SIMPLE-MINDED SUBCATEGORIES AND T-STRUCTURES

OLAF M. SCHNÜRER

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1. Introduction

Let $\mathcal{T}$ be a triangulated category. The aim of this short note is to establish a bijection between the set of bounded t-structures on $\mathcal{T}$ whose heart is a finite-length category and the set of full simple-minded subcategories of $\mathcal{T}$ (see Definition 4.1). A t-structure as above corresponds to the full subcategory of $\mathcal{T}$ whose objects are the simple objects of its heart.

The results presented here are essentially contained in [AN09, KN13, KY14, RR17, SY19] with special emphasis on [AN09]. We do not give precise references to these sources later on. This note was originally written around 2013 when we were not aware of [AN09]. Our motivation was to generalize some results of [Sch11]. This note may also be helpful for understanding [EQ16, Theorem 2.17]. The main results are Theorems 4.4 and 4.6.

2. Notation

Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{U}$ be a full subcategory of $\mathcal{T}$. We denote by $\operatorname{tria}(\mathcal{U})$ (resp. $\operatorname{thick}(\mathcal{U})$) the smallest strict full triangulated (resp. thick) subcategory of $\mathcal{T}$ that contains $\mathcal{U}$.

We denote by $\operatorname{susp}(\mathcal{U})$ (resp. $\operatorname{susp}^-(\mathcal{U})$) the smallest strict full subcategory containing $\mathcal{U} \cup \{0\}$ which is closed under extensions and positive (resp. negative) shifts (it is in particular closed under finite direct sums). Its closure under direct summands will be denoted $\operatorname{susp}^{\oplus}(\mathcal{U})$ (resp. $\operatorname{susp}^{-,\oplus}(\mathcal{U})$).

3. Filtrations and cofiltrations

Let $\mathcal{T}$ be a triangulated category.

Definition 3.1. Let M be an object of $\mathcal{T}$, and let $S_1, S_2, \dots, S_m$ be objects of $\mathcal{T}$. An $(S_1, \dots, S_m)$**-filtration** of M is a finite sequence

$$0 = M_0 \xrightarrow{0=f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{m-2}} M_{m-1} \xrightarrow{f_{m-1}} M_m = M$$

in $\mathcal{T}$ such that each morphism f_{i-1} fits into a triangle

$$M_{i-1} \xrightarrow{f_{i-1}} M_i \to S_i \to [1]M_{i-1},$$

for $1 \leqslant i \leqslant m$; in other words a/any cone of f_{i-1} is isomorphic to S_i. (Note that $M_1 \cong S_1$.)

An $(S_1, \dots, S_m)$**-cofiltration** of M is a finite sequence

$$M = M^0 \xrightarrow{g^0} M^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{m-2}} M^{m-1} \xrightarrow{g^{m-1}} M^m = 0$$

in $\mathcal{T}$ such that each morphism g^{i-1} fits into a triangle

$$S_i \to M^{i-1} \xrightarrow{g^{i-1}} M^i \to [1]S_i,$$

for $1 \leqslant i \leqslant m$. (Note that $S_m \cong M^{m-1}$.) (If $m = 0$ then $M = 0$ in both cases.)

We sometimes refer to the S_i appearing in an $(S_1, \dots, S_m)$-(co)filtration of M as the **subquotients** of the (co)filtration.

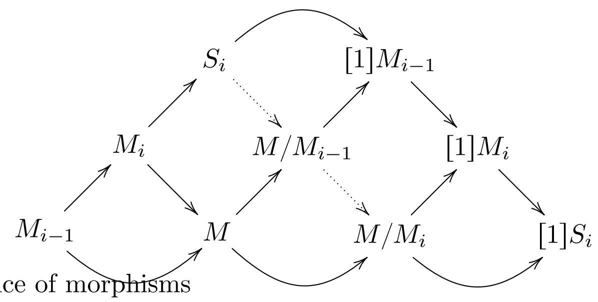
Lemma 3.2. *Let M be an object of $\mathcal{T}$, and let $S_1, S_2, \ldots, S_m$ be objects of $\mathcal{T}$. Then M has an $(S_1, \ldots, S_m)$-filtration if and only if it has an $(S_1, \ldots, S_m)$-cofiltration.*

Moreover, the cone of $M_j \xrightarrow{f_{m-1}\cdots f_j} M$ has an $(S_{j+1}, \ldots, S_m)$-filtration, for any $0 \leqslant j \leqslant m$.

Proof. We start with an $(S_1, \ldots, S_m)$-filtration of M as above. For any $0 \leqslant i \leqslant m$ let

$$M_i \xrightarrow{f_{m-1}\cdots f_i} M \to M/M_i \to [1]M_i$$

be a triangle (here M/M_i is just suggestive notation). If $i > 0$, the octahedral axiom applied to $M_{i-1} \to M_i \to M$ provides an octahedral diagram



Hence we obtain a sequence of morphisms

$$M \xrightarrow{\sim} M/M_0 \to M/M_1 \to M/M_2 \to \cdots \to M/M_{m-1} \to M/M_m = 0$$

and triangles

$$S_i \to M/M_{i-1} \to M/M_i \to [1]S_i.$$

This means that M has a $(S_1, \ldots, S_m)$-cofiltration.

Similarly, one passes from a cofiltration to a filtration.

This argument also shows that the cone M/M_j of $M_j \xrightarrow{f_{m-1}\cdots f_j} M$ has an $(S_{j+1}, \ldots, S_m)$-cofiltration and hence an $(S_{j+1}, \ldots, S_m)$-filtration. □

We now explain when it is possible to swap two consecutive subquotients.

Lemma 3.3 (Swapping consecutive subquotients)**.** *Let M be an object of $\mathcal{T}$, and let $S_1, S_2, \ldots, S_m$ be objects of $\mathcal{T}$. Let*

$$0 = M_0 \xrightarrow{0=f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{m-2}} M_{m-1} \xrightarrow{f_{m-1}} M_m = M$$

be an $(S_1, \ldots, S_m)$-filtration of M, with $m \geqslant 2$, and let $1 \leqslant i < m$. Assume that one of the following two conditions is satisfied (the first condition clearly implies the second one)

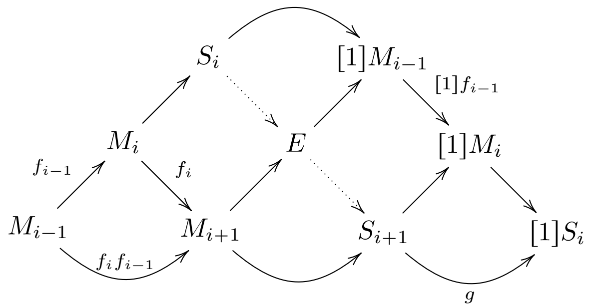
- $\operatorname{Hom}(S_{i+1}, [1]S_i) = 0$
- *If $M_{i-1} \xrightarrow{f_{i-1}} M_i \to S_i \to [1]M_{i-1}$ and $M_i \xrightarrow{f_i} M_{i+1} \to S_{i+1} \to [1]M_i$ is a/any choice of triangles (cf. Definition 3.1), then the composition $S_{i+1} \to [1]M_i \to [1]S_i$ vanishes.*

Then M has an $(S_1, \ldots, S_{i-1}, S_{i+1}, S_i, S_{i+2}, \ldots, S_m)$-filtration.

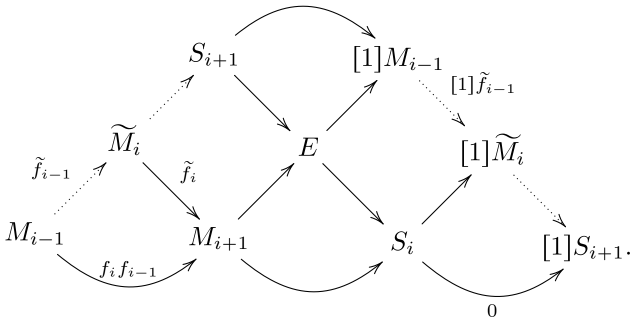
Proof. By assumption each morphism f_{j-1} fits into a triangle

$$M_{j-1} \xrightarrow{f_{j-1}} M_j \to S_j \to [1]M_{j-1},$$

for $1 \leqslant j \leqslant m$. Consider the octahedron



By assumption we have $g = 0$. Hence we can assume that $E = S_i \oplus S_{i+1}$ and that the two dotted arrows in the above octahedron are the obvious inclusion and projection morphisms. The octahedral axiom applied to the composition $M_{i+1} \to E \to S_i$, where the second map is the projection, yields an octahedron



Hence by replacing the part $M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1}$ of the filtration of M by $M_{i-1} \xrightarrow{\tilde{f}_{i-1}} \widetilde{M}_i \xrightarrow{\tilde{f}_i} M_{i+1}$ we obtain the filtration we want. ☐

Definition 3.4. Let $\mathcal{U}$ be a full subcategory of $\mathcal{T}$, and let $(d_1, \dots, d_m)$ be a sequence of integers. A **$(d_1, \dots, d_m)$-$\mathcal{U}$-(co)filtration** of an object M is a $([d_1]U_1, \dots, [d_m]U_m)$-(co)filtration where $U_1, \dots, U_m$ are objects of $\mathcal{U}$.

Lemma 3.5. *An object of $\mathcal{T}$ is in $\operatorname{tria}(\mathcal{U})$ if and only if it has a $(d_1, \dots, d_m)$-$\mathcal{U}$-(co)filtration, for some sequence $(d_1, \dots, d_m)$ of integers.*

Proof. All objects having such a (co)filtration certainly are in $\operatorname{tria}(\mathcal{U})$. Moreover, the subcategory of these objects is a strict full triangulated subcategory (use [BBD82, Lemma 1.3.10]). Hence it is equal to $\operatorname{tria}(\mathcal{U})$. ☐

4. Simple-minded subcategories and t-structures

Let $\mathcal{T}$ be a triangulated category.

Definition 4.1. A **simple-minded subcategory** of $\mathcal{T}$ is a strict full subcategory $\mathcal{S}$ of $\mathcal{T}$ that satisfies the following three conditions.

(S1) No object of $\mathcal{S}$ is (isomorphic to) zero.
(S2) Any non-zero morphism between two objects of $\mathcal{S}$ is an isomorphism.
(S3) $\operatorname{Hom}(S, [i]S') = 0$ for all S, $S' \in \mathcal{S}$ and all $i < 0$.

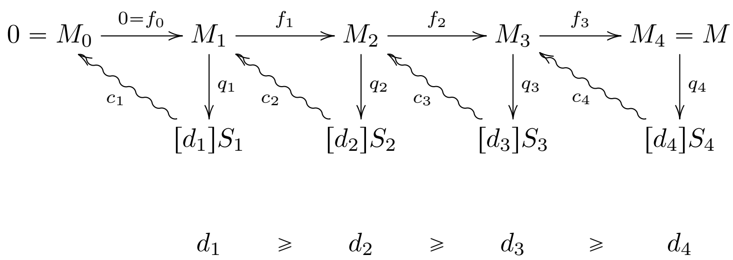
We say that a simple-minded subcategory $\mathcal{S}$ of $\mathcal{T}$ is **full** if $\operatorname{tria}(\mathcal{S}) = \mathcal{T}$.

Let $\mathcal{S}$ be a simple-minded subcategory of $\mathcal{T}$. Note that conditions (S1) and (S2) imply that the endomorphism ring of each object of $\mathcal{S}$ is a division ring, so in particular each object of $\mathcal{S}$ is indecomposable.

Our aim is to construct a t-structure on $\operatorname{tria}(\mathcal{S})$ (which will turn out to coincide with $\operatorname{thick}(\mathcal{S})$). We first show that any object of $\operatorname{tria}(\mathcal{S})$ has a nice filtration.

Lemma 4.2. *Given an object M of $\operatorname{tria}(\mathcal{S})$ there is a decreasing sequence $d_1 \geqslant d_2 \geqslant \dots \geqslant d_m$ of integers such that M has a $(d_1, \dots, d_m)$-$\mathcal{S}$-filtration.*

Here is an illustration in case $m = 4$:



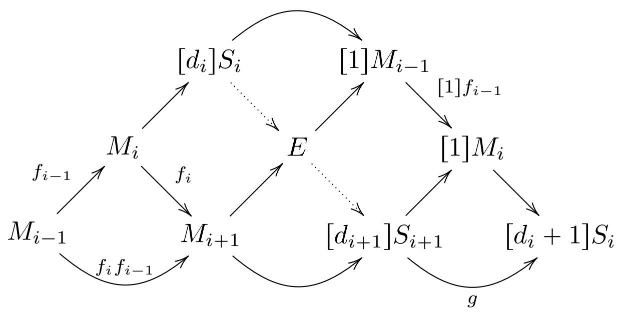
Proof. We already know by Lemma 3.5 that M has a $(d_1, \dots, d_m)$-$\mathcal{S}$-filtration

$$0 = M_0 \xrightarrow{0=f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{m-2}} M_{m-1} \xrightarrow{f_{m-1}} M_m = M, \tag{4.1}$$

so there are objects $S_1, \dots, S_m \in \mathcal{S}$ such that

$$M_{i-1} \xrightarrow{f_{i-1}} M_i \to [d_i]S_i \to [1]M_{i-1} \tag{4.2}$$

are triangles. Assume now that there exists an integer i with $1 \leqslant i < m$ such that $d_i < d_{i+1}$. Consider the octahedron



If $g = 0$ we can swap our filtration of M to a $([d_1]S_1, \dots, [d_{i+1}]S_{i+1}, [d_i]S_i, \dots, [d_m]S_m)$-filtration by Lemma 3.3.

Assume now that g is non-zero. Then condition (S3) together with $d_i < d_{i+1}$ show that $d_{i+1} = d_i + 1$, and condition (S2) implies that g is an isomorphism. Hence E is zero and $f_i f_{i-1} : M_{i-1} \to M_{i+1}$ is an isomorphism. As a consequence, if $i + 2 \leqslant m$, we can replace the filtration (4.1) by the shorter filtration

$$0 = M_0 \xrightarrow{0=f_0} \cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i+1}f_i f_{i-1}} M_{i+2} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_{m-1}} M_m = M.$$

Otherwise we have $i + 1 = m$. Then in case $m = 2$ (and $i = 1$) the claim is trivially true since $0 = M_0 \xrightarrow{\sim} M_2 = M$, and otherwise we can replace the filtration (4.1) by the shorter filtration

$$0 = M_0 \xrightarrow{0=f_0} \cdots \xrightarrow{f_{i-3}} M_{i-2} \xrightarrow{f_i f_{i-1} f_{i-2}} M_{i+1} = M.$$

So, under the assumption $g \neq 0$, we have deleted in all cases the two subquotients $[d_i]S_i$ and $[d_{i+1}]S_{i+1}$ of our original filtration.

Now it is clear how we can rearrange resp. modify the filtration of M in finitely many steps till it has the form we want. □

Lemma 4.3. *Let (4.1) be a $(d_1, \dots, d_m)$-$\mathcal{S}$-filtration of an object $M \in \mathcal{T}$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$. We fix triangles (4.2) with $S_i \in \mathcal{S}$ for all i and assume that $m \geqslant 1$. Then both morphisms $M_1 \xrightarrow{f_{m-1} \dots f_2 f_1} M$ and $M \to [d_m]S_m$ are non-zero.*

Proof. We fit $f_{m-1} \dots f_2 f_1 : M_1 \to M$ into a triangle

$$M_1 \to M \to M/M_1 \to [1]M_1$$

and assume that $M_1 \to M$ vanishes. Then $M/M_1 \cong M \oplus [1]M_1$, in particular $[1]M_1 \cong [d_1 + 1]S_1$ is a direct summand of M/M_1. By Lemma 3.2 we know that M/M_1 has a $([d_2]S_2, \dots, [d_m]S_m)$-filtration. By assumption (S3) we have for all $2 \leqslant i \leqslant m$ (and also for $i = 1$)

$$\operatorname{Hom}([d_1 + 1]S_1, [d_i]S_i) = \operatorname{Hom}(S_1, [\underbrace{d_i - d_1 - 1}_{<0}]S_i) = 0.$$

This implies that $\operatorname{Hom}([d_1 + 1]S_1, M/M_1)$ vanishes. Since $[d_1 + 1]S_1$ is a direct summand of M/M_1, this implies that S_1 vanishes; this contradicts condition (S1).

Assume now that the map α in the triangle

$$M_{m-1} \xrightarrow{f_{m-1}} M \xrightarrow{\alpha} [d_m]S_m \to [1]M_{m-1}$$

vanishes. Then M_{m-1} is isomorphic to $[d_m - 1]S_m \oplus M$. Obviously M_{m-1} has a $([d_1]S_1, \dots, [d_{m-1}]S_{m-1})$-filtration. For all $1 \leqslant i \leqslant m - 1$ (and also for $i = m$) we have

$$\operatorname{Hom}([d_i]S_i, [d_m - 1]S_m) = \operatorname{Hom}(S_i, [\underbrace{d_m - d_i - 1}_{<0}]S_m) = 0$$

by condition (S3). Hence $\mathrm{Hom}(M_{m-1},[d_m-1]S_m)$ vanishes which yields a contradiction as above. □

Let $n \in \mathbb{Z}$. We denote by $\mathcal{D}^{\leqslant n}$ (resp. $\mathcal{D}^{\geqslant n}$) the full subcategory of $\mathcal{T}$ consisting of objects that have a $(d_1,\dots,d_m)$-$\mathcal{S}$-filtration for some $m \in \mathbb{N}$ and some sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$ of integers such that $-d_m \leqslant n$ (resp. $-d_1 \geqslant n$). By convention, the last condition is empty if $m = 0$. Note that the zero object is both in $\mathcal{D}^{\leqslant n}$ and in $\mathcal{D}^{\geqslant n}$.

Note that $\mathcal{D}^{\leqslant n} = [-n]\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant n} = [-n]\mathcal{D}^{\geqslant 0}$.

Theorem 4.4. *Let $\mathcal{S}$ be a simple-minded subcategory of a triangulated category $\mathcal{T}$.*

(a) We have $\mathrm{tria}(\mathcal{S}) = \mathrm{thick}(\mathcal{S})$, *and*

$$t_{\mathcal{S}} := (\mathrm{susp}(\mathcal{S}), \mathrm{susp}^{-}(\mathcal{S}))$$

defines a bounded (and hence non-degenerate) t-structure $t_{\mathcal{S}}$ on this category. We have $\mathrm{susp}(\mathcal{S}) = \mathrm{susp}(\mathcal{S})^{\oplus} = \mathcal{D}^{\leqslant 0}$ *and* $\mathrm{susp}^{-}(\mathcal{S}) = \mathrm{susp}^{-,\oplus}(\mathcal{S}) = \mathcal{D}^{\geqslant 0}$.

(b) The heart $\heartsuit(t_{\mathcal{S}})$ of this t-structure $t_{\mathcal{S}}$ is a finite-length category and its simple objects are precisely the objects of $\mathcal{S}$.

Proof. Part (a): We claim that $t := (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ defines a t-structure on $\mathrm{tria}(\mathcal{S})$.

The condition $\mathrm{Hom}(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1}) = 0$ follows from assumption (S3). The conditions $\mathcal{D}^{\leqslant 0} \subset \mathcal{D}^{\leqslant 1}$ and $\mathcal{D}^{\geqslant 1} \subset \mathcal{D}^{\geqslant 0}$ are obvious.

Finally, let $M \in \mathrm{tria}(\mathcal{S})$ be given. We want to construct a "truncation triangle" for M and can certainly assume that $M \neq 0$. Lemma 4.2 provides a decreasing sequence of integers $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$ (where $m \geqslant 1$) such that M has a $(d_1,\dots,d_m)$-$\mathcal{S}$-filtration (4.1). Let $i \in \{0,\dots,m\}$ be maximal such that $-d_j \leqslant 0$ for all $1 \leqslant j \leqslant i$. If we fit the morphism $M_i \xrightarrow{f_{m-1}\cdots f_i} M$ into a triangle

$$M_i \to M \to B \to [1]M_i,$$

then $M_i \in \mathcal{D}^{\leqslant 0}$ and $B \in \mathcal{D}^{\geqslant 1}$ by Lemma 3.2. This shows that t is a t-structure on $\mathrm{tria}(\mathcal{S})$.

This t-structure is obviously bounded: For M as above we have $M \in \mathcal{D}^{\geqslant -d_1} \cap \mathcal{D}^{\leqslant -d_m}$. This also implies that t is a non-degenerate t-structure.

The main theorem of [LC07] now shows that $\mathrm{tria}(\mathcal{S})$ is Karoubian and hence $\mathrm{tria}(\mathcal{S}) = \mathrm{thick}(\mathcal{S})$.

Note that $\mathcal{S} \subset \mathcal{D}^{\leqslant 0} \subset \mathrm{susp}(\mathcal{S}) \subset \mathrm{susp}^{\oplus}(\mathcal{S})$. Since any left aisle of a t-structure is closed under extensions, positive shifts and direct summands, the last two inclusions are equalities. Similarly we obtain $\mathcal{D}^{\geqslant 0} = \mathrm{susp}^{-}(\mathcal{S}) = \mathrm{susp}^{-,\oplus}(\mathcal{S})$. So $t = t_{\mathcal{S}}$. This proves part (a).

Part (b): We claim that any object of the heart $\heartsuit(t)$ of our t-structure $t = t_{\mathcal{S}}$ on $\mathrm{thick}(\mathcal{S}) = \mathrm{tria}(\mathcal{S})$ is a finite extension of objects of $\mathcal{S}$.

Let M be in $\heartsuit(t)$ with $M \neq 0$. Lemmata 4.2 and 4.3 yield a $(d_1,\dots,d_m)$-$\mathcal{S}$-filtration of M with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$ (and subquotients $S_1,\dots,S_m \in \mathcal{S}$) and non-zero morphisms $[d_1]S_1 \to M$ and $M \to [d_m]S_m$ for objects S_1 and S_m in $\mathcal{S}$. This implies that $d_1 \leqslant 0$ (since otherwise $[d_1]S_1 \in \mathcal{D}^{\leqslant -1}$ has no non-zero morphism to $M \in \mathcal{D}^{\geqslant 0}$) and $d_m \geqslant 0$ (since otherwise $[d_m]S_m \in \mathcal{D}^{\geqslant 1}$ receives no non-zero morphism from $M \in \mathcal{D}^{\leqslant 0}$). Hence $0 = d_1 = d_2 = \cdots = d_m$. This shows our claim.

We show that the simple objects are precisely the objects of $\mathcal{S}$.

Let $0 \neq M \in \heartsuit(t)$ be filtered as above. Recall that short exact sequences in the heart correspond bijectively to triangles with terms in the heart. Then we have a short exact sequence $M_{m-1} \to M \to S_m$ in $\heartsuit(t)$. Recall that S_m is nonzero by condition (S1). Hence if M is simple then $M \to S_m$ must be an isomorphism (and $m = 1$). Hence any simple object of the heart is in $\mathcal{S}$.

Conversely, let $S \in \mathcal{S}$. Let $M \to S$ be a non-zero subobject in $\heartsuit(t)$. We have seen above that M has a $(0,\dots,0)$-$\mathcal{S}$-filtration. This yields a morphism $S_1 \to M$ with $S_1 \in \mathcal{S}$ whose cone is in $\heartsuit(t)$ by Lemma 3.2. This means that $S_1 \to M$ is a subobject. The composition $S_1 \to M \to S$ of monomorphisms in the heart is then a monomorphism. Since $S_1 \neq 0$ by (S1), this monomorphism is non-zero and hence an isomorphism by assumption (S2). In particular $M \to S$ is an isomorphism, so S is a simple object of the heart. Hence any object of $\mathcal{S}$ is a simple object of the heart.

It is now clear that $\heartsuit(t)$ is a finite-length category. □

Corollary 4.5. *Let $n \in \mathbb{Z}$. Then*

$$\mathcal{D}^{\leqslant n} = \{M \in \mathrm{thick}(\mathcal{S}) \mid \mathrm{Hom}(M,[-l]S) = 0 \textit{ for all } S \in \mathcal{S} \textit{ and } l > n\},$$
$$\mathcal{D}^{\geqslant n} = \{M \in \mathrm{thick}(\mathcal{S}) \mid \mathrm{Hom}([-l]S,M) = 0 \textit{ for all } S \in \mathcal{S} \textit{ and } l < n\}.$$

Proof. Using [BBD82, 1.3.4] this is immediate from Theorem 4.4. □

Theorem 4.6. *Let $\mathcal{T}$ be a triangulated category. Mapping a full simple-minded subcategory $\mathcal{S}$ of $\mathcal{T}$ to the t-structure $t_{\mathcal{S}} = (\mathrm{susp}(\mathcal{S}), \mathrm{susp}^{-}(\mathcal{S}))$ from Theorem 4.4 defines a bijection between the sets of*

(a) *full simple-minded subcategories of $\mathcal{T}$ and*

(b) *bounded t-structures on $\mathcal{T}$ whose heart is a finite-length category.*

The inverse map maps a bounded t-structure t with finite-length heart to the full subcategory $\mathrm{Simples}(\heartsuit(t))$ of $\mathcal{T}$ formed by the simple objects of its heart $\heartsuit(t)$.

Proof. Theorem 4.4 shows that the map $\mathcal{S} \mapsto t_{\mathcal{S}}$ is well-defined and satisfies $\mathrm{Simples}(\heartsuit(t_{\mathcal{S}})) = \mathcal{S}$.

Conversely, let $t = (t^{\leqslant 0}, t^{\geqslant 0})$ be a bounded t-structure on $\mathcal{T}$ with finite-length heart. Then $\mathcal{S} := \mathrm{Simples}(\heartsuit(t))$ certainly satisfies the conditions (S1) and (S2) of Definition 4.1, and condition (S3) follows from $\mathrm{Hom}(t^{\leqslant 0}, t^{\geqslant 1}) = 0$. Since t is bounded and $\heartsuit(t)$ is finite-length it is clear that $\mathrm{tria}(\mathcal{S}) = \mathcal{T}$. This shows that $\mathcal{S}$ is a full simple-minded subcategory of $\mathcal{T}$. Obviously we have $\mathrm{susp}(\mathcal{S}) \subset t^{\leqslant 0}$ and $\mathrm{susp}^{-}(\mathcal{S}) \subset t^{\geqslant 0}$. So the two aisles of $t_{\mathcal{S}}$ are contained in the corresponding aisles of t. This implies that $t = t_{\mathcal{S}} = t_{\mathrm{Simples}(\heartsuit(t))}$. □

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