l)$ is non-zero. For this, we observe that x, f_3, and f_5 do not generate the monomial y^2z^4. However, G contains this monomial by its very definition. $\square$

4.5. Example. —— Consider the $K3$ surface S over $\mathbb{Q}$, given by $w^2 = f_6(x,y,z)$ for

$$\begin{aligned} f_6(x,y,z) = {} & 4x^6 + 2x^5y + 12x^5z + 2x^4y^2 + 4x^4yz + 12x^4z^2 + 24x^3y^3 - 57x^3y^2z \\ & - 9x^3yz^2 + 6x^3z^3 + 8x^2y^4 - 5x^2y^3z - 72x^2y^2z^2 + 7x^2yz^3 + 4x^2z^4 \\ & + 20xy^4z - 52xy^3z^2 - 57xy^2z^3 + 7xyz^4 + 4y^5z - 7y^4z^2 - 18y^3z^3 \\ & + 7y^2z^4 + 12yz^5 + 2z^6 \, . \end{aligned}$$

Then, $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 3$.

Proof. We have

$$\begin{aligned} f_6 = {} & (2x^3 + 2x^2z + 2y^2z + yz^2 + z^3)^2 \\ & + (2x^2 + 2xz + yz + z^2)(x^3y + 2x^3z + x^2y^2 + x^2yz + 2x^2z^2 + 12xy^3 \\ & - 34xy^2z - 9xyz^2 - 2xz^3 + 4y^4 - 15y^3z - 7y^2z^2 + 9yz^3 + z^4) \end{aligned}$$

and

$$\begin{aligned} f_6 = {} & 4(x^3 + 2x^2y + 2x^2z + xy^2 + xyz + xz^2 + y^2z + yz^2 + z^3)^2 \\ & - (x^2 + xz + yz + z^2)(14x^3y + 4x^3z + 22x^2y^2 + 22x^2yz + 8x^2z^2 - 8xy^3 \\ & + 61xy^2z + 9xyz^2 + 6xz^3 - 4y^4 + 15y^3z + 11y^2z^2 - 6yz^3 + 2z^4) \, . \end{aligned}$$

Hence, there are two conics C_1 and C_2, each of which is six times tangent to the ramification locus of S. The irreducible components of their pull-backs yield the intersection matrix

$$\begin{pmatrix} -2 & 6 & 1 & 3 \\ 6 & -2 & 3 & 1 \\ 1 & 3 & -2 & 6 \\ 3 & 1 & 6 & -2 \end{pmatrix},$$

which is of rank 3. Hence, $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \geq 3$.

On the other hand, S has good reduction at the prime $p = 3$. Point counting over extensions of $\mathbb{F}_3$ shows that the characteristic polynomial of the Frobenius operating on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l(1))$ is

$$\begin{aligned} & \frac{1}{3}(t-1)^4(3t^{18} + 3t^{17} + 2t^{16} + 2t^{15} + 4t^{14} + 5t^{13} + 4t^{12} + 3t^{11} + 6t^{10} + 8t^9 \\ & \qquad + 6t^8 + 3t^7 + 4t^6 + 5t^5 + 4t^4 + 2t^3 + 2t^2 + 3t + 3) \, . \end{aligned}$$

Consequently, we have $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_3}) \leq 4$.

In particular, the assumption $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) > 3$ implies $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = \operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_3})$. Theorem 3.6 guarantees that the specialization map $\operatorname{sp}_{\overline{\mathbb{Q}}}\colon \operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \to \operatorname{Pic}(S_{\overline{\mathbb{F}}_3})$ must be bijective. Giving one invertible sheaf $\mathscr{L} \in \operatorname{Pic}(S_{\overline{\mathbb{F}}_3})$ with a non-trivial obstruction will be enough to yield a contradiction.

For this, observe that the ramification locus of S_3 has a tritangent line given by $x + y + z = 0$. Indeed,

$$\begin{aligned} f_6(x,y,z) \equiv{}& (x^3 + x^2y + xy^2 + y^3)^2 + (x+y+z)(2x^3y^2 + x^3yz + 2x^2yz^2 + 2xy^4 \\ &+ xy^3z + xy^2z^2 + 2xyz^3 + xz^4 + 2y^5 + 2y^4z + yz^4 + 2z^5) \pmod{3}\,. \end{aligned}$$

Modulo the ideal $(3, x + y + z)$, we have $f_3 \equiv x^3 + x^2y + xy^2 + y^3$, $f_5 \equiv -(x^5 + x^3y^2 + x^2y^3 + xy^4 + y^5)$, and $G \equiv x^6 + 2x^5y + x^4y^2 + 2xy^5 + y^6$. Trying to generate G by 3, $x + y + z$, f_3, and f_5 now leads to a system of seven linear equations in six unknowns that is easily seen to be unsolvable. $\square$

4.6. Remarks. —— i) It is not at all hard to generate more examples similar to 1.7 and 4.4. Choosing the coefficients in $\mathbb{F}_p$ at random, one usually finds Picard rank 2 over $\overline{\mathbb{F}}_p$ after a few trials. One may work with small primes, only, say $p \leq 7$.

Clearly, for our arguments, it is of importance to have explicit generators for $\operatorname{Pic}(S_{\overline{\mathbb{F}}_p})$. In practice, it turns out that a second generator may often be found. We have no formal reason for this. However, [Ko] might give an indication.

In example 4.4, we applied a linear transform in order to make the obstruction depend only on a single coefficient. In general, one would have a linear form in the coefficients.

ii) Example 4.5 is a bit more particular. Both conics, which are six times tangent to the ramification sextic, simultaneously lift to $\mathbb{Q}$. This is not at all the generic behaviour.

iii) It seems to be substantially more difficult to construct examples, for which $\operatorname{rk}\operatorname{Pic}(X) \leq \operatorname{rk}\operatorname{Pic}(X_p) - 2$ may be shown. To understand the problem, recall the obstruction homomorphism $\delta\colon \operatorname{Pic}(X_\mathfrak{p}) \to H^2(X, \mathscr{O}_X)$, introduced in Remark 2.3. In Proposition 4.1, we calculated $\delta(\mathscr{O}(l))$ at a precision of one p-adic digit.

In order to verify $\operatorname{rk}\operatorname{Pic}(X) \leq \operatorname{rk}\operatorname{Pic}(X_p) - 2$, one would have to ensure that $\operatorname{rk}_\mathbb{Z}(\operatorname{im}\delta) \geq 2$. This, however, is impossible as long as only p-adic approximations of finitely many values $\delta(\mathscr{L})$ are known.

Observe that there are methods known to show $\operatorname{rk}\operatorname{Pic}(X) \leq \operatorname{rk}\operatorname{Pic}(X_{p_1}) - 2$ and $\operatorname{rk}\operatorname{Pic}(X) \leq \operatorname{rk}\operatorname{Pic}(X_{p_2}) - 2$ when one works with two primes [EJ3].

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