ON THE COMPUTATION OF THE PICARD GROUP FOR CERTAIN SINGULAR QUARTIC SURFACES

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ABSTRACT. We test the methods for computing the Picard group of a $K3$ surface in a situation of high rank. The examples chosen are resolutions of quartics in $\mathbf{P}^3$ having 14 singularities of type A_1. Our computations show that the method of R. van Luijk works well when sufficiently large primes are used.

1. INTRODUCTION

1.1. The methods to compute the Picard rank of a $K3$ surface V are limited up to now. As shown, for example in [vL] or [EJ1], it is possible to construct a $K3$ surface with a prescribed Picard group. But when a $K3$ surface is given, say, by an equation with rational coefficients, then it is not entirely clear whether its geometric Picard rank may be determined using the methods presently known.

1.2. Generally speaking, it is always possible to give upper and lower bounds. For the lower bound, it is necessary to specify divisors explicitly and to verify that their intersection matrix is nondegenerate. This part is definitely problematic. It might happen that a nontrivial divisor is hidden somewhere and very difficult to find.

The general strategy for the computation of upper bounds is to use reduction modulo p. The idea to use characteristic p methods here is due to R. van Luijk [vL]. We will describe this approach in more detail below.

Observe, however, that the Picard rank of a $K3$ surface over $\overline{\mathbb{F}}_p$ is conjectured to be always even. In particular, if $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$ is odd then there is no prime p such that $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p}) = \operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$. Even more, the rank over $\overline{\mathbb{Q}}$ being even or

odd, there is no obvious reason why there should exist a prime number p such that $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$ is at least close to $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$.

1.3. The goal of this article is to test van Luijk's method on a randomly chosen sample of $K3$ surfaces. We will focus on surfaces of Picard rank ≥ 15. The reason for this is a practical one. For surfaces of small Picard rank, one is forced to work with very small primes such as 2 or 3 as, otherwise, the computations run out of time. This would make it impossible to systematically study the behaviour of a single surface at various primes. When the Picard rank is larger, prime numbers in a bigger range may be used.

Concretely, our sample consists of the resolutions of quartic surfaces having only A_1 singularities. We chose 1600 quartic surfaces with 14 singularities. For each of the surfaces, we computed the upper bounds which were found at all the primes $p < 50$. In some cases, we continued the computations using larger primes up to $p = 103$.

We could compute all the geometric Picard ranks, including those that are odd, using the method described below.

The method of van Luijk in detail. The Picard group of a $K3$ surface is isomorphic to $\mathbb{Z}^n$ where n may range from 1 to 20. An upper bound for the Picard rank of a $K3$ surface may be computed as follows. One has the inequality

$$\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}}) \leq \operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$$

which is true for every smooth variety V over $\mathbb{Q}$ and every prime p of good reduction [Fu, Example 20.3.6].

Further, for a $K3$ surface $\mathscr{V}$ over the finite field $\mathbb{F}_p$, one has the first Chern class homomorphism

$$c_1 \colon \operatorname{Pic}(\mathscr{V}_{\overline{\mathbb{F}}_p}) \longrightarrow H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$$

into l-adic cohomology. There is a natural operation of the Frobenius endomorphism Frob on $H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$. All eigenvalues are of absolute value 1. The Frobenius operation on the Picard group is compatible with the operation on cohomology.

Every divisor is defined over a finite extension of the ground field. Consequently, on the subspace $\operatorname{Pic}(\mathscr{V}_{\overline{\mathbb{F}}_p}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_l \hookrightarrow H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$, all eigenvalues are roots of unity. These correspond to eigenvalues of the Frobenius operation on $H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$ which are of the form $p\zeta$ for ζ a root of unity. One may therefore estimate the rank of the Picard group $\operatorname{Pic}(\mathscr{V}_{\overline{\mathbb{F}}_q})$ from above by counting how many eigenvalues are of this particular form.

Doing this for one prime, one obtains an upper bound for $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$ which is always even. The Tate conjecture asserts that this bound is actually sharp. For this reason, one tries to combine information from two primes. The assumption that the surface would have Picard rank $2l$ over $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}}_p$ implied that the

discriminants of both Picard groups, $\mathrm{Pic}(V_{\overline{\mathbb{Q}}})$ and $\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})$, were in the same square class. Note here that reduction modulo p respects the intersection product. When combining information from two primes, it may happen that one finds the rank bound $2l$ twice, but the square classes of the discriminants are different. Then, these data are incompatible with Picard rank $2l$ over $\overline{\mathbb{Q}}$. One gets the rank bound $(2l - 1)$.

Remark 1.4. There are refinements of the method of van Luijk described in [EJ3] and [EJ5]. We will not test these refinements here.

Question 1.5. As mentioned above, the central point is the existence of good primes. For a prime p, being *good* means here that the geometric Picard rank of the reduction modulo p does not exceed the Picard rank over $\overline{\mathbb{Q}}$ by more than one. Do there exist good primes for all $K3$ surfaces over $\mathbb{Q}$?

Example 1.6. Let V be a $K3$ surface of Picard rank 1. We denote by

$$V^n := \underset{i=1}{\overset{n}{\times}} V$$

the n-fold cartesian product. Then, the Picard rank of V^n is equal to n. Assuming the Tate conjecture, one sees that the Picard rank of the reduction at an arbitrary prime is at least $2n$.

This shows that, for $n \geq 2$, there is no good prime for V^n. Not knowing the decomposition of V^n into a direct product, we could not determine its Picard rank.

Analytic discriminants – The Artin-Tate formula. For the final step in the argument above, one needs to know the discriminant of the Picard lattice. One possibility to compute this is to use the Artin-Tate formula.

Conjecture 1.7 (Artin-Tate). *Let V be a $K3$ surface over a finite field $\mathbb{F}_q$. Denote by ρ the rank and by Δ the discriminant of the Picard group of V, defined over $\mathbb{F}_q$. Then,*

$$|\Delta| = \frac{\lim\limits_{T \to q} \frac{\Phi(T)}{(T-q)^\rho}}{q^{21-\rho} \# \mathrm{Br}(V)} .$$

Here, Φ denotes the characteristic polynomial of Frob on $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_{q'}}, \mathbb{Q}_l)$. Finally, $\mathrm{Br}(V)$ is the Brauer group of V.

Remarks 1.8. i) The Artin-Tate conjecture is proven for most $K3$ surfaces. Most notably, the Tate conjecture implies the Artin-Tate conjecture [Mi1, Theorem 6.1]. In these cases, $\# \mathrm{Br}(V)$ is a perfect square.

On its part, the Tate conjecture is proven for $K3$ surfaces under various additional assumptions. For example, it is true for elliptic $K3$ surfaces [ASD].

ii) In such a case, the Artin-Tate formula allows to compute the square class of the discriminant of the Picard group over a finite field. No knowledge of explicit generators is necessary.

2. Singular quartics

Singular quartic surfaces were extensively studied by the classical geometers of the 19th century, particularly by E. Kummer and A. Cayley. For example, the concept of a trope is due to this period [Je].

Definition 2.1. Let $V \subset \mathbf{P}^3$ be any quartic surface. Then, by a *trope* on V, we mean a plane E such that $V \cap E$ is a double conic. This is equivalent to the condition that the equation defining V becomes a perfect square on E.

Remark 2.2. Tropes lead to singular points on the surface $V^\vee \subset (\mathbf{P}^3)^\vee$ dual to V.

Lemma 2.3 (Kummer)**.** *A quartic surface without singular curves may have at most 16 singular points.* □

Fact 2.4. *Let V be a normal quartic surface.*

a) *Then, not more than three singular points on V may be collinear.*

b) *If three singular points on V are collinear then the line connecting them lies on V.*

Proof. b) Otherwise, this line would meet V in each of the three singular points with multiplicity at least two.

a) Suppose $k \geq 4$ singular points are collinear. Then, the line connecting them is contained in V. Choose a plane through this line not meeting any other singularity. The intersection consists of the line and a possibly degenerate cubic curve. They may not have more than three points in common. □

A classical family. A classification of the singular quartic surfaces with at least eight singularities of type A_1 was given by K. Rohn [Ro]. In this article, we will deal with one of the most important classical families.

Lemma 2.5 (Cayley, Rohn)**.** *A family of quartics in $\mathbf{P}^3$ such that the generic member has 14 singularities of type A_1 and no others is given by*

$$\det \begin{pmatrix} 0 & l_1 & l_2 & l_3 \\ l_1 & 0 & l'_3 & l'_2 \\ l_2 & l'_3 & 0 & l'_1 \\ l_3 & l'_2 & l'_1 & 0 \end{pmatrix} = 0 \,.$$

Here, $l_1, l_2, l_3, l'_1, l'_2, l'_3 \in \mathbb{C}[x, y, z, w]$ are linear forms. □

Remark 2.6. According to K. Rohn [Ro], every quartic surface with exactly 14 singular points, each being of type A_1, is contained in this family. Cf. [Je, Ch. I, §12].

Remarks 2.7. i) Evaluating the determinant, we find the explicit equation

$$l_1^2 {l_1'}^2 + l_2^2 {l_2'}^2 + l_3^2 {l_3'}^2 - 2l_1 l_2 l_1' l_2' - 2l_1 l_3 l_1' l_3' - 2l_2 l_3 l_2' l_3' = 0\,.$$

ii) A singular point is given by $l_1 = l_2 = l_3 = 0$. As the equation has the three independent symmetries $l_i \leftrightarrow l_i'$, there are eight singularities of this type.

Two further singular points are given by $l_1 = l_1' = l_2 l_2' - l_3 l_3' = 0$. As the roles of the indices are interchangeable, there are a total of six singularities of this form.

iii) Each of the six planes $l_1 = 0, l_2 = 0, l_3 = 0, l_1' = 0, l_2' = 0, l_3' = 0$ is a trope. Generically, these are the only tropes on such a quartic. Each trope passes through six of the 14 singular points.

3. The desingularization

Lemma 3.1. *Let $\pi\colon \widetilde{V} \to V$ be the desingularization of a normal quartic surface V with only A_1 singularities. Then, $\widetilde{V}$ is a K3 surface.*

Proof. On the smooth part of V, the adjunction formula [GH, Sec. 1.1, Example 3] may be applied as usual. For the canonical sheaf, one has $K_{\mathbf{P}^3} = \mathscr{O}(-4)$. This shows that the invertible sheaf $\Omega^2_{V^{\mathrm{reg}}}$ is trivial. Consequently, $K_{\widetilde{V}}$ is given by a linear combination of the exceptional curves.

However, for an exceptional curve E, we have $E^2 = -2$. Hence, according to the adjunction formula, $K_{\widetilde{V}} E = 0$, which shows that $K_{\widetilde{V}}$ is trivial. The classification of algebraic surfaces [Be] assures that $\widetilde{V}$ is either a K3 surface or an abelian surface.

Further, a standard application of the theorem on formal functions implies that $R^1 \pi_* \mathscr{O}_{\widetilde{V}} = 0$. Hence, $\chi_{\mathrm{alg}}(\widetilde{V}) = \chi_{\mathrm{alg}}(V) = 2$. This shows that $\widetilde{V}$ is in fact a K3 surface. $\square$

Remarks 3.2. i) For the assertion of the lemma, it is actually sufficient to assume that the singularities of V are of types A, D, or E [Li].

ii) In general, the desingularization of a normal quartic surface is a K3 surface, a rational surface, a ruled surface over an elliptic curve, or a ruled surface over a curve of genus three [IN]. The latter possibility is caused by a quadruple point. The existence of a triple point implies that the surface is rational. It is, however, also possible that there is a double point, not of type A, D, or E. Then, $\widetilde{V}$ is rational or a ruled surface over an elliptic curve.

Blowing up one singular point. A generic line intersects a quartic $V \subset \mathbf{P}^3$ in precisely four points. Assume that P is a double point on V which is not

contained in a line lying on V. Then, the generic line through P intersects V in two further points. As the lines through P are parametrized by $\mathbf{P}^2$, this leads to a double cover of $\mathbf{P}^2$ birational to V.

Definition 3.3. We will call this scheme the *degree two model* corresponding to V.

Remarks 3.4. i) It is not hard to make the construction explicit. For this, suppose that $P = (0 : 0 : 0 : 1)$. Then, V is given by an equation of the form $Q(x, y, z)w^2 + K(x, y, z)w + F(x, y, z) = 0$ for a quadratic form Q, a cubic form K, and a quartic form F. Multiplying by Q and substituting W for Qw yields

$$W^2 + K(x, y, z)W + F(x, y, z)Q(x, y, z) = 0\,.$$

The ramification locus is the sextic curve given by $4FQ - K^2 = 0$. Actually, when there are lines through P lying on V, this transformation works, too.

ii) If there is no line on V containing P then the degree two model is simply the blow-up of V in P. Indeed, there is a morphism from the blow-up to the degree two model which is finite and generically one-to-one. As the degree two model is a normal scheme, Zariski's main theorem applies.

In general, the degree two model is the blow-up of V in P with the lines containing P blown down.

iii) Observe that the conic "$Q = 0$" is tangent to the ramification sextic. Hence, this conic splits in the double cover. Actually, the exceptional divisor produced by blowing up the singular point is one component of the preimage.

Remark 3.5. When we apply this construction to the particular singular quartics described above, the ramification sextic must have exactly 13 singular points. According to Plücker, such a highly singular degree-six curve is necessarily reducible. It is the union of three lines and a singular cubic or the union of two lines and two conics.

4. Point counting

In order to determine the eigenvalues of the Frobenius on $H^2_{\text{ét}}(\widetilde{V}_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l)$, the usual method is to count the points on V defined over $\mathbb{F}_q$ and extensions and to apply the Lefschetz trace formula [Mi2, Ch. VI, Theorem 12.3].

Fact 4.1 (Elliptic fibration). *Let $V \subset \mathbf{P}^3$ be an irreducible quartic surface having at least two singular points. Then, $\widetilde{V}$ is a rational surface or has an elliptic fibration.*

Proof. Intersect V with the pencil of hyperplanes through the two singular points. This yields a fibration of a surface birationally equivalent to V. The assumption implies that the generic fiber is an irreducible curve. Depending on whether or not the line connecting the two singularities lies on V, it is

either a cubic curve or a quartic curve with at least two singular points. In both cases, Plücker's formulas show that its genus is at most one. The existence of a fibration into curves of genus zero implies that $\widetilde{V}$ is rational. □

To count the points on a singular quartic surface V over $\mathbb{F}_q$, we have at least the following possibilities.

Algorithms 4.2 (Point counting). i) Count points directly. This means, intersect V with a 2-dimensional family of lines. For each line, determine the number of points on it. This last step amounts to solving an equation of degree four in $\mathbb{F}_q$.

ii) Use the elliptic fibration. Enumerate all fibers, defined over $\mathbb{F}_q$. On each fiber, count the number of points.

iii) Compute a degree two model of the surface and count the points there. This means, one has to evaluate a sextic form on $\mathbf{P}^2$ and run an is-square routine at each step.

Remarks 4.3. a) If the surface is defined over $\mathbb{F}_p$ then it suffices to count the points on a fundamental domain of the absolute Frobenius. This leads to a significant speed-up for all three methods.

b) In our examples, it turned out that the degree two model approach was the fastest one. We used it except for those surfaces where there were lines through each singularity defined over $\mathbb{F}_p$.

5. Lower bounds for the Picard rank

Lemma 5.1. *Let* $\pi\colon \widetilde{V} \to V$ *be the desingularization of a proper surface* V *having only* A_1-*singularities.*

a) *Then, the exceptional curves define a non-degenerate orthogonal system in* $\mathrm{Pic}(\widetilde{V})$.

b) *In particular, the Picard rank of* $\widetilde{V}$ *is strictly bigger than the number of singularities of* V.

Proof. a) The exceptional curves have self-intersection number (-2) each and do not meet each other.

b) For H the hyperplane section, $\pi^*\mathscr{O}_V(H)$ is orthogonal to the exceptional curves. □

Remark 5.2. A strategy to calculate the square class of the discriminant of $\mathrm{Pic}(\widetilde{V})$ is thus as follows. Consider in $\mathrm{Pic}(\widetilde{V})$ the orthogonal complement $P := \langle E_1, \ldots, E_n \rangle^\perp$. Then, $\mathrm{disc}\,\mathrm{Pic}(\widetilde{V}) \in 2^n(\mathrm{disc}\,P)\mathbb{Q}^{*2}$.

5.3. The only method known to prove a non-trivial lower bound for the Picard rank is to write down divisors explicitly. We always have the hyperplane section. For special quartics from the Cayley-Rohn family, we observed two types of additional divisors.

i) **Lines.** One could search for lines on the surfaces by a Gröbner base calculation. However, in our particular situation, every line connects at least two singular points. We will show this in Proposition 5.7 below.

ii) **Conics.** There is a special case in that there exists a plane containing exactly four singularities, no three of which are collinear. Then, the quartic curve on this plane splits into two conics. The same may happen when a plane through three singularities is tangent to the surface at another point.

5.4. In both these situations, one may directly calculate the corresponding intersection matrices.

i) Let $k = 2, 3$ be the number of singularities connected by the line l. Choose a plane through l such that the intersection curve splits into l and a smooth cubic curve. Then, on $\widetilde{V}$, we have two divisors L and C such that $L^2 = -2$, $C^2 = 0$, and $CE = 3 - k$. For $E_1, \ldots, E_k$ the exceptional divisors met by L, $L' := L + \frac{1}{2}E_1 + \ldots + \frac{1}{2}E_k$ and $C' := C + \frac{1}{2}E_1 + \ldots + \frac{1}{2}E_k$ are in the orthogonal complement of the exceptional divisors (after tensoring by $\mathbb{Q}$). Indeed, this is an immediate consequence of Lemma 5.6, shown below. We find the intersection matrix $\begin{pmatrix} -2+k/2 & 3-k/2 \\ 3-k/2 & k/2 \end{pmatrix}$ of determinant $2k - 9$.

ii) Here, there are two conics meeting in four points, k of which are singular on V. This yields two divisors Q_1 and Q_2 on $\widetilde{V}$ such that $Q_1^2 = Q_2^2 = -2$ and $Q_1 Q_2 = 4 - k$. In a manner analogous to i), we end up with the intersection matrix $\begin{pmatrix} -2+k/2 & 4-k/2 \\ 4-k/2 & -2+k/2 \end{pmatrix}$ of determinant $2k - 12$.

Remark 5.5. Observe the following rules of thumb that apply as long as there are no multiplicities >1. If D meets exactly k singular points then $D'^2 = D^2 + \frac{k}{2}$. If $D_1 \neq D_2$ are irreducible curves having k singular and k' smooth points in common then $D_1 D_2 = k' + \frac{k}{2}$.

Lemma 5.6. *Let C be a curve on a surface V having an A_1-singularity in P. Suppose $P \in C$ and that P is smooth on C. Then, on the desingularization $\pi \colon \widetilde{V} \to V$, the strict transform of C meets the exceptional curve $\pi^{-1}(P)$ of order one.*

Proof. Indeed, the model case for this situation is given by a conical quadric V in $\mathbf{P}^3$ and a line l on V. Then, the desingularization is a Hirzebruch surface Σ_2 which is a ruled surface over $\mathbf{P}^1$ with exactly one (-2)-curve B. The strict transform of l is a line F from the ruling. It is well known that $BF = 1$. □

Proposition 5.7 (Lines on special quartics)**.** *Let V be a quartic surface with 14 singular points. Then, every line on V contains two or three singular points.*

Proof. Let l be a line on V. By Fact 2.4, l cannot contain more than three singularities. Suppose first that l is contained in one of the tropes. Then, this is a degenerate trope, the conic splitting into two lines. As a trope contains six singular points, there must be three on each line.

Otherwise, l meets each trope in a single point. We claim that these six points of intersection are all singular. Then, the assertion follows, since a point cannot be contained in more than three tropes.

To show the claim, assume that l would meet a trope in a smooth point p. As l is supposed to be contained in V, it is everywhere tangential to V. But for a point on a trope, the tangent plane is the trope itself. Hence, l would be contained within the trope. This is a contradiction. $\square$

6. Computations and numerical data

6.1. Consider the Cayley-Rohn family of determinantal quartics as described in Lemma 2.5. Then, over a Zariski open subset of the base, one may normalize to $l_1 = x$, $l_2 = y$, $l_3 = z$, and $l_1' = w$. We will write $l_2' = c_1 x + c_2 y + c_3 z + c_4 w$ and $l_3' = c_5 x + c_6 y + c_7 z + c_8 w$ for a coefficient vector $[c_1, \ldots, c_8]$. Over a possibly smaller Zariski open subset, one has $c_1, c_2, c_3 \neq 0$ in which case these coefficients may be normalized to 1.

The computations carried out. We chose a sample of 1600 singular quartics from the Cayley-Rohn family. We worked with the normal form as described in 6.1. The coefficient vectors were produced by a random number generator. The coefficients themselves were integers in the range $-20, \ldots, 20$. We always put $c_1 = c_2 = c_3 = 1$.

For each surface V in the sample and each prime $p < 50$ of good reduction, we counted the number of points in $V(\mathbb{F}_p)$, $V(\mathbb{F}_{p^2})$, and $V(\mathbb{F}_{p^3})$. From these data, we tried to compute the characteristic polynomial of the Frobenius. For the determination of the sign in the functional equation, we followed the strategy described in [EJ4]. We used the explicitly known 15-dimensional sublattice of the Picard group generated by the hyperplane section and the exceptional curves in order to adapt the conditions to our situation. In 430 cases, we computed, in addition, $\#V(\mathbb{F}_{p^4})$. Here, p was up to 23. From the characteristic polynomial, we read off the rank of $\mathrm{Pic}(V_{\overline{\mathbb{F}_p}})$ and, using the Artin-Tate formula, computed its discriminant.

Remark 6.2. In the cases that remained with an unknown sign, we worked with the pair of possible characteristic polynomials. This means, we took the maximum of the predicted ranks as an upper bound. In the case that both upper bounds were equal to 16, we got a pair of possible square classes for the discriminants. Combining information from different primes then amounted to forming the intersection of these sets. To give a typical example, for $p = 31$, these sign

problems occurred in 139 of 1299 cases of good reduction. Other primes led to similar rates.

Remark 6.3. According to Fact 4.1, every surface in the sample is elliptic. This is enough to show that the Artin-Tate formula 1.7 for the discriminant is applicable.

The average value for a prime. The probability to obtain a good rank bound increases when the prime numbers increase. Let us visualize this by a diagram.
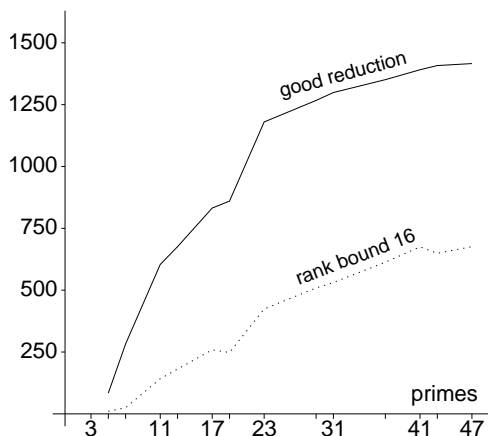

FIGURE 1. Number of surfaces with good reduction and rank bound 16 for $p < 50$

The discriminants. We computed the discriminant in all cases of Picard rank 16. In 4690 cases, we obtained a rank bound of 16 and the determination of the sign in the functional equation was possible. These data led to 59 distinct square classes for the discriminant. The most frequent square class was (-1) with 819 repetitions. The next one was (-2) having 608 repetitions. On the other hand, each of the discriminants (-47), (-59), (-67), (-71), (-82), (-101), (-118), (-141), (-149), and (-177) occurred only once.

The ranks over $\mathbb{Q}$. On each surface in the sample, we searched for additional divisors. It turned out that 1504 of the surfaces contained no line and no plane through four singular points. For these, we tried to prove that the Picard rank is 15. For the others, we tried to prove Picard rank 16. The statistics over the primes used is given by the table below.

prime	#cases finished	#cases left
11	2	1502
13	15	1487
17	36	1451
19	57	1394
23	151	1243
29	181	1062
31	219	843
37	214	629
41	173	456
43	136	320
47	118	202
53	80	122
59	44	78
61	36	42
67	20	22
71	12	10
73	6	4
79	2	2
103	1	1

Rank 15 expected

prime	#cases finished	#cases left
5	1	95
7	3	92
11	3	89
13	5	84
17	2	82
19	4	78
23	11	67
29	7	60
31	6	54
37	8	46
41	12	34
43	7	27
47	6	21
53	4	17
59	1	16
61	3	13
67	3	10
73	1	9
79	1	8
83	3	5
97	1	4

Rank 16 expected

TABLE 1. Progress of the upper bounds

Observe that there were a few cases where the data for $p < 50$ were not sufficient. For these, we continued the point count, in an extreme case up to $p = 103$.

Testing isomorphy. As a byproduct of the computations, we proved that the surfaces in our sample are pairwise non-isomorphic. For this, it was sufficient to show that, for each pair of surfaces, there existed a prime where both have good reduction, but the geometric Picard groups differ in rank or discriminant. In order to do this, we had to continue the point count in a few cases. In fact, the data for $p \leq 61$ contained enough information.

The five examples left.

Example 6.4. Let S_1 be the surface given by the coefficient vector $[1, 1, 1, -7, 16, 6, -9, 12]$. Here, there is a plane through three singularities which is tangent to S_1 at a fourth point. The intersection curve splits into two conics. The Picard rank is thus at least 16. On the other hand, we found the rank bound 16 for $p = 61, 71, 83$, and 101.

Examples 6.5. Let S_2, S_3, and S_4 be the surfaces given by the vectors $[1, 1, 1, -1, -16, 7, 10, -10]$, $[1, 1, 1, 3, -16, 2, 4, 15]$, and $[1, 1, 1, -1, 13, -11, 1, 15]$, respectively. For each surface, we found rank 18 at several primes with various discriminants. Hence, in each case, there was an upper bound of 17 for the Picard rank.

i) On S_2, we found a plane E through four singular points $P_1, \ldots, P_4$. On E, the quartic splits into two conics Q_1, Q_2. Further, there are two lines L_1, L_2 through P_1 and P_2 on V which meet in a smooth point. Actually, L_1 and L_2 form a degenerate trope. Arguing as in 5.4, we find the intersection matrix

$$\begin{pmatrix} 0 & 2 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

of rank three. This confirms Picard rank 17.

ii) On S_3, the situation is analogous to that on S_2. The only difference is that the plane E meets three singular points and is tangent to V at a fourth point. L_1 and L_2 meet E in singular points. We find the intersection matrix

$$\begin{pmatrix} -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

of rank three. Again, this confirms Picard rank 17.

iii) Here, we have a plane E through five singular points. On E, the quartic splits into a conic Q and two lines L_1, L_2. There are two further lines L_3, L_4 through three singularities. L_1 and L_3 meet in a smooth point. Together, they form a degenerate trope. The same is true for L_2 and L_4. Finally, L_3 and L_4 have a singular point in common. We find the intersection matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

of rank three. Again, this confirms Picard rank 17.

Example 6.6. Let S_5 be the surface given by the coefficient vector $[1, 1, 1, -1, -13, 0, 11, -11]$. We got a rank bound of 18 for $p = 23$, 31, 61, 79, 89, 97, and 101.

On the other hand, we found quite a number of particular divisors on this surface. There are six planes through exactly four singular points. On each of these planes, the quartic splits into two conics. The combinatorial structure is rather interesting. In a somewhat arbitrary numbering, the table below describes which plane meets which singular points.

E_1	P_1	P_2	P_9	P_{11}
E_2	P_3	P_4	P_{13}	P_{14}
E_3	P_3	P_7	P_{12}	P_{14}
E_4	P_4	P_8	P_{10}	P_{13}
E_5	P_5	P_6	P_9	P_{11}
E_6	P_7	P_8	P_{10}	P_{12}

TABLE 2. Planes through four singular points

Further, we have the two lines, L_1 through P_9, P_{10} and P_{14}, and L_2 through P_{11}, P_{12} and P_{13}. The lines L_1 and L_2 have a smooth point in common. They form a degenerate trope. The intersection matrix of L_1, the two conics in E_2, and one of the conics in E_3 alone is

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 2 & 1 \\ \frac{1}{2} & 2 & 0 & 1 \\ \frac{1}{2} & 1 & 1 & 0 \end{pmatrix}$$

of rank four. This proves that the Picard rank is 18.

Summary 6.7. We considered the resolutions of 1600 randomly chosen Cayley-Rohn quartics with exactly 14 singularities of type A_1. The corresponding $K3$ surfaces were mutually non-isomorphic. It turned out that all the Picard ranks could be determined. However, at several examples rather large primes up to $p = 103$ had to be considered. We found Picard rank fifteen 1503 times and Picard rank sixteen 93 times. Further, there were three surfaces of Picard rank seventeen and one surface of Picard rank eighteen in the sample.

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