Finite symplectic matrix groups

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Contents

Chapter 1

Introduction

The finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ are those subgroups that fix a full lattice in $\mathbb{Q}^{1\times m}$ together with some positive definite symmetric form (see Chapter 2 for precise definitions).

A subgroup of $\mathrm{GL}_m(\mathbb{Q})$ is called *symplectic*, if it fixes a nondegenerate skewsymmetric form. Such groups only exist if m is even. A symplectic subgroup of $\mathrm{GL}_m(\mathbb{Q})$ is called maximal finite symplectic if it is not contained in another finite symplectic subgroup of $\mathrm{GL}_m(\mathbb{Q})$.

This thesis classifies all conjugacy classes of maximal finite symplectic matrix groups in $\mathrm{GL}_m(\mathbb{Q})$ for $m \leq 22$.

Such classifications have a long tradition. Minkowski [Min87] gave upper bounds on the orders of finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ using the theory of quadratic forms. Later, Schur [Sch05] gave bounds on the orders of finite subgroups of $\mathrm{GL}_m(\mathbb{C})$ having rational traces using character theory. Serre extended this work to arbitrary fields (see [GTT07]). In [Bli17] Blichfeldt classified all finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{PGL}_3(\mathbb{C})$. The finite (quasiprimitive) subgroups of $\mathrm{GL}_m(\mathbb{C})$ for $m \leq 10$ have been determined by Blichfeld, Brauer, Lindsey, Wales and Feit (see [Fei76]). These results do not depend on the classification of all finite simple groups. Using the classification of all finite simple groups, the maximal finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ have been determined in [BBNZ77, Ple91, NP95, Neb95, Neb96] for $m \leq 31$. Similarly, the maximal finite subgroups of $\mathrm{GL}_m(\mathcal{Q})$ have been classified in [Neb98a] for all totally definite quaternion algebras $\mathcal{Q}$ with $m \cdot \dim_{\mathbb{Q}}(\mathcal{Q}) \leq 40$.

Each conjugacy class of symplectic matrix groups contains a representative G in $\mathrm{Sp}_{2n}(\mathbb{Q}) = \{g \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid gJ_ng^{\mathrm{tr}} = J_n\}$ where $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. So one might ask how the orbit $\{G^x \mid x \in \mathrm{GL}_{2n}(\mathbb{Q}) \text{ such that } G^x < \mathrm{Sp}_{2n}(\mathbb{Q})\}$ decomposes into $\mathrm{Sp}_{2n}(\mathbb{Q})$-conjugacy classes. If the commuting algebra of G is a quadratic number field, then Lemma 2.1.12 shows that there exists a parametrization of these classes using norm groups. In particular, it turns out that each orbit decomposes into infinitely many $\mathrm{Sp}_{2n}(\mathbb{Q})$-conjugacy classes. Hence only $\mathrm{GL}_{2n}(\mathbb{Q})$-conjugacy classes of symplectic matrix groups are considered in this thesis.

The natural representation of a maximal finite symplectic matrix group is a sum of pairwise nonisomorphic rationally irreducible representations that yield maximal finite symplectic matrix groups. Thus it suffices to classify only the (conjugacy classes of) symplectic irreducible maximal finite (s.i.m.f.) matrix groups.

Each s.i.m.f. group $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is contained in a rationally irreducible maximal finite (r.i.m.f.) group $H \leq \mathrm{GL}_{2n}(\mathbb{Q})$. But in practice, this fact cannot be used to classify all s.i.m.f. groups by computing proper subgroups of such groups H since the index $[H : G]$ can be very large. For example, the s.i.m.f. group $QD_{64} < \mathrm{GL}_{16}(\mathbb{Q})$ is contained in the r.i.m.f. group $\mathrm{Aut}(B_{16}) < \mathrm{GL}_{16}(\mathbb{Q})$ with index $2^{10} \cdot 16!$.

Hence one has to proceed as in the classification of the r.i.m.f. matrix groups. The r.i.m.f. or s.i.m.f. matrix groups are full automorphism groups of some lattices. Further, two such groups are conjugate if and only if certain lattices are isometric.

The concept of primitivity is the key ingredient to these classifications since it has some important consequences. A symplectic matrix group is called *symplectic primitive* if it is not contained (up to conjugacy) in a wreath product of some symplectic matrix group. The restriction of the natural character of a symplectic primitive irreducible maximal finite (s.p.i.m.f.) matrix group $G < \mathrm{GL}_{2n}(\mathbb{Q})$ to a normal subgroup N is a multiple of a single rationally irreducible character of N. Furthermore, if G is s.p.i.m.f. then there exists a finite list of candidates (depending only on n) for the generalized Fitting subgroup $F^*(G)$ of G. The possible Fitting subgroups are given by a theorem of Hall. The possible layers (central products of quasisimple groups) can be taken from Hiss and Malle [HM01] which is based of the atlas of finite simple groups [CCN+85]. So the completeness of this list depends on the classification of the finite simple groups.

Then it remains to construct all possible extensions G of $F^*(G)$ up to conjugacy. There are several shortcuts to find G or at least a large (normal) subgroup of G. A very useful tool is the so-called generalized Bravais group. If N is a normal subgroup of a s.p.i.m.f. matrix group $G < \mathrm{GL}_{2n}(\mathbb{Q})$, then the generalized Bravais group $\mathcal{B}^o(N)$ contains N and can be computed directly from N. Further, N and $\mathcal{B}^o(N)$ have the same commuting algebras and $\mathcal{B}^o(N)$ is a normal subgroup of G.

It turns out that, like in the classification of r.i.m.f. matrix groups, the number of conjugacy classes of s.i.m.f. subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$ varies greatly depending on whether n is divisible by a large power of 2 or not. This is due to the fact that the list of possible Fitting subgroups is much larger in the first case. For example, there are 91 conjugacy classes of s.i.m.f. groups in $\mathrm{GL}_{16}(\mathbb{Q})$, but there are only 5 conjugacy classes in dimension 14.

This classification relies on calculations (computations of automorphism groups and invariant forms, ideal arithmetic, . . .) that require the use of a computer algebra system. All these calculations were performed in MAGMA [BCP97] since this system is extensible and it provides almost all necessary algorithms for lattices, group theory, number fields and (quaternion) algebras. In particular, MAGMA contains an implementation of the algorithm of Plesken and Souvignier [PS97] for computing automorphism groups and isometries of lattices.

This thesis is organized as follows. Section 2.1 starts with basic definitions and gives a first overview of symplectic matrix groups. It also recalls the definition of generalized Bravais groups and elaborates the general outline of the classification. Section 2.2

explains the so-called "m-parameter argument" which allows us to construct all s.i.m.f. supergroups of a given irreducible matrix group U whose commuting algebra is a field. We also give an algorithm which computes all s.i.m.f. supergroups G of U if the commuting algebra of U is a quaternion algebra provided that $U \trianglelefteq G$ and $[G : U]$ is a power of 2. This section also contains some methods that can be used to rule out several candidates for normal subgroups. Section 2.3 explains the sublattice algorithm. Section 2.4 describes several constructions and notational conventions for maximal finite matrix groups. Finally, Section 2.5 contains a complete list of all possible generalized Fitting subgroups of s.p.i.m.f. matrix groups up to dimension $2n = 22$.

Chapter 3 describes some infinite families of s.i.m.f. matrix groups. In particular, all s.i.m.f. subgroups of $\mathrm{GL}_{p-1}(\mathbb{Q})$ and $\mathrm{GL}_{p+1}(\mathbb{Q})$ whose orders are divisible by a prime $p \geq 5$ are determined.

Chapter 4 deals with the classification of the s.i.m.f. subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$ for $1 \leq n \leq 11$. For each dimension, the classification of the conjugacy classes of s.i.m.f. subgroups is given as a table. For each class, it contains a name that describes how a representative of that class can be constructed (from p-subgroups, quasisimple groups or smaller maximal finite matrix groups by taking generalized Bravais groups, tensor products, wreath products or group extensions). The table also contains the following invariants of each conjugacy class: group order, commuting algebra, number of isomorphism classes of invariant lattices and further information on certain invariant lattices. Together these invariants provide an easy method for recognizing the conjugacy class of a given s.i.m.f. matrix group.

For each conjugacy class, the appendix contains a symmetric positive definite and a skewsymmetric form such that the automorphism group of the standard lattice with respect to these two forms represents that particular class. These forms are also available in a MAGMA [BCP97] readable format from

`http://www.math.rwth-aachen.de/~Markus.Kirschmer/symplectic/`.

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Chapter 2

Methods

2.1 Definitions

2.1.1 Symplectic matrix groups

This thesis classifies the conjugacy classes of all maximal finite symplectic subgroups of $\mathrm{GL}_m(\mathbb{Q})$ for $1 \leq m \leq 22$. Two very important tools for the classification are the form spaces and the commuting algebras:

Definition 2.1.1 Let $G \leq \mathrm{GL}_m(\mathbb{Q})$.

(a) The $\mathbb{Q}$-space of *G-invariant forms* is given by

$$\mathcal{F}(G) := \{F \in \mathbb{Q}^{m\times m} \mid gFg^{\mathrm{tr}} = F \text{ for all } g \in G\}.$$

Further $\mathcal{F}_{sym}(G)$, $\mathcal{F}_{>0}(G)$ and $\mathcal{F}_{skew}(G)$ denote the subset of symmetric, symmetric positive definite and skewsymmetric G-invariant forms respectively.

The group G is called *symplectic* if $\mathcal{F}_{skew}(G)$ contains an invertible element and G is said to be *uniform* if $\dim_{\mathbb{Q}}(\mathcal{F}_{sym}(G)) = 1$.

(b) The *enveloping algebra* $\overline{G}$ of G is the subspace of $\mathbb{Q}^{m\times m}$ generated by the matrices in G. Further

$$\mathrm{End}(\overline{G}) := C_{\mathbb{Q}^{m\times m}}(G) := \{X \in \mathbb{Q}^{m\times m} \mid Xg = gX \text{ for all } g \in G\}$$

is the *endomorphism ring* or *commuting algebra* of G.

Remark 2.1.2 Let $G < \mathrm{GL}_m(\mathbb{Q})$.

(a) If $F \in \mathcal{F}(G)$ is invertible, then $\mathrm{End}(\overline{G}) \to \mathcal{F}(G),\ e \mapsto eF$ is an isomorphism of $\mathbb{Q}$-spaces. Its inverse is given by $\mathcal{F}(G) \to \mathrm{End}(\overline{G}),\ F' \mapsto F'F^{-1}$.

(b) If G is finite, then $\sum_{g\in G} gg^{\mathrm{tr}} \in \mathcal{F}_{>0}(G)$. In particular, $\mathcal{F}(G) \simeq \mathrm{End}(\overline{G})$.

Remark 2.1.3 Let $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{Q})$.

(a) If $G < \mathrm{GL}_m(\mathbb{Q})$ is symplectic, them m is even.

(b) $\mathrm{Sp}_{2n}(\mathbb{Q}) := \{g \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid gJ_ng^{\mathrm{tr}} = J_n\}$ is a subgroup of $\mathrm{SL}_{2n}(\mathbb{Q})$.

(c) An invertible matrix $S \in \mathrm{GL}_{2n}(\mathbb{Q})$ is skewsymmetric if and only if $S = J_n^x$ for some $x \in \mathrm{GL}_{2n}(\mathbb{Q})$. In particular, a finite subgroup $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is symplectic if and only if there exists some $x \in \mathrm{GL}_{2n}(\mathbb{Q})$ such that $G^x < \mathrm{Sp}_{2n}(\mathbb{Q})$.

Proof: See for example [Art57, Theorems 3.7 and 3.25]. □

So any conjugacy class of (maximal) finite symplectic subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$ has a representative in $\mathrm{Sp}_{2n}(\mathbb{Q})$.

The most important computational tool for the enumeration of the maximal finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ are the G-invariant lattices and automorphism groups. They are defined as follows.

Definition 2.1.4 Let R be a Dedekind ring such that its quotient field K is a number field.

(a) An R-lattice is a finitely generated R-module in some vector space over K.

(b) An R-order is a subring of a finite dimensional K-algebra that is also an R-lattice.

(c) If Λ is a $\mathbb{Z}$-order in $\mathbb{Q}^{m\times m}$ then

$$\mathcal{Z}(\Lambda) := \{L \subset \mathbb{Q}^{1\times m} \mid L \text{ is } \mathbb{Z}\text{-lattice of rank } m \text{ with } Lx \subseteq L \text{ for all } x \in \Lambda\}$$

denotes the set of all Λ-*invariant lattices*.

Similarly if $G < \mathrm{GL}_m(\mathbb{Q})$, then

$$\mathcal{Z}(G) := \{L \subset \mathbb{Q}^{1\times m} \mid L \text{ is a } \mathbb{Z}\text{-lattice of rank } m \text{ with } Lx \subseteq L \text{ for all } x \in G\}$$

is the set of all G-invariant lattices.

(d) For a $\mathbb{Z}$-lattice $L \subset \mathbb{Q}^{1\times m}$ of rank m, a set $\mathcal{F} \subseteq \mathbb{Q}^{m\times m}$ and some subfield K of $\mathbb{Q}^{m\times m}$ let

$$\mathrm{Aut}_K(L, \mathcal{F}) = \{g \in \mathrm{GL}_m(\mathbb{Q}) \mid Lg = L, gFg^{\mathrm{tr}} = F, gc = cg \text{ for all } F \in \mathcal{F}, c \in K\}$$

be the group of K-linear automorphisms of L with respect to $\mathcal{F}$. If $\mathcal{F} = \{F\}$ consists only of one form, we write $\mathrm{Aut}_K(L, F)$ instead of $\mathrm{Aut}_K(L, \{F\})$ and if $K \simeq \mathbb{Q}$, we will omit the subscript K.

Note that, if $\mathcal{F}$ contains a positive definite symmetric matrix, then

$$\mathrm{Aut}_K(L, \mathcal{F}) = \mathrm{Aut}(L, \{xF \mid x \in K, F \in \mathcal{F}\})$$

and we will switch frequently between these two notations in the sequel.

We are now ready to give a characterization of (maximal) finite rational matrix groups.

Remark 2.1.5

(a) Let $L \subseteq \mathbb{Q}^{1\times m}$ be a $\mathbb{Z}$-lattice of rank m and $F \in \mathbb{Q}^{m\times m}$ be symmetric and positive definite. Then $\mathrm{Aut}(L, F)$ is finite.

(b) A group $G < \mathrm{GL}_m(\mathbb{Q})$ is finite if and only if $\mathcal{F}_{>0}(G)$ and $\mathcal{Z}(G)$ are nonempty.

(c) If $G < \mathrm{GL}_m(\mathbb{Q})$ is finite then $S := \{\mathrm{Aut}(L, F) \mid (L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)\}$ contains all maximal finite supergroups of G.

In particular, G is maximal finite if and only if $S = \{G\}$. The maximal finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ have been classified in [BBNZ77, Ple91, NP95, Neb95, Neb96] for all $m < 32$.

Proof: (a) The norm induced by F on $\mathbb{R}^{1\times m}$ is equivalent to the maximum norm. So there exist only finitely many vectors in L of a given length. Hence there exist only finitely many possible images for some fixed basis vectors of L under an automorphism. (b) If G is finite then $\sum_{g\in G} gg^{\mathrm{tr}} \in \mathcal{F}_{>0}(G)$ and $\sum_{g\in G} Lg \in \mathcal{Z}(G)$ for any $\mathbb{Z}$-lattice L of rank m. Conversely, if $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ then $G \leq \mathrm{Aut}(L, F)$ is finite. □

In the same spirit, we want to characterize the maximal finite symplectic subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$. First we will give this characterization for rationally irreducible matrix groups, where irreducibility is defined as follows:

Definition 2.1.6 A matrix group $G < \mathrm{GL}_m(K)$ is called *K-irreducible* (or just irreducible) if the natural representation of G is irreducible over K. In the case $K = \mathbb{Q}$ we also use the phrase “rationally irreducible”.

As Remark 2.1.2 shows, there is a tight connection between the form space $\mathcal{F}(G)$ and the commuting algebra $\mathrm{End}(\overline{G})$. In particular, symplectic matrix groups can also be characterized by their endomorphism rings as Lemma 2.1.9 shows.

But before we state this lemma, we recall two well known facts.

Definition and Remark 2.1.7 Let $G < \mathrm{GL}_m(\mathbb{Q})$ be irreducible and finite. Then $E := \mathrm{End}(\overline{G})$ is a skewfield of dimension $e := \dim_{\mathbb{Q}}(E)$ say. Suppose that $S \subseteq E$ is a simple subalgebra with $s := \dim_{\mathbb{Q}}(S)$. By the double centralizer property, we have a sequence of $\mathbb{Q}$-algebra monomorphisms

$$\overline{G} = C_{\mathbb{Q}^{m\times m}}(E) \simeq (E^o)^{\frac{m}{e}\times\frac{m}{e}} \hookrightarrow (S^o)^{\frac{m}{s}\times\frac{m}{s}} \overset{\triangle^{\mathbb{Q}}}{\hookrightarrow} \mathbb{Q}^{m\times m}$$

where the superscript ^{o} denotes the opposite algebra. Let $\triangle_S$ be the composition of the first two morphisms. Then the character of $\triangle_S(G) < \mathrm{GL}_{\frac{m}{s}}(S^o)$ is not uniquely determined by G, but the composition $\triangle^{\mathbb{Q}} \circ \triangle_S$ is conjugation by some $x \in \mathrm{GL}_m(\mathbb{Q})$ according to the Skolem-Noether theorem. In particular, G and $\triangle^{\mathbb{Q}}(\triangle_S(G))$ are conjugate and $\triangle_S(G)$ is irreducible.

Remark 2.1.8 Let K be a number field of degree $d = \dim_{\mathbb{Q}}(K)$. Further let $H_1, H_2 < \mathrm{GL}_m(K)$ be irreducible and finite. If H_1 and H_2 are conjugate in $\mathrm{GL}_m(K)$ then $\triangle^{\mathbb{Q}}(H_1)$ and $\triangle^{\mathbb{Q}}(H_2)$ are also conjugate in $\mathrm{GL}_{md}(\mathbb{Q})$. Conversely, if $\triangle^{\mathbb{Q}}(H_1)$ and $\triangle^{\mathbb{Q}}(H_2)$ are conjugate then the natural characters of H_1 and H_2 must be algebraically conjugate.

Lemma 2.1.9 *Let $G < \mathrm{GL}_m(\mathbb{Q})$ be irreducible and finite. Further let $E := \mathrm{End}(\overline{G})$ and denote by K the center of E.*

(a) Let $F \in \mathcal{F}_{>0}(G)$ and $e \in E$. If $eF \in \mathcal{F}(G)$ is symmetric (skewsymmetric) then the subfield $\mathbb{Q}(e) \leq E$ is totally real (totally complex).

Conversely, if $\mathbb{Q}(e) \leq K$ is totally real, then eF is symmetric.

(b) The following statements are equivalent:

(1) G is symplectic.

(2) E contains a (minimal) totally complex subfield.

(3) There exists a (minimal) totally complex number field K' of degree $d \mid m$ and some $H < \mathrm{GL}_{\frac{m}{d}}(K')$ such that G is conjugate to $\triangle^{\mathbb{Q}}(H)$ in $\mathrm{GL}_m(\mathbb{Q})$.

In particular, G is a symplectic irreducible maximal finite (s.i.m.f.) subgroup of $\mathrm{GL}_m(\mathbb{Q})$ if and only if $G = \mathrm{Aut}_{K'}(L, F)$ for all $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ and for all minimal totally complex subfields K' of $\mathrm{End}(\overline{G})$.

(c) Each $\tilde{F} \in \mathcal{F}_{>0}(G)$ induces involutions on E, $\overline{G}$ and K via $x \mapsto x^\circ := \tilde{F} x^{\mathrm{tr}} \tilde{F}^{-1}$.

The involutions on $\overline{G}$ and K do not dependent on the form $\tilde{F}$ and the fixed field of $^\circ \colon K \to K$ is the maximal totally real subfield K^+ of K.

Further, K is either totally real or a CM-field (i.e. K is totally complex and $[K : K^+] = 2$). In particular, $^\circ$ is the (unique) complex conjugation on K.

Proof:

(a) If eF is symmetric, then $eF = Fe^{\mathrm{tr}}$ shows that e is a selfadjoint automorphism of the Euclidean space $(\mathbb{R}^{1\times m}, F)$. So it generates a totally real field. A similar argument holds for skewsymmetric forms.

Suppose now $e \in K$ is totally real. Since $\mathcal{F}(G)$ is closed under taking transposes, it decomposes into $\mathcal{F}_{sym}(G) \oplus \mathcal{F}_{skew}(G)$. Hence $eF = e_1F + e_2F$ with e_1F symmetric and e_2F skewsymmetric. In particular e_1 is totally real and e_2 totally complex by the above. But $e_2 = e - e_1 \in \mathbb{Q}(e, e_1)$ is contained in a totally real field. So $e_2 = 0$.

(b) Part (a) shows (1) $\Rightarrow$ (2). For the converse fix $F \in \mathcal{F}_{>0}(G)$ and note that $E \otimes_{\mathbb{Q}} \mathbb{R}$ cannot be a sum of copies of $\mathbb{R}$. Thus G fixes at least one real valued skewsymmetric form. Hence $\mathcal{F}_{skew}(G) \subset \{eF \mid e \in E\}$ contains a nonzero element, S say. Since E is a skewfield, the form S is already invertible. So G is symplectic. For (2) $\Rightarrow$ (3) one can choose $H := \triangle_{K'}(G)$ where K' is a minimal totally complex subfield of E. For the converse, note that $\mathrm{End}(\overline{\triangle^{\mathbb{Q}}(H)}) \simeq E$ has a subfield isomorphic to K'.

(c) It is clear that $\circ$ is an involution on $\overline{G}$ and E. Thus it is an automorphism on $E \cap \overline{G} = K$. Since any $\tilde{F} \in \mathcal{F}_{>0}(G)$ is of the form eF for some $e \in E$, it follows that $\circ\colon \overline{G} \to \overline{G}$ does not depend on $\tilde{F}$. By part (a) it also follows that K^+ is the fixed field of $\circ\colon K \to K$.

The field K is the character field of some complex constituent of the natural representation of G. So $K/\mathbb{Q}$ is Galois. In particular, if K has an embedding into $\mathbb{R}$ then all embeddings $K \to \mathbb{C}$ would be real. So K is either totally real or totally complex and the index $[K : K^+]$ equals the order of $\circ\colon K \to K$. □

From this result, we immediately obtain the following corollary. It shows that we only have to classify the conjugacy classes of s.i.m.f. matrix groups to get the classification of the conjugacy classes for all maximal finite symplectic matrix groups.

Corollary 2.1.10 *If $G < \mathrm{GL}_m(\mathbb{Q})$ is maximal finite symplectic, then the natural representation $\triangle\colon G \to \mathrm{GL}_m(\mathbb{Q})$ splits into a sum of pairwise nonisomorphic irreducible representations $\triangle_i\colon G \to \mathrm{GL}_{m_i}(\mathbb{Q})$ and each group $\triangle_i(G)$ is s.i.m.f..*

Proof: We have a decomposition $\triangle = \sum_{i=1}^{s} n_i \triangle_i$ into irreducible and pairwise nonisomorphic representations $\triangle_i\colon G \to \mathrm{GL}_{m_i}(\mathbb{Q})$. Hence we may assume that $G < \{\mathrm{Diag}(x_1, \ldots, x_m) \mid x_i \in \mathrm{GL}_{n_i m_i}(\mathbb{Q})\}$. Hence $\mathrm{End}(\overline{G})$ and thus $\mathcal{F}(G)$ are given by block diagonal matrices and each group $n_i\triangle_i(G)$ is maximal finite symplectic since G fixes an invertible skewsymmetric form S.
Suppose now $n_i > 1$ for some i. If $n_i > 2$ then $(n_i - 2)\triangle_i(G)$ is symplectic. This is clearly true if $\triangle_i(G)$ fixes a skewsymmetric form. In the other case, $E_i := \mathrm{End}(\overline{\triangle(G)})$ is a totally real field and S is the tensor product of an invertible skewsymmetric matrix in $E_i^{n_i \times n_i}$ with some $F \in \mathcal{F}_{>0}(\triangle_i(G))$. Thus n_i is even and $(n_i - 2)\triangle_i(G)$ is symplectic. So we may suppose that $n_i = 2$. But then $2\triangle_i(G)$ is properly contained in $H := \left\langle 2\triangle_i(G), \left(\begin{smallmatrix} 0 & \zeta \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$ where $\zeta \in E_i$ is a torsion unit of maximal order. One checks that H is irreducible and thus symplectic by the previous lemma since its endomorphism ring contains a cyclotomic subfield. □

Remark 2.1.11 Suppose K is a minimal totally complex number field. Lemma 2.1.9 and Remark 2.1.8 show that the classification of all s.i.m.f. subgroups of $\mathrm{GL}_{m \dim_{\mathbb{Q}}(K)}(\mathbb{Q})$ yields all conjugacy classes of maximal finite K-irreducible subgroups $H < \mathrm{GL}_m(K)$ satisfying $\mathrm{End}(\overline{\triangle^{\mathbb{Q}}(H)}) \simeq K$. If $K \not\simeq \mathrm{End}(\overline{\triangle^{\mathbb{Q}}(H)})$, then two problems may arise:

- If $H < \mathrm{GL}_m(K)$ is K-irreducible and maximal finite then $\triangle^{\mathbb{Q}}(H)$ might be reducible over $\mathbb{Q}$ as the example $H = \langle \pm 1 \rangle < \mathrm{GL}_1(\mathbb{Q}(\sqrt{-d}))$ for every squarefree $d \in \mathbb{Z}_{>0} \setminus \{1, 3\}$ shows.
- Even if $H < \mathrm{GL}_m(K)$ is K-irreducible and maximal finite such that $\triangle^{\mathbb{Q}}(H)$ is rationally irreducible, it might not be maximal finite symplectic, as the following example shows:

  Let $\mathcal{Q}_{\infty,2}$ be the quaternion algebra over $\mathbb{Q}$ that is only ramified at 2 and the infinite place. Denote by $\mathfrak{M}$ a maximal order (it is unique up to conjugacy). Then the torsion subgroup $\mathfrak{M}^{*,1}$ is isomorphic to $\mathrm{SL}_2(3)$. We denote by ${}_{\infty,2}[\mathrm{SL}_2(3)]_1 := \triangle^{\mathbb{Q}}(\mathfrak{M}^{*,1})$ the corresponding subgroup of $\mathrm{Sp}_4(\mathbb{Q})$. By Theorem 4.3.1 this group has (up to conjugacy) three s.i.m.f. supergroups namely ${}_i[(D_8 \otimes C_4).S_3]_2$, ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$ and ${}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$. These groups have $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$ as commuting algebras respectively. Let $K = \mathbb{Q}(\sqrt{-d})$ be any splitting field of $\mathcal{Q}_{\infty,2}$ such that $d \notin \{1,2,3\}$ (for example $K = \mathbb{Q}(\sqrt{-5})$). Then $H := \triangle^K(\mathfrak{M}^{*,1})$ is a K-irreducible maximal finite subgroup of $\mathrm{GL}_2(K)$ but ${}_{\infty,2}[\mathrm{SL}_2(3)]_1 = \triangle^{\mathbb{Q}}(H)$ has $\mathcal{Q}_{\infty,2}$ as commuting algebra and this group is not s.i.m.f..

By Remark 2.1.3 any conjugacy class of maximal finite symplectic matrix groups contains a representative in $\mathrm{Sp}_{2n}(\mathbb{Q})$ for some $n \in \mathbb{Z}$. So one might ask to find all (maximal) finite subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ up to conjugacy in $\mathrm{Sp}_{2n}(\mathbb{Q})$. The following remark shows that there are infinitely many of these classes:

Lemma 2.1.12 *Let $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ be finite such that $E := \mathrm{End}(\overline{G})$ is a field. Denote by E^+ its maximal totally real subfield.*

(a) Let $t_1, \dots, t_s$ be representatives of $N_{\mathrm{GL}_{2n}(\mathbb{Q})}(G)/\langle G, E^* \rangle \le \mathrm{Out}(G)$. For $1 \le i \le s$ let $e_i := t_i J_n t_i^{\mathrm{tr}} J_n^{-1} \in (E^+)^*$. Then $\mathcal{S} := \bigcup_{i=1}^s e_i \mathrm{Nr}_{E/E^+}(E^*)$ is independent of the choice of the t_i.

(b) Let $H := \{x \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid G^x \le \mathrm{Sp}_{2n}(\mathbb{Q})\}$. Then $\varphi\colon H \to (E^+)^*,\ x \mapsto xJ_nx^{\mathrm{tr}}J_n^{-1}$ is surjective.

(c) Let $x \in H$. Then G and G^x are conjugate in $\mathrm{Sp}_{2n}(\mathbb{Q})$ if and only if $\varphi(x) \in \mathcal{S}$.

(d) Suppose E is an imaginary quadratic number field. Then H and $\mathcal{S}$ are groups and φ is a homomorphism of groups. Let $x, y \in H$. Then G^x and G^y are conjugate in $\mathrm{Sp}_{2n}(\mathbb{Q})$ if and only if $\varphi(x)\mathcal{S} = \varphi(y)\mathcal{S}$.

Moreover, the $\mathrm{GL}_{2n}(\mathbb{Q})$ conjugacy class of G intersected with $\mathrm{Sp}_{2n}(\mathbb{Q})$ (i.e. the set $\{G^x \mid x \in H\}$) decomposes into infinitely many $\mathrm{Sp}_{2n}(\mathbb{Q})$ conjugacy classes and there is a bijection between these classes and $\mathbb{Q}^*/\mathcal{S}$.

Proof:

(a) Each t_i normalizes G, thus it acts on $\mathcal{F}_{skew}(G) = \{eJ_n \mid e \in E^+\}$. Hence $e_i \in E^+$. Moreover if t_i' and t_i represent the same coset, then $t_i' = get_i$ for some $g \in G$ and $e \in E$. In particular $t_i'J_nt_i'^{\mathrm{tr}} = gee_iJ_ne^{\mathrm{tr}}g^{\mathrm{tr}} = e_i\mathrm{Nr}_{E/E^+}(e)J_n$.

(b) Let $x \in \mathrm{GL}_{2n}(\mathbb{Q})$. Then $x \in H$ if and only if $J_n \in \mathcal{F}_{skew}(G^x) = \{x^{-1}eJ_nx^{-\mathrm{tr}} \mid e \in E^+\}$. Thus $x \in H$ implies $\varphi(x) \in (E^+)^*$. So φ is well defined. Finally, it follows from Remark 2.1.3(c) and the above that φ is surjective.

(c) Suppose $G^x = G^y$ for some $y \in \mathrm{Sp}_{2n}(\mathbb{Q})$. Then xy^{-1} normalizes G. So $xy^{-1} = egt_i$ for some $1 \le i \le s$, $e \in E$ and $g \in G$. Then $\varphi(x)J_n = xJ_nx^{\mathrm{tr}} = egt_iyJ_ny^{\mathrm{tr}}t_i^{\mathrm{tr}}g^{\mathrm{tr}}e^{\mathrm{tr}} = \mathrm{Nr}_{E/E^+}(e)e_iJ_n$ shows that $\varphi(x) \in \mathcal{S}$.

Conversely, if $\varphi(x) \in \mathcal{S}$ then $\varphi(x) = \mathrm{Nr}_{E/E^+}(e)e_i$ for some $1 \le i \le s$ and $e \in E$. Hence $xJ_nx^{\mathrm{tr}} = et_iJ_nt_i^{\mathrm{tr}}e^{\mathrm{tr}}$ implies that $y := t_i^{-1}e^{-1}x \in \mathrm{Sp}_{2n}(\mathbb{Q})$. Further $G^y = (G^{t_i^{-1}e^{-1}})^x = G^x$.

(d) Since $E^+ = \mathbb{Q}$ consists only of scalar matrices, one checks then H and $\mathcal{S}$ are groups and φ is a morphism. Further $G^x = (G^y)^z$ for some $z \in \mathrm{Sp}_{2n}(\mathbb{Q})$ if and only if $\varphi(xz^{-1}y^{-1}) \in \mathcal{S}$. Since $\varphi(z) = 1$ this is equivalent to $\varphi(x)\mathcal{S} = \varphi(y)\mathcal{S}$. It remains to show that $\mathbb{Q}^*/\mathcal{S}$ is infinite. This follows from $[\mathcal{S} : \mathrm{Nr}_{E/\mathbb{Q}}(E^*)] \leq |\mathrm{Out}(G)|$ and the fact that $\mathbb{Q}^*/\,\mathrm{Nr}_{E/E^+}(E^*)$ is always an infinite group. A proof of this statement is given in [Ste89, pg. 208] and I would like to thank Hans Opolka for pointing out this reference. □

From now on, conjugacy means conjugacy in $\mathrm{GL}_{2n}(\mathbb{Q})$. Further, since we want to classify the conjugacy classes of maximal finite symplectic matrix groups, we may w.l.o.g. suppose that a given symplectic matrix group is contained in $\mathrm{Sp}_{2n}(\mathbb{Q})$. I.e. we write $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ to indicate that $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is symplectic.

2.1.2 Primitivity

To classify all maximal finite symplectic matrix groups, it suffices to classify only s.i.m.f. matrix groups as Corollary 2.1.10 shows. In this section we will reduce the number of groups to consider even further.

Definition 2.1.13 Let K be a number field. A K-irreducible subgroup $G < \mathrm{GL}_m(K)$ is called *primitive*, if G is not conjugate to a subgroup of the *wreath product*

$$H \wr S_k := \left\langle \mathrm{Diag}(h_1, \ldots, h_k), P \otimes I_{\frac{m}{k}} \mid h_i \in H, P \text{ a } k \times k \text{ permutation matrix} \right\rangle$$

for some $H < \mathrm{GL}_{\frac{m}{k}}(K)$ where k is a divisor of m.
Similarly, a $\mathbb{Q}$-irreducible symplectic subgroup $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is called *symplectic primitive*, if G is not conjugate to a subgroup of $H \wr S_k$ for some $H < \mathrm{Sp}_{\frac{2n}{k}}(\mathbb{Q})$ where $k \mid n$.

Remark 2.1.14 A rationally irreducible symplectic subgroup $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is symplectic primitive if and only if $\triangle_K(G)$ is primitive for all minimal totally complex subfields K of $\mathrm{End}(\overline{G})$.

The concept of primitivity is a key ingredient in the determination of all irreducible finite matrix groups. It has some important consequences for normal subgroups.

Theorem 2.1.15 ([NP95, Lemma (III.1)]) *Let $G < \mathrm{GL}_m(K)$ be a rationally irreducible primitive matrix group and $N \trianglelefteq G$. Then $\langle N \rangle_K \leq K^{m\times m}$ is a simple algebra or equivalently, the natural KN-module $K^{1\times m}$ splits into a direct sum of k isomorphic KN-modules of dimension $\frac{m}{k}$.*

Proof: The group G acts on N by conjugation. Hence it also acts on the set of central primitive idempotents of $\langle N\rangle_K$. Thus G permutes the homogeneous components of the natural KN-module $K^{1\times m}$. But since G is primitive, there can only be one such component. □

Corollary 2.1.16 ([NP95, (III.1)-(III.3)]) *Let $G < \mathrm{GL}_{2n}(\mathbb{Q})$ be rationally irreducible and symplectic primitive. Further let p be a prime divisor of $|G|$.*

(a) If $N \trianglelefteq G$ then $\overline{N} \leq \mathbb{Q}^{2n\times 2n}$ is a simple subalgebra.

(b) If $O_p(G) \neq 1$ then there exists some $k \geq 0$ such that $p^k(p-1)$ divides $2n$.

(c) All abelian characteristic subgroups of $O_p(G)$ are cyclic.

Proof:

(a) Let $K < \mathrm{End}(\overline{G})$ be any minimal totally complex subfield. Let $\{f_1,\dots,f_r\}$ and $\{e_1,\dots,e_s\}$ be the central primitive idempotents of the enveloping algebras $\overline{N}$ and $\langle G\rangle_K$ respectively. Let χ_i denote the character corresponding to a simple $\langle G\rangle_K e_i$ module and let $L = \mathbb{Q}(\chi_1,\dots,\chi_s) \subseteq K$ be their character field. Then $L/\mathbb{Q}$ is Galois and $e_i = \frac{\chi_i(1)}{|G|}\sum_{g\in G}\chi_i(g^{-1})g \in \langle G\rangle_L$. Since G is irreducible, $\{e_1,\dots,e_s\}$ is a Galois orbit under $\mathrm{Gal}(L/\mathbb{Q})$. For any $1 \leq j \leq r$ there exists some i such that $e_if_j \neq 0$. Since $f_j \in \overline{N}$ is fixed under $\mathrm{Gal}(L/\mathbb{Q})$, we get that $e_if_j \neq 0$ for all i,j. The enveloping algebra $\langle \triangle_K(G)\rangle_K$ is isomorphic to $\langle G\rangle_K e_i$ for some i. Now $\{e_if_1,\dots,e_if_r\}$ is a set of central idempotents of $\langle N\rangle_K e_i \simeq \langle \triangle_K(N)\rangle_K$. But $\triangle_K(G)$ is primitive and therefore $\langle \triangle_K(N)\rangle_K$ is a simple algebra by Theorem 2.1.15. This shows $r=1$, since no e_if_j vanishes.

(b) By (a), $O_p(G)$ has a rationally irreducible representation of degree d for some divisor d of $2n$. But for any p-group, d is of the form $p^k(p-1)$ for some $k \geq 0$.

(c) Any characteristic subgroup U of $O_p(G)$ is a normal subgroup of G. Thus by (a), the abelian group U admits a faithful irreducible representation. Therefore U is cyclic. □

Suppose N is a normal subgroup of an irreducible and symplectic primitive group $G < \mathrm{GL}_{2n}(\mathbb{Q})$. Then the natural character χ of N is sufficient to recover the conjugacy class of N. If N has several $\mathbb{Q}$-irreducible faithfull representations, we will use the phrase "G contains N with character χ" to distinguish the conjugacy classes of matrix groups isomorphic to N.
If $\tilde{N} < \mathrm{GL}_m(\mathbb{Q})$ denotes an $\mathbb{Q}$-irreducible constituent of N, we will identify N with $\tilde{N}$ since the precise notation $\tilde{N} \otimes I_{\frac{2n}{m}}$ is not very handy.

The following theorem of Philip Hall classifies all finite p-groups whose abelian characteristic subgroups are cyclic. In particular, together with the above result, this classifies all possibible candidates for the Fitting subgroups of symplectic primitive irreducible maximal finite (s.p.i.m.f.) matrix groups.

Theorem 2.1.17 (P. Hall) *If P is a finite p-group with no noncyclic abelian characteristic subgroups, then P is the central product of subgroups P_1 and P_2 where*

(a) P_1 is an extraspecial 2-group and P_2 is either a cyclic, dihedral, quasidihedral or generalized quaternion 2-group.

(b) p is odd and P_1 is an extraspecial p-group of exponent p and P_2 is cyclic.

Proof: See for example [Hup67, Satz 13.10, p. 357]. □

We close this section by showing that symplectic imprimitive matrix groups can easily be recognized. Further, the wreath products of symplectic primitive irreducible maximal finite (s.p.i.m.f.) matrix groups are usually again maximal finite symplectic. So we can restrict the classification to s.p.i.m.f. matrix groups.

Definition 2.1.18 Let $\mathcal{F}$ be a nonempty family of bilinear forms on $\mathbb{R}^n$. A lattice L in $\mathbb{R}^n$ is called *indecomposable* w.r.t. $\mathcal{F}$, if L cannot be written as a direct sum $L = L_1 \oplus L_2$ where $b(L_1, L_2) = \{0\}$ for all $b \in \mathcal{F}$. A vector $x \in L$ is called *indecomposable* in L w.r.t. $\mathcal{F}$ if it cannot be written as $x = y + z$ with $y, z \in L \setminus \{0\}$ and $b(y, z) = 0$ for all $b \in \mathcal{F}$.

Theorem 2.1.19 *Let $\mathcal{F}$ be a family of bilinear forms on $\mathbb{R}^n$ that contains at least one positive definite form f. Then each lattice L in $\mathbb{R}^n$ admits a decomposition $L = \oplus_{i=1}^k L_i$ where each L_i is indecomposable w.r.t. $\mathcal{F}$ and $b(L_i, L_j) = \{0\}$ for all $b \in \mathcal{F}$ and all $1 \le i < j \le k$. This decomposition is unique up to permutation of the L_i.*

Proof: We adapt [Kne02, Satz (27.2)] slightly. Let $L = \oplus_{i=1}^l L_i'$ be any decomposition such that $b(L_i', L_j') = 0$ for all $b \in \mathcal{F}$ and all $i \neq j$. If $x \in L$ is indecomposable w.r.t. $\mathcal{F}$ then $x \in L_i'$ for some i. Thus two indecomposable elements x and y with $b(x, y) \neq 0$ for some $b \in \mathcal{F}$ are in the same component L_i'. Two indecomposable elements $x, y \in L$ are said to be equivalent if and only if there exists some indecomposable elements $x = x_1, \dots, x_r = y \in L$ and some $b_1, \dots, b_r \in \mathcal{F}$ such that $b_i(x_i, x_{i+1}) \neq 0$ for all $1 \le i < r$. This defines an equivalence relation on the set of indecomposable elements of L. Since the equivalence classes give rise to a orthogonal decomposition of the Euclidean space $(\mathbb{R}^n, f)$ there are at most n such classes $K_1, \dots, K_k$ say. Denote by L_i the sublattice of L generated by K_i. For $1 \le i < j \le k$ we have $b(L_i, L_j) = \{0\}$ for all $b \in \mathcal{F}$ by construction. Further, every nonzero $x \in L$ can be written as a finite sum of indecomposable elements in L. If x is decomposable, we find some $r, s \in L$ such that $x = r + s$ and $b(r, s) = 0$ for all $b \in \mathcal{F}$. In particular $0 < f(r, r), f(s, s) < f(x, x)$. Hence this decomposition procedure must end. Therefore $L = \oplus_{i=1}^k L_i$ is a decomposition of L which has the desired properties. Each component L_i is indecomposable and contained in L_j' for some j.

To proof the uniqueness, assume that all L_j' are also indecomposable. For $1 \le j \le l$ let $I_j = \{1 \le i \le k \mid L_i \subseteq L_j'\}$ and set $M_j := \oplus_{i \in I_j} L_i \subseteq L_j'$. We are done if we can show $L_j' = M_j$ for all j since then $|I_j| = 1$. Let $x \in L_j'$. Write $x = \sum_{i=1}^l x_i$ with $x_i \in M_i \subseteq L_i'$ for all i. Since $\oplus_{i=1}^l L_i' = L$ this implies $x_i = 0$ for all $i \neq j$. So $M_j = L_j'$ as claimed. □

Remark 2.1.20 Let $G < \mathrm{GL}_m(\mathbb{Q})$ be finite and $L \in \mathcal{Z}(G)$. Then every automorphism in $\mathrm{Aut}(L, \mathcal{F}(G))$ permutes the components of the unique indecomposable orthogonal decomposition of L wrt. $\mathcal{F}(G)$.
Hence a finite irreducible subgroup $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ is symplectic primitive if and only if each $L \in \mathcal{Z}(G)$ is indecomposable w.r.t. $\mathcal{F}(G)$.

Lemma 2.1.21 ([Ple91, Proposition II.7]) *Let $H < \mathrm{Sp}_{2n}(\mathbb{Q})$ be s.p.i.m.f. such that $E := \mathrm{End}(\overline{H})$ is a minimal totally complex number field. If the 2-modular trivial Brauer character is no constituent of the natural 2-modular character of H, then the wreath product $H \wr S_k < \mathrm{Sp}_{2nk}(\mathbb{Q})$ is s.i.m.f. for all $k \geq 1$.*

Proof: Since $-I_n \in H$ we have $\mathrm{End}(\overline{H \wr S_k}) = \{I_k \otimes c \mid c \in E\} \simeq E$ and $\mathcal{F}(H \wr S_k) = \{I_k \otimes F \mid F \in \mathcal{F}(H)\}$.
Let $L = L_1 \oplus \cdots \oplus L_k$ for some $L_i \in \mathcal{Z}(G)$. View the L_i as a $\tilde{H}$-module where $\tilde{H}$ is the direct product of k copies of H. By our assumption L_i and L_j have no common p-modular constituent for $i \neq j$ as $\tilde{H}$-modules. By [Ple78, Theorem I.1] we get $\mathcal{Z}(\tilde{H}) = \{\oplus_{i=1}^k L_i \mid L_i \in \mathcal{Z}(H)\}$. Hence $\mathcal{Z}(H \wr S_k) = \{\oplus_{i=1}^k L \mid L \in \mathcal{Z}(H)\}$. The result now follows since $E \simeq \mathrm{End}(\overline{H \wr S_k})$ is minimal totally complex and $H \wr S_k = \mathrm{Aut}(L, \mathcal{F}(H \wr S_k))$ for all $L \in \mathcal{Z}(H \wr S_k)$. □

The assumption on the 2-modular constituents is necessary. The group $H := \langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle < \mathrm{Sp}_2(\mathbb{Q})$ is s.p.i.m.f. but $H \wr S_2 \leq \mathrm{Sp}_4(\mathbb{Q})$ is not maximal finite (see Theorem 4.3.1). In fact, this is the only example that we will encounter.

2.1.3 Generalized Bravais groups

If N is a normal subgroup of a s.p.i.m.f. matrix group G then

$$\mathcal{B}^o(N) := \{x \in G \mid x \text{ centralizes } \mathrm{End}(\overline{N})\}$$

is also a normal subgroup of G which contains N. We will show that $\mathcal{B}^o(N)$ can be computed from the group N without knowledge of G.

The natural $\overline{N}$-module $\mathbb{Q}^{1\times m}$ decomposes into a direct sum of copies of a single irreducible $\overline{N}$-module V as we have seen in Corollary 2.1.16.

The $\mathbb{Z}$-span $\Lambda_1(N) := \langle N \rangle_{\mathbb{Z}}$ is invariant under conjugation by G and we recursively define an ascending chain of orders in $\overline{N}$ having the same property. This is the so called *radical idealizer process*:

- If $\Lambda_i(N)$ has already been defined, then let R_i be the arithmetic radical of $\Lambda_i(N)$ i.e. the intersection of all maximal ideals of $\Lambda_i(N)$ that contain the (reduced) discriminant of $\Lambda_i(N)$.
- Let $\Lambda_{i+1}(N)$ be the right order of R_i in $\overline{N}$.

If $\Lambda_i(N)$ is G-invariant then R_i is also G-invariant by definition. So $g \in G$ and $x \in \Lambda_{i+1}(N)$ imply $R_i(g^{-1}xg) = g^{-1}R_ixg \subseteq R_i$. This shows that $\Lambda_{i+1}(N)$ is G-invariant as claimed.

Like any ascending chain of orders having full rank in $\overline{N}$, this chain stabilizes at some order $\Lambda_\infty(N)$ say. It follows from [Rei03, Theorems (39.11), (39.14) and (40.5)] that the above chain stabilizes at $\Lambda_i(N)$ if and only if $\Lambda_i(N)$ is hereditary. So in particular $\Lambda_\infty(N)$ is hereditary.

Since $\Lambda_\infty(N)$ is G-invariant, G acts on $\mathcal{Z}(\Lambda_\infty(N))$. Thus G fixes at least one of the $\Lambda_\infty(N)$-lattices. This leads to the following definition.

Definition 2.1.22 With the above notation, let $F \in \mathcal{F}_{>0}(N)$. The *generalized Bravais group* of N is

$$\mathcal{B}^o(N) := \{g \in \overline{N} \mid L_i g = L_i \text{ for all } 1 \leq i \leq s \text{ and } gFg^{\mathrm{tr}} = F\}$$

where $\{L_1, \ldots, L_s\}$ represents the isomorphism classes of $\Lambda_\infty(N)$-lattices in V.

By construction $\mathcal{B}^o(N)$ is a finite subgroup of $\overline{N}^*$ containing N. Moreover N and $\mathcal{B}^o(N)$ have the same commuting algebras and thus the same invariant forms. In particular, $\mathcal{B}^o(N)$ does not depend on the choice of F.

Lemma 2.1.23 *Let $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ be s.p.i.m.f.. If $N \trianglelefteq G$ then*

(a) $N \trianglelefteq \mathcal{B}^o(N) \trianglelefteq G$

(b) If $X < \overline{N}^*$ is a finite subgroup such that $N \trianglelefteq X$ then $X \leq \mathcal{B}^o(N)$.

(c) $\mathcal{B}^o(N) = \{x \in G \mid x$ centralizes $\mathrm{End}(\overline{N})\}$

Proof: The parts (a) and (b) are proven in [NP95, Proposition II.10, p. 82]. Part (c) follows from (a), (b) and the double centralizing property. □

Part (b) can be used to find a large subgroup of $\mathcal{B}^o(N)$. Part (a) rules out some normal subgroups. For example, suppose $G < \mathrm{Sp}_{10k}$ is s.p.i.m.f. such that G contains a normal subgroup $N \simeq \mathrm{Alt}_5$ where the character of N is a multiple of the 5-dimensional irreducible character of Alt_5. Then (by Table 2.5.1) $N \ntrianglelefteq \mathcal{B}^o(N) \simeq C_2 \times \mathrm{Alt}_6$ gives a contradiction.

2.1.4 General outline of the classification

We recall the definition of the generalized Fitting subgroup and its self-centralizing property. In this section, let G be a finite group.

Definition 2.1.24 A finite perfect group H is called *quasisimple* if $H/Z(H)$ is simple. A *component* of G is a subnormal quasisimple subgroup of G. The subgroup generated by all components of G is called the *layer* of G and is denoted by $E(G)$. Finally, the *generalized Fitting subgroup* $F^*(G)$ of G is the subgroup generated by the layer $E(G)$ and the *Fitting subgroup* $F(G) = \prod_{p||G|} O_p(G)$.

Lemma 2.1.25 *The layer $E(G)$ is a central product of its components. The generalized Fitting subgroup $F^*(G)$ is the central product of $F(G)$ and $E(G)$.*

Proof: See for example [Asc00, 31.7 and 31.12]. □

Theorem 2.1.26 *The generalized Fitting subgroup $F^*(G)$ is self-centralizing in G. In particular, $G/F^*(G)$ is isomorphic to a subgroup of* $\mathrm{Out}(F^*(G))$.

Proof: See for example [Asc00, 31.13]. □

Remark 2.1.27 The general outline of the classification of the s.i.m.f. subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is now as follows:

(a) The symplectic imprimitive matrix groups come from the classifications of $\mathrm{Sp}_{2d}(\mathbb{Q})$ where d runs through all divisors of n. These groups are usually s.i.m.f. (see Lemma 2.1.21). Suppose now $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ is s.p.i.m.f..

(b) There are only finitely many candidates for the Fitting subgroup $F(G)$ according to Theorem 2.1.17. These are listed in Table 2.5.2.

(c) There are only finitely many candidates for the layer $E(G)$. These are described in [HM01] which is based on the ATLAS [CCN$^+$85] and listed in Table 2.5.1 up to $n = 11$. (Note that this step depends on the completeness of the classification of all finite simple groups).

(d) For two such candidates $F(G)$ and $E(G)$ we know that $G/F^*(G)$ is isomorphic to a subgroup of $\mathrm{Out}(F^*(G))$. So we have to find all such possible extensions of $F^*(G)$ up to conjugacy in $\mathrm{GL}_{2n}(\mathbb{Q})$.

The last step is the crucial one. Although it is a cohomological task to find all abstract extensions G, we are interested in finding all matrix group extension of $F^*(G)$. Of course, one can always replace $F^*(G)$ by its generalized Bravais group.

In the next section, we will describe methods that construct G from $F^*(G)$ under certain assumptions on $F^*(G)$. We will also give some more criteria that eliminate some candidates for $F^*(G)$.

2.2 Methods

2.2.1 Normal subgroups of index 2^k

For many s.i.m.f. matrix groups G the quotient $G/\mathcal{B}^o(F^*(G))$ is an elementary abelian 2-group as Tables 2.5.1 and 2.5.2 show. Thus we give two results that rule out some candidates for normal subgroups N having index 2^k in G.

Part (c) of the next lemma is an analogon to [NP95, Corollary III.4]. It will be used frequently in the classification.

Lemma 2.2.1 *Let $N \lhd G$ with $[G : N] = 2$ where $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ is irreducible and symplectic primitive. Let $g \in G \setminus N$ and suppose that N is reducible in $\mathrm{GL}_{2n}(\mathbb{Q})$.*

(a) The restriction $\triangle|_N$ of the natural representation of G onto N splits into $\triangle_1 + \triangle_1^g$ where $\triangle_1 \colon N \to \mathrm{GL}_n(\mathbb{Q})$ is irreducible.

(b) $\dim_{\mathbb{Q}} \mathcal{F}_{skew}(\triangle_1(N)) = 0$.

(c) $L := \mathrm{End}(\overline{\triangle_1(N)}) \subseteq \mathbb{Q}^{n\times n}$ *is a totally real number field.*

Proof: (a) By Clifford theory.

(b) If $\dim_{\mathbb{Q}} \mathcal{F}_{skew}(\triangle_1(N)) > 0$ then all nonzero elements in $\mathcal{F}_{skew}(\triangle_1(N))$ are invertible, since $\triangle_1$ is irreducible. Thus $\triangle_1(N)$ is symplectic. The representation $\triangle$ is induced by $\triangle_1$, hence we may suppose that $\triangle(N)$ is given by blockdiagonal matrices and $\triangle(g) = \begin{pmatrix} 0 & I_n \\ \triangle_1(g^2) & 0 \end{pmatrix}$. But then $G \leq \triangle_1(N) \wr C_2$ is symplectic imprimitive.

(c) $\triangle_1$ can also be seen as a real representation $\delta \colon N \to \mathrm{GL}_n(\mathbb{R})$. Let δ decompose into $\mathbb{R}$-irreducible representations $\delta_1, \dots, \delta_s$ say. Then $\dim_{\mathbb{R}} \mathcal{F}_{skew}(\delta(N)) = \dim_{\mathbb{Q}} \mathcal{F}_{skew}(\triangle_1(N)) = 0$ and hence $\dim_{\mathbb{R}} \mathcal{F}_{skew}(\delta_i(N)) = 0$ for all i.

So $\mathrm{End}_{\mathbb{R}}(\delta_i(N)) \simeq \mathbb{R}$ for all i. Thus $\mathrm{End}_{\mathbb{R}}(\delta(N)) \simeq \mathbb{R} \otimes_{\mathbb{Q}} L \simeq \oplus_{i=1}^s \mathbb{R}$ shows that L must be a ring direct sum of totally real number fields. Since L is simple, the result follows. □

Lemma 2.2.2 *Let A be a simple $\mathbb{Q}$-algebra. Suppose that $\alpha \in \mathrm{Aut}_{\mathbb{Q}}(A)$ is a $\mathbb{Q}$-algebra automorphism of order 2. If $A^+ := \{x \in A \mid \alpha(x) = x\}$ is a simple ring, then $\dim_{\mathbb{Q}}(A) = 2\dim_{\mathbb{Q}}(A^+)$.*

Proof: Let $A^- = \{x \in A \mid \alpha(x) = -x\}$. Then $A = A^+ \oplus A^-$. The automorphism α maps the center K of A onto itself and hence induces an automorphism of K. If $\alpha|_K$ is not trivial, there exists some $x \in K$ such that $x \neq \alpha(x)$. But then $a := x - \alpha(x) \in K^* \cap A^-$. In particular, $A^+ \to A^-, x \mapsto ax$ is an isomorphism of $\mathbb{Q}$-spaces and the result follows.
So we may now suppose that $\alpha \in \mathrm{Aut}_K(A)$. In particular, A^+ is a central simple K-algebra and by Skolem-Noether, α is conjugation by some invertible $g \in A^+$. The

Wedderburn theorem allows us to identify A with $\mathcal{Q}^{n\times n}$ for some skewfield $\mathcal{Q}$ with center K. Let $s := g^2 \in K$. If $s \in K^2$, then we may assume that $g^2 = 1$ thus g is conjugate to $\mathrm{Diag}(I_k, -I_l)$ in $K^{n\times n}$ for some $k+l=n$. But then $A^+ = \mathcal{Q}^{k\times k} \oplus \mathcal{Q}^{l\times l}$ is simple if and only if g is central. This contradicts the assumptions. So $\tilde{K} := K(\sqrt{s})$ is a proper extension of K. Let $\tilde{g} := \frac{1}{\sqrt{s}} g$ and $\tilde{A} := A \otimes_K \tilde{K}$. Again, since $\tilde{A}$ is a central simple $\tilde{K}$-algebra, we may assume that $\tilde{A} = \tilde{Q}^{\tilde{n}\times\tilde{n}}$ for some skewfield $\tilde{Q}$ with center $\tilde{K}$. As above $\tilde{g} := \frac{1}{\sqrt{s}} g$ is conjugate to $\mathrm{Diag}(I_k, -I_l)$ in $\tilde{K}^{\tilde{n}\times\tilde{n}}$ with $k+l=\tilde{n}$. Thus $\tilde{A}^+ = C_{\tilde{A}}(g) = \tilde{\mathcal{Q}}^{k\times k} \oplus \tilde{\mathcal{Q}}^{l\times l}$. But on the other hand, $e := \frac{1}{2}(1+\tilde{g})$ and $f := \frac{1}{2}(1-\tilde{g})$ are (the unique) central primitive idempotents of $\tilde{A}^+$. Let $\langle\sigma\rangle = \mathrm{Gal}(\tilde{K}/K) \simeq C_2$. By acting on the structure constants of $\tilde{A}^+$, σ extends to an K-algebra automorphism of $\tilde{A}^+$ such that $\sigma(\sqrt{s}) = -\sqrt{s}$ and $\sigma(x) = x$ for all $x \in A^+$. In particular $\sigma(e) = f$. Thus σ interchanges the ring direct summands $\tilde{\mathcal{Q}}^{k\times k}$ and $\tilde{\mathcal{Q}}^{l\times l}$. This shows $k = l = \frac{\tilde{n}}{2}$ and the result follows, since $\dim_K(A^+) = \dim_{\tilde{K}}(\tilde{A}^+) = \frac{\tilde{n}^2}{2} \cdot \dim_{\tilde{K}}(\tilde{Q}) = \frac{1}{2}\dim_{\tilde{K}}(\tilde{A}) = \frac{1}{2}\dim_K(A)$. □

If n is not a power of 2, the following corollary is used to rule out some candidates for normal subgroups. See [Neb95, (III.4)] for a similar result in $\mathrm{GL}_n(\mathbb{Q})$.

Corollary 2.2.3 *Let $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ be irreducible and symplectic primitive. If $N \lhd G$ with $|G/N| = 2^k$, then $\dim_{\mathbb{Q}}(\mathrm{End}(\overline{G})) = 2^l \dim_{\mathbb{Q}}(\mathrm{End}(\overline{N}))$ for some $0 \le l \le k$.*

Proof: Let $N = N_1 \lhd N_2 \lhd \dots N_{r-1} \lhd N_r = G$ be a normal series of G such that $N_{i+1}/N_i \simeq C_2$. By Corollary 2.1.16 all the commuting algebras $E_i := \mathrm{End}(\overline{N_i})$ are simple. Let $g \in N_2 \setminus N_1$. Then g induces an automorphism on E_2 of order at most 2. So by the result above, $E_2 = C_{E_1}(g)$ either equals E_2 or has dimension $\frac{1}{2}\dim_{\mathbb{Q}}(E_1)$. The result follows by induction. □

2.2.2 Primitive, normalized and normal critical lattices

Suppose $G < \mathrm{GL}_m(\mathbb{Q})$. To find the r.i.m.f. or s.i.m.f. supergroups of G, one has to consider automorphism groups of $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}(G)$. Since the number of such pairs (L, F) is infinite, we have to reduce this number. In this section we start with some general results that do not depend on $\mathrm{End}(\overline{G})$.

The following definitions will be used frequently in the sequel.

Definition 2.2.4 Let $L', L \subset \mathbb{Q}^{1\times m}$ be $\mathbb{Z}$-lattices of full rank and let $F', F \in \mathbb{Q}^{m\times m}$ be symmetric and positive definite.

(a) $L^{\#,F} := \{x \in \mathbb{Q}^{1\times m} \mid xFy^{\mathrm{tr}} \in \mathbb{Z} \text{ for all } y \in L\}$ is the *dual lattice* of L wrt. F.

(b) F is said to be *integral* on L if $L \subseteq L^{\#,F}$.

(c) F is said to be *primitive* on L if $L \subseteq L^{\#,F}$ but $L \nsubseteq kL^{\#,F}$ for all $k \in \mathbb{Z}_{>1}$.

(d) (L, F) is *normalized* if $L \subseteq L^{\#,F}$ and the abelian group $L^{\#,F}/L$ is of squarefree exponent with rank at most $\frac{m}{2}$.

(e) The pairs (L, F) and (L', F') are said to be *isometric* if there exists some $x \in \mathrm{GL}_m(\mathbb{Q})$ such that $L' = Lx$ and $F' = x^{-1}Fx^{-\mathrm{tr}}$. In this case x is called an *isometry* between L and L'.

The next remark shows that every finite matrix group G fixes a normalized pair (L, F).

Remark 2.2.5 Let $G < \mathrm{GL}_m(\mathbb{Q})$ be finite.

(a) The set $\mathcal{Z}(G)$ is closed under the following operations

(1) $\mathcal{Z}(G) \times \mathcal{F}_{>0}(G) \to \mathcal{Z}(G),\ (L, F) \mapsto L^{\#,F}$

(2) $\mathcal{Z}(G) \times \mathrm{End}(\overline{G})^* \to \mathcal{Z}(G),\ (L, c) \mapsto Lc$

(3) $\mathcal{Z}(G) \times \mathcal{Z}(G) \to \mathcal{Z}(G),\ (L, L') \mapsto L + L'$

(4) $\mathcal{Z}(G) \times \mathcal{Z}(G) \to \mathcal{Z}(G),\ (L, L') \mapsto L \cap L'$

(5) $\mathcal{Z}(G) \times N_{\mathrm{GL}_m(\mathbb{Q})}(G) \to \mathcal{Z}(G),\ (L, h) \mapsto Lh$

(b) Let $F \in \mathcal{F}_{>0}(G)$ be integral on $L \in \mathcal{Z}(G)$. If (L, F) is not normalized, then there exist some prime divisor p of $\det(L, F)$ such that $(L \cap pL^{\#,F}, \frac{1}{p}F)$ is an integral lattice of smaller determinant. In particular, iterating this process results in some normalized $(L', \frac{1}{d}F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ with $d \mid \det(L, F)$.

Now we want to reduce the number of lattices L that we have to consider.

Remark 2.2.6 If Λ is a $\mathbb{Z}$-order in $\mathbb{Q}^{m\times m}$ then $L, L' \in \mathcal{Z}(\Lambda)$ are *isomorphic* as Λ-right modules if and only if there exists some $x \in C_{\mathbb{Q}^{m\times m}}(\Lambda)$ such that $Lx = L'$. The number of isomorphism classes is finite by the Jordan-Zassenhaus theorem (see [Rei03, Chapter 26]).

Let $G < \mathrm{GL}_m(\mathbb{Q})$ be finite. Then $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}(G)$ is isometric to $(Lx, x^{-1}Fx^{-\mathrm{tr}})$ for all $x \in \mathrm{End}(\overline{G})^*$. Since isometric pairs have conjugate automorphism groups, it suffices to consider pairs (L, F) where L runs through a (finite) system of representatives of the isomorphism classes of $\mathcal{Z}(G)$.

If one wants to find all r.i.m.f. or s.p.i.m.f. groups that contain G as a normal subgroup, one can usually reduce the number of lattices L that one has to consider even further.

Definition 2.2.7 Let $G < \mathrm{GL}_m(\mathbb{Q})$ be finite. A finite subset $S \subset \mathcal{Z}(G)$ is called *G-normal critical*, if for every finite supergroup $H < \mathrm{GL}_m(\mathbb{Q})$ with $G \trianglelefteq H$ there exists some $x \in \mathrm{GL}_m(\mathbb{Q})$ such that $\mathcal{Z}(H^x) \cap S \neq \emptyset$. If $S = \{L\}$, then L is called G-normal critical.

Remark 2.2.8 Let $G < \mathrm{GL}_m(\mathbb{Q})$ be finite such that $E := \mathrm{End}(\overline{G})$ is a field. When one wants to find a set of G-normal critical lattices, one usually has to deal with one of the following situations:
Let $\Lambda := \langle G, \mathbb{Z}_E \rangle_{\mathbb{Z}}$. The fractional $\mathbb{Z}_E$-ideals act on $\mathcal{Z}(\Lambda)$. Let $L_1, \ldots, L_s$ represent the orbits.

(a) If there exists some $1 \leq i \leq s$ such that every $h \in N_{\mathrm{GL}_m(\mathbb{Q})}(G)$ of finite order fixes the set $\mathcal{L}_i := \{L_i\mathfrak{a} \mid \mathfrak{a} \text{ a fractional } \mathbb{Z}_E\text{-ideal}\}$ then $\mathcal{S} = \{L_i\mathfrak{a} \mid [\mathfrak{a}] \in \mathcal{C}l(\mathbb{Z}_E)\}$ is G-normal critical.

This situation can arise as follows:

(1) Suppose there exists some $1 \leq i \leq s$ such that for all $j \neq i$ and all fractional ideals $\mathfrak{a}$ of $\mathbb{Z}_E$ the determinant of a base change matrix from L_i to $L_j\mathfrak{a}$ does not equal ± 1. Then $\mathcal{L}_i$ has the above property.

(2) Choose one of the following options to definite m_i and M_i for $1 \leq i \leq s$:

- Let Λ' be either Λ or $\langle G \rangle_{\mathbb{Z}}$. Let m_i be the number of minimal Λ'-invariant sublattices of L_i whose index in L_i has only prime divisors in a given fixed set.
- Let $m_i = |\{L \in \mathcal{Z}(G) \setminus \mathcal{Z}(\Lambda) \mid L \text{ a minimal sublattice of } L_i\}|$.

Similarly one defines $M_1, \ldots, M_s$ by taking superlattices in the definitions above. If there exists some $1 \leq i \leq s$ such that $(m_i, M_i) \neq (m_j, M_j)$ for all $j \neq i$ then $\mathcal{L}_i$ has the above property.

(b) Suppose that $s = 4$ and suppose that the L_i can be chosen such that $L_1 = L_2 + L_3$, $L_4 = L_2 \cap L_3$ and there are no elements of $\mathcal{Z}(\Lambda)$ between L_1/L_i, L_i/L_4 for $i = 2, 3$. Then $\mathcal{S} = \{L_i\mathfrak{a} \mid i \in \{1,2\},\ [\mathfrak{a}] \in \mathcal{C}l(\mathbb{Z}_E)\}$ is a G-normal critical set.

Proof: Let $H < N_{\mathrm{GL}_m(\mathbb{Q})}(G)$ be finite.

(a) Let $O := L_i \cdot H$ be the orbit of L_i under the action of H. Then $L := \sum_{L' \in O} L' \in \mathcal{Z}(\Lambda)$. By the assumption, $O \subset \mathcal{L}_i$. So for each $L' \in O$ there exists a fractional $\mathbb{Z}_E$-ideal $\mathfrak{a}_{L'}$ such that $L' = L_i\mathfrak{a}_{L'}$. But then $L = L_i\mathfrak{a}$ for $\mathfrak{a} = \sum_{L' \in O} \mathfrak{a}_{L'}$. So $\mathcal{S}$ is G-normal critical.

Part (a1) is obvious since $\det(h) \in \{\pm 1\}$ for all $h \in H$ and (a2) follows from the fact that the action of H on $\mathcal{Z}(G)$ preserves inclusions, endomorphism rings and the index of sublattices.

(b) Summing over H shows that H fixes $L_i\mathfrak{a}$ for some fractional $\mathbb{Z}_E$-ideal $\mathfrak{a}$ and some $1 \leq i \leq 4$. If $i = 4$ then H fixes $L_1\mathfrak{a} = L_2\mathfrak{a} + L_3\mathfrak{a}$ since $L_2\mathfrak{a}$ and $L_3\mathfrak{a}$ are the unique minimal superlattices of $L_4\mathfrak{a}$ in $\mathcal{Z}(\Lambda)$ which are not of the form $L_4\mathfrak{a}'$ for some fractional $\mathbb{Z}_E$-ideal $\mathfrak{a}'$. Using the same argument one shows that if H fixes $L_3\mathfrak{a}$ then it also fixes $L_2\mathfrak{a}$. □

2.2.3 Fields as endomorphism rings (m-parameter argument)

In this section, we give an algorithm (the so called m-parameter argument) that constructs all r.i.m.f. or s.i.m.f. supergroups G of an irreducible matrix group U if $\mathrm{End}(\overline{U})$ is a field. In particular, this includes irreducible cyclic matrix groups U.

By the previous section, the problem is to reduce the number of forms that one has to consider. To do so, we need all possible prime divisors of $|G|$. If we have no other assumptions on G, we can always fall back on the Minkowski bound:

Lemma 2.2.9 (Minkowski's bound, [Min87]) *The least common multiple of the orders of all finite subgroups of* $\mathrm{GL}_n(\mathbb{Q})$ *is given by*

$$\prod_p p^{\left\lfloor \frac{n}{(p-1)} \right\rfloor + \left\lfloor \frac{n}{p(p-1)} \right\rfloor + \left\lfloor \frac{n}{p^2(p-1)} \right\rfloor + \dots}$$

where the product is taken over all primes $p \leq n+1$*.*

Further, the m-parameter argument needs a set of primes $\tilde{\Pi}(\mathrm{End}(\overline{U}), |G|)$ depending on $\mathrm{End}(\overline{U})$ and $|G|$ as follows:

Definition 2.2.10 Let K be an algebraic number field.

(a) Let $\sigma_i \colon K \to \mathbb{R} \quad (1 \leq i \leq d)$ be the real embeddings of K. If we fix the order of the σ_i, we get a group homomorphism

$$s \colon K^* \to \mathbb{F}_2^m, \quad x \mapsto (x_1, \dots, x_d) \text{ where } x_i = \begin{cases} 0 & \text{if } \sigma_i(x) > 0, \\ 1 & \text{if } \sigma_i(x) < 0 \end{cases}.$$

(b) For $k \in \mathbb{Z}$ define $\Pi(k)$ to be the set of all primes dividing k.

(c) We define a finite set of primes $\Pi(K)$ such that

(1) In each class of $\mathcal{C}l(\mathbb{Z}_K)$ there exists an integral ideal which contains $\prod_{p \in \Pi(K)} p^{a_p}$ with some $a_p \in \mathbb{N}_0$.

(2) The group $s(K^*)$ is generated by $s(x_1), \dots, s(x_\ell)$ for some $x_i \in \mathbb{Z}_K$ satisfying $\Pi(\mathrm{Nr}_{K/\mathbb{Q}}(x_i)) \subseteq \Pi(K)$.

(d) For $k \in \mathbb{Z}$ set $\tilde{\Pi}(K,k) := \Pi(k) \cup \bigcup_{F \leq K} \Pi(F)$ where the union is taken over all subfields F of K. Note that this set is not unique.

We follow [Neb95, Satz III.2, p. 16] to give an algorithm which constructs all r.i.m.f. or s.i.m.f. supergroups of a given irreducible matrix group of the same dimension.

Theorem 2.2.11 *Let $G < \mathrm{GL}_n(\mathbb{Q})$ be finite and irreducible. Let L be a $\mathbb{Z}G$-lattice. Assume that $C := C_{\mathbb{Q}^{n\times n}}(G)$ is either commutative or a positive definite quaternion algebra. Then there exists a $F \in \mathcal{F}_{>0}(G)$ such that F is primitive on L and $\Pi(\det(L,F)) \subseteq \Pi(K) \cup \Pi(|G|)$ where K denotes the maximal real subfield of $Z(C)$.*

Proof: If C is commutative, a proof is given in [Neb95, Satz III.2, p. 16]. So we may assume that C is a positive definite quaternion algebra. Choose any $F \in \mathcal{F}_{>0}(G)$ which is primitive on L. Suppose that there exists some prime $p \notin \Pi(K)$ such that $p \mid \det(L,F)$ but $p \nmid |G|$. It suffices to show that there exists some $c \in K$ such that cF is integral on L and the primes dividing $\det(L,cF)$ are contained in $(\Pi(|G| \cdot \det(L,F)) \cup \Pi(K))) \setminus \{p\}$.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ be the prime ideals of $\mathbb{Z}_K$ over p. Then $K_p := K \otimes_\mathbb{Q} \mathbb{Q}_p = \bigoplus_{i=1}^{\ell} K_{\mathfrak{p}_i}$. Denote by $\varepsilon_1, \ldots, \varepsilon_\ell$ the primitive Idempotents of K_p such that $K_p\varepsilon_i = K_{\mathfrak{p}_i}$.

Let $L_p := L \otimes_\mathbb{Z} \mathbb{Z}_p$ and $C_p := C \otimes_\mathbb{Q} \mathbb{Q}_p$. Since $p \nmid |G|$, p is not ramified in $\overline{G}$ and the order $\Lambda_p := \langle G \rangle_{\mathbb{Z}_p}$ is maximal in $(\overline{G})_p$ (see [Rei03, Theorems 41.1 and 41.7]). Therefore $\mathrm{End}_{\Lambda_p}(L_p)$ is maximal in C_p and p is not ramified in C_p. Hence $C_p \simeq \bigoplus_{i=1}^{\ell} K_{\mathfrak{p}_i}^{2\times 2}$.

So each Λ_p-lattice $L_p\varepsilon_i$ decomposes into two irreducible Λ_p-lattices $X_{i,1} \oplus X_{i,2}$. From $\varepsilon_i \in K \subseteq Z(C)$ we get $\varepsilon_i F (1-\varepsilon_i)^{tr} = 0$ which shows that the lattices $L_p\varepsilon_i$ are orthogonal to each other.

Since $\mathrm{End}_{K_{\mathfrak{p}_i}G}(X_{i,1}) \simeq K_{\mathfrak{p}_i}$, there exists no symmetric G-invariant nonzero form on $X_{i,1}$. Hence F induces an embedding $\varphi_i \colon X_{i,1} \hookrightarrow {X_{i,2}}^*$, $x \mapsto Fx^{tr}$ for each i. So there exists some $k_i \in \mathbb{Z}$ such that $\varphi_i(X_{i,1}) = p^{k_i} \cdot {X_{i,2}}^*$. Therefore the Gram matrix of (L_p, F) is of the form $\mathrm{diag}(p^{k_1}G_1, \ldots, p^{k_\ell}G_\ell)$ for some $G_i \in \mathrm{GL}_{n_i}(\mathbb{Z}_p)$ (wrt. a proper choice of a basis).

By property (1) of the definition of $\Pi(K)$, there exists some $\mathbb{Z}_K$-ideal $\mathfrak{a}_i$ whose norm is only divisible by primes in $\Pi(K)$ and some $y_i \in K$ such that $\mathfrak{p}_i \cdot \mathfrak{a}_i = y_i\mathbb{Z}_K$. Then $\mathfrak{p}_i = \langle p, y_i \rangle$ since this identity holds locally everywhere. In particular $y_i \in \mathbb{Z}_K$ and $\Pi(\mathrm{Nr}_{K/\mathbb{Q}}(y_i)) \subseteq \{p\} \cup \Pi(K)$. By property (2) of the definition of $\Pi(K)$, there exists some $x \in K$ such that $\mathrm{Nr}_{K/\mathbb{Q}}(x) \in \Pi(K)$ and $y := x \cdot \prod_{i=1}^{\ell} y_i^{-k_i}$ is totally positive.

Let $F' := y \cdot F$. Then $y_i\varepsilon_i$ and $p\varepsilon_i$ are both primitive elements of $K_{\mathfrak{p}_i}$. Hence (L_p, F') is self-dual. It might happen that F' is no longer integral on L. But then there exists some $k \in \mathbb{N}$ such that kF' is primitive on L and the primes dividing k are divisors of $|G|$. □

From this theorem we finally obtain

Corollary 2.2.12 (m-parameter argument, [Neb95, Korollar III.3, p. 17]) *Let $U \leq G < \mathrm{GL}_n(\mathbb{Q})$ be finite subgroups such that $C := \mathrm{End}(\overline{U})$ is a either a field or a positive definite quaternion algebra. Suppose $L \in \mathcal{Z}(G) \subseteq \mathcal{Z}(U)$. Then there exists some $F \in \mathcal{F}_{>0}(G)$ that is primitive on L with $\Pi(\det(L,F)) \subseteq \tilde{\Pi}(K,|G|)$ where K denotes the maximal real subfield of the center of C.*

Proof: The maximal totally real subfield K' of the center of $\mathrm{End}(\overline{G})$ is contained in K. By the theorem above, there exists some $F \in \mathcal{F}_{>0}(G)$ such that F is primitive on L and $\Pi(\det(L,F)) \subseteq \Pi(K',|G|) \subseteq \tilde{\Pi}(K,|G|)$. □

The following rather technical remark shows how this corollary will be used later.

Remark 2.2.13 Let $U < \mathrm{GL}_m(\mathbb{Q})$ be finite such that $C := \mathrm{End}(\overline{U})$ is a field. Denote by K the maximal totally real subfield of C. Further suppose that $L_1, \ldots, L_s$ represent the isomorphism classes of U-invariant lattices. Finally fix $F_i \in \mathcal{F}_{>0}(U)$ and let $R_i = \mathrm{End}_{\mathbb{Z}U}(L_i) \subseteq \mathbb{Z}_C$ (in most cases $R_i = \mathbb{Z}_C$ and there exists at least one i where equality holds). Denote by $R_i^+ := R_i \cap K$. The following algorithm finds (up to conjugacy) all finite supergroups G of U of order dividing a given $\ell \in \mathbb{N}$:

For $1 \leq i \leq s$ let

$$N_i := \{x \in N_{\mathrm{GL}_m(\mathbb{Q})}(U) \mid L_i x = L_i \text{ and } x F_i x^{\mathrm{tr}} F_i^{-1} \in R_i^*\} \text{ and}$$
$$P_i := \{a R_i^+ \mid a \in K_{>0},\ (L_i, aF_i) \text{ is normalized and } \Pi(\det(L_i, aF_i)) \subseteq \tilde{\Pi}(K, \ell)\}\,.$$

Then the group N_i acts on C, K and R_i via conjugation. Moreover, P_i consists of full orbits under this action. Let S_i be a set of representatives of these orbits.

Finally let U_i be a coset of $(R_i^+)^*_{>0} / \mathrm{Nr}_{C/K}(R_i^*)$ and let

$$\mathcal{S} := \{(L_i, uaF_i) \mid u \in U_i, aR_i^+ \in S_i, a \in K_{>0}, 1 \leq i \leq s\}\,.$$

Then every finite supergroup of U of order dividing ℓ is conjugate in $N_{\mathrm{GL}_m(\mathbb{Q})}(U)$ to some group that fixes one of the lattices in the finite set $\mathcal{S}$.
In particular, the r.i.m.f. supergroups of U are elements of $\{\mathrm{Aut}(L, F) \mid (L, F) \in \mathcal{S}\}$ and the s.i.m.f. supergroups of U are elements of $\{\mathrm{Aut}_{K_j}(L, F) \mid (L, F) \in \mathcal{S}, 1 \leq j \leq r\}$ where $K_1, \ldots, K_r$ denote the minimal totally complex subfields of C.

Proof: Let $G < \mathrm{GL}_m(\mathbb{Q})$ be a finite supergroup of U with $|G|$ dividing ℓ. By Corollary 2.2.12, the group G fixes some $(L', F') \in \mathcal{Z}(U) \times \mathcal{F}_{>0}(U)$ such that F' is integral on L' and $\Pi(\det(L', F')) \subseteq \tilde{\Pi}(K, \ell)$. Applying the process described in Definition 2.2.4 yields a normalized lattice $(\tilde{L'}, \tilde{F'})$ with $\Pi(\det(\tilde{L'}, \tilde{F'})) \subseteq \tilde{\Pi}(K, \ell)$.
Now $\tilde{L'} = L_i c$ for some $1 \leq i \leq s$ and $c \in C$. After replacing G by $G^{c^{-1}}$, G fixes (L_i, F) where $F := c\tilde{F'}c^{\mathrm{tr}}$. So there exists some $aR_i^+ \in P_i$ such that $F = aF_i$ (and thus $a \in K_{>0}$). By definition, there exists some $x \in N_i$ such that $a^x R_i^+ \in S_i$. After replacing G by $G^{x^{-1}}$, it fixes $L_i x^{-1} = L_i$ and $xaF_i x^{\mathrm{tr}} = a^{x^{-1}}(xF_i x^{\mathrm{tr}} F_i^{-1})F_i = \tilde{a}F_i$ for some $\tilde{a} \in K_{>0}$ with $aR_i^+ = \tilde{a}R_i^+$. Now a or $\tilde{a}$ are defined by the ideal aR_i^+ only up to some element of $(R_i^+)^*_{>0}$. For $y \in R_i^*$ it follows from Lemma 2.1.9 that $G^{y^{-1}}$ fixes $(L_i y^{-1}, y\tilde{a}F_i y^{\mathrm{tr}}) = (L_i, \mathrm{Nr}_{C/K}(y)\tilde{a}F_i)$.
So we have shown that G is conjugate (in $N_{\mathrm{GL}_m(\mathbb{Q})}(U)$) to some group that fixes a lattice in the set $\mathcal{S}$.
The result now follows, if we can show that $\mathcal{S}$ is finite. The set U_i is finite by Dirichlet's unit theorem (note that if $K \neq C$ then $(R_i^+)^2 \leq \mathrm{Nr}_{C/K}(R_i^*)$). The number of isomorphism classes of $\mathbb{Z}U$-invariant lattices is finite by the Jordan-Zassenhaus theorem.
So it remains to prove that P_i is finite: Fix $bR_i^+ \in P_i$ and let $J_i := \mathrm{Ann}_{R_i^+}(L_i^{\#, bF_i}/L_i)$. If $x \in K_{>0}$ such that xbF_i is integral on L_i, then $L_i x J_i \subseteq L_i^{\#, bF_i} J_i \subseteq L_i$. Thus $x \in J^{-1}$. □

To simplify the definition of the N_i in the previous remark, one can use the following

Remark 2.2.14 Assume the situation of the previous remark.

(a) If F_i is integral on L_i and $\det(L_i, F_i) = 1$ then

$$\{F \in \mathcal{F}_{>0}(U) \mid (L_i, F) \text{ is integral}\} = \{cF_i \mid c \in R_i \cap K_{>0}\}\,.$$

(b) If $\{F \in \mathcal{F}_{>0}(U) \mid (L_i, F) \text{ is integral}\} = \{cF_i \mid c \in R_i \cap K_{>0}\}$ then

$$N_i = N_{\mathrm{GL}_m(\mathbb{Q})}(U) \cap \mathrm{GL}(L) = \{x \in N_{\mathrm{GL}_m(\mathbb{Q})}(U) \mid L_i x = L_i\}\,.$$

Proof:

(a) Suppose $c \in K_{>0}$ such that cF_i is integral on L_i. Then $xcF_i y^{\mathrm{tr}} \in \mathbb{Z}$ for all $x, y \in L_i$. Hence $L_i c \subseteq L_i^{\#, F_i} = L_i$ and therefore $c \in R_i$.

(b) Let $x \in N_{\mathrm{GL}_m(\mathbb{Q})}(U) \cap \mathrm{GL}(L)$. Then $xF_i x^{\mathrm{tr}} \in \mathcal{F}_{>0}(U)$. Hence there exists some $c \in K_{>0}$ such that $xF_i x^{\mathrm{tr}} = cF_i$. Now xFx^{tr} is integral on $L_i x^{-1} = L_i$. This shows $c \in R_i^+$. From $\det(c) = 1$ it follows that $\mathrm{Nr}_{K/\mathbb{Q}}(c) = 1$ and thus $c \in (R_i^+)^*$. This proves $x \in N_i$. □

Note that there does not always exist some F_i such that the condition of (b) holds.

2.2.4 Quaternion algebras as endomorphism rings

We now turn to the case where a s.i.m.f. matrix group G contains an irreducible normal subgroup N such that $\mathrm{End}(\overline{N})$ is a quaternion algebra $\mathcal{Q}$ with center K.

If $N = F^*(G)$, then $\mathcal{Q}$ will frequently be a totally definite quaternion algebra (i.e. $\mathcal{Q}$ is ramified at all infinite places of K). In this case the structure of G is rather limited.

Theorem 2.2.15 *Let $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ be s.p.i.m.f.. Suppose that $E := \mathrm{End}(\overline{F^*(G)})$ is a totally definite quaternion algebra with center K. Then G acts on E and K by conjugation. Let*

$$S := \{g \in G \mid gx = xg \text{ for all } x \text{ in } K\} \text{ and}$$
$$B := \mathcal{B}^o(F^*(G)) = \{g \in G \mid gx = xg \text{ for all } x \text{ in } E\}$$

be the kernels of these actions. Then

$$1 \trianglelefteq F^*(G) \trianglelefteq B \trianglelefteq S \trianglelefteq G\,.$$

Further G/S is isomorphic to a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ *and S/B has exponent* 1 *or* 2 *(and is abelian).*

Proof: Only the claim that S/B has exponent 1 or 2 is not obvious. The proof given in [Neb98b, Theorem 4] applies mutatis mutandis. □

With the notation from above, the next lemma can be used to extend B by some element from S.

Lemma 2.2.16 *Let $N = \mathcal{B}^o(N) < \mathrm{GL}_m(\mathbb{Q})$ be finite such that $Q := \mathrm{End}(\overline{N})$ is a quaternion algebra with center K. Suppose $x, y \in \mathrm{GL}_m(\mathbb{Q})$ induce the same outer automorphism on N and $x^2, y^2 \in N$. If x commutes with K then*

(a) $\mathrm{End}(\overline{\langle N, x\rangle}) = C_Q(x) \simeq K[X]/(X^2 - d)$ for some $d \in K$.

(b) There exist some $g \in N$ and $c \in Q$ such that $y = gcx$. In particular, y commutes with K. Then $\mathrm{End}(\overline{\langle N, y\rangle}) \simeq K[X]/(X^2 - \mathrm{nr}_{Q/K}(c)d)$ where $\mathrm{nr}_{Q/K}$ denotes the reduced norm on Q. Further $\mathrm{nr}_{Q/K}(c)$ is a root of unity in $\mathbb{Z}_K$.

(c) If $C_Q(x)$ is a field and $\mathrm{nr}_{Q/K}(c) \in (\mathbb{Z}_K^*)^2$ then $\langle N, x\rangle$ and $\langle N, y\rangle$ are conjugate.

Proof:

(a) For any $z \in Q$ we have $z^x \in \mathrm{End}(\overline{N^x}) = \mathrm{End}(\overline{N}) = Q$. Hence x induces a K-automorphism on Q. By the Skolem-Noether theorem there exists some $a \in Q^*$ such that $z^a = z^x$ for all $z \in Q$. It follows from $x \notin C_{\mathrm{GL}_n\mathbb{Q}}(Q)$ that

$$K \subsetneq K[a] \subseteq C_Q(a) = C_Q(x) \subsetneq Q\,.$$

Thus $K[a] = C_Q(x)$. Since $x^2 \in N$, it induces the identity on Q. Hence $a^2 \in K$ and therefore $K[a] \cong K[X]/(X^2 - d)$ where $d := -\,\mathrm{nr}_{Q/K}(a) \in K$.

(b) Since yx^{-1} induces an inner automorphism on N, it is contained in NQ^*. It follows from part (a) that

$$\mathrm{End}(\overline{\langle N, y\rangle}) = \mathrm{End}(\overline{\langle N, cx\rangle}) \simeq K[X]/(X^2 + \mathrm{nr}_{Q/K}(ca))$$

as claimed.

Further $cxcx^{-1} \in (g^{-1}y)^2x^{-2} \in N$ has finite order. Hence the reduced norm $\mathrm{nr}_{Q/K}(c)^2 = \mathrm{nr}_{Q/K}(c \cdot (xcx^{-1})) \in K^*$ also has finite order.

(c) Let $u \in \mathbb{Z}_K^*$ such that $\mathrm{nr}_{Q/K}(c) = u^2$. Thus $u \in \mathbb{Z}_K^*$ has finite order which implies $u \in \mathcal{B}^o(N) = N$. By the above, both a and $u^{-1}ca$ have vanishing traces. So $\mathrm{nr}_{Q/K}(u^{-1}c) = 1$ implies that a and $u^{-1}ca$ have the same minimal polynomial (over K). It follows from the Skolem-Noether theorem that there exists some $t \in Q^*$ such that $a^t = u^{-1}ca$. Finally

$$x^t = t^{-1}xtx^{-1}x = t^{-1}ata^{-1}x = u^{-1}caa^{-1}x = (gu)^{-1}y$$

shows $\langle N, x\rangle^t = \langle N, (gu)^{-1}y\rangle = \langle N, y\rangle$. □

In most cases, we will apply this result to the following situation.

Remark 2.2.17 Let $N < \mathrm{Sp}_{2n}(\mathbb{Q})$ be finite such that $Q := \mathrm{End}(\overline{N})$ is a quaternion skewfield with center K. If $\mathcal{B}^o(N)$ is not s.p.i.m.f., then the above lemma allows us to construct (up to conjugacy) all s.p.i.m.f. supergroups $G \rhd N$ satisfying one of the following conditions:

- $C_G(N) \not\subseteq \mathcal{B}^o(N)$
- $\mathcal{B}^o(N)$ has even index in $\{g \in G \mid gx = xg \text{ for all } x \in K\}$.
(Note that this condition holds for example if $N = F^*(G)$ is assumed and Q is a totally definite quaternion algebra over $\mathbb{Q}$ as Theorem 2.2.15 shows).

More precisely, one can use the following algorithm:

(a) Set $X := \emptyset$.

(b) If $N = F^*(G)$ is assumed skip this step. Otherwise let $\mathfrak{M}_1, \dots, \mathfrak{M}_r$ be representatives of the conjugacy classes of maximal $\mathbb{Z}_K$-orders in Q. Further let $\mathfrak{M}_i^{*,1} = \{x \in \mathfrak{M}_i \mid \mathrm{nr}_{Q/K}(x) = 1\}$ be the torsion subgroup of $\mathfrak{M}_i^*$. (See [KV] for algorithms to compute these objects).

Include to the set X all elements $x \in \mathfrak{M}_i^{*,1}$ whose order is a prime power greater than 2, provided that X does not already have an element of the same order.

(c) Let U be the torsion subgroup of $\mathbb{Z}_K^* \cap \mathrm{nr}_{Q/K}(Q^*)$. For any uU^2 in U/U^2 find some c_u with $u = \mathrm{nr}_{Q/K}(c)$. Let C be the set of the c_u. (In most cases, Q will be a totally definite quaternion algebra, so one can choose $C = \{1\}$).

(d) For each class of outer automorphisms in $\mathrm{Out}(N)$ that is not realized in $\mathcal{B}^o(N)$ but its square is, compute one $x \in \mathrm{GL}_{2n}(\mathbb{Q})$ that realizes this automorphism.

If such an x exists, then include $\{cx \mid c \in C\}$ to X.

(e) Each group G satisfying the hypothesis contains (up to conjugacy) a subgroup $\langle N, x\rangle$ for some $x \in X$.

Since $\mathrm{End}(\overline{\langle \mathcal{B}^o(N), x\rangle})$ is a field, one can use the m-parameter argument (Corollary 2.2.12) to construct a representative of the conjugacy class of G.

Proof: We have to show that each G satisfying the hypothesis contains a subgroup conjugate to $\langle \mathcal{B}^o(N), x\rangle$ for some $x \in X$.
Suppose first that there exists some $g \in C_G(N) = C_G(\mathcal{B}^o(N))$ such that $g \notin \mathcal{B}^o(N)$. Taking appropriate powers, we can assume that the order of g is a prime power different from 2 since $-I_{2n} \in \mathcal{B}^o(N)$. So g is conjugate (in Q) to some $x \in X$. Hence $\langle \mathcal{B}^o(N), g\rangle$ and $\langle \mathcal{B}^o(N), x\rangle$ are conjugate in Q.
Suppose now $C_G(N) = C_G(\mathcal{B}^o(N)) \subseteq \mathcal{B}^o(N)$. Then by the assumption, there exists some $g \in G \setminus \mathcal{B}^o(N)$ that commutes with K and $g^2 \in \mathcal{B}^o(N)$. So the claim follows from Lemma 2.2.16. □

2.3 Algorithms

Let $G < \mathrm{GL}_m(\mathbb{Q})$ be finite. After a change of bases, we may suppose that $G < \mathrm{GL}_m(\mathbb{Z})$. We first explain how to compute representatives for the isomorphism classes of G-invariant lattices. Clearly $\mathcal{Z}(G)$ is closed under taking sums and intersections.
Suppose $L' \subseteq L \in \mathcal{Z}(G)$. Let $M_1, \ldots, M_s$ be the nontrivial p-Sylow groups of L/L'. Then G acts on L/L' as automorphisms of groups. In particular $L' = \bigcap_{i=1}^s L_i$ is the intersection of the G-invariant lattices $L_i = \oplus_{j \neq i}(L' + M_j)$ whose index in L is a prime power. Moreover, this intersection is unique.

Algorithm 2.3.1 ([PH84]) Let $\Lambda \subseteq \mathbb{Z}^{m\times m}$ be a $\mathbb{Z}$-order in a simple subalgebra A of $\mathbb{Q}^{m\times m}$. Further let $L = \mathbb{Z}^{1\times m}$ be the natural Λ-lattice. This algorithm returns all Λ-invariant sublattices of L that contain Lp^k for some $k \geq 1$ and a fixed prime p.
Input: Generators of the natural representation of Λ and a prime p.

(a) Using the meataxe (see [Par84]), find all p-modular constituents of the natural representation of Λ and the corresponding simple $\Lambda/p\Lambda$-modules $S_1, \ldots, S_k$.

(b) For each Λ - invariant lattice M found so far, compute all maximal Λ - invariant sublattices of M as kernels of $\Lambda/p\Lambda$-epimorphisms $M \to S_i$ for some $1 \leq i \leq s$.

(c) One continues the algorithm with these newly constructed lattices, provided they are not a scalar multiple of a lattice computed before.

Remark 2.3.2 ([Neb95, III.11]) In general, the above algorithm does not terminate. One has to specify some additional stopping conditions as follows.
Let $\Lambda := \langle G\rangle_{\mathbb{Z}}$ for some finite rationally irreducible subgroup $G < \mathrm{GL}_m(\mathbb{Z})$ where $E := \mathrm{End}(\overline{G})$ is a field. Since $\Lambda' := \langle \Lambda, \mathbb{Z}_E\rangle_{\mathbb{Z}}$ is an order, there exists some $L \in \mathcal{Z}(G)$ such that $\mathrm{End}_\Lambda(L) = \mathbb{Z}_K$. So without loss of generality $L = \mathbb{Z}^{1\times m}$.
Denote by $p_1, \ldots, p_s$ the prime divisors of $|G|$ and let $\mathfrak{p}_{i,1}, \ldots, \mathfrak{p}_{i,n_i}$ be the prime ideals over $p_i\mathbb{Z}_K$. These prime ideals act on the Λ'-invariant sublattices of L whose index is a power of p_i. Using the above algorithm, we find representatives $L_{i,1}, \ldots, L_{i,m_i}$ of the corresponding orbits.
Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be representatives of $\mathcal{C}l(\mathbb{Z}_K)$. Then

$$\mathcal{S}' := \{(\bigcap_{i=1}^{s} L_{i,j_i})\mathfrak{a}_j \mid 1 \leq j \leq h,\ 1 \leq i_j \leq m_i\}$$

is finite and it contains a system of representatives for the isomorphism classes of Λ'-invariant lattices.
Using the above algorithm again, for each $L' \in \mathcal{S}'$ one computes the finite set

$$\mathcal{Z}_o(L') := \{M = \bigcap_{i=1}^{s} M_i \mid M_i \in \mathcal{Z}(G), M_i \leq L', L'/M_i \text{ a } p_i\text{-group and } M\mathbb{Z}_E = L'\}\,.$$

Then $\mathcal{S} := \bigcup_{L'\in\mathcal{S}'} \mathcal{Z}_o(L')$ contains a representative of each isomorphism class of $\mathcal{Z}(G)$.

Proof: By the Jordan-Zassenhaus theorem ([Rei03, Chapter 26]) the number of isomorphism classes of Λ'-invariant lattices is finite. So $\mathcal{S}'$ is finite. Suppose $L' \leq L$ is Λ'-invariant, then [Rei03, Theorems 41.1 and 41.7] show that each prime p not dividing $|G|$, does not split $\overline{G}$ and the completion $\Lambda'_p := \Lambda' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order. Hence the completions $L' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are locally isomorphic. So there exists some fractional ideal $\mathfrak{a}$ of $\mathbb{Z}_E$ such that $L'\mathfrak{a} \leq L$ and $|L/L'\mathfrak{a}|$ is a product of the p_i. Then it follows from the definition of $\mathcal{S}'$ that $L'\mathfrak{a}$ is isomorphic to some lattice in $\mathcal{S}'$.
Suppose now $L' \in \mathcal{S}'$ and $M \in \mathcal{Z}_o(L')$. Then there are only finitely many isomorphism classes of such lattices M. Further $x \in E$ satisfies $Mx \in \mathcal{Z}_o(L')$ if and only if $x \in \mathbb{Z}_E^*$. Since $[\mathbb{Z}_E^* : \mathrm{End}_\Lambda(M)^*]$ is finite, it follows that $\mathcal{Z}_o(L')$ is finite.
Suppose now $L' \in \mathcal{Z}(G)$. Then there exists some $x \in E$ such that $L'\mathbb{Z}_E x \in \mathcal{S}'$. Furthermore $|L'\mathbb{Z}_E x / L'x|$ is a product of the p_i since $\mathrm{End}_{\Lambda_p}(L'_p)$ is maximal for all primes p not dividing $|G|$. Hence $L'x \in \mathcal{Z}_o(L'\mathbb{Z}_E x) \subseteq \mathcal{S}$ as claimed. □

In [PS97], Plesken and Souvignier describe an algorithm which computes automorphism groups of lattices. Together with the Minkowski bound (see Lemma 2.2.9), the m-parameter argument (see Remark 2.2.13) and the above algorithm we can thus find (up to conjugacy) all r.i.m.f. or s.i.m.f. supergroups of any finite matrix group G where $\mathrm{End}(\overline{G})$ is a field. Provided that we can test whether two given groups are conjugate:

Algorithm 2.3.3 The following algorithm tests whether two given s.i.m.f. groups $G_1, G_2 < \mathrm{Sp}_{2n}(\mathbb{Q})$ are conjugate. If so, it returns some $x \in \mathrm{GL}_{2n}(\mathbb{Q})$ such that $G_2 = G_1^x$. For $i = 1, 2$ let $E_i := \mathrm{End}(\overline{G_i})$.

- Let $d := \min\{\det(L, F) \mid F \in \mathcal{F}_{>0}(G_1) \text{ is primitive on } L \in \mathcal{Z}(G_1)\}$ and fix some primitive pair $(L, F) \in \mathcal{Z}(G_1) \times \mathcal{F}_{>0}(G_1)$ such that $\det(L, F) = d$.
- Let $L'_1, \ldots, L'_s$ be representatives the isomorphism classes of $\mathcal{Z}(G_2)$.
- Two elements in
$$\mathcal{L}' := \bigcup_{1=1}^{s} \{(L'_i, F') \mid F' \in \mathcal{F}_{>0}(G_2) \text{ is primitive on } L'_i \text{ and } \det(L'_i, F') = d\}$$
are said to be equivalent if there exists some isometry t between them such that $E_2^t = E_2$. Let S be a set of representatives of the equivalence classes.
- If there exists some isometry x between (L, F) and some element of S such that $E_1^x = E_2$ then return x otherwise return *false*.

An algorithm for the required isometry tests is also given in [PS97].

Proof: It follows from the proof of Remark 2.2.13 that $\mathcal{L}'$ is finite. Thus the algorithm terminates. Suppose first that there exists some $x \in \mathrm{GL}_{2n}(\mathbb{Q})$ such that $(Lx, x^{-1}Fx^{-\mathrm{tr}}) \in S$ and $E_1^x = E_2$. Then by maximality

$$G_1^x = (\mathrm{Aut}_{E_1}(L, F))^x = \mathrm{Aut}_{E_1^x}(Lx, x^{-1}Fx^{-\mathrm{tr}}) = G_2 \,.$$

Conversely suppose that $G_2 = G_1^y$ for some $y \in \mathrm{GL}_{2n}(\mathbb{Q})$. Then $(Ly, y^{-1}Fy^{-\mathrm{tr}}) \in \mathcal{Z}(G_2) \times \mathcal{F}_{>0}(G_2)$ is primitive and has determinant d. So there exists some $e \in E_2$ such that $(Lye, (ye)^{-1}F(ye)^{-\mathrm{tr}}) \in \mathcal{L}'$. By definition of S, there exists some $t \in \mathrm{GL}_{2n}(\mathbb{Q})$ such that $E_2^t = E_2$ and $(Lyet, (yet)^{-1}F(yet)^{-\mathrm{tr}}) \in S$. So $x := yet$ furnishes an isometry between (L, F) and some element of S satisfying $E_1^x = E_2$ as claimed. □

2.4 Some notation

Definition and Remark 2.4.1 For $i = 1, 2$ let $G_i < \mathrm{GL}_{m_i}(\mathbb{Q})$ be finite irreducible matrix groups and let $E_i := \mathrm{End}(\overline{G_i})$. The tensor product

$$G_1 \otimes G_2 := \{g_1 \otimes g_2 \mid g_i \in G_i\} < \mathrm{GL}_{m_1 m_2}(\mathbb{Q})$$

is isomorphic to some central product $G_1 \curlyvee_C G_2$. This matrix group is usually not irreducible, since $E_1 \otimes E_2$ might not be a skewfield. The following definition is used to construct a (usually irreducible) direct summand of $G_1 \otimes G_2$.

Let Q be a maximal common subalgebra of E_1 and E_2 of dimension $d = \dim_{\mathbb{Q}}(Q)$. Then $G_1 \underset{Q}{\otimes} G_2 := \triangle^{\mathbb{Q}}(\triangle_Q(G_1) \otimes \triangle_Q(G_2)) < \mathrm{GL}_{\frac{m_1 m_2}{d}}(\mathbb{Q})$ is isomorphic to $G_1 \curlyvee_C G_2$ with $C = G_1 \cap Q \cap G_2$. Moreover $G_1 \underset{Q}{\otimes} G_2$ contains a normal subgroup H_i isomorphic to G_i where the restriction of the natural character onto H_i is a multiple of the natural character of G_i. To simplify the notation, we use the following conventions: If $Q \simeq \mathbb{Q}(\alpha)$ is a field, we write $\underset{\alpha}{\otimes}$ instead of $\underset{\mathbb{Q}(\alpha)}{\otimes}$. If $Q \simeq \mathcal{Q}_{\alpha, P_1, \dots, P_r}$ is a quaternion algebra with center K ramified only at the (finite or infinite) places P_i, we write $\underset{P_1, \dots, P_r}{\otimes}$ if $K = \mathbb{Q}$ and we write $\underset{\alpha, P_1, \dots, P_r}{\otimes}$ if $K \simeq \mathbb{Q}(\alpha)$. Finally, if $d = m_2$, then G_2 embeds into E_1^*. In this case, we use $\circ$ instead of $\underset{Q}{\otimes}$.

Note that this construction does not always give irreducible matrix groups. We will only encounter the following examples (in dimension 16) where this is not the case: In these cases (after exchanging the G_i) we have $G_1 \simeq C_{10}$, $m_1 = 4$ and $E_1 \simeq \mathbb{Q}(\zeta_{10})$. Further E_2 is a totally definite quaternion algebra which is split by E_1. Thus the maximal common subalgebra of E_1 and E_2 equals $\mathbb{Q}$ and $G_1 \otimes G_2$ is reducible. In these cases, we make a slight abuse of notation and denote by $G_1 \underset{\sqrt{5}'}{\otimes} G_2$ an irreducible direct summand of $G_1 \otimes G_2$ (this was first introduced in [NP95, page 91]). This notation has the advantage, that in all our examples the dimension formula still holds, that is: $G_1 \underset{\sqrt{5}'}{\otimes} G_2$ denotes a subgroup of $\mathrm{GL}_k(\mathbb{Q})$ where $k = 2m_2 = \frac{m_1 m_2}{\dim_{\mathbb{Q}}(\mathbb{Q}(\sqrt{5}))}$.

Definition and Remark 2.4.2 We will use the following conventions for the names (of the conjugacy classes) of rational/symplectic/quaternionic matrix groups.

- If $G < \mathrm{GL}_m(Q)$ for some skewfield Q, then $\pm G := \langle G, -I_m \rangle < \mathrm{GL}_m(Q)$.
- The symbols $A_n, B_n, F_4 = D_4, E_6, E_7, E_8$ denote (the automorphism groups of) the corresponding root lattices.
- If a maximal finite irreducible rational/symplectic/quaternionic primitive matrix group $G < \mathrm{GL}_m(\mathbb{Q})$ is not a tensor product of rational, symplectic or quaternionic matrix groups of smaller dimension, it is denoted by ${}_E[Con]_k$ where $E \simeq \mathrm{End}(\overline{G})$ and $m = k \cdot \dim_{\mathbb{Q}}(E)$ since G can be identified with a subgroup of $\mathrm{GL}_k(E)$. Further, Con describes some construction of G from building blocks (O_p and components; see Tables 2.5.1 and 2.5.2) or smaller matrix groups by taking generalized Bravais groups, tensor products or group extensions.

  Again, if $E \simeq \mathbb{Q}$ we omit the subscript E. If $E \simeq \mathbb{Q}(\sqrt{\alpha})$ is a field, we write ${}_\alpha[Con]_k$ and if $E \simeq \mathcal{Q}_{\alpha,P_1,\ldots,P_r}$ is a quaternion algebra over K ramified only at the places P_i we write ${}_{P_1,\ldots,P_r}[Con]_k$ if $K = \mathbb{Q}$ and ${}_{\alpha,P_1,\ldots,P_r}[Con]_k$ if $K \simeq \mathbb{Q}(\alpha)$.
- If G is symplectic imprimitive, then it is conjugate to the wreath product of some s.p.i.m.f. subgroup $H < \mathrm{Sp}_{\frac{2n}{k}}(\mathbb{Q})$ with S_k. In this case, we write H^k.

These conventions are consistent with the ones of [Ple91, NP95, Neb95, Neb96, Neb98a]. Moreover, for matrix groups described in loc. cit. we use the names given there.

Example 2.4.3

- C_{30} denotes the torsion subgroup of $\mathbb{Q}(\zeta_{30})^*$ which gives rise to a subgroup of $\mathrm{GL}_8(\mathbb{Q})$ with $\mathbb{Q}(\zeta_{30})$ as commuting algebra. Clearly, there exists only one split extension of this matrix group by C_4 that fixes $\mathbb{Q}(\sqrt{-15})$. One finds that this group is s.p.i.m.f. and we denote it by ${}_{\sqrt{-15}}[C_{30}\!:\!C_4]_4 < \mathrm{Sp}_8(\mathbb{Q})$.
- The group $D_8 \otimes C_4$ is a subgroup of $\mathrm{GL}_4(\mathbb{Q})$ and it is one of the building blocks described in Table 2.5.2. Taking its generalized Bravais group, one obtains a s.p.i.m.f. subgroup of $\mathrm{Sp}_4(\mathbb{Q})$ denoted by its isomorphism type ${}_i[(D_8 \otimes C_4).S_3]_2$.

  Tensoring this group with $\mathrm{Aut}(A_2) < \mathrm{GL}_2(\mathbb{Q})$ gives a s.p.i.m.f. subgroup of $\mathrm{Sp}_8(\mathbb{Q})$ which we call ${}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_2$.

  Lemma 2.1.21 shows that taking wreath products of these two s.p.i.m.f. groups also yields maximal finite groups denoted by ${}_i[(D_8 \otimes C_4).S_3]_2^k < \mathrm{Sp}_{4k}(\mathbb{Q})$ and $({}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_2)^\ell < \mathrm{Sp}_{8\ell}(\mathbb{Q})$.

As in the classification of the maximal finite rational and quaternionic matrix groups, we frequently have to construct a matrix group H that contains an irreducible subgroup $G_1 \underset{Q}{\otimes} G_2$ of index 2. Most of these extensions come from one of the following three constructions:

Definition and Remark 2.4.4 ([NP95, (II.4) Proposition]) For $i = 1, 2$ let $G_i < \mathrm{GL}_{m_i}(\mathbb{Q})$ be two finite irreducible matrix groups and let $E_i := \mathrm{End}(\overline{G_i})$. Further let Q be a maximal common subalgebra of E_1 and E_2 such that $G_1 \underset{Q}{\otimes} G_2 < \mathrm{GL}_m(\mathbb{Q})$ is irreducible. We view G_i as a subgroup of $G_1 \underset{Q}{\otimes} G_2$ and $\overline{G_i}$ as a subalgebra of $\overline{G_1 \underset{Q}{\otimes} G_2}$.

(a) Suppose there exist some units $a_i \in \overline{G_i} \setminus G_i$ and a positive squarefree integer p such that $G_i^{a_1} = G_i$ and $p^{-1}a_i^2 \in G_i$. Then

$$G_1 \overset{2(p)}{\underset{Q}{\otimes}} G_2 := \left\langle G_1 \underset{Q}{\otimes} G_2,\ p^{-1}a_1a_2 \right\rangle$$

is an irreducible subgroup of $\mathrm{GL}_m(\mathbb{Q})$ that contains $G_1 \underset{Q}{\otimes} G_2$ with index 2.

(b) Suppose $G_1 \underset{Q}{\otimes} G_2 \subseteq A \subset B \subseteq \mathbb{Q}^{m\times m}$ is a chain of simple algebras with $B = A \oplus xA$ for some $x \in B$ such that $x^2 = \pm 1$, $xA = Ax$ and $x\overline{G_i} = \overline{G_i}x$. If there exist some units $a_i \in \overline{G_i}$ and a positive squarefree integer p such that $G_i^{a_i x} = G_i$ and $p^{-1}(a_i x)^2 \in G_i$ then

$$G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2 := \left\langle G_1 \underset{Q}{\otimes} G_2,\ p^{-1}a_1a_2x \right\rangle$$

is an irreducible subgroup of $\mathrm{GL}_m(\mathbb{Q})$ that contains $G_1 \underset{Q}{\otimes} G_2$ with index 2.

(c) Suppose $A \subseteq C_{\mathbb{Q}^{m\times m}}(\overline{G_2})$ is a simple subalgebra that contains $\overline{G_1}$. If there exist units $a_1 \in A$, $a_2 \in \overline{G_2}$ and a positive squarefree integer p such that $G_i^{a_i} = G_i$ and $p^{-1}a_i^2 \in G_i$, then

$$G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2 := \left\langle G_1 \underset{Q}{\otimes} G_2,\ p^{-1}a_1a_2 \right\rangle$$

is an irreducible subgroup of $\mathrm{GL}_m(\mathbb{Q})$ that contains $G_1 \underset{Q}{\otimes} G_2$ with index 2.

In addition, the symbols are simplified as follows:

- If $m_2 = \dim_{\mathbb{Q}}(Q)$, we write $\overset{2(p)}{\circ}$, $\overset{2(p)}{\square}$ and $\overset{2(p)}{\sqsupset}$ instead of $\overset{2(p)}{\underset{Q}{\otimes}}$, $\overset{2(p)}{\underset{Q}{\boxtimes}}$ and $\overset{2(p)}{\underset{Q}{\boxtimes}}$ respectively.
- If $Q \simeq \mathbb{Q}$, the subscript Q is completely omitted. If $Q \simeq \mathbb{Q}(\alpha)$ is a field, we write α instead of Q. If $Q \simeq \mathcal{Q}_{\alpha,P_1,\ldots,P_s}$ is a quaternion algebra with center K ramified only at the places $P_1, \ldots, P_r$, we replace the subscript Q by $P_1, \ldots, P_s$ if $K = \mathbb{Q}$ or by $\alpha, P_1, \ldots, P_s$ if $K \simeq \mathbb{Q}(\alpha)$.
- If $p = 1$, we omit (p) in the above symbols.

In most cases, two different extensions of $G_1 \underset{Q}{\otimes} G_2$ by C_2 can be distinguished by these symbols together with their commuting algebras. Note however, this is not always the case due to the solvability of certain relative norm equations:

Example 2.4.5 Let $G_1 = {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ be the torsion subgroup of any maximal order of the quaternion algebra $\mathcal{Q}_{\sqrt{5},\infty}$ with center $\mathbb{Q}(\sqrt{5})$ ramified only at the two infinite places. Since its commuting algebra is another copy of $\mathcal{Q}_{\sqrt{5},\infty}$, we find some $x \in E_1$ of order 4. Let G_2 be the subgroup generated by x. Thus $G := G_1 \circ G_2$ denotes an irreducible subgroup of $\mathrm{GL}_8(\mathbb{Q})$ with commuting algebra $E \simeq \mathbb{Q}(i, \sqrt{5})$. Since $\mathrm{Out}(G_1) \simeq \mathrm{Out}(G_2) \simeq C_2$ there exists only one class of outer automorphism that acts nontrivially on G_1 and G_2. It can be represented by an automorphism of order 2, thus there exists only two possible such extensions.
Construction (b) can be used to find some $\alpha = a_1 a_2 x$ that acts like the nontrivial outer automorphism on the G_i. Clearly $H_1 := \langle G, \alpha \rangle$ has commuting algebra $\mathbb{Q}(\sqrt{-5})$. But then $u := \frac{1+\sqrt{5}}{2} \in E$ satisfies $\mathrm{Nr}_{E/\mathbb{Q}(\sqrt{-5})}(u) = u \cdot u^\alpha = -1$. In particular $H_2 := \langle G, u\alpha \rangle$ is an extension of G, not isomorphic to H_1 but with the same commuting algebra.

Remark 2.4.6 If (like in the previous example) there exist only two s.i.m.f. extensions one split and one not, we write $2_+(p)$ for the split and $2_-(p)$ for the nonsplit extension. Thus the groups H_i from above are labeled ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_+}{\square} C_4$ and ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_-}{\square} C_4$ (see Theorem 4.5.1).

If there are several such extensions yielding s.i.m.f. groups that cannot be distinguished by the above, we describe the automorphisms induced by $p^{-1}a_1a_2$ or $p^{-1}a_1a_2x$ explicitly when the groups are introduced for the first time.

Frequently, one has some s.p.i.m.f. matrix group G that contains an irreducible subgroup $N := G_1 \otimes_Q G_2$ of index 2 and one wants to know whether G can be obtained from N by one of the constructions above. We want to discuss one example, to give some idea how this can be done. Clearly (b) is the most difficult construction, since it requires to find 4 parameters p, a_1, a_2, x whereas the others only require 3.

Example 2.4.7 Suppose $G < \mathrm{Sp}_8(\mathbb{Q})$ is s.p.i.m.f. and it contains a normal subgroup $N := {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes D_{10}$ with index 2 and $\mathrm{End}(\overline{G}) \simeq \mathbb{Q}(\sqrt{-10})$. Then $E := \mathrm{End}(\overline{N}) \simeq \mathbb{Q}(\sqrt{-2}, \sqrt{5})$ and every $g \in G \setminus N$ acts on E by $\sqrt{-2}^g = -\sqrt{-2}$ and $\sqrt{5}^g = -\sqrt{5}$. In particular, this rules out constructions (a) and (c) from Definition 2.4.4.
Let $\varphi\colon G_1 := {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \xrightarrow{\sim} \langle \mathrm{SL}_2(3), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \subset \mathbb{F}_3^{2\times 2}$ and set $z := \varphi^{-1}(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$. Further let $G_2 := D_{10}$ be generated by some $a, b \in N$ that satisfy $a^5 = b^2 = (ab)^2 = 1$.
Since $\mathrm{Out}(G_i) \simeq C_2$ we find some $g \in G$ which satisfies $a^g = g^2 = b, z^g = -z$ and $g \in C_G(\mathcal{B}^o(O_2(G)))$ where $\mathcal{B}^o(O_2(G)) \simeq \mathrm{SL}_2(3)$. (Note that we might have to exchange b by zb).
Now suppose $g = p^{-1}a_1a_2x$ as described in Definition 2.4.4 (b). Then

$$b = g^2 = p^{-2}a_1a_2x^2a_1^xa_2^x = \pm p^{-2}(a_1a_1^x)(a_2a_2^x)$$

since the $a_i \in \overline{G_i}$ commute and $x^2 = \pm 1$ by our assumption. The action of x on $\overline{G_2}$ is explicitly known and one finds (using Groebner bases in MAGMA) that $a_2 := (1 + a^2 + a^3) + b$ satisfies $a_2a_2^x = b$. In particular, if we set $p = a_1 = 1$ and $x := a_2^{-1} \cdot g$, then $x^2 = 1$. One easily checks that these elements meet all requirements and thus $G \simeq {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$ (see Theorem 4.9.1).

2.5 Some tables

2.5.1 Candidates for the layer

Table 2.5.1 *This table lists all quasisimple finite rational irreducible matrix groups N in $\mathrm{GL}_m(\mathbb{Q})$ for $2 \le m \le 22$.*
The representations with $m \ge 12$ and Schur index + are omitted since they can not be a subgroup of $E(G)$ for some s.p.i.m.f. subgroup $G \le \mathrm{Sp}_{2n}(\mathbb{Q})$ for $1 \le n \le 11$.

N	$\mathcal{B}^o(N)$	m	*character*	$\mathrm{End}(\overline{N})$	*omit*
Alt_5	$\pm S_5 = A_4$	4	χ_4	$\mathbb{Q}$	$-$
Alt_5	$\pm S_6 = A_5$	5	χ_5	$\mathbb{Q}$	$+$
Alt_6	$\pm S_6 = A_5$	5	χ_{5a} *or* χ_{5b}	$\mathbb{Q}$	$-$
Alt_5	$\pm\mathrm{Alt}_5$	6	χ_{3ab}	$\mathbb{Q}(\sqrt{5})$	$-$
$\mathrm{L}_2(7)$	$\pm\mathrm{L}_2(7)$	6	χ_{3ab}	$\mathbb{Q}(\sqrt{-7})$	$-$
$\mathrm{L}_2(7)$	$\pm\mathrm{L}_2(7)$	6	χ_6	$\mathbb{Q}$	$-$
Alt_7	$\pm S_7 = A_6$	6	χ_6	$\mathbb{Q}$	$-$
$U_4(2)$	$\pm U_4(2){:}2 = E_6$	6	χ_6	$\mathbb{Q}$	$-$
Alt_8	$\pm S_8 = A_7$	7	χ_7	$\mathbb{Q}$	$-$
$U_3(3)$	$\pm S_6(2) = E_7$	7	χ_7	$\mathbb{Q}$	$+$
$\mathrm{L}_2(7)$	$\pm S_6(2) = E_7$	7	χ_7	$\mathbb{Q}$	$+$
$\mathrm{L}_2(8)$	$\pm S_6(2) = E_7$	7	χ_{7a}	$\mathbb{Q}$	$+$
$S_6(2)$	$\pm S_6(2) = E_7$	7	χ_7	$\mathbb{Q}$	$-$
$\mathrm{SL}_2(5)$	$\mathrm{SL}_2(5)$	8	$2\chi_{2ab}$	$\mathcal{Q}_{\sqrt{5},\infty}$	$-$
$\mathrm{SL}_2(5)$	$\mathrm{SL}_2(9)$	8	$2\chi_4$	$\mathcal{Q}_{\infty,3}$	$+$
$\mathrm{L}_2(7)$	$\pm\mathrm{L}_2(7).2 = M_{8,3}$	8	χ_8	$\mathbb{Q}$	$-$
$\mathrm{SL}_2(7)$	$\mathrm{SL}_2(7)$	8	χ_{4ab}	$\mathbb{Q}(\sqrt{-7})$	$-$
$\mathrm{SL}_2(9)$	$\mathrm{SL}_2(9)$	8	$2\chi_{4a}$ *or* $2\chi_{4b}$	$\mathcal{Q}_{\infty,3}$	$-$
$\mathrm{L}_2(8)$	$2.O_8^+(2).2 = E_8$	8	χ_8	$\mathbb{Q}$	$+$
$2.\mathrm{Alt}_7$	$2.\mathrm{Alt}_7$	8	χ_{4ab}	$\mathbb{Q}(\sqrt{-7})$	$-$
$2.\mathrm{Alt}_8$	$2.O_8^+(2).2 = E_8$	8	χ_8	$\mathbb{Q}$	$+$
$\mathrm{Sp}_4(3)$	$\mathrm{Sp}_4(3) \circ C_3$	8	χ_{4ab}	$\mathbb{Q}(\sqrt{-3})$	$-$
Alt_9	$2.O_8^+(2).2 = E_8$	8	χ_8	$\mathbb{Q}$	$+$
$2.\mathrm{Alt}_9$	$2.O_8^+(2).2 = E_8$	8	χ_8	$\mathbb{Q}$	$+$
$2.\mathrm{Sp}_6(2)$	$2.O_8^+(2).2 = E_8$	8	χ_8	$\mathbb{Q}$	$+$
$2.O_8^+(2)$	$2.O_8^+(2).2 = E_8$	8	χ_8	$\mathbb{Q}$	$-$
Alt_6	$\pm S_{10} = A_9$	9	χ_9	$\mathbb{Q}$	$+$
Alt_{10}	$\pm S_{10} = A_9$	9	χ_9	$\mathbb{Q}$	$-$
$\mathrm{L}_2(11)$	$\pm\mathrm{L}_2(11)$	10	χ_{5ab}	$\mathbb{Q}(\sqrt{-11})$	$-$
$U_4(2) = S_4(3)$	$\pm S_4(3) \circ C_3$	10	χ_{5ab}	$\mathbb{Q}(\sqrt{-3})$	$-$
$\mathrm{L}_2(11)$	$\pm\mathrm{L}_2(11){:}2 = A_{10}^{(2)}$	10	χ_{10a}	$\mathbb{Q}$	$-$
$\mathrm{L}_2(11)$	$\pm\mathrm{L}_2(11)$	10	χ_{10b}	$\mathbb{Q}$	$-$
Alt_6	$\pm S_6$	10	χ_{10}	$\mathbb{Q}$	$-$
Alt_{11}	$\pm S_{11} = A_{10}$	10	χ_{10}	$\mathbb{Q}$	$-$
M_{11}	$\pm S_{11} = A_{10}$	10	χ_{10a}	$\mathbb{Q}$	$+$

Alt_{12}	$\pm S_{12} = A_{11}$	11	χ_{11}	$\mathbb{Q}$	$-$
M_{11}	$\pm M_{11}$	11	χ_{11}	$\mathbb{Q}$	$-$
$\mathrm{L}_2(11)$	$\pm S_{12} = A_{11}$	11	χ_{11}	$\mathbb{Q}$	$+$
M_{12}	$\pm S_{12} = A_{11}$	11	χ_{11a} *or* χ_{11b}	$\mathbb{Q}$	$+$
$\mathrm{SL}_2(5)$	$\mathrm{SL}_2(5)$	12	$2\chi_6$	$\mathcal{Q}_{\infty,2}$	$-$
$6.U_4(3)$	$6.U_4(3).2$	12	χ_{12ab}	$\mathbb{Q}(\sqrt{-3})$	$-$
$3.\mathrm{Alt}_6$	$\pm 3.\mathrm{Alt}_6$	12	$\chi_{3ab} + \chi'_{3ab}$	$\mathbb{Q}(\sqrt{-3}, \sqrt{5})$	$-$
$3.\mathrm{Alt}_6$	$\pm 3.\mathrm{Alt}_6$	12	$\chi_6 + \chi'_6$	$\mathbb{Q}(\sqrt{-3})$	$-$
$\mathrm{SL}_2(11)$	$\mathrm{SL}_2(11)$	12	χ_{6ab}	$\mathbb{Q}(\sqrt{-11})$	$-$
$3.\mathrm{Alt}_7$	$6.U_4(3).2$	12	$\chi_6 + \chi'_6$	$\mathbb{Q}(\sqrt{-3})$	$+$
$U_3(3)$	$\pm U_3(3)$	12	$2\chi_6$	$\mathcal{Q}_{\infty,3}$	$-$
$6.L_4(3)$	$6.L_4(3)$	12	$\chi_6 + \chi'_6$	$\mathbb{Q}(\sqrt{-3})$	$-$
$U_3(3)$	$U_3(3) \circ C_4$	14	χ_{7ab}	$\mathbb{Q}(i)$	$-$
$\mathrm{SL}_2(7)$	$\mathrm{SL}_2(7)$	16	$2\chi_8$	$\mathcal{Q}_{\infty,3}$	$-$
$\mathrm{L}_2(17)$	$\pm \mathrm{L}_2(17).2$	16	χ_{16}	$\mathbb{Q}$	$-$
$\mathrm{L}_2(16)$	$\pm S_{17}$	16	χ_{16}	$\mathbb{Q}$	$+$
$2.\mathrm{Alt}_{10}$	$2.\mathrm{Alt}_{10}$	16	χ_{16}	$\mathbb{Q}$	$-$
Alt_{17}	$\pm S_{17}$	16	χ_{16}	$\mathbb{Q}$	$-$
$3.\mathrm{Alt}_6$	$\pm 3.M_{10}$	18	χ_{9ab}	$\mathbb{Q}(\sqrt{-3})$	$-$
$\mathrm{L}_2(19)$	$\pm \mathrm{L}_2(19)$	18	χ_{9ab}	$\mathbb{Q}(\sqrt{-19})$	$-$
$\mathrm{SL}_2(11)$	$\mathrm{SL}_2(11)$	20	$2\chi_{10}$	$\mathcal{Q}_{\infty,2}$	$-$
Alt_7	$\pm \mathrm{Alt}_7$	20	χ_{10ab}	$\mathbb{Q}(\sqrt{-7})$	$-$
M_{11}	$2.M_{12}\!:\!2$	20	χ_{10bc}	$\mathbb{Q}(\sqrt{-2})$	$+$
$2.M_{12}$	$2.M_{12}\!:\!2$	20	χ_{10ab}	$\mathbb{Q}(\sqrt{-2})$	$-$
$2.M_{22}$	$2.M_{22}\!:\!2$	20	χ_{10ab}	$\mathbb{Q}(\sqrt{-7})$	$-$
$U_4(2)$	$\pm U_4(2) \circ C_3$	20	χ_{10ab}	$\mathbb{Q}(\sqrt{-3})$	$-$
$U_5(2)$	$\pm U_5(2)$	20	$2\chi_{10}$	$\mathcal{Q}_{\infty,2}$	$-$
$2.L_3(4)$	$2.L_3(4)\!:\!2_2$	20	χ_{10ab}	$\mathbb{Q}(\sqrt{-7})$	$-$
$\mathrm{SL}_2(19)$	$\mathrm{SL}_2(19)$	20	χ_{10ab}	$\mathbb{Q}(\sqrt{-19})$	$-$
$U_5(2)$	$\pm U_5(2) \circ C_3$	22	χ_{11ab}	$\mathbb{Q}(\sqrt{-3})$	$-$
$\mathrm{L}_2(23)$	$\pm \mathrm{L}_2(23)$	22	χ_{11ab}	$\mathbb{Q}(\sqrt{-23})$	$-$

If N is not normal in $\mathcal{B}^o(N)$, then N is not a normal subgroup of any s.p.i.m.f. matrix group (see Lemma 2.1.23). These cases are indicated in the last row with a $+$.
Note also that the above characters are indexed by their degrees and not the numbers given in the ATLAS [CCN+85].

Proof: This table can be taken from [Neb98a, Table 9.1]. Alternatively, the representations can be taken from [Nic06] which is based on [HM01]. For the generalized Bravais groups, one computes the order $\Lambda_\infty(N) \subset \mathbb{Q}^{m \times m}$ (cf. Section 2.1.3) using linear algebra over $\mathbb{Z}$. Then one computes some $\Lambda_\infty(N)$-invariant lattices and takes the intersection A of their automorphism groups wrt. the full form space $\mathcal{F}(N)$. If $N \trianglelefteq A$ then $\mathcal{B}^o(N) = A$. Otherwise, one computes $\mathcal{B}^o(N)$ using the isomorphism classes of $\mathcal{Z}(\Lambda_\infty(N))$ using Remark 2.3.2. This method works for all the groups N from above except $\mathrm{SL}_2(5)$ with character $2\chi_4$ since here $N \ntrianglelefteq \mathcal{B}^o(N)$ and $\mathrm{End}(\overline{N})$ is not a field.

In this case, one can use the algorithm [Neb95, III.11] to compute the isomorphism classes of $\Lambda_\infty(N)$-invariant lattices. □

2.5.2 Candidates for the Fitting subgroup

Table 2.5.2 *All candidates for $N := O_p(G)$ of an irreducible symplectic primitive matrix group G are given by*

N	$\mathcal{B}^o(N)$	$\dim_{\mathbb{Q}}(\overline{N})$	$\mathrm{End}(\overline{N})$
C_{p^m}	$\pm N$	$p^{m-1}(p-1)$	$\mathbb{Q}(\zeta_{p^m})$
$p_+^{1+2n}, \quad (p>2)$	$\pm N.\mathrm{Sp}_{2n}(p)$	$p^n(p-1)$	$\mathbb{Q}(\zeta_p)$
2_+^{1+2n}	$N.O_{2n}^+(2)$	2^n	$\mathbb{Q}$
2_-^{1+2n}	$N.O_{2n}^-(2)$	2^{n+1}	$\mathcal{Q}_{\infty,2}$
$p_+^{1+2n} \curlyvee C_{p^m}, \quad (m>1)$	$\pm N.\mathrm{Sp}_{2n}(p)$	$p^{m+n-1}(p-1)$	$\mathbb{Q}(\zeta_p)$
$2_+^{1+2n} \curlyvee D_{2^m}, \quad (m>3)$	$N.\mathrm{Sp}_{2n}(2)$	2^{n+m-2}	$\mathbb{Q}(\theta_{2^{m-1}})$
$2_+^{1+2n} \curlyvee Q_{2^m}, \quad (m>3)$	$N.\mathrm{Sp}_{2n}(2)$	2^{n+m-1}	$\mathcal{Q}_{\theta_{2^{m-1}},\infty}$
$2_+^{1+2n} \curlyvee QD_{2^m}, \quad (m>3)$	$N.\mathrm{Sp}_{2n}(2)$	2^{n+m-2}	$\mathbb{Q}(\zeta_{2^{m-1}} - \zeta_{2^{m-1}}^{-1})$

In the last three rows $n = 0$ is allowed. In these cases N is D_{2^m}, Q_{2^m} or QD_{2^m} respectively.
Note that $\mathcal{B}^o(N)$ is only correct under the assumption that $N \trianglelefteq G$.

Proof: All abelian characteristic subgroups of N are cyclic by Corollary 2.1.16. Thus N must be isomorphic to a subgroup from above by a theorem of Ph. Hall (see Theorem 2.1.17). These groups have only one faithfull rational irreducible representation. The generalized Bravais groups $\mathcal{B}^o(N)$ of these representations have been computed in [Neb98a, Chapter 8] (under the assumption that $N \trianglelefteq G$). □

2.5.3 Tables for number fields

Table 2.5.3 *This table contains information on cyclotomic number fields that is needed to construct the s.i.m.f. supergroups of some irreducible cyclic subgroups of order m.*

The second column contains the set $\mathcal{S}_m := \tilde{\Pi}(\mathbb{Q}(\theta_m), m) \setminus \Pi(m)$ of "additional primes"; if not empty. The third column contains the decomposition of $p\mathbb{Z}[\theta_m]$ into prime ideals for various primes p. The last column contains generators for all minimal totally complex subfields of $\mathbb{Q}(\zeta_m)$.

The fourth column contains the narrow class group $\mathcal{C}l^+(\mathbb{Z}[\theta_m])$; if not trivial. It is given as follows: Let $\psi_m \colon \mathcal{C}l^+(\mathbb{Z}[\theta_m]) \xrightarrow{\sim} \bigoplus_i \mathbb{Z}/a_{m,i}\mathbb{Z}$ be an isomorphism where the $a_{m,i}$ are elementary divisors. The first row lists the $a_{m,i}$. The following rows list the nonzero images of the prime ideals from above under ψ_m. The superscript $(*)$ means all ideals of a given norm.

m	$\mathcal{S}_m$	*prime decomposition in* $\mathbb{Z}[\theta_m]$	$\mathcal{C}l^+(\mathbb{Z}[\theta_m])$		*minimal totally complex subfields of* $\mathbb{Q}(\zeta_m)$
8		$(2)=\mathfrak{p}_2^2$, (3), (5)			i, $\sqrt{-2}$
12		$(2)=\mathfrak{p}_2^2$, $(3)=\mathfrak{p}_3^2$, (5)		2	i, $\sqrt{-3}$
			$\mathfrak{p}_2,\mathfrak{p}_3$	1	
14		(2), (3), (5), $(7)=\mathfrak{p}_7^3$			$\sqrt{-7}$
16		$(2)=\mathfrak{p}_2^4$, (3), (5)			i, $\sqrt{-2}$, $\zeta_{16}-\zeta_{16}^{-1}$
18		(2), $(3)=\mathfrak{p}_3^3$, (5), (7)			$\sqrt{-3}$
20		$(2)=\mathfrak{p}_2^2$, (3), $(5)=\mathfrak{p}_5^4$		2	i, $\sqrt{-5}$, ζ_{10}
			$\mathfrak{p}_2$	1	
24		$(2)=\mathfrak{p}_2^4$, $(3)=\mathfrak{p}_3^2$, $(5)=\mathfrak{p}_5\mathfrak{p}_5'$		2	i, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-6}$
			$\mathfrak{p}_2$	1	
28		$(2)=\mathfrak{p}_2^2$, $(3)=\mathfrak{p}_3\mathfrak{p}_3'$, (5), $(7)=\mathfrak{p}_7^6$		2	i, $\sqrt{-7}$
			$\mathfrak{p}_3^{(*)},\mathfrak{p}_7$	1	
30		(2), $(3)=\mathfrak{p}_3^2$, $(5)=\mathfrak{p}_5^4$		2	$\sqrt{-3}$, $\sqrt{-15}$, ζ_{10}
			$\mathfrak{p}_3,\mathfrak{p}_5$	1	
32		$(2)=\mathfrak{p}_2^8$, (3), (5), $(7)=\mathfrak{p}_7\mathfrak{p}'_7$, (11), (13)			i, $\sqrt{-2}$, $\zeta_{16}-\zeta_{16}^{-1}$, $\zeta_{32}-\zeta_{32}^{-1}$
36		$(2)=\mathfrak{p}_2^2$, $(3)=\mathfrak{p}_3^6$, (5), (7)		2	i, $\sqrt{-3}$
			$\mathfrak{p}_2,\mathfrak{p}_3$	1	
40		$(2)=\mathfrak{p}_2^4$, $(3)=\mathfrak{p}_3\mathfrak{p}_3'$, $(5)=\mathfrak{p}_5^4$		2	i, $\sqrt{-2}$, $\sqrt{-5}$, $\sqrt{-10}$,
			$\mathfrak{p}_2$	1	ζ_{10}, $\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})$
42		(2), $(3)=\mathfrak{p}_3^2$, $(5)=\mathfrak{p}_5\mathfrak{p}_5'$, $(7)=\mathfrak{p}_7^6$		2	$\sqrt{-3}$, $\sqrt{-7}$
			$\mathfrak{p}_3,\mathfrak{p}_5^{(*)}$	1	
44		$(2)=\mathfrak{p}_2^2$, (3), $(5)=\mathfrak{p}_5\mathfrak{p}_5'$, $(11)=\mathfrak{p}_{11}^{10}$		2	i, $\sqrt{-11}$
			$\mathfrak{p}_2,\mathfrak{p}_{11}$	1	
48		$(2)=\mathfrak{p}_2^8$, $(3)=\mathfrak{p}_3^2$		2	i, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-6}$,
			$\mathfrak{p}_2$	1	$\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})$, $\zeta_{16}-\zeta_{16}^{-1}$
50		(2), (3), $(5)=\mathfrak{p}_5^{10}$			ζ_{10}
54		(2), $(3)=\mathfrak{p}_3^9$, (5), (7), (11), (13)			$\sqrt{-3}$
60	$\{11,59\}$	$(2)=\mathfrak{p}_2^2$, $(3)=\mathfrak{p}_3^2$, $(5)=\mathfrak{p}_5^4$,		2	i, $\sqrt{-3}$, $\sqrt{-5}$, $\sqrt{-15}$,
		$(11)=\mathfrak{p}_{11}\mathfrak{p}_{11}'\mathfrak{p}_{11}''\mathfrak{p}_{11}'''$	$-$	1	ζ_{10}, $\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})$
66		$(2)=\mathfrak{p}_2\mathfrak{p}_2'$, $(3)=\mathfrak{p}_3^2$, (5), (7), $(11)=\mathfrak{p}_{11}^{10}$		2	$\sqrt{-3}$, $\sqrt{-11}$
			$\mathfrak{p}_2^{(*)},\mathfrak{p}_{11}$	1	

All cyclotomic fields $\mathbb{Q}(\zeta_m)$ from above have class number 1. This implies that their maximal totally real subfields $\mathbb{Q}(\theta_m)$ also have class number 1 (see [Was96, Theorem 4.10]). Moreover $\mathrm{Nr}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}(\theta_m)}(\mathbb{Z}[\zeta_m]^*)=\mathbb{Z}[\theta_m]^*_{>0}$ in all cases. So one only has to consider a single class of totally positive units in the m-parameter argument (see Remark 2.2.13).

Proof: Follows from explicit calculations with MAGMA [BCP97]. □

Table 2.5.4 *The following table contains information on some totally real number fields K needed for the m-parameter argument in the sequel. The second column contains the set $\tilde{\Pi}(K,2)\backslash\{2\}$. The fourth column describes the narrow class group $\mathcal{C}l^+(\mathbb{Z}_K)$ in the notation of the previous table.*

K		*prime decomposition in* $\mathbb{Z}_K$	$\mathcal{C}l^+(\mathbb{Z}_K)$		$\mathbb{Z}^*_{K,>0}/(\mathbb{Z}^*_K)^2$
$\mathbb{Q}(\sqrt{2})$	$\emptyset$	$(2)=\mathfrak{p}_2^2,\ (3),\ (5)$	1		1
$\mathbb{Q}(\sqrt{3})$	$\emptyset$	$(2)=\mathfrak{p}_2^2,\ (3)=\mathfrak{p}_3^2,\ (5),\ (7)$		2	C_2
			$\mathfrak{p}_2,\mathfrak{p}_3$	1	
$\mathbb{Q}(\sqrt{5})$	$\emptyset$	$(2),\ (3),\ (5)=\mathfrak{p}_5^2$	1		1
$\mathbb{Q}(\sqrt{6})$	$\emptyset$	$(2)=\mathfrak{p}_2^2,\ (3)=\mathfrak{p}_3^2,\ (5)=\mathfrak{p}_5\mathfrak{p'}_5$		2	C_2
			$\mathfrak{p}_2,\mathfrak{p}_5^{(*)}$	1	
$\mathbb{Q}(\sqrt{7})$	$\{3\}$	$(2)=\mathfrak{p}_2^2,\ (3)=\mathfrak{p}_3\mathfrak{p}_3',\ (5),\ (7)=\mathfrak{p}_7^2$		2	C_2
			$\mathfrak{p}_2,\mathfrak{p}_3^{(*)}$	1	
$\mathbb{Q}(\sqrt{21})$	$\{3\}$	$(2),\ (3)=\mathfrak{p}_3^2,\ (5)=\mathfrak{p}_5\mathfrak{p}_5',\ (7)=\mathfrak{p}_7^2$		2	C_2
			$\mathfrak{p}_3,\mathfrak{p}_5^{(*)}$	1	
$\mathbb{Q}(\sqrt{-2}\cdot(\zeta_{10}-\zeta_{10}^{-1}))$	$\emptyset$	$(2)=\mathfrak{p}_2^2,\ (3),\ (5)=\mathfrak{p}_5^4$		2	C_2
			$\mathfrak{p}_2,\mathfrak{p}_5$	1	
$\mathbb{Q}(\sqrt{2},\sqrt{5})$	$\emptyset$	$(2)=\mathfrak{p}_2^2,\ (3)=\mathfrak{p}_3\mathfrak{p}_3',\ (5)=\mathfrak{p}_5^2$	1		1
$\mathbb{Q}(\sqrt{3},\sqrt{5})$	$\{11\}$	$(2)=\mathfrak{p}_2^2,\ (3)=\mathfrak{p}_3^2,\ (5)=\mathfrak{p}_5^2$		2	C_2
			–	1	

Proof: Follows from explicit calculations with MAGMA [BCP97]. □

Chapter 3

Some infinite s.i.m.f. families

3.1 Some subgroups of $\mathrm{Sp}_{p-1}(\mathbb{Q})$

Let $p \geq 5$ be a prime and write $p - 1 = 2^a \cdot o$ with o odd. In the spirit of [NP95, chapter V] we describe all s.i.m.f. supergroups G of C_p in dimension $p-1$ where C_p denotes the (up to conjugacy) unique cyclic matrix group of order p in $\mathrm{GL}_{p-1}(\mathbb{Q})$.

Clearly, one possibility is that G contains a normal subgroup conjugate to C_p. Since the commuting algebra of C_p is isomorphic to $\mathbb{Q}(\zeta_p)$ we have $G/\pm C_p \leq C_{2^a} \times C_o$. By Galois theory, G is symplectic if and only if $G/\pm C_p \leq C_o$ and therefore $G \simeq \pm C_p{:}C_o$ by maximality. (The group $\pm C_p{:}C_o$ has only one irreducible rational representation of degree $p-1$ and we identify the group with this representation.)

Another class of candidates are extensions of $\mathrm{L}_2(p)$. The smallest faithful irreducible complex representations of $\mathrm{L}_2(p)$ are of degree $\frac{p-1}{2}$ and algebraically conjugate. The corresponding character field is $\mathbb{Q}(\sqrt{\pm p})$ with the $-$ sign if and only if $p \equiv_4 -1$ (see [Dor71, §38]).
If $p \equiv_4 -1$ then $\mathrm{L}_2(p)$ contains a subgroup U isomorphic to $C_p{:}C_{\frac{p-1}{2}}$. The restriction of the natural representation of $\mathrm{L}_2(p)$ on U is irreducible and has the same character field $\mathbb{Q}(\sqrt{-p})$ ([Dor71, §38]). By [Lor71, Satz 1.2.1, p. 67], the Schur index of $\mathrm{L}_2(p)$ is equal to the Schur index of U which is 1. Thus $C_2 \times \mathrm{L}_2(p)$ has a unique $p-1$ dimensional rationally irreducible representation (denoted by ${}_{\sqrt{-p}}[\pm\mathrm{L}_2(p)]_{\frac{p-1}{2}}$ in the sequel) with commuting algebra $\mathbb{Q}(\sqrt{-p})$.

The next result shows that there are no further possibilites. More precisely:

Theorem 3.1.1 *Let $p \geq 11$ be prime and $G < \mathrm{Sp}_{p-1}(\mathbb{Q})$ such that p divides $|G|$. Write $p-1 = 2^a \cdot o$ with o odd.*
Then G is s.i.m.f. if and only if G is conjugate to $\begin{cases} \pm C_p{:}C_o & \text{if } p \equiv_4 +1 \\ {}_{\sqrt{-p}}[\pm\mathrm{L}_2(p)]_{\frac{p-1}{2}} & \text{if } p \equiv_4 -1 \end{cases}$.

Proof: Let $\pm I_{p-1} < G < \mathrm{Sp}_{p-1}(\mathbb{Q})$ such that p divides $|G|$. Further let $P \in \mathrm{Syl}_p(G)$. Then by Minkowski's bound, $|P| = p$. So the commuting algebra E of P is isomorphic to $\mathbb{Q}(\zeta_p)$. Thus $\pm I_{p-1} \leq Z(G) \leq C_G(P) = \pm P$. Since G must have a faithful irreducible complex character of degree $\leq \frac{p-1}{2}$ it follows from a theorem of H. Blau [Fei82, VIII Theorem 7.2, p. 365] that either $P \trianglelefteq G$ or $G/Z(G) \simeq \mathrm{L}_2(p)$. In the first case, G is conjugate to a subgroup of $\pm C_p : C_o$. In the second case, $Z(G) = \pm P$ would imply that $G = \pm P$ since $\pm P$ is self-centralizing in G. So $Z(G) = \pm I_{p-1}$. Since $\mathrm{L}_2(p)$ is perfect, G is either isomorphic to $\pm\mathrm{L}_2(p)$ or $\mathrm{SL}_2(p)$. So by [Dor71, Th. 38.1] this implies $p \equiv_4 -1$. But if $p \equiv_4 -1$ the real Schur indices of the $\frac{p-1}{2}$ dimensional complex characters of $\mathrm{SL}_2(p)$ are 2 (which can be computed from the character table of $\mathrm{SL}_2(p)$). Thus G is conjugate to ${}_{\sqrt{-p}}[\pm\mathrm{L}_2(p)]_{\frac{p-1}{2}}$. Since ${}_{\sqrt{-p}}[\pm\mathrm{L}_2(p)]_{\frac{p-1}{2}}$ contains a subgroup conjugate to $\pm C_p : C_o$ the result follows. □

Note that the above result is also true for $p = 5$ and 7 as the explicit calculations in Lemmas 4.3.3 and 4.4.2 show.

3.2 Some subgroups of $\mathrm{Sp}_{p+1}(\mathbb{Q})$

Let $p \geq 5$ be a prime. If $p \equiv_4 -1$ then $G := \mathrm{SL}_2(p)$ has only two algebraically conjugate complex representations of degree $\frac{p+1}{2}$ as the generic character table [Dor71, Th. 38.1] shows. Let χ denote one of the corresponding characters and let $P \in \mathrm{Syl}_p(G)$. An explicit calculation shows $(1_P^G, \chi)_G = (1_P, \chi|_P)_P = 1$. Thus (by [Isa94, Corollary (10.2) part (c)]) χ is realizable over its character field, which is $\mathbb{Q}(\sqrt{-p})$. So χ gives rise to a subgroup of $\mathrm{Sp}_{p+1}(\mathbb{Q})$ denoted by ${}_{\sqrt{-p}}[\mathrm{SL}_2(p)]_{\frac{p+1}{2}}$.

Theorem 3.2.1 *Let $p \geq 11$ be prime and $G < \mathrm{Sp}_{p+1}(\mathbb{Q})$ such that p divides $|G|$. Then G is s.i.m.f. if and only if $p \equiv_4 -1$ and G is conjugate to ${}_{\sqrt{-p}}[\mathrm{SL}_2(p)]_{\frac{p+1}{2}}$.*

Proof: Let $P \in \mathrm{Syl}_p(G)$. Then again, by Minkowski's bound, $|P| = p$. Thus the natural representation of P splits into twice the trivial one and the $\mathbb{Q}$-irreducible of degree $p-1$. Since there exists an embedding $\delta\colon G \to \mathrm{GL}_m(K)$ for some totally complex number field K with $m \cdot [K : \mathbb{Q}] = p+1$ there is only the possibility $K = \mathbb{Q}(\sqrt{-p})$ and $m = \frac{p+1}{2}$. So the commuting algebra of $\delta(P)$ in $K^{m\times m}$ is isomorphic to $\mathbb{Q}(\zeta_p) \times \mathbb{Q}(\sqrt{-p})$. In particular, $Z(G)$ is isomorphic to a subgroup of $C_{2p} \times C_2$. By Corollary 2.1.16 we know that $P \ntrianglelefteq G$ and therefore $Z(G) \leq C_2 \times C_2$. Since G cannot be imprimitive by Minkowski's bound, $Z(G)$ is not isomorphic to $C_2 \times C_2$. This shows $Z(G) = \pm I_{p+1}$. From Blau's theorem [Fei82, VIII Theorem 7.2, p. 365] it follows that G must be isomorphic to $\pm\mathrm{L}_2(p)$ or $\mathrm{SL}_2(p)$. Since $K \simeq \mathbb{Q}(\sqrt{-p})$ it follows from the character table of $\mathrm{SL}_2(p)$ [Dor71, Th. 38.1] that $p \equiv_4 -1$ and that $\pm\mathrm{L}_2(p)$ has no faithful complex character of degree $\frac{p+1}{2}$. So G must be conjugate to ${}_{\sqrt{-p}}[\mathrm{SL}_2(p)]_{\frac{p+1}{2}}$ and the result follows. □

Remark 3.2.2 Theorem 4.4.1 shows that the above result also holds for $p = 5$. But according to Theorem 4.5.1, the unique s.i.m.f. subgroup of $\mathrm{Sp}_8(\mathbb{Q})$ whose order is divisible by 7 is ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$ (which contains a subgroup conjugate to $\mathrm{SL}_2(7)$).

3.3 The group QD_{2^n}

Let $n \geq 4$. The group $QD_{2^n} = \left\langle x, y \mid x^{2^{n-1}}, y^2, x^y = x^{2^{n-2}-1} \right\rangle$ has one rationally irreducible representation of degree 2^{n-2}. This representation has $\mathbb{Q}(\zeta_{2^{n-1}} - \zeta_{2^{n-1}}^{-1})$ as commuting algebra, so we denote it by ${}_{\zeta_{2^{n-1}}-\zeta_{2^{n-1}}^{-1}}[QD_{2^n}]_2$.

Lemma 3.3.1 *If $n \geq 5$ then ${}_{\zeta_{2^{n-1}}-\zeta_{2^{n-1}}^{-1}}[QD_{2^n}]_2$ is a s.i.m.f. subggroup of $\mathrm{Sp}_{2^{n-2}}(\mathbb{Q})$.*

Proof: The commuting algebra of $H := {}_{\zeta_{2^{n-1}}-\zeta_{2^{n-1}}^{-1}}[QD_{2^n}]_2$ is isomorphic to $K := \mathbb{Q}(\zeta_{2^{n-1}} - \zeta_{2^{n-1}}^{-1})$. This is the fixed field of the automorphism of $\mathbb{Q}(\zeta_{2^{n-1}})$ induced by $\zeta_{2^{n-1}} \mapsto -\zeta_{2^{n-1}}^{-1} = \zeta_{2^{n-1}}^{2^{n-2}-1}$. The subfields of K are linearly ordered by Galois theory and the maximal subfield is the fixed-field of complex conjugation i.e. totally real. Thus each s.i.m.f. supergroup G of H embeds into $\mathrm{GL}_2(K)$. Therefore $\bar{G} := G/Z(G) = G/\langle \pm I_{2^{n-2}} \rangle$ embeds into $\mathrm{PGL}_2(K)$. Since $n \geq 5$ is assumed we get $\bar{G}$ is a dihedral group (of order 2^{n-1}) according to Blichfeldt's classification [Bli17]. This shows $G = H$ as claimed. □

3.4 The group 2_+^{1+2n}

In this section, which is heavily based on Section 5 of [NRS01], let $T_n \simeq 2_+^{1+2n}$ be the n-fold tensor product of $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \simeq D_8$. We describe the construction of $\mathcal{B}^o(T_n)$ and define a maximal finite subgroup of $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$.

Let $(b_0, \ldots, b_{2^n-1})$ be the standard basis of $\mathbb{Q}^{1\times 2^n}$. We identify $v \in \mathbb{F}_2^n$ with $j = \sum_i v_i 2^{i-1}$ and thereby we index the basis vectors b_j with elements of $\mathbb{F}_2^n$. For an affine subspace U of $\mathbb{F}_2^n$ let $\chi_U = \sum_{u\in U} b_u$. Then L_n and L_n' are the $\mathbb{Z}$-lattices in $\mathbb{Q}^{1\times 2^n}$ spanned by

$$\left\{ 2^{\lfloor (n-\dim(U)+\delta)/2 \rfloor} \chi_U \mid U \text{ an affine subspace of } \mathbb{F}_2^n \right\}$$

where $\delta = 0$ for L_n and $\delta = 1$ for L_n'.

In [Wal62, Theorem 3.2] it is shown that $H_n := \mathrm{Aut}(L_n, I_{2^n}) \cap \mathrm{Aut}(L_n', I_{2^n})$ is isomorphic to $2_+^{1+2n}.O_{2n}^+(2)$ and further $O_2(H_n)$ is conjugate to T_n.

By [Win72] we have $\mathrm{Out}(2_+^{1+2n}) \simeq O_{2n}^+(2):2 \simeq GO_{2n}^+(2)$ the full orthogonal group of a quadratic form of Witt defect 0. Conjugation by $h_n := \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right) \otimes I_{2^{n-1}}$ induces an outer automorphism on T_n which is not realized by H_n. Since $\mathrm{End}(\overline{T}_n) \simeq \mathbb{Q}$ and $h_n^2 = 2I_{2^n}$, there exists no element in $\mathrm{GL}_{2^n}(\mathbb{Q})$ of finite order which induces the same automorphism on T_n. Thus H_n is conjugate to $\mathcal{B}^o(T_n)$. Moreover $\frac{1}{\sqrt{-2}} h_n$ normalizes T_n and therefore $\mathcal{B}^o(T_n)$. Hence $\mathcal{H}_n := \left\langle H_n, \frac{1}{\sqrt{-2}} h_n \right\rangle$ is the unique extension of H_n in $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$. The group $\mathcal{H}_n$ gives rise to a finite symplectic matrix group in $\mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$ which will be denoted by ${}_{\sqrt{-2}}[2_+^{1+2n}.(O_{2n}^+(2):2)]_{2^n}$ in the sequel.

Finally, by extending scalars $M_n := \sqrt{-2}L'_n + L_n$ is a $\mathbb{Z}[\sqrt{-2}]$-lattice generated by

$$\left\{ \sqrt{-2}^{n-\dim(U)} \chi_U \mid U \text{ an affine subspace of } \mathbb{F}_2^n \right\} .$$

Lemma 3.4.1 *The lattice M_n is the n-fold tensor product $M_1 \underset{\mathbb{Z}[\sqrt{-2}]}{\otimes} \cdots \underset{\mathbb{Z}[\sqrt{-2}]}{\otimes} M_1$.*

Proof: We use induction on n. Let $V_{n-1} = \langle e_1, \dots, e_{n-1} \rangle$ and $V_1 = \langle e_n \rangle$. One checks that $b_x \otimes b_y = b_{x+y}$ for all $x \in V_{n-1}, y \in V_1$.
Let U be a d-dimensional affine subspace of V_{n-1}. For y in V_1 it follows that $\sqrt{-2}^{n-1-d} \chi_U \otimes \sqrt{-2} b_y = \sqrt{-2}^{n-d} \chi_{y+U} \in M_n$. Similarly, since $U + V_1$ has dimension $d+1$, we get $\sqrt{-2}^{n-1-d} \chi_U \otimes \chi_{V_1} = \sqrt{-2}^{n-(d+1)} \chi_{U+V_1} \in M_n$. Thus $M_{n-1} \otimes M_1 \subseteq M_n$.
Conversely, suppose U is a d-dimensional affine subspace of $\mathbb{F}_2^n$. Write $U = x + y + U_0$ where U_0 is a subspace of $\mathbb{F}_2^n$ and $x \in V_{n-1}$, $y \in V_1$.
If $U_0 \le V_{n-1}$ then $\sqrt{-2}^{n-d} \chi_U = \sqrt{-2}^{n-1-d} \chi_{x+U} \otimes \sqrt{-2} b_y \in M_{n-1} \otimes M_1$. Otherwise $U_{n-1} := U_0 \cap V_{n-1}$ is a $(d-1)$-dimensional subspace and $U_0 = U_{n-1} \cup (z + e_n + U_{n-1})$ for some $z \in V_{n-1}$.
If $z \in U_{n-1}$, then $\sqrt{-2}^{n-d} \chi_U = \sqrt{-2}^{n-1-(d-1)} \chi_{x+U_{n-1}} \otimes \chi_{V_1}$ otherwise we have

$$\begin{aligned} \sqrt{-2}^{n-d} \chi_U &= \sqrt{-2}^{n-1-d} \chi_{x+U_{n-1}+\langle z \rangle} \otimes \sqrt{-2} b_y \\ &+ \sqrt{-2}^{n-1-(d-1)} \chi_{x+z+U_{n-1}} \otimes \chi_{V_1} \\ &- \sqrt{-2} (\sqrt{-2}^{n-1-(d-1)} \chi_{x+z+U_{n-1}} \otimes \sqrt{-2} b_y) \end{aligned}$$

This shows $M_n \subseteq M_{n-1} \otimes M_1$. □

Lemma 3.4.2 *The group $\mathcal{H}_n$ is conjugate to the Hermitian automorphism group* $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(M_n) := \mathrm{Stab}_{U_{2^n}(\mathbb{Q}(\sqrt{-2}))}(M_n)$ *in* $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$.

Proof: For $n = 3$ the result can be checked explicitly. So we may assume that $n \neq 3$. Let $(v_1, \dots, v_{2^n})$ be a $\mathbb{Z}$-basis of L'_n such that $(2v_1, \dots, 2v_{2^{n-1}}, v_{2^{n-1}+1}, \dots, v_{2^n})$ is a $\mathbb{Z}$-basis of L_n. Then $(\sqrt{-2}v_1, \dots, \sqrt{-2}v_{2^n}, 2v_1, \dots, 2v_{2^{n-1}}, v_{2^{n-1}+1}, \dots, v_{2^n})$ is a $\mathbb{Z}$-basis of $M_n = \sqrt{-2}L'_n \oplus L_n$. In particular, the $\mathbb{Z}$-lattices L_n and $\sqrt{-2}L'_n$ are perpendicular with respect to the scalar product $(x, y) \mapsto \frac{1}{2} \mathrm{Tr}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(x\bar{y}^{\mathrm{tr}})$. Thus the group $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(M_n)$ is the subgroup of $\mathrm{Aut}(\sqrt{-2}L'_n \perp L_n, I_{2^{n+1}})$ which commutes with $\sqrt{-2}$. Since $n \neq 3$ is assumed, the automorphism groups of L_n and L'_n equal H_n [Wal62, Theorem 3.2]. Hence using appropriate bases, $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(M_n)$ contains a subgroup G_n of index at most two where

$$G_n = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \mid g_1, g_2 \in H_n \right\} \cap \mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(M_n)$$

Now $\sqrt{-2}$ interchanges $\sqrt{-2}L'_n$ and L_n, i.e. it operates as a block matrix $\left(\begin{smallmatrix} 0 & w \\ -2w^{-1} & 0 \end{smallmatrix}\right)$ for some $w \in \mathrm{GL}_{2^n}(\mathbb{Q})$. So $\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right) \in G_n$ if and only if $g_2 = g_1^w$ and therefore $G_n \simeq H_n$.

On the other hand, $\frac{1}{\sqrt{-2}}h_n$ acts on $M_1 \otimes M_{n-1} = M_n$. So $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(M_n)$ is isomorphic to $\mathcal{H}_n$. Both groups have conjugate Fitting subgroups and therefore their generalized Bravais groups must be conjugate as well. But then the whole groups must be conjugate, since $\mathcal{H}_n$ is the unique extension of H_n in $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$ by the extra automorphism induced by h_n. □

Lemma 3.4.3 *If $n \geq 2$, then ${}_{\sqrt{-2}}[2_+^{1+2n}.(O_{2n}^+(2):2)]_{2^n}$ is the (up to conjugacy) unique s.i.m.f. subgroup of $\mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$ with Fitting group 2_+^{1+2n}.*

Proof: Since $\mathbb{Z}[\sqrt{-2}]$ is a PID, M_n has a $\mathbb{Z}[\sqrt{-2}]$-basis. With respect to this basis, its automorphism group G_n is a finite subgroup of $\mathrm{GL}_{2^n}(\mathbb{Z}[\sqrt{-2}])$. By explicit calculations in MAGMA one checks that for $n \in \{2, 3\}$ the $\mathbb{Z}$-span $\langle G_n \rangle_{\mathbb{Z}}$ equals $\mathbb{Z}[\sqrt{-2}]^{2^n \times 2^n}$. For $n \geq 4$ it follows from $M_n = M_{n-2} \underset{\mathbb{Z}[\sqrt{-2}]}{\otimes} M_2$ that $G_2 \otimes G_{n-2} \subset G_n$ (using appropriate bases). By induction, we get

$$\mathbb{Z}[\sqrt{-2}]^{2^n \times 2^n} = \langle G_{n-2} \rangle_{\mathbb{Z}} \underset{\mathbb{Z}[\sqrt{-2}]}{\otimes} \langle G_2 \rangle_{\mathbb{Z}} \subseteq \langle G_n \rangle_{\mathbb{Z}} \subseteq \mathbb{Z}[\sqrt{-2}]^{2^n \times 2^n} .$$

In particular, each G_n-invariant $\mathbb{Z}[\sqrt{-2}]$-lattice is a multiple of M_n, since $\mathbb{Z}[\sqrt{-2}]$ has class number 1. Thus G_n is a maximal finite subgroup of $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$. But any finite symplectic supergroup of ${}_{\sqrt{-2}}[2_+^{1+2n}.(O_{2n}^+(2):2)]_{2^n}$ comes from a finite supergroup of $\mathcal{H}_n \simeq G_n < \mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$. Thus ${}_{\sqrt{-2}}[2_+^{1+2n}.(O_{2n}^+(2):2)]_{2^n}$ is s.i.m.f.. The second statement follows from the construction of $\mathcal{H}_n$ and Remark 2.2.17. □

3.5 The group p_+^{1+2n}

In this section, let p be an odd prime. We will describe a family of irreducible symplectic matrix groups in dimension $p^n(p-1)$ which will be maximal finite in the case that p is a Fermat prime, i.e. $p-1$ is a power of two.

Let $T_n^{(p)} \simeq p_+^{1+2n}$ be the n-fold tensor product of $T_1^{(p)}$ where $T_1^{(p)}$ is the subgroup of $\mathrm{GL}_p(\mathbb{Q}(\zeta_p))$ generated by the diagonal matrix $\mathrm{Diag}(1, \zeta_p, \ldots, \zeta_p^{p-1})$ and the permutation matrix corresponding to the p-cycle $(1, \ldots, p)$. Further let $H_n^{(p)} = N_{U_{p^n}(\mathbb{Q}(\zeta_p))}(T_n^{(p)})$. By [Win72] $H_n^{(p)}$ is isomorphic to a subgroup of $C_2 \times p_+^{1+2n}.\mathrm{Sp}_{2n}(p)$, since the group of outer automorphisms which act trivially on the center of $T_n^{(p)}$ is isomorphic to $\mathrm{Sp}_{2n}(p)$.

In [Wal62, Section 4] Wall constructs a $(p-1)p^n$ dimensional rational lattice on which $C_2 \times p_+^{1+2n}.\mathrm{Sp}_{2n}(p)$ acts. Thus $H_n^{(p)} = \mathcal{B}^o(T_n^{(p)}) \simeq C_2 \times p_+^{1+2n}.\mathrm{Sp}_{2n}(p)$ (if we assume that $T_n^{(p)} \trianglelefteq \mathcal{B}^o(T_n^{(p)})$). Now $H_n^{(p)}$ gives rise to a finite subgroup of $\mathrm{Sp}_{p^n(p-1)}(\mathbb{Q})$ which will be denoted by ${}_{\zeta_p}[\pm p_+^{1+2n}.\mathrm{Sp}_{2n}(p)]_{p^n}$.

Theorem 3.5.1 *If p is a Fermat prime, then ${}_{\zeta_p}[\pm p_+^{1+2n}.\mathrm{Sp}_{2n}(p)]_{p^n}$ is a s.i.m.f. subgroup of* $\mathrm{Sp}_{p^n(p-1)}(\mathbb{Q})$.

Proof: The commuting algebra C of ${}_{\zeta_p}[\pm p_+^{1+2n}.\mathrm{Sp}_{2n}(p)]_{p^n}$ is isomorphic to $\mathbb{Q}(\zeta_p)$. Since p is a Fermat prime, C has only one maximal subfield, which must be totally real. Thus any symplectic supergroup must come from a finite subgroup of $\mathrm{GL}_{p^n}(\mathbb{Q}(\zeta_p))$. But $H_n^{(p)}$ is maximal finite in $\mathrm{GL}_{p^n}(\mathbb{Q}(\zeta_p))$ according to [NRS01, Theorem 7.3]. □

Chapter 4

The classification

In this final chapter, we classify all s.i.m.f. subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$ for $1 \le n \le 11$. The first section handles the generic case $n > 3$ prime. The subsequent sections each deal with one particular dimension $2n$. The s.i.m.f. subgroups $G < \mathrm{GL}_{2n}(\mathbb{Q})$ are listed in tables as follows:

The s.i.m.f. groups are grouped together by their commuting algebras. Groups having the same commuting algebras are sorted by group order (descending) and minimal determinant d (ascending). The invariant d is explained below.

The first column contains a number for the group for easy referencing. The primitive groups are numbered consecutively. For the imprimitive groups we use triples $[i, j, k]$ with $jk = 2n$ which stands for the wreath product of the i-th s.p.i.m.f. matrix group in dimension j with S_k.

The second column contains the name of G using the conventions from Definition 2.4.2. The next two columns list the group order $|G|$ and the number of isomorphism classes of G-invariant lattices. These two numbers and the endomorphism ring (which can be read off the second column) are invariants of the conjugacy class of G. But usually this does not identify the class of G uniquely.

Thus the fifth column (labeled L_{min}) contains some more invariants: Let

$$S := \{(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G) \mid F \text{ integral on } L\} \quad \text{and}$$
$$d := \min\{\det(L, F) \mid (L, F) \in S\} .$$

Then $\{(L, F) \in S \mid \det(L, F) = d\}$ partitions into several isometry classes. If $(L_1, F_1), \dots, (L_s, F_s)$ represent these classes, then the third column lists the distinct triples $[d, \min(L_i, F_i), |\mathrm{SV}(L_i, F_i)|]$ for $(1 \le i \le s)$. Here $\min(L, F)$ and $\mathrm{SV}(L, F)$ denote the minimum and the set of shortest vectors of (L, F) respectively.

It turns out that in almost all cases, these invariants determine the conjugacy class of G uniquely. (In fact, the only exceptions are $({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5) \otimes A_2$ and ${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2 < \mathrm{Sp}_{16}(\mathbb{Q})$. But they can easily be distinguised by their Fitting subgroups.)

Finally the last column lists the r.i.m.f. supergroups of G (up to conjugacy).

4.1 Dimension 2p

Let G be a s.p.i.m.f. matrix group of degree $2p$ for some prime $p > 3$.

Lemma 4.1.1 *The Fitting subgroup $F(G)$ is cyclic of order $2, 4$ or 6.*

Proof: By Corollary 2.1.16 $O_l(G) = 1$ for all primes ℓ with $\ell - 1 \nmid 2p$. Suppose first $q := 2p + 1$ is prime and $q \mid |G|$. Then $q \geq 11$ and $q \equiv_4 -1$. So Theorem 3.1.1 shows that $G \simeq {}_{\sqrt{-q}}[\pm \mathrm{L}_2(q)]_p$ and thus $F(G) \simeq C_2$.
So we may now assume that $F(G) = O_2(G)O_3(G)$. In particular G cannot be cyclic since otherwise G is either reducible or $O_q(G) \neq 1$. Thus G embeds irreducibly into $\mathrm{GL}_p(K)$ for some imaginary quadratic number field K. So the result follows from Table 2.5.2, since D_8 does not have such an embedding. □

Corollary 4.1.2 *The group G is not soluble. Moreover, $K := \mathrm{End}(\overline{G})$ is an imaginary quadratic number field such that $\triangle_K(E(G)) < \mathrm{GL}_p(K)$ is absolutely irreducible.*

Thus $E(G)$ is either reducible with $\mathbb{Q}^{2\times 2}$ as commuting algebra or it is irreducible and its commuting algebra is an imaginary quadratic number field. In both cases, G can easily be recovered from $F^*(G)$ (cf. Sections 2.2.3 and 2.2.4).

4.2 Dimension 2

Theorem 4.2.1 *The s.i.m.f. subgroups G of $\mathrm{Sp}_2(\mathbb{Q})$ are*

#	G	$\lvert G\rvert$	$\lvert \mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
1	${}_i[C_4]_1$	2^2	1	$[1,1,4]$	B_2
2	${}_{\sqrt{-3}}[C_6]_1$	$2\cdot 3$	1	$[3,2,6]$	A_2

Proof: Every finite subgroup of $\mathrm{Sp}_2(\mathbb{Q})$ is cyclic since it admits a faithful representation of degree 1 over some imaginary quadratic number field. □

4.3 Dimension 4

Theorem 4.3.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_4(\mathbb{Q})$ *are*

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
1	${}_i[(D_8 \otimes C_4).S_3]_2$	$2^5{\cdot}3$	1	$[2^2, 2, 24]$	F_4
2	${}_i[C_4]_1 \otimes A_2$	$2^3{\cdot}3$	2	$[3^2, 2, 12]$	A_2^2
3	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$	$2^4{\cdot}3$	1	$[2^2, 2, 24]$	F_4
4	${}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$	$2^3{\cdot}3^2$	2	$[2^2, 2, 24]$	F_4
$[2, 2, 2]$	${}_{\sqrt{-3}}[C_6]_1^2$	$2^3{\cdot}3^2$	1	$[3^2, 2, 12]$	A_2^2
5	${}_{\zeta_{10}}[C_{10}]_1$	$2{\cdot}5$	1	$[5, 2, 20]$	A_4

Proof: It follows from explicit calculations, that the above table is correct and yields s.i.m.f. groups. The r.i.m.f. supergroups are easily constructed, since all these groups except ${}_{\zeta_{10}}[C_{10}]_1$ are uniform. Further ${}_{\zeta_{10}}[C_{10}]_1$ can only be contained in $\mathrm{Aut}(A_4)$ by comparison of orders. The imprimitive s.i.m.f. matrix groups can only be conjugate to ${}_{\sqrt{-3}}[C_6]_1^2$ or ${}_i[C_4]_1^2$. The first group if s.i.m.f. by Lemma 2.1.21. The group $H := {}_i[C_4]_1^2$ has $\mathbb{Q}(i)$ as commuting algebra. Further it fixes two lattices L_1, L_2. One checks that one of the groups $\mathrm{Aut}(L_j, \mathcal{F}(H))$ $(j = 1, 2)$ is conjugate to ${}_i[(D_8 \otimes C_4).S_3]_2$. Thus H is not maximal finite.
So it remains to prove that each s.p.i.m.f. subgroup of $\mathrm{Sp}_4(\mathbb{Q})$ is conjugate to one of the groups from above. This is done in Section 4.3.2. □

4.3.1 Irreducible cyclic subgroups

Before we prove the completeness, we classify all finite subgroups $G < \mathrm{Sp}_4(\mathbb{Q})$ that contain an irreducible cyclic subgroup U. Then $\lvert \pm U\rvert \in \{8, 10, 12\}$.

Lemma 4.3.2 *If* $\lvert U\rvert = 8$ *then G is conjugate to* ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$ *or* ${}_i[(D_8 \otimes C_4).S_3]_2$.

Proof: It follows from Minkowski's bound that $\Pi(G) \subseteq \{2, 3, 5\}$. The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_8)$ and has class number 1. So U fixes only one lattice L. One finds some $F \in \mathcal{F}_{>0}(U)$ that is integral on L such that $\det(L, F) = 1$. By Table 2.5.3 G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^a F)$ for some $a \in \{0, 1\}$ and $d \in \{1, 2\}$. This leaves the candidates:

	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	$\lneq {}_i[(D_8 \otimes C_4).S_3]_2$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$
$\mathfrak{p}_2 F$	${}_i[(D_8 \otimes C_4).S_3]_2$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$

□

Lemma 4.3.3 *If* $\pm U$ *is of order* 10 *then* $G \simeq {}_{\zeta_{10}}[C_{10}]_1$.

Proof: The commuting algebra $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{10})$ has only one proper subfield, which is $\mathbb{Q}(\sqrt{5})$. So $\mathrm{End}(\overline{U}) = \mathrm{End}(\overline{G})$. Further U fixes up to isomorphism only one lattice L. Let $F \in \mathcal{F}_{>0}(U)$, then $\mathrm{Aut}_{\mathbb{Q}(\zeta_{10})}(L, F) = \pm U$ shows that $\pm U$ is s.i.m.f.. □

Lemma 4.3.4 *If U is of order 12 then G is conjugate to ${}_i[(D_8 \otimes C_4).S_3]_2$, ${}_i[C_4]_1 \otimes A_2$, ${}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$ or ${}_{\sqrt{-3}}[C_6]_1^2$.*

Proof: Again by Minkowski's bound $\Pi(G) \subseteq \{2, 3, 5\}$. The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{12})$ and has class number 1. So U fixes only one lattice L. Further, one finds some $F \in \mathcal{F}_{>0}(U)$ that is integral on L such that $\det(L, F) = 4$. By Table 2.5.3, G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^{-a}\mathfrak{p}_3^{a}F)$ for some $a \in \{0, 1\}$ and $d \in \{1, 3\}$. This leaves the four candidates:

	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-3})$
F	${}_i[(D_8 \otimes C_4).S_3]_2$	${}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$
$\mathfrak{p}_2^{-1}\mathfrak{p}_3 F$	${}_i[C_4]_1 \otimes A_2$	${}_{\sqrt{-3}}[C_6]_1^2$

So the result follows. □

4.3.2 Proof of Theorem 4.3.1

Let $G < \mathrm{Sp}_4(\mathbb{Q})$ be s.p.i.m.f.. Then $E(G) = 1$ according to Table 2.5.1. Thus $F(G)$ is selfcentralizing.
By Corollary 2.1.16 we know that $O_p(G) = 1$ for all $p \geq 7$. The case $O_5(G) \neq 1$ is handled in Lemma 4.3.3. The remaining cases are handled in the two subsequent lemmas.

Lemma 4.3.5 *If $O_3(G) \neq 1$ then G is conjugate to ${}_i[C_4]_1 \otimes A_2$ or ${}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$.*

Proof: By Table 2.5.2 we get $O_3(G) = C_3$ and $O_2(G)$ is one of C_2, C_4, D_8 or Q_8. If $O_2(G) = C_2$ then $[G : C_6] \leq |\mathrm{Out}(C_6)| = 2$ contradicts Lemma 2.2.1. In all other cases G contains an irreducible cyclic subgroup of order 12, so the result follows from Lemma 4.3.4. □

Lemma 4.3.6 *If $F(G)$ is a 2-group, then G is conjugate to ${}_i[(D_8 \otimes C_4).S_3]_2$ or ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$.*

Proof: We already know that $\mathcal{B}^o(D_8 \otimes C_4) = {}_i[(D_8 \otimes C_4).S_3]_2$ is maximal finite. Assume now that $O_2(G)$ is cyclic or isomorphic to D_8, then $G = O_2(G)$ since $E(G) = 1$ and $\mathrm{Out}(F(G))$ is a 2-group. Thus $G = O_2(G)$ is either reducible or not maximal. If $O_2(G) \simeq QD_{16}$ then G contains an irreducible cyclic subgroup of order 8. This contradicts Lemma 4.3.2. So there remains the case that $O_2(G) \simeq Q_8$. Then $B := \mathcal{B}^o(O_2(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ and $[G : B] = 2$ since B is not maximal. By Remark 2.2.17 there is only one such extension, which is ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$. □

4.4 Dimension 6

Theorem 4.4.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_6(\mathbb{Q})$ *are*

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
$[2,1,3]$	${}_{i}[C_4]_1^3$	$2^7\cdot 3$	3	$[1,1,12]$	B_6
1	${}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3$	$2^4\cdot 3^4$	2	$[3,2,72]$	E_6
$[2,2,3]$	${}_{\sqrt{-3}}[C_6]_1^3$	$2^4\cdot 3^4$	1	$[3^3,2,18]$	A_2^3
2	${}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3$	$2^4\cdot 3\cdot 7$	1	$[7^3,4,42]$	$A_6^{(2)}$

We need two lemmas for the proof. The first lemma shows that Theorem 3.1.1 also holds for $p=7$.

Lemma 4.4.2 *Let $G<\mathrm{Sp}_6(\mathbb{Q})$ be s.i.m.f. such that $\lvert G\rvert$ is divisible by 7. Then G is conjugate to ${}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3$.*

Proof: As in the proof of Theorem 3.1.1, it follows that G is symplectic primitive and contains an irreducible subgroup U isomorphic to C_7. By Minkowski's bound we get $\Pi(\lvert G\rvert)\subseteq\{2,3,5,7\}$. The group U fixes only one lattice L since $\mathrm{End}(\overline{U})\simeq\mathbb{Q}(\zeta_{14})$ has class number 1. Further there exists some $F\in\mathcal{F}_{>0}(U)$ that is integral on L with $\det(L,F)=7$. Table 2.5.3 shows that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-7})}(L,\mathfrak{p}_7^kF)$ for some $k\in\{0,1\}$. So the result is easily verified. □

Lemma 4.4.3 *Let $G<\mathrm{Sp}_6(\mathbb{Q})$ be s.p.i.m.f. such that $\lvert O_3(G)\rvert>3$ or G contains an irreducible subgroup $U\simeq C_{18}$. Then $O_3(G)\simeq 3_+^{1+2}$ and $G\simeq{}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3$.*

Proof: If $\lvert O_3(G)\rvert>3$ then, by Table 2.5.2, $O_3(G)$ is either cyclic of order 9 or isomorphic to 3_+^{1+2}. In the latter case $\mathcal{B}^o(O_3(G))\simeq{}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3$ and this group is s.i.m.f. by Theorem 3.5.1. So we may assume that G contains an irreducible subgroup $U\simeq C_{18}$. The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{18})$ and has class number 1. Thus U fixes only one lattice L. One finds some $F\in\mathcal{F}_{>0}(U)$ that is integral on L such that $\det(L,F)=3$. Table 2.5.3 shows that G is conjugate to $\mathrm{Aut}_{\sqrt{-3}}(L,\mathfrak{p}_3^kF)$ for some $k\in\{0,1\}$. These two automorphism groups are conjugate to ${}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3$ and ${}_{\sqrt{-3}}[C_6]_1^3$ respectively. □

Proof (of Theorem 4.4.1): One checks explicitely that the table given in Theorem 4.4.1 is correct. (Note that the r.i.m.f. supergroups are easily constructed sind all s.i.m.f. groups are uniform.) The group ${}_{\sqrt{-3}}[C_6]_1^3$ is s.i.m.f. by Lemma 2.1.21. The group ${}_{i}[C_4]_1^3$ fixes three lattices and has $\mathbb{Q}(i)$ as commuting algebra. So one checks that it is maximal. Thus it remains to show that each s.p.i.m.f. group $G<\mathrm{Sp}_6(\mathbb{Q})$ is conjugate to ${}_{\sqrt{-3}}[\pm 3_+^{1+2}:\mathrm{SL}_2(3)]_3$ or ${}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3$. By Table 2.5.2 and the preceding lemmas we may assume that $O_3(G)\le C_3$ and $O_p(G)=1$ for all primes $p>3$. If $E(G)=1$ then $F(G)\in\{\pm I_6,C_4,\pm C_3,D_8\}$. In any case $[G:F(G)]\le 2$, which contradicts Lemma 2.2.1. Hence $E(G)\neq 1$ and from Table 2.5.1 it follows that $E(G)\simeq\mathrm{L}_2(7)$. But then $G\simeq{}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3$ by the above. □

4.5 Dimension 8

Theorem 4.5.1 *The s.i.m.f. subgroups G of $\mathrm{Sp}_8(\mathbb{Q})$ are:*

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
1	${}_{i}[(2_+^{1+4}\otimes C_4).S_6]_4$	$2^{10}\cdot3^2\cdot5$	1	$[1,2,240]$	E_8
[4, 1, 2]	${}_{i}[(D_8\otimes C_4).S_3]_2^2$	$2^{11}\cdot3^2$	1	$[2^4,2,48]$	F_4^2
[2, 1, 4]	${}_{i}[C_4]_1^4$	$2^{11}\cdot3$	3	$[1,1,16]$	B_8
[4, 2, 2]	$({}_{i}[C_4]_1\otimes A_2)^2$	$2^7\cdot3^2$	2	$[3^4,2,24]$	A_2^4
2	${}_{i}[(D_8\otimes C_4).S_3]_2\otimes A_2$	$2^6\cdot3^2$	2	$[2^4\cdot3^4,4,72]$	$A_2\otimes F_4$
[4, 3, 2]	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^2$	$2^9\cdot3^2$	1	$[2^4,2,48]$	F_4^2
3	${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2:2]_4$	$2^8\cdot3\cdot5$	1	$[1,2,240]$	E_8
4	${}_{\sqrt{-2}}[2_+^{1+4}.(O_4^+(2):2)]_4$ $= {}_{\sqrt{-2}}[F_4:2]_4$	$2^8\cdot3^2$	1	$[2^4,2,48]$	F_4^2
5	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4$	$2^7\cdot3^5\cdot5$	1	$[1,2,240]$	E_8
[2, 2, 4]	${}_{\sqrt{-3}}[C_6]_1^4$	$2^7\cdot3^5$	1	$[3^4,2,24]$	A_2^4
[4, 4, 2]	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)^2$	$2^7\cdot3^4$	2	$[2^4,2,48]$	F_4^2
6	${}_{\sqrt{-3}}[C_6]_1\otimes F_4$	$2^7\cdot3^3$	2	$[2^4\cdot3^4,4,72]$	$A_2\otimes F_4$
7	${}_{\infty,5}[\mathrm{SL}_2(5):2]_2\circ C_3$	$2^4\cdot3^2\cdot5$	2	$[5^4,4,120]$	S
8	${}_{\sqrt{-3}}[C_6]_1\otimes A_4$	$2^4\cdot3^2\cdot5$	2	$[3^4\cdot5^2,4,60]$	$A_2\otimes A_4$
9	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\overset{2-}{\square}C_4]_4$	$2^5\cdot3\cdot5$	2	$[1,2,240]$	E_8
10	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\overset{2+}{\square}C_4]_4$	$2^5\cdot3\cdot5$	2	$[5^4,4,120]$	S
11	${}_{\sqrt{-5}}[C_{20}:C_4]_4$	$2^4\cdot5$	4	$[5^2,2,40]$	A_4^2
12	${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1\overset{2-}{\square}C_3]_4$	$2^5\cdot3^2$	2	$[1,2,240]$	E_8
13	${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1\overset{2+}{\square}C_3]_4$	$2^5\cdot3^2$	2	$[2^4,2,48]$	F_4^2
14	${}_{\sqrt{-6}}[D_{16}\overset{2}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_4$	$2^5\cdot3$	4	$[3^4,2,24]$	$A_2^4, A_2\otimes F_4$
15	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$	$2^4\cdot3^2\cdot5\cdot7$	1	$[1,2,240]$	E_8
16	${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\overset{2-}{\square}C_3]_4$	$2^4\cdot3^2\cdot5$	2	$[1,2,240]$	E_8
17	${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\overset{2+}{\square}C_3]_4$	$2^4\cdot3^2\cdot5$	2	$[5^4,4,120]$	S
18	${}_{\sqrt{-15}}[C_{30}:C_4]_4$	$2^3\cdot3\cdot5$	4	$[3^4\cdot5^2,4,60]$	$A_2\otimes A_4$
19	${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_5$	$2^3\cdot3\cdot5^2$	1	$[1,2,240]$	$E_8,\ S$
[4, 5, 2]	${}_{\zeta_{10}}[C_{10}]_1^2$	$2^3\cdot5^2$	1	$[5^2,2,40]$	A_4^2
20	${}_{\zeta_{10}}[C_{10}]_1\otimes A_2$	$2^2\cdot3\cdot5$	2	$[3^4\cdot5^2,4,60]$	$A_2\otimes A_4$
21	${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2$	2^5	2	$[1,1,16]$	B_8, F_4^2

where S denotes the r.i.m.f. matrix group $[(\mathrm{SL}_2(5)\overset{2}{\square}\mathrm{SL}_2(5)):2]_8$.

To distinguish the groups ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1\overset{2-}{\square}C_3]_4$ and ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1\overset{2+}{\square}C_3]_4$ we make the following convention. Both groups are generated by $N := {}_{\sqrt{2},\infty}[\tilde{S}_4]_1\circ C_3$ and some $\alpha\in G$ with $\alpha^2\in N$ that centralizes $\mathcal{B}^o(O_2(G))\simeq{}_{\infty,2}[\mathrm{SL}_2(3)]_1$ and induces the nontrivial *outer automorphisms on $\tilde{S}_4$ and C_3. Up to isomorphism there are two choices for α^2. The name ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4$ stands for $\alpha^2 = I_8$ and we write ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2-}{\square} C_3]_4$ if $\alpha^2 = -I_8$.*

The proof of the theorem is given in Section 4.5.2.

4.5.1 Irreducible cyclic subgroups

If $G < \mathrm{Sp}_8(\mathbb{Q})$ is s.i.m.f. and contains an irreducible cyclic subgroup U then $|\pm U| \in \{16, 20, 24, 30\}$. In this section, we construct all such groups G.

Lemma 4.5.2 *Let $G < \mathrm{Sp}_8(\mathbb{Q})$ be s.i.m.f. with an irreducible subgroup $U \simeq C_{16}$. Suppose that $\Pi(|G|) \subseteq \{2, 3, 5\}$ or that there exists some $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L and $\Pi(\det(L, F)) \subseteq \{2, 3, 5\}$. Then G is conjugate to one of the following groups:*

$$ {}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2,\ {}_i[C_4]_1^4,\ {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^2,\ {}_i[(D_8 \otimes C_4).S_3]_2^2 . $$

Proof: The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{16})$ and has class number 1. Thus U fixes only one lattice L. One finds some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L, F) = 1$. By Table 2.5.3, we have to consider the groups $\mathrm{Aut}_K(L, \mathfrak{p}_2^k F)$ for some $0 \le k \le 2$ and $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\zeta_{16} - \zeta_{16}^{-1})\}$.

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\zeta_{16} - \zeta_{16}^{-1})$
F	${}_i[C_4]_1^4$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^2$	${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2$
$\mathfrak{p}_2 F$	${}_i[C_4]_1^4$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^2$	${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2$
$\mathfrak{p}_2^2 F$	${}_i[(D_8 \otimes C_4).S_3]_2^2$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^2$	${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2$

So the result follows. □

Lemma 4.5.3 *Let $G < \mathrm{Sp}_8(\mathbb{Q})$ be s.i.m.f. with an irreducible subgroup $U \simeq C_{20}$. Suppose that $\Pi(|G|) \subseteq \{2, 3, 5\}$ or that there exists some $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L and $\Pi(\det(L, F)) \subseteq \{2, 3, 5\}$. Then G is conjugate to ${}_{\zeta_{10}}[C_{10}]_1^2$, ${}_{\sqrt{-5}}[C_{20} : 4]_4$, ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$, ${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4$, ${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4$ or ${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$.*

Proof: The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{20})$ and has class number 1. Thus U fixes only one lattice L. One finds some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L, F) = 1$. By Table 2.5.3, we have to consider the groups $\mathrm{Aut}_K(L, \mathfrak{p}_5^k F)$ for some $0 \le k \le 2$ and $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\zeta_{10})\}$.

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Q}(\zeta_{10})$
F	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4$	${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$
$\mathfrak{p}_5 F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_{\sqrt{-5}}[C_{20} : 4]_4$	${}_{\zeta_{10}}[C_{10}]_1^2$
$\mathfrak{p}_5^2 F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4$	${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$

This proves the claim. □

Lemma 4.5.4 *Let $G < \mathrm{Sp}_8(\mathbb{Q})$ be s.i.m.f. with an irreducible subgroup $U \simeq C_{24}$. Suppose that $\Pi(|G|) \subseteq \{2,3,5\}$ or that there exists some $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L and $\Pi(\det(L,F)) \subseteq \{2,3,5\}$. Then G is conjugate to one of:*

$$
{}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\square} C_3]_4,\ {}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\square} C_3]_4,\ {}_i[((D_8 \otimes C_4).S_3)]_2 \otimes A_2,
$$
$$
{}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4,\ {}_i[(2_+^{1+4} \otimes C_4).S_6]_4,\ {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4, ({}_i[C_4]_1 \otimes A_2)^2,
$$
$$
{}_{\sqrt{-3}}[C_6]_1^4,\ ({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)^2,\ {}_i[((D_8 \otimes C_4).S_3)]_2^2,\ {}_{\sqrt{-3}}[C_6]_1 \otimes F_4,
$$
$$
{}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3,\ {}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4,\ {}_{\sqrt{-2}}[F_4 : 2]_4 .
$$

Proof: The commuting algebra $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{30})$ has class number 1, so U fixes up to isomorphism one lattice L. One finds some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L,F) = 1$. By Table 2.5.3 we have to consider the groups $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^{2a}\mathfrak{p}_3^b\mathfrak{p}_5^c{\mathfrak{p}'_5}^{c'}F)$ for $a,b,c,c' \in \{0,1\}$ and $d \in \{1,2,3,6\}$.
Since there exists some $x \in N_{\mathrm{GL}_8(\mathbb{Q})}(U) \cap \mathrm{GL}(L)$ such that $\mathfrak{p}_5^x = \mathfrak{p}'_5$ (and necessarily $\mathfrak{p}_2^x = \mathfrak{p}_2$, $\mathfrak{p}_3^x = \mathfrak{p}_3$) we may assume that $c' \leq c$. Since $\mathfrak{p}_5\mathfrak{p}'_5 = 5\mathbb{Z}[\theta_{24}]$ we may even assume that $c' = 0$. So we have 32 possibilities:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_i[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$
$\mathfrak{p}_2^2F$	${}_i[(D_8 \otimes C_4).S_3]_2^2$	${}_{\sqrt{-2}}[F_4 : 2]_4$
$\mathfrak{p}_3F$	$({}_i[C_4]_1 \otimes A_2)^2$	$\lneq {}_i[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$
$\mathfrak{p}_2^2\mathfrak{p}_3F$	${}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_2$	$\lneq {}_{\sqrt{-2}}[F_4 : 2]_4$
$\mathfrak{p}_5F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	$\lneq {}_{\sqrt{-2}}[F_4 : 2]_4$
$\mathfrak{p}_2^2\mathfrak{p}_5F$	$\lneq ({}_i[C_4]_1 \otimes A_2)^2$	$\lneq {}_i[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$
$\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	$\lneq {}_{\sqrt{-2}}[F_4 : 2]_4$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq ({}_i[C_4]_1 \otimes A_2)^2$	$\lneq {}_i[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$
form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-6})$
F	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$	${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\square} C_3]_4$
$\mathfrak{p}_2^2F$	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)^2$	${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\square} C_3]_4$
$\mathfrak{p}_3F$	${}_{\sqrt{-3}}[C_6]_1^4$	${}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$
$\mathfrak{p}_2^2\mathfrak{p}_3F$	${}_{\sqrt{-3}}[C_6]_1 \otimes F_4$	${}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$
$\mathfrak{p}_5F$	${}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3$	$\lneq {}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\square} C_3]_4$
$\mathfrak{p}_2^2\mathfrak{p}_5F$	$\lneq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$	$\lneq {}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$
$\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3$	$\lneq {}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\square} C_3]_4$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$	$\lneq {}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$

This proves the claim. □

Lemma 4.5.5 *Let $G < \mathrm{Sp}_8(\mathbb{Q})$ be s.i.m.f. with an irreducible subgroup $U \simeq C_{30}$. Suppose that $\Pi(|G|) \subseteq \{2,3,5\}$ or that there exists some $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L and $\Pi(\det(L,F)) \subseteq \{2,3,5\}$. Then G is conjugate to one of the following groups:*

$$_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4,\ {}_{\zeta_{10}}[C_{10}]_1 \otimes A_2,\ {}_{\sqrt{-15}}[C_{30}\,{:}\,4]_4,\ {}_{\sqrt{-3}}[C_6]_1 \otimes A_4,\ {}_{\infty,5}[\mathrm{SL}_2(5)\,{:}\,2]_2 \circ C_3,$$
$$_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4,\ {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4,\ {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5\,.$$

Proof: The commuting algebra $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{30})$ has class number 1, so U fixes up to isomorphism one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L,F) = 1$. By Table 2.5.3 we have to consider the groups $\mathrm{Aut}_K(L, \mathfrak{p}_3^a\mathfrak{p}_5^b F)$ where $a \in \{0,1\}$, $b \in \{0,1,2\}$ with $a \equiv_2 b$ and $K \in \{\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-15}), \mathbb{Q}(\zeta_{10})\}$. So we have the following 9 automorphism groups to check:

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-15})$	$\mathbb{Q}(\zeta_{10})$
F	$_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$	$_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4$	$_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$
$\mathfrak{p}_3\mathfrak{p}_5 F$	$_{\sqrt{-3}}[C_6]_1 \otimes A_4$	$_{\sqrt{-15}}[C_{30}\,{:}\,4]_4$	$_{\zeta_{10}}[C_{10}]_1 \otimes A_2$
$\mathfrak{p}_5^2 F$	$_{\infty,5}[\mathrm{SL}_2(5)\,{:}\,2]_2 \circ C_3$	$_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4$	$_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$

These groups are all maximal finite. □

4.5.2 Proof of Theorem 4.5.1

We have to prove the completeness of the list given in Theorem 4.5.1. The candidates for the maximal finite symplectic imprimitive groups come from the classification of the s.p.i.m.f. subgroups of $\mathrm{Sp}_2(\mathbb{Q})$ and $\mathrm{Sp}_4(\mathbb{Q})$. By Lemma 2.1.21, we only have to check the group $_i[C_4]_1^4$. It fixes up to isomorphism three lattices and has $\mathbb{Q}(i)$ as commuting algebra. One verifies that it is s.i.m.f..

So it remains to prove the completeness for the s.p.i.m.f. subgroups of $\mathrm{Sp}_8(\mathbb{Q})$. Thus, for the remainder of this section let $G < \mathrm{Sp}_8(\mathbb{Q})$ be s.p.i.m.f..

First we handle the case that G contains a quasisimple normal subgroup N. By Table 2.5.1, N must be conjugate to Alt_5, $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{SL}_2(9)$, $2.\mathrm{Alt}_7$ or $\mathrm{Sp}_4(3)$. These cases are handled in the next two lemmas.

Lemma 4.5.6 *Let N be a normal subgroup of G.*

(a) If N is conjugate to Alt_5 then $G \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes A_4$.

(b) If N is conjugate to $2.\mathrm{Alt}_7$, then $G = N$.

(c) If N is conjugate to $\mathrm{Sp}_4(3)$, then $G = \mathcal{B}^o(N) \simeq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$.

(d) N is not conjugate to $\mathrm{SL}_2(7)$.

(e) N is not conjugate to $\mathrm{SL}_2(9)$.

Proof:

(a) Let $H := \mathcal{B}^o(N) \simeq \pm S_5 \simeq \mathrm{Aut}(A_4)$. So H cannot be self centralizing since $\mathrm{Out}(H)$ is trivial and $\mathrm{End}(\overline{H}) \simeq \mathbb{Q}^{2\times 2}$. Thus G must contain a subgroup conjugate to $C_k \otimes A_4$ with $k \in \{6, 4\}$. These groups have $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(i)$ as commuting algebras and they fix 2 and 4 lattices respectively. One easily checks that ${}_{\sqrt{-3}}[C_6]_1 \otimes A_4$ is s.i.m.f. whereas $C_4 \otimes A_4$ is properly contained in ${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$.

(b) Up to isomorphism, N fixes only one lattice L and it has $\mathbb{Q}(\sqrt{-7})$ as commuting algebra. So the claim is easily verified.

(c) Again, $\mathcal{B}^o(N) \simeq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$ fixes only one lattice L and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. The result follows as above.

(d) N fixes 5 lattices and has $\mathbb{Q}(\sqrt{-7})$ as commuting algebra. Let $F \in \mathcal{F}_{>0}(N)$. One checks that $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-7})}(L, F)$ is either conjugate to N or ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$ for all $L \in \mathcal{Z}(N)$.

(e) The commuting algebra of N is isomorphic to $\mathcal{Q}_{\infty,3}$ and $\mathrm{Out}(N) \simeq C_2 \times C_2$ but only one class of outer automorphisms can be realized in $\mathrm{GL}_8(\mathbb{Q})$. Using Remark 2.2.17, one finds that G contains a subgroup conjugate to $N \circ C_4$, $N \circ C_3$ or $N.2$. The first group fixes 2 lattices and is only contained in ${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$ the other two groups fix only one lattice and they are only contained in ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$. So the result follows. □

Lemma 4.5.7 *If G contains a normal subgroup N conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, then G is conjugate to one of*

$$ {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4,\ {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4,\ {}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3, $$
$$ {}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4,\ {}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4 \ \textit{or}\ {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5\,. $$

Proof: Let $Q := \mathrm{End}(\overline{N}) \simeq \mathcal{Q}_{\sqrt{5},\infty}$ and let $K \simeq \mathbb{Q}(\sqrt{5})$ be its center. Further denote by $\mathfrak{M}$ a maximal $\mathbb{Z}_K$-order of Q. If $C_G(N) \subset N$, then $[G : N] = |\mathrm{Out}(N)| = 2$. The outer automorphism of N does not centralize K. So $\mathrm{End}(\overline{G}) < Q$ is a totally definite quaternion algebra with center $\mathbb{Q}$ ramified at a subset of $\Pi(|G|) = \{2, 3, 5\}$. Hence it contains nontrivial torsion units which contradicts the assumption $C_G(N) \subset N$.
Thus there exists some $g \in C_G(N) \setminus N$. We may assume that g is contained in the torsion group $\mathfrak{M}^{*,1} \simeq \mathrm{SL}_2(5)$, since all maximal $\mathbb{Z}_K$-orders of Q are conjugate. This leaves three cases and in any case $\Pi(|G|) = \{2, 3, 5\}$.

- $U := N \circ C_3 \leq G$. Then K is the maximal totally real subfield of $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{5})$. Further U fixes only one lattice L and there exists some $F \in \mathcal{F}_{>0}(U)$ which is integral on L with $\det(L, F) = 1$. By the 2-parameter argument (see Table 2.5.4), G must either fix (L, F) or $(L, \mathfrak{p}_5 F)$. The minimal totally complex subfields of K are isomorphic to $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-15})$, this gives rise to four candidates:

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-15})$
F	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$	${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4$
$\mathfrak{p}_5 F$	${}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3$	${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4$

- $U := N \circ C_4 \leq G$. Then $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(i, \sqrt{5})$. As above, U fixes only one lattice L and there exists some $F \in \mathcal{F}_{>0}(U)$ which is integral on L and $\det(L, F) = 1$. Again we have four candidates

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-5})$
F	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4$
$\mathfrak{p}_5 F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4$

- $U := N \circ C_5 \leq G$. Then $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{10})$ is minimal totally complex. Further U fixes only one lattice L. One checks that U is already s.i.m.f..

After omitting the groups which do not contain a normal subgroup conjugate to N, one gets the claimed result. □

We now turn to a case to case discussion of the various Fitting subgroups.

Lemma 4.5.8 *If $O_5(G) \neq 1$ then G is conjugate to one of the following groups:*

$$ {}_{\zeta_{10}}[C_{10}]_1 \otimes A_2,\ {}_{\sqrt{-15}}[C_{30}:C_4]_4,\ {}_{\sqrt{-4}}[C_{20}:C_4]_4 \ \textit{or}\ {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5 . $$

Proof: Table 2.5.2 shows $O_5(G) \simeq C_5$. If $O_3(G) \simeq C_3$ then G contains an irreducible cyclic normal subgroup of order 30. Since $|\mathrm{Aut}(C_{30})| = 8$, it follows from Lemma 4.5.5 that $G \simeq {}_{\zeta_{10}}[C_{10}]_1 \otimes A_2$ or ${}_{\sqrt{-15}}[C_{30}:C_4]_4$.
Suppose now $O_3(G) = 1$. By Table 2.5.2, $O_2(G)$ must be conjugate to one of C_2, C_4, D_8 or Q_8 (note that $\mathbb{Q}(\zeta_{10})$ splits $\mathcal{Q}_{\infty,2}$). In the latter three cases, G contains an irreducible cyclic subgroup of order 20 and $\Pi(|G|) \subseteq \{2, 3, 5\}$. Hence $G \simeq {}_{\sqrt{-4}}[C_{20} : C_4]_4$ by Lemma 4.5.3.
If $O_2(G) \simeq C_2$, then $C_G(F(G))$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{10}))$. If $E(G) \neq 1$, then by Table 2.5.1, $E(G)$ is conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$. In this case, Lemma 4.5.7 shows that $G \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$. It remains the case that $F^*(G) \simeq C_{10}$. Then $G/F^*(G) \simeq \mathrm{Aut}(C_{10}) \simeq C_4$ by Lemma 2.2.1. There are two such extensions. The group $C_{10} : C_4$ is reducible, whereas $C_{10} \cdot C_4$ has $\mathcal{Q}_{\infty,5}$ as commuting algebra. This group cannot be maximal finite since $\mathcal{Q}_{\infty,5}$ contains nontrivial torsion units. □

$O_3(G) \neq 1$ **and** $O_5(G) = 1$

We now suppose that $O_5(G) = 1$ and $O_3(G) \neq 1$. By Table 2.5.2 we have $O_3(G) \simeq C_3$ and $O_2(G) \in \{C_2, C_4, C_8, D_8, Q_8, 2_+^{1+4}, 2_-^{1+4}, D_8 \otimes C_4, D_{16}, QD_{16}, Q_{16}\}$.
These cases are handled in the following lemmas.

Lemma 4.5.9

(a) If $O_2(G) \simeq D_{16}$ *then* $G \simeq {}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$.

(b) $O_2(G)$ *is not conjugate to* C_8, QD_{16} *or* Q_{16}.

Proof: In all these cases G contains an irreducible cyclic subgroup of order 24 and $\Pi(|G|) = \{2, 3, 5\}$. So the result follows from Lemma 4.5.4. □

Lemma 4.5.10 *If* $O_2(G) \simeq 2_+^{1+4}$, *then* $G \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes F_4$.

Proof: $\mathcal{B}^o(F(G))$ is conjugate to ${}_{\sqrt{-3}}[C_6]_1 \otimes F_4$ and fixes only one lattice. Its commuting algebra is isomorphic to $\mathbb{Q}(\sqrt{-3})$. Hence the claim is easily verified. □

Lemma 4.5.11 $O_2(G) \not\simeq 2_-^{1+4}$.

Proof: Suppose $O_2(G) \simeq 2_-^{1+4}$. Then G contains the normal subgroup $N := \mathcal{B}^o(F(G))$ conjugate to ${}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \circ C_3$. The group N fixes 2 lattices and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. One easily checks that it is only contained in ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$ (which has the wrong Fitting subgroup). □

Lemma 4.5.12 *If* $O_2(G) \simeq D_8 \otimes C_4$, *then* $G \simeq {}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_2$.

Proof: The subgroup $N := \mathcal{B}^o(F(G)) \simeq {}_i[(D_8 \otimes C_4).S_3]_2 \otimes C_3$ of G contains an irreducible cyclic subgroup of order 24 and $\Pi(|G|) = \{2, 3\}$. So the result follows from Lemma 4.5.4. □

Lemma 4.5.13 $O_2(G) \not\simeq D_8$ *and* $O_2(G) \not\simeq C_4$.

Proof: Suppose $O_2(G)$ is one of these groups. In either case G contains an irreducible normal subgroup $N \leq F(G)$ which is isomorphic to C_{12}. Then $C := C_G(N)$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{12}))$ and contains N. Thus $E(G) = 1$ and $G/F(G) \leq \mathrm{Out}(F(G)) \simeq C_2 \times C_2$. If $O_3(G) \not\subseteq Z(G)$ then there exists some $g \in G$ such that g induces the nontrivial outer automorphism on $O_3(G)$ and $g^2 \in O_2(G)$. By maximality we get $g \in O_2(G)$ which contradicts $O_2(G) \subset C_G(O_3(G))$. So $G/F(G) \leq C_2$. But this contradicts Lemma 2.2.1. □

Lemma 4.5.14 *If $O_2(G) \simeq C_2$ then G is conjugate to ${}_{\sqrt{-3}}[C_6]_1 \otimes A_4$, ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$, ${}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3$, ${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4$ or ${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4$.*

Proof: If $E(G) = 1$, then $G/F(G) \leq \mathrm{Out}(C_6) \simeq C_2$ contradicts Lemma 2.2.1. So $E(G) \neq 1$. But all these cases have already been handled in Lemmas 4.5.6 and 4.5.7. □

Lemma 4.5.15 *If $O_2(G) \simeq Q_8$ then G is either conjugate to ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4$ or ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2-}{\square} C_3]_4$.*

Proof: Let $N := \mathcal{B}^o(F(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$. Then $C := C_G(N)$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-3}))$ which implies that C is soluble. Therefore $G/N \leq \mathrm{Out}(N) \simeq C_2 \times C_2$. By Lemma 2.2.1, we know that $[G : N] = 4$. So there is some $\alpha \in G \setminus N$ such that α induces the outer automorphism on $\mathrm{SL}_2(3)$ and acts trivially on $O_3(G)$. Hence α commutes with the center of $E := \mathrm{End}(N) \simeq \mathbb{Q}(\sqrt{-3})^{2\times 2}$. By Lemma 2.2.16, $H := \langle N, \alpha \rangle$ is (up to conjugacy) uniquely determined by the isomorphism type of $K := C_E(\alpha)$. There are two possibilities, namely $K \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{2})$ or $K \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{-2})$.

- $K \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{-2})$: The maximal totally real subfield of K is isomorphic to $\mathbb{Q}(\sqrt{6})$. Further H fixes only one lattice L and there exists some $F \in \mathcal{F}_{>0}(H)$ that is integral on L such that $\det(L, F) = 16$. Since $\mathrm{Nr}_{K/\mathbb{Q}(\sqrt{6})}(\mathbb{Z}_K^*) = \mathbb{Z}[\sqrt{6}]_{>0}^*$ we may restrict ourselves to one class of totally positive units. By Table 2.5.4, this leaves the following four candidates.

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-2})$
F	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)^2$	${}_{\sqrt{-2}}[F_4:2]_2$
$\mathfrak{p}_3 F$	${}_{\sqrt{-3}}[C_6]_1 \otimes F_4$	$\lneq {}_{\sqrt{-2}}[F_4:2]_2$

But none of these groups has the correct Fitting subgroup.

- $K \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{2})$: The maximal totally real subfield of K is isomorphic to $\mathbb{Q}(\sqrt{2})$. Further H fixes only one lattice L and there exists some $F \in \mathcal{F}_{>0}(H)$ which is integral on L such that $\det(L, F) = 1$. By Table 2.5.4, there are four candidates:

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-6})$
F	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$	${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2-}{\square} C_3]_4$
$\mathfrak{p}_2 F$	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)^2$	${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4$

The result follows if one checks the Fitting subgroups of these candidates. □

$O_p(G) = 1$ for all odd primes p

In this last section, suppose that $F(G) = O_2(G)$. By Table 2.5.2, $O_2(G)$ is isomorphic to one of $C_2, C_4, C_8, C_{16}, D_8, Q_8, 2_+^{1+4}, 2_-^{1+4}, D_8 \otimes C_4, D_8 \otimes C_8, D_8 \otimes QD_{16}$, $2_+^{1+4} \otimes C_4$, D_{16}, QD_{16}, Q_{16} or QD_{32}. These cases are handled below. This concludes the classification of the s.i.m.f. matrix groups of degree 8.

Lemma 4.5.16 *If $F(G) \simeq 2_+^{1+4} \otimes C_4$, then $G = \mathcal{B}^o(F(G)) \simeq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4$.*

Proof: The group $\mathcal{B}^o(F(G))$ fixes up to isomorphism only one lattice L and has $\mathbb{Q}(i)$ as commuting algebra. Hence the claim is easily verified. □

Lemma 4.5.17 *If $F(G) \simeq QD_{32}$, then $G = F(G)$.*

Proof: Follows from Lemma 3.3.1. □

Lemma 4.5.18 *$F(G)$ is not isomorphic to C_8, D_{16}, QD_{16} or Q_{16}.*

Proof: In all these cases G would contain a normal cyclic subgroup N of order 8. Then $C := C_G(N)$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_8))$. Hence $E(G) = 1$ and this implies that $G/F(G) \leq \mathrm{Out}(F(G))$ is a 2-group. So $G = F(G)$ is reducible. □

Lemma 4.5.19 *If $F(G) \simeq 2_-^{1+4}$, then G is conjugate to ${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$.*

Proof: The normal subgroup $N := \mathcal{B}^o(F(G)) \simeq {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$ is self centralizing in G and $\mathrm{End}(\overline{N}) \simeq Q_{\infty,2}$. Hence $[G : N] \leq 2$. Thus by Remark 2.2.17, G must contain a subgroup U conjugate to $N.2$. Since U fixes up to isomorphism a unique lattice and has $\mathbb{Q}(\sqrt{-2})$ as commuting algebra, the claim is easily verified. □

Lemma 4.5.20 *If $F(G) \simeq 2_+^{1+4}$, then G is conjugate to ${}_{\sqrt{-2}}[F_4 : 2]_4$.*

Proof: Let $N := \mathcal{B}^o(F(G)) \simeq F_4$. Then $C := C_G(N)$ embeds into $\mathbb{Q}^{2\times 2}$. So C and G are soluble. Thus again $[G : N] \leq 2$ and by Remark 2.2.17 we conclude that $G \simeq {}_{\sqrt{-2}}[F_4 : 2]_4$. Finally one checks that this group is s.i.m.f.. □

Lemma 4.5.21 *$F(G) \not\simeq C_{16}$.*

Proof: Follows from Lemma 4.5.2. □

Lemma 4.5.22 *$F(G)$ is neither isomorphic to $D_8 \otimes C_8$ nor $D_8 \otimes QD_{16}$.*

Proof: In both cases, G would contain an irreducible normal subgroup $N \simeq D_8 \otimes C_8$. Then $B := \mathcal{B}^o(N) \simeq N.S_3$ fixes only one lattice L and has $\mathbb{Q}(\zeta_8)$ as commuting algebra. One finds some $F \in \mathcal{F}_{>0}(B)$ that is integral on L such that $\det(L, F) = 1$. Since $\Pi(|G|) = \{2, 3\}$ it follows from Table 2.5.4 that we have to check the following four candidates:

	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4$	${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$
$\mathfrak{p}_2 F$	${}_i[(D_8 \otimes C_4).S_3]_2^2$	${}_{\sqrt{-2}}[F_4 : 2]_4$

None of these groups has the correct Fitting subgroup. □

Lemma 4.5.23 *$F(G) \not\simeq D_8 \otimes C_4$.*

Proof: Suppose $F(G) \simeq D_8 \otimes C_4$. Then $N := \mathcal{B}^o(F(G)) \simeq (D_8 \otimes C_4).S_3$ has $\mathbb{Q}(i)^{2\times 2}$ as commuting algebra. Thus $E(G) = 1$. But then $[G : N] \leq 2$ contradicts Lemma 2.2.1. □

Lemma 4.5.24 *$F(G) \not\simeq Q_8$.*

Proof: Suppose $F(G) \simeq Q_8$. Then $N := \mathcal{B}^o(F(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ and $C_G(N)$ embeds into $\mathrm{GL}_2(\mathcal{Q}_{\infty,2})$. Hence it follows from Table 2.5.1 that $E(G) = 1$. Thus $G/N \leq \mathrm{Out}(\mathrm{SL}_2(3)) \simeq C_2$ contradicts Lemma 2.2.1. □

Lemma 4.5.25

(a) If $F(G)$is isomorphic to C_4, then G is conjugate to ${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4$ or ${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4$.

(b) The Fitting group $F(G)$ is not isomorphic to D_8.

Proof: If $E(G) = 1$ then $[G : F(G)] \leq 2$ contradicts Lemma 2.2.1. So by Table 2.5.1, $E(G)$ must be conjugate to Alt_5 or $\mathrm{SL}_2(5)$. Thus the result follows from Lemmas 4.5.6 and 4.5.7. □

Lemma 4.5.26 *If $F(G) \simeq C_2$, then G is conjugate to ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$.*

Proof: By Table 2.5.1, $E(G)$ is conjugate to one of Alt_5, $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{SL}_2(9)$ or $2.\mathrm{Alt}_7$ (note that $O_3(\mathcal{B}^o(\mathrm{Sp}_4(3))) \simeq C_3$). We have already classified these groups G in Lemmas 4.5.6 and 4.5.7. □

4.6 Dimension 10

Theorem 4.6.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_{10}(\mathbb{Q})$ *are*

#	G	$\|G\|$	$\|\mathcal{Z}(G)\|$	L_{min}	*r.i.m.f. supergroups*
$[2,1,5]$	${}_i[C_4]_1^5$	$2^{13}\cdot 3\cdot 5$	3	$[1,1,20]$	B_{10}
1	${}_i[C_4]_1 \otimes A_5$	$2^6\cdot 3^2\cdot 5$	6	$[3^2,2,60]$	A_5^2
$[2,2,5]$	${}_{\sqrt{-3}}[C_6]_1^5$	$2^8\cdot 3^6\cdot 5$	1	$[3^5,2,30]$	A_2^5
2	${}_{\sqrt{-3}}[\pm S_4(3)\circ C_3]_5$	$2^7\cdot 3^5\cdot 5$	2	$[2^2\cdot 3^5,4,270]$	$[(C_6\times S_4(3)).2]_{10}$
3	${}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5$	$2^3\cdot 3\cdot 5\cdot 11$	1	$[11^5,6,110]$	$A_{10}^{(3)}$

Proof: We know that that ${}_{\sqrt{-3}}[C_6]_1^5$ is maximal finite by Lemma 2.1.21. The group ${}_i[C_4]_1^5$ fixes three lattices and has $\mathbb{Q}(i)$ as commuting algebra. One checks that it is also maximal finite. So we may now suppose that G is s.p.i.m.f..

Then G cannot be soluble according to Corollary 4.1.2. So Table 2.5.1 shows that $E(G)$ is isomorphic to Alt_6, $\mathrm{L}_2(11)$ or $S_4(3)$.

The group ${}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5$ is s.i.m.f. by Theorem 3.1.1.

If $E(G)\simeq S_4(3)$, then $\mathrm{End}(\overline{N})\simeq\mathbb{Q}(\sqrt{-3})$ and $\mathcal{B}^o(E(G))\simeq {}_{\sqrt{-3}}[\pm S_4(3)\circ C_3]_5$ fixes up to isomorphism one lattice L. One immediately verifies that it is maximal finite.

Suppose now $E(G)\simeq\mathrm{Alt}_6$. Then $N := \mathcal{B}^o(E(G))\simeq\mathrm{Aut}(A_5)\simeq\pm S_6$. If $F(G) = \pm I_{10}$ then $G = N$ is reducible, since the exceptional outer automorphism of S_6 cannot be realized in $\mathrm{GL}_5(\mathbb{C})$. So $F(G)$ is cyclic of order 6 or 4 by Lemma 4.1.1. In the first case G would be properly contained in ${}_{\sqrt{-3}}[\pm S_4(3)\circ C_3]_5$. In the latter case $G = {}_i[C_4]_1\otimes A_5$ which one easily verifies to be maximal finite by computing the automorphism groups of all six G-invariant lattices.

The r.i.m.f. supergroups are easily constructed since G is uniform in any case. □

4.7 Dimension 12

Theorem 4.7.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_{12}(\mathbb{Q})$ *are*

#	G	$\|G\|$	$\|\mathcal{Z}(G)\|$	L_{min}	*r.i.m.f. supergroups*
[4, 1, 3]	$_i[(D_8 \otimes C_4).S_3]_2^3$	$2^{16}{\cdot}3^4$	1	$[2^6, 2, 72]$	F_4^3
[2, 1, 6]	$_i[C_4]_1^6$	$2^{16}{\cdot}3^2{\cdot}5$	3	$[1, 1, 24]$	B_{12}
1	$_i[C_4]_1 \otimes E_6$	$2^9{\cdot}3^4{\cdot}5$	2	$[3^2, 2, 144]$	E_6^2
[4, 2, 3]	$({}_i[C_4]_1 \otimes A_2)^3$	$2^{10}{\cdot}3^4$	2	$[3^6, 2, 36]$	A_2^6
2	$_{\infty,3}[\pm U_3(3)]_3 \circ C_4$	$2^7{\cdot}3^3{\cdot}7$	2	$[3^6, 4, 756]$	$[6.U_4(3).2^2]_{12}$
3	$_i[C_4]_1 \otimes A_6$	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	2	$[7^2, 2, 84]$	A_6^2
4	$_i[S \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$	$2^6{\cdot}3^4$	4	$[2^6{\cdot}3^4, 4, 216]$	$[S \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{12}$
5	$_i[\mathrm{L}_2(7) \overset{2(2)}{\otimes} {}_i[C_4]_1]_6$	$2^6{\cdot}3{\cdot}7$	2	$[2^6{\cdot}7^2, 4, 336]$	$[\mathrm{L}_2(7) \overset{2(2)}{\otimes} D_8]_{12}$
6	$_i[C_4]_1 \otimes A_6^{(2)}$	$2^6{\cdot}3{\cdot}7$	2	$[7^6, 4, 84]$	$(A_6^{(2)})^2$
7	$_i[{}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$	$2^6{\cdot}3{\cdot}7$	2	$[2^6{\cdot}7^6, 8, 336]$	$[\mathrm{L}_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}$
[4, 3, 3]	$_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^3$	$2^{13}{\cdot}3^4$	1	$[2^6, 2, 72]$	F_4^3
8	$_{\sqrt{-2}}[\pm\mathrm{L}_2(7){\cdot}2]_6$	$2^5{\cdot}3{\cdot}7$	2	$[7^2, 2, 84]$	A_6^2
9	$_{\sqrt{-2}}[{}_{\infty,2}[\mathrm{SL}_2(5)]_3 : 2]_6$	$2^4{\cdot}3{\cdot}5$	6	$[2^2{\cdot}5, 3, 80]$	$[\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}$
10	$_{\sqrt{-3}}[6.U_4(3).2]_6$	$2^9{\cdot}3^7{\cdot}5{\cdot}7$	1	$[3^6, 4, 756]$	$[6.U_4(3).2^2]_{12}$
[2, 2, 6]	$_{\sqrt{-3}}[C_6]_1^6$	$2^{10}{\cdot}3^8{\cdot}5$	1	$[3^6, 2, 36]$	A_2^6
[6, 1, 2]	$_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3^2$	$2^9{\cdot}3^8$	2	$[3^2, 2, 144]$	E_6^2
[4, 4, 3]	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)^3$	$2^{10}{\cdot}3^7$	2	$[2^6, 2, 72]$	F_4^3
11	$S \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$	$2^6{\cdot}3^5$	4	$[2^6{\cdot}3^4, 4, 216]$	$[S \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{12}$
12	$_{\sqrt{-3}}[\pm 3.M_{10}]_6$	$2^5{\cdot}3^3{\cdot}5$	2	$[3^6{\cdot}5^6, 8, 270]$	$[\pm 3.\mathrm{Alt}_{12}.2^2]_{12}$
13	$_{\sqrt{-3}}[C_6]_1 \otimes M_{6,2}$	$2^4{\cdot}3^2{\cdot}5$	6	$[3^6{\cdot}5^6, 6, 60]$	$A_2 \otimes M_{6,2}$
14	$_{\infty,2}[\mathrm{SL}_2(5)]_3 \circ C_3$	$2^3{\cdot}3^2{\cdot}5$	8	$[2^2{\cdot}5^4, 4, 360]$	$[\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}$
15	$_{\sqrt{-5}}[{}_i[C_4]_1 \overset{2_-}{\boxtimes} \mathrm{Alt}_5]_6$	$2^5{\cdot}3{\cdot}5$	14	$[1, 1, 24]$ $[1, 2, 264]$	B_{12}
16	$_{\sqrt{-5}}[{}_i[C_4]_1 \overset{2_+}{\boxtimes} \mathrm{Alt}_5]_6$	$2^5{\cdot}3{\cdot}5$	14	$[5^6, 4, 60]$ $[5^6, 3, 40]$	$M_{6,2}^2$
[6, 2, 2]	$_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3^2$	$2^9{\cdot}3^2{\cdot}7^2$	1	$[7^2, 4, 84]$	$(A_6^{(2)})^2$
17	$_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$	$2^5{\cdot}3^2{\cdot}7$	2	$[3^6, 4, 756]$	$[6.U_4(3).2^2]_{12}$
18	$_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \otimes A_2$	$2^5{\cdot}3^2{\cdot}7$	2	$[3^6{\cdot}7^6, 8, 126]$	$A_2 \otimes A_6^{(2)}$
19	$_{\sqrt{-11}}[\mathrm{SL}_2(11)]_6$	$2^3{\cdot}3{\cdot}5{\cdot}11$	5	$[1, 1, 24]$ $[1, 2, 264]$	B_{12}
20	$_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{\cdot}2_1]_6$	$2^5{\cdot}3^3{\cdot}5$	2	$[3^6, 4, 756]$	$[6.U_4(3).2^2]_{12}$
21	$_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6 : 2_1]_6$	$2^5{\cdot}3^3{\cdot}5$	2	$[3^6{\cdot}5^6, 8, 270]$	$[\pm 3.\mathrm{Alt}_{12}.2^2]_{12}$
[4, 5, 3]	$_{\zeta_{10}}[C_{10}]_1^3$	$2^4{\cdot}3{\cdot}5^3$	1	$[5^3, 2, 60]$	A_4^3

22	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$	$2^3 \cdot 3 \cdot 5^2$	2	$[2^4 \cdot 5^3, 4, 420]$	$[D_{20} \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{12}$
23	${}_{\zeta_{26}+\zeta_{26}^3+\zeta_{26}^9}[C_{26} : C_3]_3$	$2 \cdot 3 \cdot 13$	9	$[13, 2, 156]$	A_{12}

Here S denotes the s.p.i.m.f. subgroup ${}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 < \mathrm{Sp}_6(\mathbb{Q})$. A proof of this theorem is given in Section 4.7.2.

4.7.1 Irreducible cyclic subgroups

Let $G < \mathrm{Sp}_{12}(\mathbb{Q})$ be s.i.m.f. such that G contains an irreducible cyclic subgroup U. Then $|\pm U| \in \{26, 28, 36, 42\}$. These groups are classified below.

Lemma 4.7.2 *If $\pm U$ has order 26 then G is conjugate to ${}_{\zeta_{26}+\zeta_{26}^3+\zeta_{26}^9}[C_{26} : C_3]_3$.*

Proof: Follows from Theorem 3.1.1. □

Lemma 4.7.3 *If U has order 28, then G is conjugate to ${}_i[C_4]_1 \otimes A_6$, ${}_i[C_4]_1 \otimes A_6^{(2)}$, ${}_i[\mathrm{L}_2(7)_6 \overset{2(2)}{\otimes} {}_i[C_4]_1]_6$, ${}_i[{}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$ or ${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$.*

Proof: The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{28})$ and has class number 1. Thus U fixes only one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L, F) = 7^2$. It follows from Minkowski's bound that $\Pi(|G|) = \{2, 3, 5, 7, 11, 13\}$. The cases that $|G|$ is divisible by 11 or 13 are handled in Theorems 3.1.1 and 3.2.1. So we have $\Pi(|G|) = \{2, 3, 5, 7\}$. By Table 2.5.3, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^k \mathfrak{p}_7^{2l} F)$ for some $k, l \in \{0, 1\}$ and $d \in \{1, 7\}$. These automorphism groups are

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-7})$
F	${}_i[C_4]_1 \otimes A_6$	$\lneq {}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$
$\mathfrak{p}_2 F$	${}_i[\mathrm{L}_2(7)_6 \overset{2(2)}{\otimes} {}_i[C_4]_1]_6$	$\lneq {}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$
$\mathfrak{p}_7^2 F$	${}_i[C_4]_1 \otimes A_6^{(2)}$	${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$
$\mathfrak{p}_2 \mathfrak{p}_7^2 F$	${}_i[{}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$	$\lneq {}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$

So the result follows. □

Lemma 4.7.4 *If U has order 36, then G is conjugate to one of:*

$$ {}_i[C_4]_1 \otimes E_6,\ {}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3^2,\ {}_i[{}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6, $$

$$ {}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \otimes {}_{\infty,2}[\mathrm{SL}_2(3)]_1,\ ({}_i[C_4]_1 \otimes A_2)^2,\ {}_{\sqrt{-3}}[C_6]_1^6 . $$

Proof: The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{36})$ and has class number 1. Thus U fixes only one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L,F) = 9$. As above, we have $\Pi(|G|) \subseteq \{2,3,5,7\}$. Restricting to normalized lattices (see Definition 2.2.4), Table 2.5.3 shows that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^k\mathfrak{p}_3^l F)$ for some $(k,l) \in \{(0,0),(1,1),(0,2)\}$ and $d \in \{1,3\}$. These groups are given below.

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-3})$
F	${}_i[C_4]_1 \otimes E_6$	${}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3^2$
$\mathfrak{p}_2\mathfrak{p}_3 F$	${}_i[{}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$	${}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3 \otimes {}_{\infty,2}[\mathrm{SL}_2(3)]_1$
$\mathfrak{p}_3^2 F$	$({}_i[C_4]_1 \otimes A_2)^2$	${}_{\sqrt{-3}}[C_6]_1^6$

This proves the result. □

Lemma 4.7.5 *If $\pm U$ has order 42, then G is conjugate to one of*

$$_{\sqrt{-3}}[6.U_4(3).2]_6,\ {}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \otimes A_2,\ {}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1 .$$

Proof: The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{42})$ and has class number 1. So U fixes only one lattice L. Further one finds some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L and $\det(L,F) = 3^6$. As above we have $\Pi(|G|) = \{2,3,5,7\}$. By Table 2.5.3, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3^{-k}\mathfrak{p}_5^k\mathfrak{p}_7^l F)$ or $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3^{-k}\mathfrak{p}_5'^k\mathfrak{p}_7^l F)$ for some $k \in \{0,1\}$, $0 \le l \le 3$ and $d \in \{3,7\}$. Since $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3^{-1}\mathfrak{p}_5'\mathfrak{p}_7^l F) = \mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3^{-1}\mathfrak{p}_5\mathfrak{p}_7^l F) = \mathrm{Aut}_{\mathbb{Q}(\sqrt{-3})}(L, \mathfrak{p}_7^l F) \cap \mathrm{Aut}_{\mathbb{Q}(\sqrt{-7})}(L, \mathfrak{p}_7^l F)$ for all l and d we have eight groups to check.

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-7})$
F	${}_{\sqrt{-3}}[6.U_4(3).2]_6$	${}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$
$\mathfrak{p}_7 F$	$\lneq {}_{\sqrt{-3}}[6.U_4(3).2]_6$	$\lneq {}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \otimes A_2$
$\mathfrak{p}_7^2 F$	$\lneq {}_{\sqrt{-3}}[6.U_4(3).2]_6$	$\lneq {}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$
$\mathfrak{p}_7^3 F$	$\lneq {}_{\sqrt{-3}}[6.U_4(3).2]_6$	${}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \otimes A_2$

So the lemma is proven. □

4.7.2 Proof of Theorem 4.7.1

We have to prove the completeness of the list given in Theorem 4.7.1 (the correctness follows from explicit calculations). The candidates for the maximal finite symplectic imprimitive groups come from the classification of the s.p.i.m.f. subgroups of $\mathrm{Sp}_2(\mathbb{Q})$, $\mathrm{Sp}_4(\mathbb{Q})$ and $\mathrm{Sp}_6(\mathbb{Q})$. By Lemma 2.1.21, we only have to check the group ${}_i[C_4]_1^6$. It fixes up to isomorphism three lattices and has $\mathbb{Q}(i)$ as commuting algebra. One verifies that it is s.i.m.f..

So it remains to prove the completeness for the s.p.i.m.f. matrix groups. Thus, for the remainder of this section let $G < \mathrm{Sp}_{12}(\mathbb{Q})$ be s.p.i.m.f..

The case that $O_{13}(G) \neq 1$ is handled in Lemma 4.7.2. Hence we may assume that $F(G) = O_2(G)O_3(G)O_5(G)$. So Theorem 4.7.1 follows by discussing all possible candidates for the Fitting subgroup of G. This is done in the subsequent lemmas.

Again, we handle the irreducible quasisimple normal subgroups first.

Lemma 4.7.6 *Suppose $G < \mathrm{Sp}_{12}(\mathbb{Q})$ is s.p.i.m.f. and G contains an irreducible quasisimple normal subgroup N. Then one of the following holds.*

(a) If $N \simeq 6.U_4(3)$ then $G = \mathcal{B}^o(N) \simeq {}_{\sqrt{-3}}[6.U_4(3).2]_6$.

(b) If $N \simeq U_3(3)$ then $G \simeq {}_{\infty,3}[\pm U_3(3)]_3 \circ C_4$.

(c) If N is conjugate to $3.\mathrm{Alt}_6$ with character $\chi_{3a}+\chi'_{3a}+\chi_{3b}+\chi'_{3b}$, then G is conjugate to ${}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{\cdot}2]_6$, ${}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{:}2]_6$ or ${}_{\sqrt{-3}}[\pm 3.M_{10}]_6$.

(d) If $N \simeq \mathrm{SL}_2(11)$ then $G = N$.

(e) If N is conjugate to ${}_{\infty,2}[\mathrm{SL}_2(5)]_3$, then G is conjugate to ${}_{\sqrt{-2}}[{}_{\infty,2}[\mathrm{SL}_2(5)]_3{:}2]_6$ or ${}_{\infty,2}[\mathrm{SL}_2(5)]_3 \circ C_3$.

Proof: Suppose first that we are in one of the cases mentioned above.

(a) $\mathcal{B}^o(N)$ fixes only one lattice and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. So the claim is easily verified.

(b) The commuting algebra of N is isomorphic to $\mathcal{Q}_{\infty,3}$ and $\mathrm{Out}(N) \simeq C_2$. Thus by Remark 2.2.17, G contains $N \circ C_4$, $N \circ C_3$ or $\pm N.2$ as a subgroup. The first group is already s.i.m.f.. The other two groups fix 1 or 4 lattices respectively. One checks that they are only contained in ${}_{\sqrt{-3}}[6.U_4(3).2]_6$.

(c) The commuting algebra C of N is isomorphic to $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$ with $\mathbb{Q}(\sqrt{5})$ as maximal totally real subfield. Further, N fixes only one lattice L and one finds some $F \in \mathcal{F}_{>0}(G)$ that is integral on L with $\det(L, F) = 3^6$. Since $\Pi(|G|) = \{2, 3, 5\}$ this leaves the following 4 candidates (see Table 2.5.4).

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-15})$
F	${}_{\sqrt{-3}}[6.U_4(3).2]_6$	${}_{\sqrt{-3}}[\pm 3.\mathrm{Alt}_6{\cdot}2]_6$
$\mathfrak{p}_5 F$	${}_{\sqrt{-3}}[\pm 3.M_{10}]_6$	${}_{\sqrt{-3}}[\pm 3.\mathrm{Alt}_6{:}2]_6$

So the result follows.

(d) Follows from Theorem 3.2.1.

(e) The centralizing algebra of N is isomorphic to $Q_{\infty,2}$ and $\mathrm{Out}(N) \simeq C_2$. Hence, by Remark 2.2.17, G contains a subgroup conjugate to: $N \circ C_3, N.2$ and $N \circ C_4$. These fix $8, 6$ and 20 lattices respectively. The first two groups are maximal finite. The remaining group is only contained in $G = {}_i[C_4]_1^6$.

If N is not conjugate to one of these groups, it follows from Table 2.5.1, that N must be conjugate to $6.L_3(4)$ or $3.\mathrm{Alt}_6$ with character $\chi_6 + \chi_6'$.
The group $6.L_3(4)$ fixes two lattices and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. One checks that it is only contained in ${}_{\sqrt{-3}}[6.U_4(3).2]_6$.
In the remaining case, N fixes 12 lattices and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. One checks that N has the maximal finite symplectic supergroups ${}_{\sqrt{-3}}[6.U_4(3).2]_6$ and ${}_{\sqrt{-3}}[C_6]_1^6$ which both do not normalize N. □

Lemma 4.7.7 *If $O_5 \neq 1$ then G is conjugate to* ${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$.

Proof: By Table 2.5.2, $O_5(G) \simeq C_5$. Thus $C_G(O_5(G))$ embeds into $\mathrm{GL}_3(\mathbb{Q}(\zeta_{10}))$. In particular, by loc. cit. $F(G) \simeq C_{10}$. But then $E(G) = 1$ contradicts Corollary 2.2.3. So Table 2.5.1 shows that $F^*(G) \simeq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$. But then $F^*(G)$ fixes 2 lattices and has $\mathbb{Q}(\zeta_{10})$ as commuting algebra. One checks that it is already s.i.m.f. by computing their automorphism group (with respect to the full form space). □

Lemma 4.7.8 *If $O_3(G) = 3_+^{1+2}$, then G is conjugate to one of*

$${}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \quad or \quad {}_i[{}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6 .$$

Proof: By Table 2.5.1 we have $E(G) = 1$. If $F(G) = \pm O_3(G)$ then $G/\mathcal{B}^o(F(G)) \leq C_2$ contradicts Lemma 2.2.1. So by Table 2.5.2 we have $F(G) \simeq 3_+^{1+2} \otimes H$ with $H = C_4, D_8$ or Q_8. If $O_2(G) \simeq Q_8$, then $G = \mathcal{B}^o(F(G)) \simeq {}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ is already s.i.m.f..
So we may now assume that G contains a normal subgroup conjugate to $N := B(O_3(G)) \otimes C_4 \simeq {}_{\sqrt{-3}}[3_+^{1+2} : \mathrm{SL}_2(3)]_3 \otimes C_4$ of index at most 4. In particular $\Pi(|G|) = \{2, 3\}$. The commuting algebra C of U is isomorphic to $\mathbb{Q}(i, \sqrt{3})$ and has $K \simeq \mathbb{Q}(\sqrt{3})$ as maximal totally real subfield. Up to isomorphism, N fixes two lattices, which are both N-normal critical. Further we find some $L \in \mathcal{Z}(N)$ and a form $F \in \mathcal{F}_{>0}(U)$ that is integral on L such that $\det(L, F) = 3^2$. Since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$ we may restrict ourselves to one class of totally positive units. By Table 2.5.4, this leaves the following cases:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-3})$
F	${}_i[C_4]_1 \otimes E_6$	${}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]^2$
$\mathfrak{p}_2\mathfrak{p}_3 F$	${}_i[{}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$	${}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$

This proves the claim. □

Lemma 4.7.9 *$O_3(G)$ is not isomorphic to C_9.*

Proof: Suppose $O_3(G) = C_9$. Then $C_G(O_3(G))$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_9))$. Thus G is soluble. If $F(G) = \pm C_9$ then, by Lemma 2.2.1, G contains a subgroup N isomorphic to $C_{18}.C_3$ of index at most 2. But then $O_3(N) = O_3(G)$ has order 27. So we may assume that $F(G) = C_9 \otimes H$ with $H \simeq C_4, D_8$ or Q_8. In any case G contains an irreducible cyclic subgroup of order 36. Hence the result follows from Lemma 4.7.4. □

Lemma 4.7.10 *If $O_3(G) = C_3$, then G is conjugate to one of* ${}_{\sqrt{-3}}[6.U_4(3).2]_6$, ${}_{\sqrt{-3}}[C_6]_1 \otimes M_{6,2}$, ${}_{\sqrt{-3}}[\pm 3.M_{10}]_6$, ${}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{\cdot}2_1]_6$, ${}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{:}2_1]_6$, ${}_{\infty,2}[\mathrm{SL}_2(5)]_3 \circ C_3$, ${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$, ${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \otimes A_2$.

Proof: By Table 2.5.1, $E(G)$ is either trivial, irreducible or isomorphic to Alt_5, $\mathrm{L}_2(7)$, $U_4(2)$. Let $N := O_3(G)E(G)$. This leaves the following cases:

(a) If $E(G) = 1$ then G is soluble and $O_2(G) \leq C_G(O_3(G))$ embeds into $\mathrm{GL}_6(\mathbb{Q}(\sqrt{-3}))$. Thus $O_2(G)$ is isomorphic to one of C_2, C_4, D_8 or Q_8. The normal subgroup $U := \mathcal{B}^o(O_2(G))O_3(G) \trianglelefteq G$ is self centralizing and embeds into $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-3}))$. But since $G/U \leq \mathrm{Out}(U)$ is a 2-group, this contradicts Corollary 2.2.3.

(b) If $E(G) \simeq \mathrm{Alt}_5$ then $C := \mathrm{End}(\overline{N}) \simeq \mathbb{Q}(\sqrt{5}, \sqrt{-3})$ and denote by $K \simeq \mathbb{Q}(\sqrt{5})$ its maximal totally real subfield. Then N fixes up to isomorphism 4 lattices whose endomorphism ring equals $\mathbb{Z}_C$. But only one of them, call it L, has minimal superlattices which have not this maximal order as endomorphism ring. So L is N-normal critical since C has class number 1 (see Remark 2.2.8). We find some $F \in \mathcal{F}_{>0}(N)$ such that $\det(L, F) = 2^4 \cdot 3^6$. Moreover $G/N \leq C_2 \times C_2$ shows that $\Pi(|G|) = \{2, 3, 5\}$. By 2.5.4 G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_5^\alpha F)$ for some $\alpha \in \{0, 1\}$ and $d \in \{3, 15\}$.

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-15})$
F	$\lneq {}_{\sqrt{-3}}[C_6]_1^6$	$\lneq {}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{\cdot}2_1]_6$
$\mathfrak{p}_5 F$	${}_{\sqrt{-3}}[C_6]_1 \otimes M_{6,2}$	$\lneq {}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{:}2_1]_6$

So $G \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes M_{6,2}$.

(c) If $E(G) \simeq \mathrm{L}_2(7)$ then G contains an irreducible cyclic subgroup of order 21. Thus G is conjugate to ${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$ or ${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \otimes A_2$ by Lemma 4.7.5.

(d) If $E(G) \simeq \mathrm{Alt}_7$ or $E(G) \simeq U_4(2)$ then N is irreducible with commuting algebra $\mathbb{Q}(\sqrt{-3})$. One checks that they have only one s.i.m.f. supergroup which is ${}_{\sqrt{-3}}[6.U_4(3).2]_6$. But this group does not contain N as a normal subgroup.

(e) Finally if $E(G)$ is irreducible (and quasisimple), Lemma 4.7.6 shows that G is conjugate to ${}_{\sqrt{-3}}[6.U_4(3).2]_6$, ${}_{\sqrt{-3}}[\pm 3.M_{10}]_6$, ${}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{\cdot}2_1]_6$, ${}_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6{:}2_1]_6$ or ${}_{\infty,2}[\mathrm{SL}_2(5)]_3 \circ C_3$. □

Lemma 4.7.11 *If $O_p(G) = 1$ for all odd primes p, then $O_2(G)$ is conjugate to one of C_2, C_4 or D_8 and $E(G) \neq 1$.*

Proof: If $O_2(G)$ is not isomorphic to C_2, C_4 and D_8, then Corollary 2.1.16 and Table 2.5.2 show that $O_2(G) \in \mathcal{O} := \{Q_8,\ C_8,\ D_{16},\ QD_{16},\ D_8 \otimes C_4,\ 2_+^{1+4}\}$.
Suppose first $E(G) = 1$. For any possible $O_2(G)$ it follows from loc. cit. that $G/\mathcal{B}^o(O_2(G))$ is a 2-group and $\mathcal{B}^o(O_2(G))$ embeds into $\mathrm{GL}_k(\mathbb{Q})$ for some $k \in \{1, 2, 4\}$. This contradicts Corollary 2.2.3. Hence $E(G) \neq 1$.
Finally, suppose $O_2(G) \in \mathcal{O}$. Then $\mathrm{End}(\overline{O_2(G)}) \simeq \mathrm{GL}_3(Q)$ where Q is isomorphic to one of $\mathbb{Q}, \mathbb{Q}(i), \mathbb{Q}(\sqrt{\pm 2})$ or $\mathcal{Q}_{\infty,2}$. But then Table 2.5.1 implies $E(G) = 1$ which is impossible (note that $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-7})$ do not split $\mathcal{Q}_{\infty,2}$). □

Lemma 4.7.12 *If $F(G)$ is conjugate to C_4 or D_8 then G is conjugate to one of*

$$_{\infty,3}[U_3(3)]_3 \circ C_4,\ _{\sqrt{-5}}[_i[C_4]_1 \overset{2_+}{\boxtimes} \mathrm{Alt}_5]_6,\ _{\sqrt{-5}}[_i[C_4]_1 \overset{2_-}{\boxtimes} \mathrm{Alt}_5]_6,\ _i[C_4]_1 \otimes A_6^{(2)},$$

$$_i[_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes}\ _i[C_4]_1]_6,\ _i[\mathrm{L}_2(7) \overset{2(2)}{\otimes}\ _i[C_4]_1]_6,\ _i[C_4]_1 \otimes A_6,\ _i[C_4]_1 \otimes E_6$$

Proof: If $E(G)$ is irreducible, then $G \simeq\ _{\infty,3}[U_3(3)]_3 \circ C_4$ according to Lemma 4.7.6. Suppose now that $E(G)$ is not irreducible. Then $E(G)$ embeds into $\mathrm{GL}_6(\mathbb{Q}(i))$ since $F(G)$ contains a characteristic subgroup U isomorphic to C_4. Table 2.5.1 shows that there are five possibilities left.

(a) If $E(G) \simeq \mathrm{Alt}_5$ then $N := \mathrm{Alt}_5 \otimes C_4 \trianglelefteq G$. The commuting algebra of N is isomorphic to $\mathbb{Q}(i, \sqrt{5})$ with $\mathbb{Q}(\sqrt{5})$ as maximal totally real subfield. Since $G/N \leq C_2 \times C_2$ we have $\Pi(|G|) = \{2, 3, 5\}$. Up to isomorphism, N fixes 3 lattices that have the maximal order as endomorphism ring. One of them is N-normal critical, L say. One finds some $F \in \mathcal{F}_{>0}(N)$ that is integral on L such that $\det(L, F) = 1$. By Table 2.5.4 this leaves 4 candidates:

form	$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Q}(i)$
F	$_{\sqrt{-5}}[_i[C_4]_1 \overset{2_-}{\boxtimes} \mathrm{Alt}_5]_6$	$\lneq\ _i[C_4]_1^6$
$\mathfrak{p}_5 F$	$_{\sqrt{-5}}[_i[C_4]_1 \overset{2_+}{\boxtimes} \mathrm{Alt}_5]_6$	$\lneq\ _i[C_4]_1^6$

So $G \simeq\ _{\sqrt{-5}}[_i[C_4]_1 \overset{2_-}{\boxtimes} \mathrm{Alt}_5]_6$ or $_{\sqrt{-5}}[_i[C_4]_1 \overset{2_+}{\boxtimes} \mathrm{Alt}_5]_6$.

(b) If $E(G)$ is conjugate to $_{\sqrt{-7}}[\mathrm{L}_2(7)]_3$ then $N :=\ _{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \otimes C_4 \trianglelefteq G$. The commuting algebra C of N is isomorphic to $\mathbb{Q}(i, \sqrt{-7})$. Let $K \simeq \mathbb{Q}(\sqrt{7})$ be its maximal totally real subfield. Since $G/N \leq \mathrm{Out}(N) \simeq C_2 \times C_2$, we have $\Pi(|G|) = \{2, 3, 7\}$. The group N fixes only one lattice L and there exists some $F \in \mathcal{F}_{>0}(H)$ that is integral on L such that $\det(L, F) = 7^6$. Since $\mathrm{End}_{\mathbb{Z}N}(L)$ is the maximal order in C and since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$ we may restrict ourselves to one class of totally positive units. Let $\langle\sigma\rangle = \mathrm{Gal}(C/K) \simeq C_2$. One finds that σ is conjugation by some $x \in N_{\mathrm{GL}_{12}(\mathbb{Q})}(N) \cap \mathrm{GL}(L)$ with $xFx^{\mathrm{tr}} = F$. So x

interchanges the two prime ideals over 3 (and necessarily fixes the unique prime ideals over 2 and 7). By Table 2.5.4 this leaves the following 8 automorphism groups:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-7})$
F	${}_i[C_4]_1 \otimes A_6^{(2)}$	${}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$
$\mathfrak{p}_2 F$	${}_i[{}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$	$\lneq {}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3^2$
$\mathfrak{p}_3\mathfrak{p}_7^{-1}F$	${}_{\infty,3}[U_3(3)]_4 \circ C_4$	${}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$
$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_7^{-1}F$	N	N

So $G \simeq {}_i[C_4]_1 \otimes A_6^{(2)}$ or ${}_i[{}_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$.

(c) If $E(G)$ is conjugate to $\mathrm{L}_2(7)$ with character $2\chi_6$ then $N := \mathrm{L}_2(7) \otimes C_4 \trianglelefteq G$ fixes up to isomorphism 6 lattices and its commuting algebra is isomorphic to $\mathbb{Q}(i)$. One checks that N is only contained in ${}_i[C_4]_1 \otimes A_6$ or ${}_i[\mathrm{L}_2(7) \overset{2(2)}{\otimes} {}_i[C_4]_1]_6$. So $G \simeq {}_i[\mathrm{L}_2(7) \overset{2(2)}{\otimes} {}_i[C_4]_1]_6$.

(d) If $E(G) \simeq \mathrm{Alt}_7$ then $\mathcal{B}^o(F^*(G)) \simeq {}_i[C_4]_1 \otimes A_6$ is already s.i.m.f..

(e) If $E(G) \simeq U_4(2)$ then $\mathcal{B}^o(F^*(G)) \simeq {}_i[C_4]_1 \otimes E_6$ is already s.i.m.f.. □

Lemma 4.7.13 *If $F(G) \simeq C_2$ then G is conjugate to ${}_{\sqrt{-2}}[\pm\mathrm{L}_2(7)\cdot 2]_6$, ${}_{\sqrt{-11}}[\mathrm{SL}_2(11)]_6$ or ${}_{\sqrt{-2}}[{}_{\infty,2}[\mathrm{SL}_2(5)]_3 : 2]_6$.*

Proof: If $E(G)$ is irreducible, then Lemma 4.7.6 shows that $G \simeq {}_{\sqrt{-11}}[\mathrm{SL}_2(11)]_6$. Otherwise Table 2.5.1 shows that $E(G)$ is isomorphic to Alt_5, Alt_7, $U_4(2)$ or $\mathrm{L}_2(7)$ (with two representations). The 6-dimensional representation of Alt_5 extends to $\pm\mathrm{Alt}_5 : 2$ in $\mathrm{GL}_6(\mathbb{Q})$ by [CCN+85]. The element $\frac{1+\sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5}) \simeq C_{\mathbb{Q}^{6\times 6}}(\mathrm{Alt}_5)$ has norm -1. Thus also $\pm\mathrm{Alt}_5 \cdot 2 < \mathrm{GL}_6(\mathbb{Q})$. So $E(G) \not\simeq \mathrm{Alt}_5$. If $E(G) \simeq \mathrm{Alt}_7$ or $U_4(2)$ then $G/\pm E(G) \leq \mathrm{Out}(E(G)) \simeq C_2 \simeq \mathcal{B}^o(E(G))/\pm E(G)$ shows that $G = \mathcal{B}^o(E(G))$ is reducible. Suppose $E(G)$ is conjugate to $\mathrm{L}_2(7)$ with character $2\chi_{3ab}$. Then $G/\pm E(G) \leq \mathrm{Out}(E(G)) \simeq C_2$ contradicts Lemma 2.2.1. Finally suppose $E(G)$ is conjugate to $\mathrm{L}_2(7)$ with character $2\chi_6$. The extension $\pm\mathrm{L}_2(7):2$ embeds into $\mathrm{GL}_6(\mathbb{Q}(\sqrt{2}))$ (see [CCN+85]). The split extension ${}_{\sqrt{-2}}[\pm\mathrm{L}_2(7){\cdot}2]_6 < \mathrm{GL}_6(\mathbb{Q}(\sqrt{-2}))$ is maximal finite as one easily verifies. □

4.8 Dimension 14

Theorem 4.8.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_{14}(\mathbb{Q})$ *are*

#	G	$\|G\|$	$\|\mathcal{Z}(G)\|$	L_{min}	*r.i.m.f. supergroups*
$[2,1,7]$	${}_i[C_4]_1^7$	$2^{18}\cdot3^2\cdot5\cdot7$	3	$[1,1,28]$	B_{14}
1	${}_i[C_4]_1\otimes E_7$	$2^{11}\cdot3^4\cdot5\cdot7$	3	$[1,2,252]$	E_7^2
2	${}_i[U_3(3)\circ C_4]_7$	$2^7\cdot3^3\cdot7$	4	$[2^6\cdot3^2,3,112]$	$[U_3(3)\overset{2}{\circ} C_4]_{14}$
$[2,2,7]$	${}_{\sqrt{-3}}[C_6]_1^7$	$2^{11}\cdot3^9\cdot5\cdot7$	1	$[3^7,2,42]$	A_2^7
3	${}_{\sqrt{-3}}[C_6]_1\otimes E_7$	$2^{10}\cdot3^5\cdot5\cdot7$	2	$[2^2\cdot3^7,4,378]$	$A_2\otimes E_7$

Proof: By explicit calculations, one verifies that the above table is correct and yields s.i.m.f. groups. Further the r.i.m.f. supergroups are easily constructed since all s.i.m.f. groups are uniform. It remains to show the completeness of the classification. The group ${}_{\sqrt{-3}}[C_6]_1^7$ is s.i.m.f. by Lemma 2.1.21. The group ${}_i[C_4]_1^7$ fixes 3 lattices and has $\mathbb{Q}(i)$ as commuting algebra. One verifies that it is s.i.m.f.. So it remains to prove that the completeness of the s.p.i.m.f. matrix groups.

Let $G < \mathrm{Sp}_{14}(\mathbb{Q})$ be s.p.i.m.f.. Then Corollary 4.1.2 shows that $E(G) \neq 1$. Thus $N := \mathcal{B}^o(E(G))$ is conjugate to A_7, E_7 or $U_3(3)\circ C_4$ by Table 2.5.1.

The group ${}_i[U_3(3)\circ C_4]_7$ is already s.i.m.f.. In the other two cases, $\mathrm{Out}(N)$ is trivial. Thus by Lemma 4.1.1, G contains a subgroup conjugate to $C_k\otimes N$ with $k\in\{4,6\}$. If $N\simeq E_7$, this yields the maximal finite subgroups stated above. The groups ${}_{\sqrt{-3}}[C_6]_1\otimes A_7$ and ${}_i[C_4]_1\otimes A_7$ are irreducible and fix 4 and 7 lattices respectively. One checks that they are only contained in ${}_{\sqrt{-3}}[C_6]_1\otimes E_7$ and ${}_i[C_4]_1\otimes E_7$ respectively. □

4.9 Dimension 16

Theorem 4.9.1 *The s.i.m.f. subgroups G of $\mathrm{Sp}_{16}(\mathbb{Q})$ are*

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
[8, 1, 2]	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	$2^{21}\cdot3^4\cdot5^2$	1	$[1, 2, 480]$	E_8^2
[2, 1, 8]	${}_i[C_4]_1^8$	$2^{23}\cdot3^2\cdot5\cdot7$	3	$[1, 1, 32]$	B_{16}
[4, 1, 4]	${}_i[(D_8 \otimes C_4).S_3]_2^4$	$2^{23}\cdot3^5$	1	$[2^8, 2, 96]$	F_4^4
1	${}_i[C_4]_1 \otimes E_8$	$2^{15}\cdot3^5\cdot5^2\cdot7$	1	$[1, 2, 480]$	E_8^2
2	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	$2^{17}\cdot3^4\cdot5\cdot7$	1	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4$
[4, 2, 4]	$({}_i[C_4]_1 \otimes A_2)^4$	$2^{15}\cdot3^5$	2	$[3^8, 2, 48]$	A_2^8
[8, 2, 2]	$({}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_2)^2$	$2^{13}\cdot3^4$	2	$[2^8\cdot3^8, 4, 144]$	$(A_2 \otimes A_4)^2$
3	${}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$2^9\cdot3^5\cdot5$	2	$[2^8\cdot3^8, 6, 960]$	$[(\mathrm{Sp}_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{16}$
4	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes A_2$	$2^{11}\cdot3^3\cdot5$	2	$[3^8, 4, 720]$	$A_2 \otimes E_8$
5	${}_i[C_4]_1 \otimes M_{8,3}$	$2^6\cdot3\cdot7$	4	$[3^2\cdot7^6, 4, 168]$	$M_{8,3}^2$
6	${}_{\infty,3}[\mathrm{SL}_2(7)]_4 \circ C_4$	$2^5\cdot3\cdot7$	4	$[3^8\cdot7^4, 6, 336]$	$[\mathrm{SL}_2(7) \overset{2(3)}{\circ} \tilde{S}_3]_{16}$
[4, 3, 4]	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^4$	$2^{19}\cdot3^5$	1	$[2^8, 2, 96]$	F_4^4
[8, 3, 2]	${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4^2$	$2^{17}\cdot3^2\cdot5^2$	1	$[1, 2, 480]$	E_8^2
[8, 4, 2]	${}_{\sqrt{-2}}[F_4 : 2]_4^2$	$2^{17}\cdot3^4$	1	$[2^8, 2, 96]$	F_4^4
7	${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 : 2]_8$	$2^{14}\cdot3^4\cdot5$	1	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4$
8	${}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8$	$2^{14}\cdot3^2\cdot5\cdot7$	1	$[1, 2, 480]$	E_8^2
9	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$	$2^7\cdot3^2\cdot5$	2	$[2^8\cdot5^4, 4, 240]$	$A_4 \otimes F_4$
10	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5) : 2]_2$	$2^7\cdot3^2\cdot5$	2	$[5^8, 4, 240]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)) : 2]_8^2$
[2, 2, 8]	${}_{\sqrt{-3}}[C_6]_1^8$	$2^{15}\cdot3^{10}\cdot5\cdot7$	1	$[3^8, 2, 48]$	A_2^8
[8, 5, 2]	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4^2$	$2^{15}\cdot3^{10}\cdot5^2$	1	$[1, 2, 480]$	E_8^2
11	${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$	$2^{14}\cdot3^6\cdot5^2\cdot7$	1	$[3^8, 4, 720]$	$A_2 \otimes E_8$
[4, 4, 4]	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)^4$	$2^{15}\cdot3^9$	2	$[2^8, 2, 96]$	F_4^4
[8, 6, 2]	$({}_{\sqrt{-3}}[C_6]_1 \otimes F_4)^2$	$2^{15}\cdot3^6$	2	$[2^8\cdot3^8, 4, 144]$	$(A_2 \otimes F_4)^2$

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
12	${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$	$2^{13}\cdot 3^5\cdot 5$	2	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4$
13	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$	$2^9\cdot 3^6\cdot 5$	2	$[2^8\cdot 3^8, 6, 960]$	$[(\mathrm{Sp}_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{16}$
[8, 7, 2]	$({}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3)^2$	$2^9\cdot 3^4\cdot 5^2$	2	$[5^8, 4, 240]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8^2$
[8, 8, 2]	$({}_{\sqrt{-3}}[C_6]_1 \otimes A_4)^2$	$2^9\cdot 3^4\cdot 5^2$	2	$[3^8\cdot 5^4, 4, 120]$	$(A_2 \otimes A_4)^2$
14	${}_{\sqrt{-3}}[C_6]_1 \otimes [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8$	$2^7\cdot 3^3\cdot 5^2$	2	$[3^8\cdot 5^8, 8, 360]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8 \otimes A_2$
15	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \otimes A_4$	$2^6\cdot 3^3\cdot 5$	4	$[2^8\cdot 5^4, 4, 240]$	$A_4 \otimes F_4$
16	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \underset{\sqrt{-3}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$	$2^6\cdot 3^3\cdot 5$	4	$[2^8\cdot 3^8\cdot 5^8, 12, 480]$	$[\mathrm{SL}_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (\mathrm{SL}_2(3) \overset{2}{\square} C_3)]_{16}$
17	${}_{\infty,5}[\mathrm{SL}_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4 \circ C_3$	$2^5\cdot 3^2\cdot 5^2$	4	$[5^4, 4, 2640]$	$[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$
18	${}_{\infty,2}[\mathrm{SL}_2(5) \overset{2(2)}{\boxtimes} D_8]_4 \circ C_3$	$2^6\cdot 3^2\cdot 5$	4	$[2^8\cdot 5^8, 8, 1200]$	$[\mathrm{SL}_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.\mathrm{Alt}_5]_{16}$
19	${}_{\sqrt{-3}}[C_{60}.(C_4 \times C_2)]_8$	$2^5\cdot 3\cdot 5$	8	$[2^8\cdot 3^8\cdot 5^4, 8, 360]$	$[D_{120}.(C_4 \times C_2)]_{16}$
[8, 9, 2]	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4^2$	$2^{11}\cdot 3^2\cdot 5^2$	2	$[1, 2, 480]$	E_8^2
[8, 10, 2]	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4^2$	$2^{11}\cdot 3^2\cdot 5^2$	2	$[5^8, 4, 240]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8^2$
20	${}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2-}{\boxtimes} {}_i[C_4]_1]_8$	$2^8\cdot 3^2\cdot 5^2$	2	$[1, 2, 480]$	E_8^2
21	${}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2+}{\boxtimes} {}_i[C_4]_1]_8$	$2^8\cdot 3^2\cdot 5^2$	2	$[5^8, 4, 240]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8^2$
[8, 11, 2]	${}_{\sqrt{-5}}[C_{20}:C_4]_4^2$	$2^9\cdot 5^2$	4	$[5^4, 2, 80]$	A_4^4
22	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	$2^8\cdot 3^2\cdot 5$	2	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4$
23	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2-}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	$2^8\cdot 3^2\cdot 5$	2	$[2^8\cdot 5^8, 8, 1200]$	$[\mathrm{SL}_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.\mathrm{Alt}_5]_{16}$
24	${}_{\sqrt{-5}}[{}_{\infty,5}[\mathrm{SL}_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4:2]_8$	$2^6\cdot 3\cdot 5^2$	4	$[5^4, 4, 2640]$	$[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)).2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$

#	G	$\|G\|$	$\|\mathcal{Z}(G)\|$	L_{min}	r.i.m.f. supergroups
25	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$	$2^7 \cdot 3 \cdot 5$	4	$[2^8 \cdot 5^4, 4, 240]$	$A_4 \otimes F_4$
[8, 12, 2]	${}_{\sqrt{-6}}[{}_{\infty,2}[\tilde{S}_4]_1 \overset{2-}{\square} C_3]_4^2$	$2^{11} \cdot 3^4$	2	$[1, 2, 480]$	E_8^2
[8, 13, 2]	${}_{\sqrt{-6}}[{}_{\infty,2}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4^2$	$2^{11} \cdot 3^4$	2	$[2^8, 2, 96]$	F_4^4
26	${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \overset{2(3)}{\boxtimes} A_2]_8$	$2^9 \cdot 3^2 \cdot 5$	2	$[1, 2, 480]$	E_8^2
27	${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \underset{i}{\overset{2(3)}{\boxtimes}} {}_{\infty,3}[\tilde{S}_3]_1]_8$	$2^9 \cdot 3^2 \cdot 5$	2	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4$
[8, 14, 2]	${}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4^2$	$2^{11} \cdot 3^2$	4	$[3^8, 2, 48]$	$A_2^8,\ (A_2 \otimes F_4)^2$
28	${}_{\sqrt{-6}}[{}_{\infty,2}[\mathrm{SL}_2(3)]_1 \underset{i}{\overset{2}{\boxtimes}} {}_{\infty,3}[\mathrm{SL}_2(9)]_2]_8$	$2^7 \cdot 3^3 \cdot 5$	2	$[3^8, 4, 720]$	$[(\mathrm{SL}_2(9) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} \mathrm{SL}_2(9)).2]_{16}$
29	${}_{\sqrt{-6}}[F_4 \overset{2}{\boxtimes} {}_{\infty,3}[\tilde{S}_3]_1]_8$	$2^9 \cdot 3^3$	2	$[3^8, 4, 720]$	$A_2 \otimes E_8$
30	${}_{\sqrt{-6}}[(F_4 \otimes A_2) : 2]_8$	$2^9 \cdot 3^3$	2	$[2^8 \cdot 3^8, 4, 144]$	$(F_4 \otimes A_2)^2$
[8, 15, 2]	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4^2$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2$	1	$[1, 2, 480]$	E_8^2
31	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes A_2$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	2	$[3^8, 4, 720]$	$E_8 \otimes A_2$
32	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	2	$[3^8 \cdot 7^8, 12, 1680]$	$[2.\mathrm{Alt}_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}$
33	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\overset{2+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	$2^7 \cdot 3^2 \cdot 5$	2	$[1, 2, 480]$	E_8^2
34	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\overset{2-}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	$2^7 \cdot 3^2 \cdot 5$	2	$[5^8, 4, 240]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)).2]_8^2$
35	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$	$2^6 \cdot 3 \cdot 5$	4	$[2^8 \cdot 5^4, 5, 240]$	$A_4 \otimes F_4$
[8, 16, 2]	${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4^2$	$2^9 \cdot 3^4 \cdot 5^2$	2	$[1, 2, 480]$	E_8^2
[8, 17, 2]	${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4^2$	$2^9 \cdot 3^4 \cdot 5^2$	2	$[5^8, 4, 240]$	$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)) : 2]_8^2$

#	G	$\|G\|$	$\|\mathcal{Z}(G)\|$	L_{min}	*r.i.m.f. supergroups*
36	${}_{\sqrt{-15}}[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\overset{2_-}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_8$	$2^7{\cdot}3^3{\cdot}5^2$	2	$[3^8,4,720]$	$A_2\otimes E_8$
37	${}_{\sqrt{-15}}[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\overset{2_+}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_8$	$2^7{\cdot}3^3{\cdot}5^2$	2	$[3^8{\cdot}5^8,8,360]$	$A_2\otimes[(\mathrm{SL}_2(5)\overset{2}{\square}\mathrm{SL}_2(5)):2]_8$
$[8,18,2]$	${}_{\sqrt{-15}}[C_{30}:C_4]_4^2$	$2^7{\cdot}3^2{\cdot}5^2$	4	$[3^8{\cdot}5^4,4,120]$	$(A_2\otimes A_4)^2$
38	${}_{\sqrt{-15}}[({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_3)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_8$	$2^5{\cdot}3^2{\cdot}5^2$	4	$[5^4,4,2640]$	$[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{16}$
$[8,19,2]$	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_5)^2$	$2^7{\cdot}3^2{\cdot}5^4$	1	$[1,2,480]$	$E_8^2,\ [(\mathrm{SL}_2(5)\overset{2}{\square}\mathrm{SL}_2(5)):2]_8^2$
$[4,5,4]$	${}_{\zeta_{10}}[C_{10}]_1^4$	$2^7{\cdot}3{\cdot}5^4$	1	$[5^4,2,80]$	A_4^4
39	${}_{\zeta_{10}}[C_{10}]_1\underset{\sqrt{5}}{\otimes}((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)$	$2^6{\cdot}3^2{\cdot}5^3$	1	$[5^4,4,2640]$	$[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{16}$
40	${}_{\zeta_{10}}[C_{10}]_1\otimes_{\sqrt{5}',\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$	$2^7{\cdot}3{\cdot}5^2$	2	$[2^8,4,4320]$	$F_4\tilde{\otimes}F_4,\ [\mathrm{SL}_2(5)\underset{\infty,2}{\overset{2(2)}{\boxtimes}}2_-^{1+4}.\mathrm{Alt}_5]_{16}$
$[8,20,2]$	$({}_{\zeta_{10}}[C_{10}]_1\otimes A_2)^2$	$2^5{\cdot}3^3{\cdot}5^2$	2	$[3^8{\cdot}5^4,4,120]$	$(A_2\otimes A_4)^2$
41	${}_{\zeta_{10}}[C_{10}]_1\otimes F_4$	$2^7{\cdot}3^2{\cdot}5$	2	$[2^8{\cdot}5^4,4,240]$	$A_4\otimes F_4$
42	${}_{\zeta_{10}}[C_{10}]_1\otimes_{\sqrt{5}',\infty,3}[\mathrm{SL}_2(9)]_2$	$2^4{\cdot}3^2{\cdot}5^2$	2	$[3^8,4,720]$	$[(\mathrm{SL}_2(9)\underset{\infty,3}{\overset{2(3)}{\otimes}}\mathrm{SL}_2(9)).2]_{16},$ $[\mathrm{SL}_2(5)\underset{\infty,3}{\overset{2(3)}{\boxtimes}}\mathrm{SL}_2(9)]_{16}$
43	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_5)\otimes A_2$	$2^4{\cdot}3^2{\cdot}5^2$	2	$[3^8,4,720]$	$A_2\otimes E_8,\ A_2\otimes[(\mathrm{SL}_2(5)\overset{2}{\square}\mathrm{SL}_2(5)):2]_8$
44	${}_{\zeta_{10}}[C_{10}]_1\otimes_{\sqrt{5}',\infty,3}[\mathrm{SL}_2(3)\overset{2}{\square}C_3]_2$	$2^4{\cdot}3^2{\cdot}5$	4	$[2^8{\cdot}3^8,6,960]$	$[\mathrm{SL}_2(5)\underset{\infty,3}{\overset{2(3)}{\boxtimes}}(\mathrm{SL}_2(3)\overset{2}{\square}C_3)]_{16},$ $[(\mathrm{Sp}_4(3)\circ C_3)\underset{\sqrt{-3}}{\overset{2}{\boxtimes}}\mathrm{SL}_2(3)]_{16}$
45	${}_{\zeta_{10}}[C_{60}.(C_2\times C_2)]_4$	$2^4{\cdot}3{\cdot}5$	4	$[2^8{\cdot}3^8{\cdot}5^4,8,360]$	$D_{120}.(C_4\times C_2)$
$[8,21,2]$	${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2^2$	2^{11}	4	$[1,1,32]$	$B_{16},\ F_4^4$
46	${}_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8\otimes QD_{32}).S_3]_4$	$2^8{\cdot}3$	2	$[1,2,480]$	$E_8^2,\ F_4^4$

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
47	${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2 \otimes A_2$	$2^6{\cdot}3$	4	$[3^8, 2, 48]$	$A_2^8,\ (A_2 \otimes F_4)^2$
48	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \underset{\sqrt{5}'}{\overset{2-}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$	$2^5{\cdot}3{\cdot}5$	2	$[1, 2, 480]$	$E_8^2,\ [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8^2$
49	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \underset{\sqrt{5}'}{\overset{2+}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$	$2^5{\cdot}3{\cdot}5$	2	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4,\ [\mathrm{SL}_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2_-^{1+4}.\mathrm{Alt}_5]_{16}$
50	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[D_{16} \overset{2}{\boxtimes} {}_{\zeta_{10}}[C_{10}]_1]_4$	$2^5{\cdot}5$	4	$[5^4, 2, 80]$	$A_4^4,\ A_4 \otimes F_4$
51	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(1,1,1)}$	$2^4{\cdot}3{\cdot}5$	4	$[1, 2, 480]$	$E_8^2,\ [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8^2$
52	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(i,1,1)}$	$2^4{\cdot}3{\cdot}5$	4	$[2^8, 4, 4320]$	$F_4 \tilde{\otimes} F_4,\ [\mathrm{SL}_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2^{1+4'}.\mathrm{Alt}_5]_{16}$
53	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(1,1,-1)}$	$2^4{\cdot}3{\cdot}5$	4	$[5^4, 4, 2640]$	$[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)).2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_{16}$
54	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(1,-1,1)}$	$2^4{\cdot}3{\cdot}5$	4	$[3^8, 4, 720]$	$A_2 \otimes E_8,\ A_2 \otimes [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8$
55	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(i,1,-1)}$	$2^4{\cdot}3{\cdot}5$	4	$[2^8{\cdot}5^4, 4, 240]$	$A_4 \otimes F_4$
56	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(i,-1,1)}$	$2^4{\cdot}3{\cdot}5$	4	$[2^8{\cdot}3^8, 6, 960]$	$[(\mathrm{Sp}_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{16}$, $[(\mathrm{SL}_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (\mathrm{SL}_2(3) \overset{2}{\square} C_3)]_{16}$
57	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(1,-1,-1)}$	$2^4{\cdot}3{\cdot}5$	4	$[3^8{\cdot}5^4, 4, 120]$	$(A_2 \otimes A_4)^2$
58	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(i,-1,-1)}$	$2^4{\cdot}3{\cdot}5$	4	$[2^8{\cdot}3^8{\cdot}5^4, 8, 360]$	$[D_{120}.(C_4 \times C_2)]_{16}$
59	${}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[{}_{\theta_{16},\infty}[Q_{32}]_1 \overset{2}{\square} C_3]_4$	$2^6{\cdot}3$	4	$[1, 2, 480]$	$E_8^2,\ F_4^4$
60	${}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[D_{32} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$	$2^6{\cdot}3$	4	$[3^8, 2, 48]$	$A_2^8,\ (A_2 \otimes F_4)^2$
61	${}_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2$	$2^3{\cdot}3{\cdot}5$	4	$[11^4, 4, 480]$	$[D_{120}.C_2]_{16,1}$, $[D_{120}.C_2]_{16,2}$
62	${}_{\zeta_{32}-\zeta_{32}^{-1}}[QD_{64}]_2$	2^6	2	$[1, 1, 32]$	$B_{16},\ F_4^4$
63	${}_{\zeta_{34}}[C_{34}]_1$	$2{\cdot}17$	1	$[17, 2, 272]$	$A_{16},\ A_{16}^{(3)}$

The eleven s.p.i.m.f. subgroups $G < \mathrm{Sp}_{16}(\mathbb{Q})$ with an irreducible normal cyclic subgroup of order 60 have the following presentations:

- ${}_{\sqrt{-3}}[C_{60}.(C_4 \times C_2)]_4 \simeq \langle x, \alpha, \beta \mid x^{60}, \alpha^4, \beta^2, x^\alpha = x^7, x^\beta = x^{19}, \alpha^\beta = x^{15}\alpha \rangle$
- ${}_{\zeta_{10}}[C_{60}.(C_2 \times C_2)]_4 \simeq \langle x, \alpha, \beta \mid x^{60}, \alpha^2, \beta^2, x^\alpha = x^{11}, x^\beta = x^{31}, \alpha^\beta = x^{15}\alpha \rangle$
- ${}_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2 \simeq \langle x, \alpha \mid x^{60}, \alpha^2 = x^{15}, x^\alpha = x^{49} \rangle$.
- ${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(a,b,e)}$ is a $\mathbb{Q}$-irreducible matrix group isomorphic to

  $$\langle x, \alpha, \beta \mid x^{60}, x^\alpha = -x^{-1}, x^\beta = x^{19}, \alpha^2 = a, \beta^2 = b, \alpha\beta = e\beta\alpha^{-1} \rangle$$

  for $e, b \in \{\pm 1\}$ and $a \in \{1, i := x^{15}\}$.

To distinguish between the matrix groups ${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2+}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$ and ${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2-}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$ we make the following convention. These two groups are generated by ${}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\otimes} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ and some α that centralizes the unique normal subgroup isomorphic to Q_8 in $O_2(G) \simeq D_8 \otimes C_4 \simeq Q_8 \circ C_4$. Up to conjugacy one has the two choices $\alpha^2 = I_{16}$ and $\alpha^2 = -I_{16}$. In the first case we write $\overset{2+}{\underset{i}{\boxtimes}}$ and in the second case we use $\overset{2-}{\underset{i}{\boxtimes}}$.

Similarly, the two s.p.i.m.f. matrix groups ${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2+}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$ and ${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2-}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$ are generated by ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ and some α that centralizes $\mathcal{B}^o(O_2(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1$. Again there are up to conjugacy two possibilities. We write $\overset{2+}{\underset{\sqrt{-2}}{\boxtimes}}$ for $\alpha^2 = I_{16}$ and $\overset{2-}{\underset{\sqrt{-2}}{\boxtimes}}$ means $\alpha^2 = -I_{16}$.

Lemma 4.9.2 *All groups listed in Theorem 4.9.1 are s.i.m.f..*

Proof: The candidates for the maximal finite symplectic imprimitive matrix groups come from the classification of the s.p.i.m.f. subgroups of $\mathrm{Sp}_{2m}(\mathbb{Q})$ with $m \in \{1, 2, 4\}$. All groups except ${}_i[C_4]_1^8$ and ${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2^2$ are s.i.m.f. by Lemma 2.1.21. These two groups have $\mathbb{Q}(i)$ and $\mathbb{Q}(\zeta_{16} - \zeta_{16}^{-1})$ as commuting algebras and they fix up to isomorphism 3 and 4 lattices respectively. One checks that they are maximal finite by computing the corresponding automorphism groups (wrt. the full form space).

So we may now assume that $G < \mathrm{Sp}_{16}(\mathbb{Q})$ is s.p.i.m.f.. According to Theorem 3.1.1, ${}_{\zeta_{34}}[C_{34}]_1$ is the unique s.i.m.f. subgroup of $\mathrm{Sp}_{16}(\mathbb{Q})$ whose order is a multiple of 17.
If G is any other group from the above table whose order is not divisible by 17 and which is not isomorphic to $C_{60}.C_2$, then $E := \mathrm{End}(\overline{G})$ is a minimal totally complex field. Thus these groups are checked to be s.i.m.f. by computing $\mathrm{Aut}_E(L, F)$ where $F \in \mathcal{F}_{>0}(G)$ is fixed and L runs through a system of representatives for the isomorphism classes of $\mathcal{Z}(G)$. Obviously, G is not a subgroup of $[D_{120}.C_2]_{16,i}$ see ([NP95, (IV.1)

Theorem]). Thus loc. cit. shows that any r.i.m.f. supergroup of G fixes a primitive lattice of determinant only divisible by $2, 3, 5$ or 7. So the r.i.m.f. supergroups can be constructed by the m-parameter argument.
Finally, suppose that $G = {}_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2$. Let $C := \mathrm{End}(\overline{G}) \simeq \mathbb{Q}(i, \sqrt{3}, \sqrt{5})$ and denote by $K \simeq \mathbb{Q}(\sqrt{3}, \sqrt{5})$ its maximal totally real subfield. Since C is not minimal totally complex, the m-parameter argument would be quite tedious. We give another proof of the maximality, using only the classification of the r.i.m.f. subgroups of $\mathrm{GL}_{16}(\mathbb{Q})$ (see [NP95, (IV.1) Theorem]).
The group G fixes up to isomorphism four lattices $L_1, \dots, L_4$ which can be chosen such that the index $[L_1 : L_i]$ is a power of 5. Further one finds some $F \in \mathcal{F}_{>0}(G)$ that is integral on L_1 with $\det(L_1, F) = 11^4$.
Suppose H is a r.i.m.f. supergroup of G. Let $(L', F') \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ be integral and of minimal determinant among all such integral pairs. Let $d = \det(L', F')$. The classification of the r.i.m.f. subgroups of $\mathrm{GL}_{16}(\mathbb{Q})$ shows that $\Pi(d) \subseteq \{2, 3, 5, 7, 11\}$. There exist $1 \le i \le 4$, $c \in C$ and $x \in K_{>0}$ such that $L' = L_i c$ and $F' = xF$. Then (L', F') is isometric to $(L_i, x\,\mathrm{Nr}_{C/K}(c^{-1})F)$. If we set $a := x\,\mathrm{Nr}_{C/K}(c^{-1}) \in K_{>0}$ then $d = \det(L_i, F)\cdot \mathrm{Nr}_{K/\mathbb{Q}}(a)^4 = 5^k \cdot 11^4 \cdot \mathrm{Nr}_{K/\mathbb{Q}}(a)^4$ for some $k \in \mathbb{Z}$. Since the prime ideals of $\mathbb{Z}_K$ over $2, 3, 5$ and 7 are generated by Elements in $K_{>0}$ and the ideals over 11 are not, this implies that d is divisible by 11. The classification of the r.i.m.f. subgroups of $\mathrm{GL}_{16}(\mathbb{Q})$ shows that H can only be conjugate to $[D_{120}.C_2]_{16,1}$ or $[D_{120}.C_2]_{16,2}$. One immediately constructs G as a subgroup of these groups. In any case $[H : G] = 2$ and H is not symplectic. This implies that G is s.i.m.f. and we have found its r.i.m.f. supergroups. □

It remains to prove that Theorem 4.9.1 contains every conjugacy class of s.p.i.m.f. matrix groups. This is accomplished in the remainder of this section. As always, we first classify the s.i.m.f. groups that contain an irreduble cyclic subgroup. Afterwards, we turn to a case by case discussion of the various Fitting subgroups.
But before we start, we find all s.p.i.m.f. groups that contain some irreducible normal subgroup N where $\mathrm{End}(\overline{N})$ is a quaternion algebra.

Lemma 4.9.3 *If $G < \mathrm{Sp}_{16}(\mathbb{Q})$ is s.p.i.m.f. and contains a normal subgroup N conjugate to ${}_{\infty,3}[\mathrm{SL}_2(7)]_4$, then G is conjugate to ${}_{\infty,3}[\mathrm{SL}_2(7)]_4 \circ C_4$.*

Proof: The commuting algebra of N is isomorphic to $\mathcal{Q}_{\infty,3}$ and $\mathrm{Out}(N) \simeq C_2$. Hence by Remark 2.2.17, G contains a subgroup conjugate to $N \circ C_4$, $N \circ C_3$ or $N.2$. The first group is maximal finite. The other two groups fix 4 and 16 lattices respectively. One checks that they both have only one s.i.m.f. supergroup, which is ${}_{\sqrt{-3}}[C_6]_1^8$. □

Lemma 4.9.4 *If $G < \mathrm{Sp}_{16}(\mathbb{Q})$ is s.p.i.m.f. and contains a normal subgroup $N \simeq 2_-^{1+6}$, then G is conjugate to ${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 : 2]_8$ or ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$.*

Proof: The normal subgroup $B := \mathcal{B}^o(N) \simeq {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4$ of G has $\mathcal{Q}_{\infty,2}$ as commuting algebra and $\mathrm{Out}(B) \simeq C_2$. Hence by Remark 2.2.17, G contains a subgroup conjugate to $B \circ C_4$, $B \circ C_3$ or $B.2$. Their commuting algebras are isomorphic to $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-2})$ respectively. Since they only fix 1 or 2 lattices, it is easily verified that $B \circ C_4$ is only contained in ${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$ whereas the other two groups are s.i.m.f.. □

4.9.1 Irreducible cyclic subgroups

In this section, we classify all s.i.m.f. groups $G < \mathrm{Sp}_{16}(\mathbb{Q})$ that contain an irreducible cyclic subgroup U (under some assumptions on $\Pi(|G|)$). There are essentially 5 such groups U since $|\pm U| \in \{k \in \mathbb{N} \mid k \equiv_2 0 \text{ and } \varphi(k) = 16\} = \{32, 34, 40, 48, 60\}$.
The group ${}_{\zeta_{34}}[C_{34}]_1$ is the unique symplectic irreducible group G whose order is divisible by 17 as we have seen in Theorem 3.1.1.

Theorem 4.9.5 *If $|U| = 32$ then G is conjugate to ${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2^2$, ${}_{\zeta_{32}-\zeta_{32}^{-1}}[QD_{64}]_2$, ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^4$, ${}_i[C_4]_1^8$ or ${}_i[(D_8 \otimes C_4).S_3]_2^4$.*

Proof: By Minkowski's bound and Theorem 3.1.1 we have $\Pi(|G|) \subseteq \{2, 3, 5, 7, 11, 13\}$. The group U fixes only one lattice L since $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{32})$ has class number 1. Further there exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L with $\det(L, F) = 1$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta_{32})/\mathbb{Q})$ such that σ interchanges the two prime ideals over 7. One finds that σ is conjugation by some $x \in N_{\mathrm{GL}_{16}(\mathbb{Q})}(U) \cap \mathrm{GL}(L)$. Hence by Table 2.5.3, G must be conjugate to $\mathrm{Aut}_K(L, \mathfrak{p}_2^a \mathfrak{p}_7^b F)$ for some $0 \le b \le 4$, $a \in \{0, 1\}$ and $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\zeta_{16} - \zeta_{16}^{-1}), \mathbb{Q}(\zeta_{32} - \zeta_{32}^{-1})\}$.
If $K = \mathbb{Q}(\zeta_{16} - \zeta_{16}^{-1})$ or $\mathbb{Q}(\zeta_{32} - \zeta_{32}^{-1})$ then G is conjugate to ${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2^2$ or ${}_{\zeta_{32}-\zeta_{32}^{-1}}[QD_{64}]_2$ respectively. Finally, $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(L, \mathfrak{p}_2^k \mathfrak{p}_7^l F) \le {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^4$ and

$$\mathrm{Aut}_{\mathbb{Q}(i)}(L, \mathfrak{p}_2^a \mathfrak{p}_7^b F) \le \begin{cases} {}_i[C_4]_1^8 & \text{if } 0 \le b \le 2 \\ {}_i[(D_8 \otimes C_4).S_3]_2^4 & \text{if } b \in \{3, 4\} \end{cases}. \qquad \square$$

Theorem 4.9.6 *Suppose $|U| = 40$ and $\Pi(|G|) = \{2, 3, 5\}$ or there exists some $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L and $\Pi(\det(L, F)) \subseteq \{2, 3, 5\}$. Then G is conjugate to one of*

$$ {}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2,\ {}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8,\ {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8, $$

$$ {}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8,\ {}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 : 2]_8,\ {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4, $$

$$ {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5) : 2]_2,\ {}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_+}{\square} C_4]_4^2,\ {}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_-}{\square} C_4]_4^2, $$

$$ {}_{\sqrt{-5}}[C_{20} : C_4]_4^2,\ {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8,\ {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2_+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8, $$

$$ {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2_-}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8,\ {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{2}}{\overset{2_+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8, $$

$$ {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{2}}{\overset{2_-}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8,\ {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8, $$

$$ ({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2,\ {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2,\ {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2, $$

$$ {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2,\ {}_{\zeta_{10}}[C_{10}]_1^4,\ {}_{\zeta_{10}}[C_{10}]_1 \otimes F_4,\ {}_{\zeta_{10}}[C_{60}.(C_2 \times C_2)]_4, $$

$$ {}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[D_{10} \overset{2}{\boxtimes} {}_{\zeta_{16}}[C_{10}]_1]_4, $$

$$ {}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \underset{\sqrt{5}'}{\overset{2_+}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4,\ {}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \underset{\sqrt{5}'}{\overset{2_-}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4 . $$

Proof: The commuting algebra $\mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{40})$ has class number 1. Thus U fixes only one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L with $\det(L, F) = 1$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta_{40})/\mathbb{Q})$ such that σ interchanges the two prime ideals over 3. One finds that σ is conjugation by some $x \in N_{\mathrm{GL}_{16}(\mathbb{Q})}(N) \cap \mathrm{GL}(L)$. Thus by Table 2.5.3, G must be conjugate to $\mathrm{Aut}_K(L, \mathfrak{p}_2^{2a}\mathfrak{p}_3^{b}\mathfrak{p}_5^{c}F)$ for some $a, b \in \{0, 1\}$, $0 \le c \le 2$ and $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-10}), \mathbb{Q}(\zeta_{10}), \mathbb{Q}(\sqrt{2}\cdot(\zeta_{10} - \zeta_{10}^{-1}))\}$.

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	${}_{\sqrt{-2}}[(2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$
$\mathfrak{p}_2^2F$	${}_i[(2_-^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4:2]_8$
$\mathfrak{p}_3F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\sqrt{-2}}[(2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$
$\mathfrak{p}_2^2\mathfrak{p}_3F$	${}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4:2]_8$
$\mathfrak{p}_5F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$
$\mathfrak{p}_2^2\mathfrak{p}_5F$	$\lneq {}_i[(2_-^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$
$\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$
$\mathfrak{p}_5^2F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$
$\mathfrak{p}_2^2\mathfrak{p}_5^2F$	$\lneq {}_i[(2_-^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$
$\mathfrak{p}_3\mathfrak{p}_5^2F$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	${}_{\sqrt{-2}}[2_-^{1+6}.(\mathrm{Alt}_8:2)]_8$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5^2F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$

form	$\mathbb{Q}(\sqrt{-5})$	$\mathbb{Q}(\zeta_{10})$
F	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4^2$	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2$
$\mathfrak{p}_2^2F$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$
$\mathfrak{p}_3F$	$\lneq {}_{\sqrt{-2}}[(2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$
$\mathfrak{p}_2^2\mathfrak{p}_3F$	$\lneq {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$
$\mathfrak{p}_5F$	${}_{\sqrt{-5}}[C_{20}:C_4]_4^2$	${}_{\zeta_{10}}[C_{10}]_1^4$
$\mathfrak{p}_2^2\mathfrak{p}_5F$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$	${}_{\zeta_{10}}[C_{10}]_1 \otimes F_4$
$\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$	${}_{\zeta_{10}}[C_{10}]_1^4$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5F$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$	${}_{\zeta_{10}}[C_{60}.(C_2 \times C_2)]_4$
$\mathfrak{p}_5^2F$	${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4^2$	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2$
$\mathfrak{p}_2^2\mathfrak{p}_5^2F$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2-}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$
$\mathfrak{p}_3\mathfrak{p}_5^2F$	$\lneq {}_{\sqrt{-2}}[(2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5^2F$	$\lneq {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$

form	$\mathbb{Q}(\sqrt{-10})$	$\mathbb{Q}(\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1}))$
F	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_+}{\underset{\sqrt{2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_2^2F$	$\lneq {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_+}{\underset{\sqrt{2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_3F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$
$\mathfrak{p}_2^2\mathfrak{p}_3F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$
$\mathfrak{p}_5F$	$\lneq {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[D_{16} \overset{2}{\boxtimes} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_2^2\mathfrak{p}_5F$	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[D_{16} \overset{2}{\boxtimes} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_i[(2_+^{1+4}\otimes C_4).S_6]_4^2$	$\lneq {}_{\zeta_{10}}[C_{10}]_1^4$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\zeta_{10}}[C_{60}.(C_2\times C_2)]_4$
$\mathfrak{p}_5^2F$	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_-}{\underset{\sqrt{2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_2^2\mathfrak{p}_5^2F$	$\lneq {}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_-}{\underset{\sqrt{2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_3\mathfrak{p}_5^2F$	$\lneq {}_i[(2_+^{1+4}\otimes C_4).S_6]_4^2$	$\lneq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$
$\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_5^2F$	$\lneq {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$	$\lneq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$

□

Theorem 4.9.7 *Suppose* $|U| = 48$ *and* $\Pi(|G|) = \{2,3\}$ *or there exists some* $(L,F) \in \mathcal{Z}(G)\times\mathcal{F}_{>0}(G)$ *such that* F *is integral on* L *and* $\Pi(\det(L,F)) \subseteq \{2,3\}$. *Then* G *is conjugate to one of*

$$
{}_i[(2_+^{1+4}\otimes C_4).S_6]_4^2,\ {}_i[(D_8\otimes C_4).S_3]_2^4,\ ({}_i[C_4]_1\otimes A_2)^4,\ ({}_i[(D_8\otimes C_4).S_3]_2\otimes A_2)^2,
$$
$$
{}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\square} C_3]_4^2,\ {}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\square} C_3]_4^2,\ {}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4^2,
$$
$$
{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4^2,\ ({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)^4,\ {}_{\sqrt{-3}}[C_6]_1^8,\ ({}_{\sqrt{-3}}[C_6]_1\otimes F_4)^2,
$$
$$
{}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]:2]_4^2,\ {}_{\sqrt{-2}}[F_4:2]_4^2,\ {}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[{}_{\theta_{16},\infty}[Q_{32}]_1 \overset{2}{\square} C_3]_4,
$$
$$
{}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[D_{32} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4,\ {}_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8\otimes QD_{32}).S_3]_4,\ {}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2\otimes A_2\ .
$$

Proof: The commuting algebra $\mathrm{End}(\overline{U})\simeq\mathbb{Q}(\zeta_{48})$ has class number 1. Thus U fixes only one lattice L. There exists some $F\in\mathcal{F}_{>0}(U)$ that is integral on L with $\det(L,F)=1$. So by Table 2.5.3, G must be conjugate to $\mathrm{Aut}_K(L,\mathfrak{p}_2^{2a}\mathfrak{p}_3^bF)$ for some $0\le a\le 2$, $b\in\{0,1\}$ and $K\in\{\mathbb{Q}(i),\mathbb{Q}(\sqrt{-2}),\mathbb{Q}(\sqrt{-3}),\mathbb{Q}(\sqrt{-6}),\mathbb{Q}(\sqrt{3}(\zeta_{16}-\zeta_{16}^{-1})),\mathbb{Q}(\zeta_{16}-\zeta_{16}^{-1})\}$.

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-6})$	$\mathbb{Q}(\sqrt{-3})$
F	$_{i}[(2_+^{1+4}\otimes C_4).S_6]_4^2$	$_{\sqrt{-6}}[_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2-}{\square} C_3]_4^2$	$_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4^2$
$\mathfrak{p}_2^2F$	$\lneq {}_{i}[(D_8\otimes C_4).S_3]_2^4$	$\lneq {}_{\sqrt{-6}}[_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4^2$	$\lneq ({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)^4$
$\mathfrak{p}_2^4F$	$_{i}[(D_8\otimes C_4).S_3]_2^4$	$_{\sqrt{-6}}[_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4^2$	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)^4$
$\mathfrak{p}_3F$	$({}_{i}[C_4]_1\otimes A_2)^4$	$_{\sqrt{-6}}[D_{16}\overset{2}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_4^2$	$_{\sqrt{-3}}[C_6]_1^8$
$\mathfrak{p}_2^2\mathfrak{p}_3F$	$\lneq ({}_{i}[C_4]_1\otimes A_2)^4$	$\lneq {}_{\sqrt{-6}}[D_{16}\overset{2}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_4^2$	$\lneq {}_{\sqrt{-3}}[C_6]_1^8$
$\mathfrak{p}_2^4\mathfrak{p}_3F$	$({}_{i}[(D_8\otimes C_4).S_3]_2\otimes A_2)^2$	$_{\sqrt{-6}}[D_{16}\overset{2}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_4^2$	$\lneq ({}_{\sqrt{-3}}[C_6]_1\otimes F_4)^2$

The remaining groups are

$$\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(L,\mathfrak{p}_2^{2a}\mathfrak{p}_3^bF)\leq\begin{cases}{}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]:2]_4^2 & \text{if } a=0\\ {}_{\sqrt{-2}}[F_4:2]_4^2 & \text{if } a\neq 0\end{cases}$$

with equality if and only if $b=0$ and $a\neq 1$

$$\mathrm{Aut}_{\mathbb{Q}(\sqrt{3}(\zeta_{16}-\zeta_{16}^{-1}))}(L,\mathfrak{p}_2^{2a}\mathfrak{p}_3^bF)\simeq\begin{cases}{}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[{}_{\theta_{16},\infty}[Q_{32}]_1\overset{2}{\square} C_3]_4 & \text{if } b=0\\ {}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[D_{32}\overset{2}{\boxtimes}{}_{\sqrt{-3}}[C_6]_1]_4 & \text{if } b=1\end{cases}$$

$$\mathrm{Aut}_{\mathbb{Q}(\zeta_{16}-\zeta_{16}^{-1})}(L,\mathfrak{p}_2^{2a}\mathfrak{p}_3^bF)\simeq\begin{cases}{}_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8\otimes QD_{32}).S_3]_4 & \text{if } b=0\\ {}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2\otimes A_2 & \text{if } b=1\end{cases}.$$

So the result follows. □

Theorem 4.9.8 *Suppose $|U|=60$ and $\Pi(|G|)=\{2,3,5\}$ or there exists some $(L,F)\in\mathcal{Z}(G)\times\mathcal{F}_{>0}(G)$ such that F is integral on L and $\Pi(\det(L,F))\subseteq\{2,3,5\}$. Then G is conjugate to one of*

$${}_{i}[C_4]_1\otimes E_8,\ {}_{i}[(2_+^{1+6}\otimes C_4).\mathrm{Sp}_6(2)]_8,\ {}_{i}[(2_+^{1+4}\otimes C_4).S_6]_4\otimes A_2,$$

$${}_{i}[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4\overset{2(2)}{\boxtimes}{}_{i}[C_4]_1]_8,\ {}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4^2,\ {}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4\circ C_3,$$

$${}_{\sqrt{-3}}[C_6]_1\otimes E_8,\ {}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4\underset{\sqrt{-3}}{\otimes}{}_{\infty,2}[\mathrm{SL}_2(3)]_1,\ {}_{\infty,5}[\mathrm{SL}_2(5)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_4\circ C_3,$$

$$({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)\otimes A_4,\ ({}_{\sqrt{-3}}[C_6]_1\otimes A_4)^2,\ {}_{\sqrt{-3}}[C_{60}.(C_4\times C_2)]_8,$$

$$({}_{\infty,5}[\mathrm{SL}_2(5):2]_2\circ C_3)^2,\ {}_{\infty,2}[\mathrm{SL}_2(5)\overset{2(2)}{\boxtimes}D_8]_4\circ C_3,\ {}_{\sqrt{-3}}[C_6]_1\otimes[(\mathrm{SL}_2(5)\overset{2}{\square}\mathrm{SL}_2(5)):2]_8,$$

$$({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)\underset{\sqrt{-3}}{\otimes}{}_{\infty,5}[\mathrm{SL}_2(5):2]_2,\ {}_{\sqrt{-5}}[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\overset{2-}{\boxtimes}{}_{i}[C_4]_1]_8,$$

$${}_{\sqrt{-5}}[{}_{\infty,5}[\mathrm{SL}_2(5)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_4:2]_8,\ {}_{\sqrt{-5}}[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\overset{2+}{\boxtimes}{}_{i}[C_4]_1]_8,$$

$$_{\sqrt{-5}}[{}_i[(D_8\otimes C_4).S_3]_2 \overset{2_+}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8,\ {}_{\sqrt{-5}}[{}_i[(D_8\otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8,$$

$$_{\sqrt{-5}}[{}_i[(D_8\otimes C_4).S_3]_2 \overset{2_-}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8,\ {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_-}{\square} C_3]_4^2,$$

$$_{\sqrt{-15}}[({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_3) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_8,\ {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_+}{\square} C_3]_4^2,$$

$$_{\sqrt{-15}}[((\mathrm{SL}_2(5)\circ \mathrm{SL}_2(5)){:}2) \overset{2_-}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8,\ {}_{\sqrt{-15}}[C_{30}:C_4]_4^2,$$

$$_{\sqrt{-15}}[((\mathrm{SL}_2(5)\circ \mathrm{SL}_2(5)){:}2) \overset{2_+}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8,\ ({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_5)^2,$$

$$_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2,\ ({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_5)\otimes A_2,\ {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2,$$

$$_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} ((\mathrm{SL}_2(5)\circ \mathrm{SL}_2(5)){:}2),\ {}_{\zeta_{10}}[C_{10}]_1\otimes F_4,\ \zeta_{10}[C_{30}:2]_2^2,\ {}_{\zeta_{10}}[C_{60}.(C_2\times C_2)]_4$$

$_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2$ *or* $_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(a,b,e)}$ *with* $a\in\{1,i\}$ *and* $b,e\in\{0,1\}$.

Proof: Let $C := \mathrm{End}(\overline{U})\simeq\mathbb{Q}(\zeta_{60})$ and $K\simeq\mathbb{Q}(\theta_{60})$ be its maximal totally real subfield. Both have class number 1. Hence U fixes up to isomorphism only one lattice L. There exists some $F\in\mathcal{F}_{>0}(U)$ that is integral on L such that $(L,F)\simeq E_8^2$. Clearly U is not maximal finite. Hence, the maximal totally real subfield of $\mathrm{End}(\overline{G})$ is properly contained in K. Thus by the proof of Corollary 2.2.12, we may assume that G fixes a form $F'\in\mathcal{F}_{>0}(U)$ that is integral on L with $\Pi(\det(L,F'))\subseteq\{2,3,5\}\cup\bigcup_{k\lneq K}\Pi(k) = \{2,3,5,11\}$. (Note that this is a huge improvement over $\tilde{\Pi}(K,60)=\{2,3,5,11,59\}$).

Denote by $\mathfrak{p}_{11}^{(i)}$ $(1\le i\le 4)$ the four prime ideals over 11. By Table 2.5.3, G is conjugate to $\mathrm{Aut}_{K'}(L,\mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^c\prod_{i=1}^4(\mathfrak{p}_{11}^{(i)})^{d_i}F)$ for some $a,b\in\{0,1\}$, $0\le c\le 2$, $d_i\ge 0$ and $K'\in\{\mathbb{Q}(i),\mathbb{Q}(\sqrt{-3}),\mathbb{Q}(\sqrt{-5}),\mathbb{Q}(\sqrt{-15}),\mathbb{Q}(\zeta_{10}),\mathbb{Q}(\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1}))\}$.

There exists some $\sigma_i\in\mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma_i(\mathfrak{p}_{11}^{(1)})=\mathfrak{p}_{11}^{(i)}$ and one finds that σ_i is conjugation by some $x_i\in N_{\mathrm{GL}_{16}(\mathbb{Q})}(N)\cap\mathrm{GL}(L)$. Since σ_i necessarily fixes $\mathfrak{p}_2$, $\mathfrak{p}_3$ and $\mathfrak{p}_5$, we may assume that $d_1\ge d_i$ for all i. Furthermore, restricting to normalized lattices (see Definition 2.2.4) yields the inequalities $d_i\le 1$ and $\sum_i d_i\le 2$.

First, we handle the cases $d_1=0$: Let $a,b\in\{0,1\}$ and $0\le c\le 2$, then

$$\mathrm{Aut}_{\mathbb{Q}(i)}(L,\mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^cF)\le\begin{cases} {}_i[C_4]_1\otimes E_8 & \text{if } a=b=0\\ {}_i[(2_+^{1+6}\otimes C_4).\mathrm{Sp}_6(2)]_8 & \text{if } (a,b)=(1,0)\\ {}_i[(2_+^{1+4}\otimes C_4).S_6]_4\otimes A_2 & \text{if } (a,b)=(0,1)\\ {}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8 & \text{if } a=b=1\end{cases}$$

and equality holds if and only if $c=0$.

The groups involving $\mathbb{Q}(\sqrt{-3})$ are

form	$\mathbb{Q}(\sqrt{-3})$
F	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4^2$
$\mathfrak{p}_2F$	${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$
$\mathfrak{p}_3F$	${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$
$\mathfrak{p}_2\mathfrak{p}_3F$	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$
$\mathfrak{p}_5F$	${}_{\infty,5}[\mathrm{SL}_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4 \circ C_3$
$\mathfrak{p}_2\mathfrak{p}_5F$	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \otimes A_4$
$\mathfrak{p}_3\mathfrak{p}_5F$	$({}_{\sqrt{-3}}[C_6]_1 \otimes A_4)^2$
$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5F$	${}_{\sqrt{-3}}[C_{60}.(C_4 \times C_2)]_8$
$\mathfrak{p}_5^2F$	$({}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \circ C_3)^2$
$\mathfrak{p}_2\mathfrak{p}_5^2F$	${}_{\infty,2}[\mathrm{SL}_2(5) \overset{2(2)}{\boxtimes} D_8]_4 \circ C_3$
$\mathfrak{p}_3\mathfrak{p}_5^2F$	${}_{\sqrt{-3}}[C_6]_1 \otimes [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8$
$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5^2F$	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \underset{\sqrt{-3}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$

which are all s.i.m.f..
Similarly

$$\mathrm{Aut}_{\mathbb{Q}(\sqrt{-5})}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^cF) \leq \begin{cases} {}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2_-}{\boxtimes} {}_i[C_4]_1]_8 & \text{if } a = c = 0 \\ {}_{\sqrt{-5}}[{}_{\infty,5}[\mathrm{SL}_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4:2]_8 & \text{if } (a,c) = (0,1) \\ {}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2_+}{\boxtimes} {}_i[C_4]_1]_8 & \text{if } (a,c) = (0,2) \\ {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2_+}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8 & \text{if } (a,c) = (1,0) \\ {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8 & \text{if } a = c = 1 \\ {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2_-}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8 & \text{if } (a,c) = (1,2) \end{cases}$$

and equality holds if and only if $b = 0$.
The groups involving $\mathbb{Q}(\sqrt{-15})$ are $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-15})}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^cF) \leq$

$$\begin{cases} {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_-}{\square} C_3]_4^2 & \text{if } b = c = 0 \\ {}_{\sqrt{-15}}[({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_3) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_8 & \text{if } (b,c) = (0,1) \\ {}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_+}{\square} C_3]_4^2 & \text{if } (b,c) = (0,2) \\ {}_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2_-}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8 & \text{if } (b,c) = (1,0) \\ {}_{\sqrt{-15}}[C_{30}:C_4]_4^2 & \text{if } b = c = 1 \\ {}_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2_+}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8 & \text{if } (b,c) = (1,2) \end{cases}$$

and equality holds if and only if $a = 0$. The remaining two fields yield

form	$\mathbb{Q}(\zeta_{10})$	$\mathbb{Q}(\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1}))$
F	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(1,-1,1)}$
$\mathfrak{p}_2 F$	${}_{\zeta_{10}}[C_{10}]_1 \otimes_{\sqrt{5}'} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(i,-1,1)}$
$\mathfrak{p}_3 F$	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)\otimes A_2$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(1,1,-1)}$
$\mathfrak{p}_2\mathfrak{p}_3 F$	${}_{\zeta_{10}}[C_{10}]_1 \otimes_{\sqrt{5}'} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(i,1,-1)}$
$\mathfrak{p}_5 F$	${}_{\zeta_{10}}[C_{10}]_1 \otimes_{\sqrt{5}} ((\mathrm{SL}_2(5)\circ \mathrm{SL}_2(5)):2)$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(1,-1,-1)}$
$\mathfrak{p}_2\mathfrak{p}_5 F$	${}_{\zeta_{10}}[C_{10}]_1 \otimes F_4$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(i,-1,-1)}$
$\mathfrak{p}_3\mathfrak{p}_5 F$	${}_{\zeta_{10}}[C_{30}:2]_2^2$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(1,1,1)}$
$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5 F$	${}_{\zeta_{10}}[C_{60}.(C_2\times C_2)]_4$	${}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2\times C_2)]_{4,(i,1,1)}$

The fields $\mathbb{Q}(\zeta_{10})$ and $\mathbb{Q}(\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1}))$ have only one proper subfield, which is isomorphic to $\mathbb{Q}(\sqrt{5})$. Thus, these groups are easily be checked to be s.i.m.f. by computing the automorphism groups of the G-invariant lattices wrt. the full form space. Moreover, we don't have to check the forms $\mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^2 F$ since $\mathfrak{p}_5^2 F = \frac{5+\sqrt{5}}{2}F$.

Suppose now $d_1 = 1$ and $d_2 = d_3 = d_4 = 0$:
For each $d \in \{1,3,5,15\}$ the group $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L_i, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^c\mathfrak{p}_{11}^{(1)}F)$ is conjugate to ${}_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2$ which we have already proven to be s.i.m.f. in Lemma 4.9.2.
For $\alpha \in \{\zeta_{10}, \sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})\}$ the groups $\mathrm{Aut}_{\mathbb{Q}(\alpha)}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^c\mathfrak{p}_{11}^{(1)}F)$ are conjugate to an extension of U by C_2 with commuting algebra $\mathbb{Q}(\zeta_{10},\sqrt{3})$. The extension is split if and only if c is odd. The nonsplit extension is properly contained in $({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1\circ C_5)^2$ and the split extension is a subgroup of ${}_{\zeta_{10}}[C_{10}]_1 \otimes_{\sqrt{5}} ((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)$.

Finally suppose $d_1 = 1$ and $\sum_{i=1}^4 d_i = 2$. Then, for every minimal totally complex subfield K' of K, $\mathrm{Aut}_{K'}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^c\prod_{i=1}^4(\mathfrak{p}_{11}^{(i)})^{d_i}F)$ is conjugate to a proper subgroup of $\mathrm{Aut}_{K'}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_5^c F)$. □

4.9.2 The case $O_{17}(G) = 1$ and $O_5(G) \neq 1$

Lemma 4.9.9 *If $E(G)$ is conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, then G is conjugate to one of*

$$ {}_{\sqrt{-5}}[{}_{\infty,5}[\mathrm{SL}_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_4 : 2]_8,\ {}_{\infty,5}[\mathrm{SL}_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_4 \circ C_3, $$

$$ {}_{\sqrt{-15}}[({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_3) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_8,\ ({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)\otimes A_2\,. $$

Proof: The group G contains a normal subgroup N conjugate to $\mathrm{SL}_2(5)\circ C_5$. Suppose N is self-centralizing. Then $\mathrm{Out}(N) \simeq \langle\alpha,\beta\rangle \simeq C_2\times C_4$. Both, α and β cannot be realized in $\mathrm{GL}_{16}(\mathbb{Q})$ alone. Hence by Lemma 2.2.1, $G = \langle N, \alpha\beta\rangle$ with $(\alpha\beta)^4 = \pm 1$.

One of these groups is realizeable in dimension 8. The other gives an irreducible group G with commuting algebra $\mathcal{Q}_{\infty,5}$. But the torsion subgroup of the (up to isomorphism unique) maximal order of $\mathcal{Q}_{\infty,5}$ is C_6. Thus N is not self-centralizing.
So we may assume that G contains an irreducible subgroup $H := N \otimes C_m$ with $m \in \{3,4\}$. In both cases G must therefore contain an irreducible cyclic subgroup of order 60. The commuting algebra of H is isomorphic to $C := \mathbb{Q}(\zeta_{10}, \zeta_m)$. The torsion subgroup of $\mathbb{Z}_C^*$ is C_{30} if $m = 3$ and C_{20} otherwise. Hence $G/C_G(N)N \leq \mathrm{Out}(N) \simeq C_2 \times C_4$ implies $\Pi(|G|) = \{2,3,5\}$. So the result follows from Theorem 4.9.8. □

Lemma 4.9.10 *If $O_3(G) \neq 1$ then G is conjugate to one of*

$$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5) \otimes A_2,\ {}_{\infty,5}[\mathrm{SL}_2(5) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_4 \circ C_3,$$

$${}_{\sqrt{-15}}[({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_3) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_8,\ {}_{\zeta_{10}}[C_{60}.(C_2 \times C_2)]_4,\ {}_{\sqrt{-3}}[C_{60}.(C_4 \times C_2)]_8,$$

$${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2,\ {}_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2$$

$$\textit{or}\ {}_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(a,b,c)}\ \textit{with}\ a \in \{1,i\}\ \textit{and}\ b,c \in \{\pm 1\}.$$

Proof: Table 2.5.2 shows that $O_3(G) \simeq C_3$ and $O_2(G)$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{15}))$. Thus $O_2(G) \simeq C_2, C_4, D_8$ or Q_8. If $O_2(G) \neq C_2$ then G contains an irreducible cyclic subgroup of order 60 and $\Pi(|G|) = \{2,3,5\}$. Theorem 4.9.8 gives precisely the claimed groups.
If $F(G) \simeq \pm C_{15}$ then $C := C_G(F(G))$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{15}))$. Table 2.5.1 shows that $E(G)$ is either trivial or conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$. The latter case has been handled in the lemma before. So we may now suppose that $F^*(G) = \pm C_{15}$. Let $C := C_G(O_3(G))$. Then $[G : C] \leq 2$ and $C/F(G) \leq \mathrm{Out}(C_5)$. Hence C is conjugate to one of $\pm C_{15}, D_{10} \otimes C_6, Q_{20} \circ C_3, (C_5 : C_4) \otimes C_6, (C_{10} \cdot C_4) \circ C_3$. In any case, C is rationally reducible and the commuting algebra of a direct summand contains a third root of unity. This contradicts Lemma 2.2.1. □

Lemma 4.9.11 *If $O_3(G) = 1$ and $O_2(G) = Q_8$ then G is conjugate to one of*

$${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8,\ {}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4,$$

$${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4\ .$$

Proof: Let $N := \mathcal{B}^o(F(G)) = {}_{\zeta_{10}}[C_{10}] \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$. Then $C_G(N)$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{10}))$. It follows from Table 2.5.1 that $E(G) = 1$. So $G/N \leq \mathrm{Out}(N) \simeq C_4 \times C_2$ and $[G : N] \geq 4$ by Lemma 2.2.1. In particular, $\Pi(|G|) = \{2,3,5\}$. Let $U := C_G(O_5(G)) \trianglelefteq G$. There are three possibilities:

- If $U \simeq {}_{\zeta_{10}}[C_{10}]_1 \otimes {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$, then U fixes up to isomorphism one lattice L. Further $C := \mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{10}, \sqrt{-2})$ has $K := \mathbb{Q}(\sqrt{-2} \cdot (\zeta_{10} - \zeta_{10}^{-1}))$ as maximal real subfield. There exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L with $\det(L, F) = 1$. Since $\mathrm{End}_{\mathbb{Z}U}(L)$ is the maximal order of C and since $\mathrm{Nr}_{C/K}(\mathbb{Z}_C^*) = \mathbb{Z}_{K,>0}^*$, we may restrict ourselves to one class of totally positive units. It follows from Table 2.5.4 that G is conjugate to one of $\mathrm{Aut}_{K'}(L, \mathfrak{p}_2^a \mathfrak{p}_5^b F)$ for some $(a, b) \in \{(0,0), (1,1), (0,2)\}$ and $K' \in \{\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-10}), \mathbb{Q}(\zeta_{10})\}$. These are

form	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\zeta_{10})$
F	${}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2$
$\mathfrak{p}_2\mathfrak{p}_5 F$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$	${}_{\zeta_{10}}[C_{10}]_1 \otimes F_4$
$\mathfrak{p}_5^2 F$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2$
form	$\mathbb{Q}(\sqrt{-10})$	
F	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_+}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	
$\mathfrak{p}_2\mathfrak{p}_5 F$	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$	
$\mathfrak{p}_5^2 F$	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_-}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$	

All groups are s.i.m.f. but only ${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$ has the correct Fitting subgroup.

- If $U \simeq {}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \underset{\sqrt{5}'}{\otimes} {}_{\zeta_{10}}[C_{10}]_1$, then again U fixes only one lattice L. The commuting algebra $C := \mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{10}, \sqrt{2})$ has $K := \mathbb{Q}(\sqrt{2}, \sqrt{5})$ as maximal real subfield. Again there exists some $F \in \mathcal{F}_{>0}(U)$ such that F is integral on L with $\det(L, F) = 1$. Further, there exists some $\sigma(\mathrm{Gal}(K/\mathbb{Q}))$ that interchanges the two prime ideals over $3\mathbb{Z}_K$ and σ is conjugation by some $x \in N_{\mathrm{GL}_{16}}(\mathbb{Q})(U) \cap \mathrm{GL}(L)$. So by Table 2.5.4, G is conjugate to $\mathrm{Aut}_{K'}(L, \mathfrak{p}_2^a \mathfrak{p}_3^b F)$ for some $a, b \in \{0, 1\}$ and $K' \in \{\mathbb{Q}(\zeta_{10}), \mathbb{Q}(\sqrt{2} \cdot (\zeta_{10} - \zeta_{10}^{-1}))\}$. Note that since K' has $\mathbb{Q}(\sqrt{5})$ as maximal totally real subfield, we do not have to check $\mathfrak{p}_5$ since $\mathfrak{p}_5 F = \frac{5+\sqrt{5}}{2} F$. These groups are:

form	$\mathbb{Q}(\zeta_{10})$	$\mathbb{Q}(\sqrt{2}\cdot(\zeta_{10} - \zeta_{10}^{-1}))$
F	$({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5)^2$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_2 F$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$	${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$
$\mathfrak{p}_3 F$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$	U
$(L, \mathfrak{p}_2\mathfrak{p}_3 F)$	${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$	U

Only ${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$ and ${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$ have the required Fitting subgroup.

- If $U = N$, then $G/N \simeq C_4$. More precisely G is one of the four groups $G = \langle N, \alpha\rangle$ where α induces an outer automorphism of order 4 on $O_5(G)$ and either
  - $\alpha^4 = I_{16}$ and $\alpha \in C_G(\mathcal{B}^o(O_2(G)))$. But then G can be extended by torsion elements in $\mathrm{End}(\overline{G}) \simeq \mathcal{Q}_{\infty,2}$.
  - $\alpha^4 = -I_{16}$ and $\alpha \in C_G(\mathcal{B}^o(O_2(G)))$. This group is for example properly contained in $({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \underset{\sqrt{-3}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$.
  - $\alpha^4 = I_{16}$ and α induces the outer automorphism of $\mathcal{B}^o(O_2(G))$. This group embeds into $\mathrm{GL}_8(\mathbb{Q})$.
  - $\alpha^4 = -I_{16}$ and α induces the outer automorphism of $\mathcal{B}^o(O_2(G))$. This group is properly contained in ${}_{\infty,5}[\mathrm{SL}_2(5) \overset{2}{\underset{\sqrt{2}}{\boxtimes}} D_{10}]_4$.

Thus the result follows. □

Lemma 4.9.12 *If $O_3(G) = 1$ then $O_2(G)$ is not isomorphic to C_4 or D_8.*

Proof: In both cases G would contain a normal subgroup $N \simeq C_{20}$. The centralizer $C := C_G(N)$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{20}))$. Hence $E(G)$ is either conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ or trivial. The first case contradicts Lemma 4.9.9.

Thus $F(G)$ is self-centralizing. Hence $G/F(G) \le \mathrm{Out}(F(G)) \simeq C_2 \times C_4$. It follows from Lemma 2.2.1 that $[G : F(G)] > 2$. If $g \in C_G(O_5(G)) \setminus F(G)$ then $g \in O_2(G)$. In particular, this shows $G/F(G) \simeq C_4$. This leaves the two cases:

- $F(G) \simeq C_{20}$. By group cohomology, MAGMA finds that G must be isomorphic to $\langle a, b, \alpha \mid a^5, b^4, [a, b], a^\alpha = a^2, b^\alpha = b^\epsilon, \alpha^4 = b^k\rangle$ for some $(\epsilon, k) \in \{(1, 0), (-1, 0), (-1, 2), (1, 2), (1, 1)\}$. The first three groups embed into $\mathrm{GL}_8(\mathbb{Q})$ with $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-5})$ as endomorphism rings. The fourth is properly contained in ${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$ and the last one only has a rational irreducible representation in dimension 32.

- $F(G) \simeq C_5 \times D_8$. In this case G must be isomorphic to

  $$\langle a, b, c, \alpha \mid a^5, b^4, [a, b], [a, c], b^c = b^{-1}, c^2, a^\alpha = a^2, b^\alpha = b^{-1}, c^\alpha = b^r c, \alpha^4 = b^{2s}\rangle$$

  for some $r, s \in \{0, 1\}$. Thus $U := \langle a, b, \alpha\rangle$ is the group discussed above (with $\epsilon = -1$ and $k \in \{0, 2\}$). In particular, U is reducible and has a imaginary quadratic field as commuting algebra. This contradicts Lemma 2.2.1.

So the result follows. □

Lemma 4.9.13

(a) *$O_2(G)$ is neither isomorphic to C_8, QD_{16} nor Q_{16}.*

(b) *If $O_2(G) \simeq D_{16}$, then G is conjugate to ${}_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[D_{16} \overset{2}{\boxtimes} {}_{\zeta_{10}}[C_{10}]_1]_4$.*

Proof: In any case, G contains an irreducible cyclic subgroup $U \leq F(G)$ of order 40. Thus $F(G)$ is self-centralizing and hence $\Pi(|G|) = \{2, 3, 5\}$. So the result follows from Theorem 4.9.6. □

Lemma 4.9.14

(a) *If $O_2(G) \simeq 2_+^{1+4}$, then G is conjugate to ${}_{\zeta_{10}}[C_{10}]_1 \otimes F_4$.*

(b) *If $O_2(G) \simeq 2_-^{1+4}$, then G is conjugate to ${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$.*

Proof: We have to prove $\mathcal{B}^o(F(G)) = G$ in both cases. The group $\mathcal{B}^o(F(G))$ is irreducible, fixes 2 lattices and has $\mathbb{Q}(\zeta_{10})$ as endomorphism ring. Hence the claim is verified by computing the automorphism groups of these lattices wrt. the full form space. □

Lemma 4.9.15 *If $O_2(G) \simeq D_8 \otimes C_4$ then $G \simeq {}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$.*

Proof: The group G contains an irreducible normal subgroup $N := \mathcal{B}^o(F(G)) = {}_i[(D_8 \otimes C_4).S_3]_2 \otimes {}_{\zeta_{10}}[C_{10}]_1$. The commuting algebra of N is isomorphic to $\mathbb{Q}(\zeta_{20})$ and N fixes only one lattice L. Thus N is self-centralizing and $\Pi(|G|) = \{2, 3, 5\}$. Further we find some $F \in \mathcal{F}_{>0}(G)$ that is integral on L with $\det(L, F) = 2^8$. Restricting to normalized lattices (see Definition 2.2.4) it follows from Table 2.5.3 that G is conjugate to $\mathrm{Aut}_K(L, \mathfrak{p}_5^a F)$ for some $0 \leq a \leq 2$ and $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\zeta_{10})\}$. These groups are

$$\mathrm{Aut}_{\mathbb{Q}(\zeta_{10})} \simeq \begin{cases} {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 & \text{if } a \text{ is even} \\ {}_{\zeta_{10}}[C_{10}]_1 \otimes F_4 & \text{if } a \text{ is odd} \end{cases}$$

and

	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-5})$
F	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_2(6)]_8$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2_+}{\boxtimes}} \mathrm{SL}_2(5)]_8$
p_5F	$\lneqq {}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_2(6)]_8$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$
p_5^2F	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_2(6)]_8$	${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \underset{i}{\overset{2_-}{\boxtimes}} \mathrm{SL}_2(5)]_8$

Only ${}_{\sqrt{-5}}[{}_i[(D_8 \otimes C_4).S_3]_2 \overset{2}{\boxtimes} D_{10}]_8$ has the correct Fitting subgroup. □

Lemma 4.9.16 *If* $F(G) \simeq C_{10}$*, then* G *is conjugate to* ${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2)$ *or* ${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$.

Proof: Since $C_G(F(G))$ embeds into $\mathbb{Q}(\zeta_{10})^{2\times 2}$, it follows from Table 2.5.1 that $E(G)$ is conjugate to Alt_5, $\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$, ${}_{\infty,3}[\mathrm{SL}_2(9)]_2$ or ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$. The latter case contradicts Lemma 4.9.9.

- If $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ then $F^*(G) \simeq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ is s.i.m.f..
- If $E(G) \simeq \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$ then $\mathcal{B}^o(F^*(G)) \simeq {}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2)$ is already s.i.m.f..
- If $E(G) \simeq \mathrm{Alt}_5$ then $\mathcal{B}^o(F^*(G)) \simeq {}_{\zeta_{10}}[C_{10}]_1 \otimes A_4$ has $\mathbb{Q}(\zeta_{10})$ as commuting algebra and fixes up to isomorphism 5 lattices. By computing their automorphism groups (wrt. the full form space), one finds that G is only contained in ${}_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2)$.

This proves the claim. □

4.9.3 The case $O_{17}(G) = O_5(G) = 1$ and $O_3(G) \neq 1$

Let $G < \mathrm{Sp}_{16}(\mathbb{Q})$ be s.p.i.m.f. such that $O_3(G) \neq 1$ and $O_p(G) = 1$ for all primes $p > 3$. Then $O_3(G) \simeq C_3$ and $O_2(G)$ is isomorphic to C_2, C_4, D_8, Q_8, C_8, D_{16}, Q_{16}, QD_{16}, $D_8 \otimes C_4$, 2_+^{1+4}, 2_-^{1+4}, C_{16}, D_{32}, Q_{32}, QD_{32}, $D_8 \otimes C_8$, $D_8 \otimes D_{16}$, $D_8 \otimes QD_{16}$, $D_8 \otimes Q_{16}$, $2_+^{1+4} \otimes C_4$, 2_+^{1+6} or 2_-^{1+6} by Table 2.5.2.

Again, we discuss all possible cases.

Lemma 4.9.17 *If* $O_2(G) \simeq 2_-^{1+6}$ *then* G *is conjugate to* ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$.

Proof: Follows from Lemma 4.9.4. □

Lemma 4.9.18 $O_2(G)$ *is not conjugate to* 2_+^{1+6}.

Proof: Otherwise G would contain $\mathcal{B}^o(F(G)) \simeq 2_+^{1+6}.\mathrm{Alt}_8 \otimes {}_{\sqrt{-3}}[C_6]_1$ as a normal subgroup. This group fixes two lattices and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. One easily checks that it is only contained in the s.i.m.f. group ${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$. □

Lemma 4.9.19 *If* $O_2(G) \simeq 2_+^{1+4} \otimes C_4$*, then* G *is conjugate to* ${}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes A_2$.

Proof: Otherwise G would contain the irreducible normal subgroup $N := \mathcal{B}^o(F(G))$ which is conjugate to ${}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes {}_{\sqrt{-3}}[C_6]_1$. Since $G/N \leq C_2 \times C_2$ we may assume $\Pi(|G|) = \{2,3,5\}$. Let $C := \mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{12})$ and $K \simeq \mathbb{Q}(\sqrt{3})$ be its totally real subfield. The group N fixes up to isomorphism only one lattice L and there exists some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L,F) = 2^8$. Since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$ we can restrict ourselves to one class of totally positive units. By Table 2.5.4, G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^{-a}\mathfrak{p}_3^a F)$ for some $a \in \{0,1\}$ and $d \in \{1,3\}$.

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-3})$
F	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$
$\mathfrak{p}_2^{-1}\mathfrak{p}_3 F$	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes A_2$	${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$

The result follows by checking the Fitting subgroups. □

Lemma 4.9.20
$O_2(G)$ *is not conjugate to* $D_8 \otimes C_8$*,* $D_8 \otimes QD_{16}$*,* $D_8 \otimes D_{16}$ *or* $D_8 \otimes Q_{16}$.

Proof: In any case G contains a normal irreducible subgroup H conjugate to $C_3 \otimes D_8 \otimes C_8$. Thus also $N := \mathcal{B}^o(H) \trianglelefteq G$ and N is conjugate to $C_3 \otimes (D_8 \otimes C_8).S_3$. The commuting algebra C of N is isomorphic to $\mathbb{Q}(\zeta_{24})$ with $K \simeq \mathbb{Q}(\theta_{24})$ as maximal totally real subfield. Further N fixes up to isomorphism only one lattice L and one finds some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L,F) = 1$. From $G/F(G) \leq \mathrm{Out}(F(G))$ it follows that $\Pi(|G|) = \{2,3\}$. Since $\mathrm{End}_{\mathbb{Z}N}(N)$ is the maximal order in C and since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$, we may restrict ourselves to one class of totally positive units. If follows from Table 2.5.3, that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^{2a}\mathfrak{p}_3^b F)$ for some $a,b \in \{0,1\}$ and $d \in \{1,2,3,6\}$. These are

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	${}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$
$\mathfrak{p}_2^2 F$	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4:2]_8$
$\mathfrak{p}_3 F$	${}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes A_2$	$\lneq {}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8:2)]_8$
$\mathfrak{p}_2^2\mathfrak{p}_3 F$	$({}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_2)^2$	$\lneq {}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4:2]_8$
	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-6})$
F	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4^2$	${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \overset{2(3)}{\underset{i}{\boxtimes}} A_2]_8$
$\mathfrak{p}_2^2 F$	${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$	${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \overset{2(3)}{\underset{i}{\boxtimes}} {}_{\infty,3}[\tilde{S}_3]_1]_8$
$\mathfrak{p}_3 F$	${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$	${}_{\sqrt{-6}}[F_4 \overset{2}{\boxtimes} {}_{\infty,3}[\tilde{S}_3]_1]_8$
$\mathfrak{p}_2^2\mathfrak{p}_3 F$	$({}_{\sqrt{-3}}[C_6]_1 \otimes F_4)^2$	${}_{\sqrt{-6}}[(F_4 \otimes A_2):2]_8$

No group has the correct Fitting subgroup. □

Lemma 4.9.21

(a) If $O_2(G) \simeq D_{32}$, then G is conjugate to ${}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[D_{32} \otimes {}_{\sqrt{-3}}[C_6]_1]_4$.

(b) If $O_2(G) \simeq Q_{32}$, then G is conjugate to ${}_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[{}_{\theta_{16},\infty}[Q_{32}]_1 \overset{2}{\square} C_3]_4$.

(c) If $O_2(G) \simeq QD_{32}$, then G is conjugate to ${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2 \otimes A_2$.

(d) $O_2(G)$ is not conjugate to C_{16}.

Proof: In all cases, $F(G)$ contains an irreducible self-centralizing cyclic subgroup of order 48. Furher, $\mathrm{Out}(F(G))$ is a 2-group which shows $\Pi(|G|) = \{2, 3\}$. So the result follows from Theorem 4.9.7. $\square$

Lemma 4.9.22 *If $O_2(G) \simeq 2_-^{1+4}$, then G is conjugate to* ${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \overset{2(3)}{\boxtimes} A_2]_8$ *or* ${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \underset{i}{\overset{2(3)}{\boxtimes}} {}_{\infty,3}[\tilde{S}_3]_1]_8$.

Proof: G contains the normal subgroup $N := \mathcal{B}^o(O_2(G)) \circ C_3 \simeq {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \circ C_3$. By Lemma 2.2.1 we get $G/N \simeq C_2 \times C_2$. Thus $C_G(O_3(G))$ is conjugate to $N_1 := {}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 : 2]_4 \otimes C_3$ or $N_2 := ({}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \cdot 2) \circ C_3$ and $\Pi(|G|) = \{2, 3, 5\}$. In any case $C_G(O_3(G))$ is irreducible and fixes only one lattice L. In both cases, there exists some $F \in \mathcal{F}_{>0}(N_i)$ that is integral on L with $\det(L, F) = 1$.

- Let $C \simeq \mathbb{Q}(\sqrt{-2}, \sqrt{-3})$ be the commuting algebra of N_1 and let $K \simeq \mathbb{Q}(\sqrt{6})$ be its maximal totally real subfield. Since $\mathrm{End}_{\mathbb{Z}N}(N)$ is the maximal order in C and since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$, we may restrict ourselves to one class of totally positive units. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that σ interchanges the two prime ideals over 5. One finds that σ is conjugation by some $x \in N_{\mathrm{GL}_{16}(\mathbb{Q})}(N_1) \cap \mathrm{GL}(L)$. By Table 2.5.4, G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3^b(\mathfrak{p}_2\mathfrak{p}_5)^a F)$ for some $a, b \in \{0, 1\}$ and $d \in \{2, 3\}$. The groups $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-2})}(L, F) \simeq {}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8$ and $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-3})}(L, F) \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes E_8$ are s.i.m.f., but have the wrong Fitting subgroups. The groups $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3 F)$ are properly contained in $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, F)$. Finally $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2\mathfrak{p}_3^b\mathfrak{p}_5 F) = N_1$. So $C_G(O_3(G)) \not\simeq N_1$.
- Let $C \simeq \mathbb{Q}(\sqrt{2}, \sqrt{-3})$ be the commuting algebra of N_2 and let $K \simeq \mathbb{Q}(\sqrt{2})$ be its maximal totally real subfield. By Table 2.5.4, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^a F)$ for some $a \in \{0, 1\}$ and $d \in \{3, 6\}$. These groups are

forms	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-6})$
F	${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4^2$	${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \overset{2(3)}{\boxtimes} A_2]_8$
$\mathfrak{p}_2 F$	${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$	${}_{\sqrt{-6}}[{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \underset{i}{\overset{2(3)}{\boxtimes}} {}_{\infty,3}[\tilde{S}_3]_1]_8$

This proves the claim. $\square$

Lemma 4.9.23 *If* $O_2(G) \simeq 2_+^{1+4}$*, then* G *is conjugate to* ${}_{\sqrt{-6}}[F_4 \overset{2}{\boxtimes} {}_{\infty,3}[\tilde{S}_3]_1]_8$ *or* ${}_{\sqrt{-6}}[(F_4 \otimes A_2):2]_8$.

Proof: G contains the normal subgroup $N := \mathcal{B}^o(O_2(G))O_3(G) \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes F_4$ and $G/N \leq C_2 \times C_2$. Thus $[G:N] = 4$ by Lemma 2.2.1. So G contains a subgroup of index two conjugate to $F_4 \otimes {}_{\infty,3}[\tilde{S}_3]_1$ or $F_4 \otimes A_2$. The result follows from Remark 2.2.17. □

Lemma 4.9.24 $O_2(G)$ *is not conjugate to* $D_8 \otimes C_4$.

Proof: Otherwise $N := \mathcal{B}^o(F(G)) \simeq {}_i[(D_8 \otimes C_4).S_3]_2 \otimes C_3$ would be a normal subgroup of G. The centralizer of N embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{12}))$. Thus again $E(G) = 1$ and $G/N \leq C_2 \times C_2$. By Lemma 2.2.1 we get $[G:N] = 4$. But MAGMA shows that no extension of N by $C_2 \times C_2$ has the correct O_2. □

Lemma 4.9.25 $O_2(G)$ *is not conjugate to* C_8*,* D_{16}*,* Q_{16} *or* QD_{16}.

Proof: In any case $C_G(F(G))$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{24}))$. So $E(G) = 1$ by Table 2.5.1. Let $C = C_G(O_3(G))$. Since $\mathrm{Out}(F(G))$ is a 2-group and $C/F(G) < \mathrm{Out}(F(G))$, we get $C = F(G)$. But then $[G:F(G)] \leq 2$ contradicts Lemma 2.2.1. □

Lemma 4.9.26 *If* $O_2(G) \simeq Q_8$ *then* G *is conjugate to* ${}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} C_4]_8$ *or* $({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \otimes A_4$.

Proof: Let $B := \mathcal{B}^o(F(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$. Then the centralizer $C_G(B)$ embeds into $\mathrm{GL}_4(\mathbb{Q}(\sqrt{-3}))$. If $E(G) = 1$ then $G/B \leq C_2 \times C_2$ shows that G would be reducible. So it follows from Table 2.5.1, that $E(G)$ is conjugate to Alt_5, ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, ${}_{\infty,3}[\mathrm{SL}_2(9)]_2$ or $\mathrm{Sp}_4(3)$. In any case, $G/F^*(G) \leq \mathrm{Out}(F^*(G))$ implies $\Pi(|G|) = \{2,3,5\}$.

- If $E(G) \simeq \mathrm{Alt}_5$, then $N := \mathcal{B}^o(F^*(G)) \simeq ({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \otimes A_4$ is a normal subgroup of G. But then $G = N$, since N is already s.i.m.f., as we have seen before.
- If $E(G) \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, then $BE(G)$ contains an irreducible cyclic subgroup of order 60. Theorem 4.9.8 implies $G \simeq {}_{\infty,5}[\mathrm{SL}_2(5):2]_2 \underset{\sqrt{-3}}{\otimes} ({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3)$.
- If $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ or $\mathrm{Sp}_4(3)$, then G contains a subgroup H conjugate to $\mathrm{SL}_2(5) \underset{\sqrt{-3}}{\otimes} C_{12}$ where the restriction of the natural character to H is $4\chi_4$. In particular G contains an irreducible cyclic subgroup of order 60. Thus $G \simeq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ by Theorem 4.9.8. □

Lemma 4.9.27

(a) *If* $O_2(G) \simeq C_4$ *then* G *is conjugate to* ${}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} C_4]_8$.

(b) *If* $O_2(G) \simeq D_8$ *then* G *is conjugate to* ${}_{\infty,2}[\mathrm{SL}_2(5) \overset{2(2)}{\boxtimes} D_8]_4 \circ C_3$.

Proof: In both cases, $F(G)$ contains a characteristic subgroup U isomorphic to C_{12}. If $E(G) = 1$ then $G/F(G) \le C_2 \times C_2$ shows that G would be reducible. So $E(G) \neq 1$ embeds into $\mathrm{GL}_4(\mathbb{Q}(\zeta_{12}))$. By Table 2.5.1, this implies that $E(G)$ is conjugate to Alt_5, ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, ${}_{\infty,3}[\mathrm{SL}_2(9)]_2$ or $\mathrm{Sp}_4(3)$. In any case $G/F^*(G) \le \mathrm{Out}(F^*(G))$ shows that $\Pi(|G|) = \{2, 3, 5\}$.

- If $E(G) \simeq \mathrm{Alt}_5$, then $N := \mathcal{B}^o(E(G)U) \simeq C_{12} \otimes A_4$ is irreducible with $C \simeq \mathbb{Q}(\zeta_{12})$ as commuting algebra. It fixes 4 lattices but there exists a N-normal critical set $\{L_1, L_2\}$. Since N is not normal in $\mathrm{Aut}_{\mathbb{Q}(\zeta_{12})}(L_1, F) \simeq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes C_3$ for some $F \in \mathcal{F}_{>0}(N)$, it follows that G must fix L_2. Clearly $\Pi(|G|) = \{2, 3, 5\}$ and one finds some $F \in \mathcal{F}_{>0}(N)$ that is integral on L_2 with $\det(L_2, F) = 2^8 \cdot 5^4$. Since $\mathrm{End}_{\mathbb{Z}N}(L_2)$ is the maximal order of C and $\mathbb{Z}[\sqrt{3}]^*_{>0} = \mathrm{Nr}_{C/\mathbb{Q}(\sqrt{3})}(\mathbb{Z}_C^*)$ we may restrict ourselves to one class of totally positive units. By Table 2.5.4, this leaves the following four groups:

lattice and form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-3})$
(L_2, F)	$\lneq {}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \otimes A_4$
$(L_2, \mathfrak{p}_2^{-1}\mathfrak{p}_3 F)$	$\lneq {}_i[(2_+^{1+4} \otimes C_4).S_6]_4 \otimes A_2$	$({}_{\sqrt{-3}}[C_6]_1 \otimes A_4)^2$

  None has the correct generalized Fitting subgroup.

- If $E(G) \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, then $UE(G)$ contains an irreducible cyclic subgroup of order 60. Theorem 4.9.8 shows that $G \simeq {}_{\infty,2}[\mathrm{SL}_2(5) \overset{2(2)}{\boxtimes} D_8]_4 \circ C_3$.

- If $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ or $\mathrm{Sp}_4(3)$, then G contains a subgroup H conjugate to $\mathrm{SL}_2(5) \underset{\sqrt{-3}}{\otimes} C_{12}$ where the restriction of the natural character to $\mathrm{SL}_2(5)$ is $4\chi_4$. In particular G contains an irreducible cyclic subgroup of order 60. Thus G is conjugate to ${}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} C_4]_8$ by Theorem 4.9.8. □

Lemma 4.9.28 *If* $O_2(G) = C_2$, *then* G *is conjugate to one of* ${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$,

$${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1,\ {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes A_2,\ {}_{\sqrt{-3}}[C_6]_1 \otimes [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8,$$

$${}_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2_+}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8,\ {}_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2) \overset{2_-}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8\ .$$

Proof: Let $N := \mathcal{B}^o(F^*(G)) \trianglelefteq G$. By Table 2.5.1 there are the following possibilities for $E(G)$:

- $E(G) \simeq \mathrm{Alt}_5$. Then $N \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes A_4$ has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra and $[G : N] \leq 2$. This contradicts Lemma 2.2.1.

- $E(G) \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$. Let $C := C_G(O_3(G))$. It follows from Lemma 2.2.1 that $[G : C] = [C : N] = 2$. Thus C is reducible and conjugate to ${}_{\infty,5}[\mathrm{SL}_2(5){:}2]_2 \circ C_3$ or ${}_{\infty,5}[\mathrm{SL}_2(5){\cdot}2]_2 \circ C_3$. This contradicts loc. cit..

- $E(G) \simeq \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$. Then $N = ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2) \otimes C_3$ is irreducible, has $\mathbb{Q}(\sqrt{5}, \sqrt{-3})$ as commuting algebra and fixes only one lattice L. One finds some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L, F) = 3^8$. Further $G/F^*(G) \leq \mathrm{Out}(F^*(G)) \simeq D_8 \times C_2$ shows $\Pi(|G|) = \{2, 3, 5\}$. By Table 2.5.4, we have to check the following groups:

	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-15})$
F	${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$	${}_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2) \overset{2_-}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8$
$\mathfrak{p}_5 F$	${}_{\sqrt{-3}}[C_6]_1 \otimes [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)){:}2]_8$	${}_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2) \overset{2_+}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8$

All groups except ${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$ have the correct generalized Fitting subgroups.

- $E(G) \simeq \mathrm{L}_2(7)$. Then $N = M_{8,3} \otimes C_3$ is irreducible with commuting algebra $\mathbb{Q}(\sqrt{-3})$ and fixes 12 lattices. By computing the corresponding automorphism groups, one checks that N is only contained in ${}_{\sqrt{-3}}[C_6]_1^8$.

- $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(7)]_4$ contradicts Lemma 4.9.3.

- $E(G) \simeq \mathrm{SL}_2(9)$. Then $E(G) \ntrianglelefteq \mathcal{B}^o(N) \simeq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$ (see Lemma 4.5.6).

- $E(G) \simeq {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$. Then $N = {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes {}_{\sqrt{-3}}[C_6]_1$ is irreducible with commuting algebra $C \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{-7})$ and fixes one lattice L. Again one concludes $\Pi(|G|) = \{2, 3, 5, 7\}$. Further there exists some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L, F) = 3^8$. Let $K \simeq \mathbb{Q}(\sqrt{21})$ be the maximal totally real subfield of C. Since $\mathrm{End}_{\mathbb{Z}N}(L)$ is the maximal order of C and since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$, we may restrict ourselves to one class of totally positive units. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that σ interchanges the two prime ideals over 5. One finds that σ is conjugation by some $x \in N_{\mathrm{GL}_{16}(\mathbb{Q})}(N) \cap \mathrm{GL}(L)$ such that $xFx^{\mathrm{tr}} = F$. Thus by Table 2.5.4, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_3^{-a}\mathfrak{p}_5^a\mathfrak{p}_7^b F)$ for some $a, b \in \{0, 1\}$ and $d \in \{3, 7\}$. If $a = 1$ then these groups equal N. The other four groups are:

form	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-7})$
F	${}_{\sqrt{-3}}[C_6]_1 \otimes E_8$	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes A_2$
$\mathfrak{p}_7 F$	$\lneq {}_{\sqrt{-3}}[C_6]_1 \otimes E_8$	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$

So G is conjugate to ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes A_2$ or ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$.

- $E(G) \simeq \mathrm{SL}_2(7)$ with character $2\chi_{4ab}$. The commuting algebra C of N is isomorphic to $\mathbb{Q}(\sqrt{-3}, \sqrt{-7})$ with $K \simeq \mathbb{Q}(\sqrt{21})$ as maximal totally real subfield. Then

N fixes 4 lattices that have $\mathbb{Z}_C$ as endomorphism ring but only one class has minimal N-invariant superlattices whose endomorphism ring is a proper suborder of $\mathbb{Z}_C$. So by Remark 2.2.8 this yields a N-normal critical lattice L. Further there exists some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L, F) = 2^4 \cdot 3^8$. Again we have $\Pi(|G|) = \{2, 3, 7\}$. Just as before we may restrict ourselves to one class of totally positive units. By Table 2.5.4, G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_7^a F)$ for some $a \in \{0, 1\}$ and $d \in \{3, 7\}$. One checks that these automorphism groups are conjugate to N if $a = 1$. And they yield proper subgroups of ${}_{\sqrt{-3}}[C_6]_1^8$ and ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes A_2$ if $a = 0$. So this case never happens.

- $E(G) \simeq \mathrm{Sp}_4(3)$. Then $N \simeq {}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$ is reducible. Again by Lemma 2.2.1 we conclude $[G : N] = 4$. Thus G contains a subgroup of index two conjugate to $\mathrm{Sp}_4(3) \underset{\sqrt{-3}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$ or $\mathrm{Sp}_4(3) \otimes A_2$. Both groups have $\mathbb{Q}(\sqrt{-3})$ as commuting algebras and fix one or three lattices respectively. One checks that they are only contained in ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4^2$.
- $E(G) \simeq \mathrm{Aut}(E_8) = 2.O_8^+(2).2$. Then $N = {}_{\sqrt{-3}}[C_6]_1 \otimes E_8$ is already s.i.m.f.. □

4.9.4 The case $O_p(G) = 1$ for all odd primes p

Let $G < \mathrm{Sp}_{16}(\mathbb{Q})$ be s.p.i.m.f. such that $F(G) = O_2(G)$. It follows from Table 2.5.2 that $O_2(G)$ is isomorphic to C_2, C_4, D_8, Q_8, C_8, $D_8 \otimes C_4$, D_{16}, Q_{16}, QD_{16}, C_{16}, $D_8 \otimes C_8$, $D_8 \otimes D_{16}$, $D_8 \otimes Q_{16}$, $D_8 \otimes QD_{16}$, 2_+^{1+4}, $2_+^{1+4} \otimes C_4$, 2_-^{1+4}, C_{32}, D_{32}, Q_{32}, QD_{32}, QD_{64}, $D_8 \otimes C_{16}$, $D_8 \otimes QD_{32}$, $D_8 \otimes D_{16}$, $2_+^{1+4} \otimes C_8$, $2_+^{1+4} \otimes D_{16}$, $2_+^{1+4} \otimes QD_{16}$, 2_-^{1+6}, 2_+^{1+6} or $2_-^{1+6} \otimes C_4$.

We discuss each possibility, which finishes the classification of the s.p.i.m.f. subgroups of $\mathrm{Sp}_{16}(\mathbb{Q})$.

Lemma 4.9.29

(a) If $F(G) \simeq 2_+^{1+6}$ then G is conjugate to ${}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8$.

(b) If $F(G) \simeq 2_+^{1+6} \otimes C_4$ then G is conjugate to ${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$.

(c) If $F(G) \simeq 2_-^{1+6}$ then G is conjugate to ${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 : 2]_8$.

Proof:

(a) Follows from Lemma 3.4.3.

(b) In this case $\mathcal{B}^o(F(G)) \simeq {}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$ fixes only one lattice and has $\mathbb{Q}(i)$ as commuting algebra. One checks that it is s.i.m.f..

(c) Follows from Lemma 4.9.4. □

Lemma 4.9.30 *$F(G)$ is not conjugate to $2_+^{1+4} \otimes C_8$, $2_+^{1+4} \otimes D_{16}$ or $2_+^{1+4} \otimes QD_{16}$.*

Proof: In any case G would contain an irreducible normal subgroup N conjugate to $2_+^{1+4} \otimes C_8$. Thus G contains a normal subgroup $B := \mathcal{B}^o(N) \simeq N.S_6$. This group fixes up to isomorphism only one lattice L and its commuting algebra is isomorphic to $\mathbb{Q}(\zeta_8)$ with maximal totally real subfield $\mathbb{Q}(\sqrt{2})$. Thus B is self-centralizing and from the structure of $\mathrm{Out}(N)$ we get $G/B \leq C_2 \times C_2$. Hence $\Pi(|G|) = \{2,3,5\}$. There exist some $F \in \mathcal{F}_{>0}(B)$ that is integral on L with $\det(L,F) = 1$. By Table 2.5.4, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^a F)$ for some $a \in \{0,1\}$ and $d \in \{1,2\}$. These groups are:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	$_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	$_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8$
$\mathfrak{p}_2 F$	$_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	$_{\sqrt{-2}}[_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 : 2]_8$

None of these groups has the correct Fitting subgroup. □

Lemma 4.9.31

(a) If $F(G) \simeq D_8 \otimes QD_{32}$ then G is conjugate to $_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8 \otimes QD_{32}).S_3]_4$.

(b) $F(G)$ is not conjugate to $D_8 \otimes C_{16}$ or $D_8 \otimes D_{16}$.

Proof: In any case, G contains an irreducible normal subgroup N conjugate to $D_8 \otimes C_{16}$. Thus G contains a normal subgroup $B := \mathcal{B}^o(N) \simeq N.S_3$. This group fixes only one lattice L and its commuting algebra is isomorphic to $\mathbb{Q}(\zeta_{16})$. Thus B is self-centralizing and from the structure of $\mathrm{Out}(N)$ we get $G/B \leq C_2 \times C_4$. Hence $\Pi(|G|) = \{2,3\}$. Let $F \in \mathcal{F}_{>0}(B)$ such that F is integral on L with $\det(L,F) = 1$. By Table 2.5.3, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\alpha)}(L, \mathfrak{p}_2^a F)$ for some $0 \leq a \leq 2$ and $\alpha \in \{i, \sqrt{-2}, \zeta_{16} - \zeta_{16}^{-1}\}$. These groups are

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\zeta_{16} - \zeta_{16}^{-1})$
F	$_i[(2_+^{1+4} \otimes C_4).S_6]_4^2$	$_{\sqrt{-2}}[_{\infty,2}[(D_8 \otimes Q_8).\mathrm{Alt}_5]_2 : 2]_4^2$	$_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8 \otimes QD_{32}).S_3]_4$
$\mathfrak{p}_2 F$	$\lneq {}_i[(D_8 \otimes C_4).S_3]_2^4$	$\lneq {}_{\sqrt{-2}}[F_4 : 2]_4^2$	$_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8 \otimes QD_{32}).S_3]_4$
$\mathfrak{p}_2^2 F$	$_i[(D_8 \otimes C_4).S_3]_2^4$	$_{\sqrt{-2}}[F_4 : 2]_4^2$	$_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8 \otimes QD_{32}).S_3]_4$

So the result follows. □

Lemma 4.9.32 *$F(G)$ is not conjugate to C_{16}, D_{32} or QD_{32}.*

Proof: Otherwise $C_G(F(G))$ embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{16}))$, so $E(G) = 1$. Thus $G/F(G) \leq \mathrm{Out}(F(G))$ is a 2-group, which implies $G = F(G)$. But then G is reducible. □

Lemma 4.9.33

(a) If $F(G) \simeq {}_{\zeta_{32}-\zeta_{32}^{-1}}[QD_{64}]_2$ *then* $G = F(G)$ *is s.i.m.f..*

(b) $F(G)$ *is not conjugate to* Q_{32} *or* C_{32}*.*

Proof: Clearly $E(G) = 1$ and thus $F(G)$ is self-centralizing in G. Since $\mathrm{Out}(F(G))$ is a 2-group, we have $G = F(G)$. The group ${}_{\zeta_{32}-\zeta_{32}^{-1}}[QD_{64}]_2$ is s.i.m.f. by Lemma 3.3.1 whereas $C_{32} \lneq QD_{64}$ and $Q_{32} \lneq Q_{32} \circ C_3$. □

Lemma 4.9.34 $F(G)$ *is not conjugate to* 2_-^{1+4}*.*

Proof: If $E(G) = 1$, then $B := \mathcal{B}^o(F(G)) \simeq {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$ and $G/B \leq C_2$ contradicts Lemma 2.2.1. So $E(G) \neq 1$ embeds into $\mathrm{GL}_2(\mathcal{Q}_{\infty,2})$ and thus must be conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ by Table 2.5.1. The group ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \underset{\infty,2}{\otimes} {}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$ is irreducible and has $\mathbb{Q}(\sqrt{5})$ as commuting algebra. In this case G would not be symplectic. □

Lemma 4.9.35 $F(G)$ *is not conjugate to* $2_+^{1+4} \otimes C_4$*.*

Proof: The centralizer $C_G(F(G))$ embeds into $\mathrm{GL}_2(\mathbb{Q}(i))$. So it follows from Table 2.5.1, that $E(G) = 1$. Thus G contains $B := \mathcal{B}^o(F(G)) \simeq F(G).S_6$ of index at most 2 and B has $\mathbb{Q}(i)^{2\times 2}$ as commuting algebra. This contradicts Lemma 2.2.1. □

Lemma 4.9.36 $F(G)$ *is not conjugate to* 2_+^{1+4}*.*

Proof: Otherwise G contains the normal subgroup $N := \mathcal{B}^o(F(G)) \simeq \mathrm{Aut}(F_4)$. Then $C_G(N)$ embeds into $\mathrm{GL}_4(\mathbb{Q})$. Table 2.5.1 implies that $E(G) = 1$ (note that $F_4 \otimes A_4$ is not symplectic). But then $G/N \leq \mathrm{Out}(N) \simeq C_2$ implies that G is reducible. □

Lemma 4.9.37 $F(G)$ *is not conjugate to* $D_8 \otimes C_8$*,* $D_8 \otimes D_{16}$*,* $D_8 \otimes Q_{16}$ *or* $D_8 \otimes QD_{16}$*.*

Proof: In any cases $E(G) = 1$ and G would contain a normal subgroup N conjugate to $D_8 \otimes C_8 = Q_8 \underset{i}{\otimes} C_8$. Let $B := \mathcal{B}^o(N) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \underset{i}{\otimes} C_8$. It follows from $G/F(G) \leq \mathrm{Out}(F(G))$ that G is soluble and $|G| = 2^k \cdot 3$ for some $k > 0$. Moreover B contains a characteristic subgroup U conjugate to ${}_{\infty,2}[\mathrm{SL}_2(3)]_1$. But $C_G(U)$ is necessarily a 2-group, so $C_G(U) \subseteq F(G)$. Hence G contains $UF(G)$ as a normal subgroup of index at most 2. Now Lemma 2.2.1 implies $F(G) \simeq D_8 \otimes Q_{16}$ or $D_8 \otimes D_{16}$. In both cases, $\mathcal{B}^o(F(G)) \simeq F(G).S_3$ implies that $\mathcal{B}^o(F(G)) = G$ since $|\mathcal{B}^o(F(G))| = 2 \cdot |UF(G)|$.
If $F(G) \simeq D_8 \otimes Q_{16}$, then $F(G)$ has $\mathcal{Q}_{\sqrt{2},\infty,\infty}$ as commuting algebra which contains nontrivial torsion units. So G cannot be s.i.m.f..
If $F(G) \simeq D_8 \otimes D_{16} = Q_{16} \circ Q_8$ then $\mathcal{B}^o(F(G)) \simeq {}_{\sqrt{2},\infty}[Q_{16}]_1 \circ {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ is reducible. □

Lemma 4.9.38 *If $F(G) \simeq D_8 \otimes C_4$ then G is conjugate to*

$$_{\sqrt{-5}}[_i[(D_8 \otimes C_4).S_3]_2 \overset{2+}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8 \text{ or } _{\sqrt{-5}}[_i[(D_8 \otimes C_4).S_3]_2 \overset{2-}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8 .$$

Proof: Let $B := \mathcal{B}^o(F(G)) \simeq {}_i[(D_8 \otimes C_4).S_3]_2$. Then $C_G(B)$ embeds into $\mathrm{GL}_4(\mathbb{Q}(i))$. By 2.5.1 $E(G)$ is conjugate to 1, Alt_5, $_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ or $_{\infty,3}[\mathrm{SL}_2(9)]_2$. Let $N := BE(G) \trianglelefteq G$.

- If $E(G) = 1$, then $[G : B] \leq 2$. This contradicts Lemma 2.2.1.
- If $E(G) \simeq \mathrm{Alt}_5$ or $_{\infty,3}[\mathrm{SL}_2(9)]_2$, then N is already irreducible, fixes 4 or 2 lattices respectively and has $\mathbb{Q}(i)$ as commuting algebra. It is easily checked that N is only contained in $_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$.
- If $E(G) \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ then N contains an irreducible cyclic subgroup of order 40. Since $\mathrm{Out}(N)$ is a 2-group, we have $\Pi(|G|) = \{2,3,5\}$. Hence G is conjugate to $_{\sqrt{-5}}[_i[(D_8 \otimes C_4).S_3]_2 \overset{2+}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$ or $_{\sqrt{-5}}[_i[(D_8 \otimes C_4).S_3]_2 \overset{2-}{\underset{i}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$ by Theorem 4.9.6.

Thus the claim follows. □

Lemma 4.9.39 *$F(G)$ is not conjugate to C_8, D_{16} or QD_{16}.*

Proof: If $E(G) = 1$, then $G/F(G) \leq \mathrm{Out}(F(G))$ is a 2-group. Which implies that $G = F(G)$ is reducible. In any case, $F(G)$ contains a characteristic subgroup $H \simeq C_8$. Thus $E(G)$ embeds into $\mathrm{GL}_4(\mathbb{Q}(\zeta_8))$. So $E(G) \neq 1$ is conjugate to Alt_5, $_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ or $_{\infty,3}[\mathrm{SL}_2(9)]_2$. In any case, $G/F^*(G) \leq \mathrm{Out}(F(G))$ shows that $\Pi(|G|) = \{2,3,5\}$. Moreover $H\mathcal{B}^o(E(G)) \trianglelefteq G$ contains an irreducible cyclic subgroup of order 40. Thus the result follows from Theorem 4.9.6. □

Lemma 4.9.40 *If $F(G) \simeq Q_8$ then G is conjugate to one of*

$$_{\sqrt{-10}}[_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2-}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8,\ _{\sqrt{-10}}[_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2+}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$$

$$_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5){:}2]_2,\ _{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4,\ _{\infty,2}[\mathrm{SL}_2(3)]_1 \overset{2}{\underset{i}{\boxtimes}} {}_{\infty,3}[\mathrm{SL}_2(9)]_2 .$$

Proof: The layer $E(G)$ embeds into $\mathrm{GL}_4(\mathcal{Q}_{\infty,2})$. Since $\mathrm{Out}(\mathcal{B}^o(Q_8)) \simeq C_2$, $E(G)$ cannot be trivial. Table 2.5.1 shows that $E(G)$ is conjugate to Alt_5, $_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ or $_{\infty,3}[\mathrm{SL}_2(9)]_2$ (note that $O_3(\mathcal{B}^o(\mathrm{Sp}_4(3))) \simeq C_3$).

- If $E(G) \simeq \mathrm{Alt}_5$ then $\mathcal{B}^o(F^*(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \otimes A_4$ is an irreducible normal subgroup of G and it has $\mathcal{Q}_{\infty,2}$ as commuting algebra. By Remark 2.2.17, G is conjugate to $_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_4$. This group is s.i.m.f..

- If $E(G) \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$, then G contains a normal subgroup $N := \mathcal{B}^o(F^*(G))$ conjugate to ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ \mathrm{SL}_2(3)$ and $G/N \leq \mathrm{Out}(N) \simeq C_2 \times C_2$.

  First we handle the case that $[G : N] = 2$. The groups ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} \mathrm{SL}_2(3)$ and ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} \mathrm{SL}_2(3)$ are reducible.
  The groups ${}_{\infty,5}[\mathrm{SL}_2(5){:}2]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ and ${}_{\infty,5}[\mathrm{SL}_2(5){\cdot}2]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ are proper subgroups of ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5) : 2]_2$ and ${}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8$ respectively. The group ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \underset{\infty,2}{\otimes} {}_{\sqrt{2},\infty}[\tilde{S}_4]_1$ is not symplectic. Finally $H := {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \underset{\sqrt{-2}}{\otimes} {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_1$ is irreducible, has $\mathbb{Q}(\sqrt{-2},\sqrt{5})$ as commuting algebra and fixes only one lattice L. There exists some $F \in \mathcal{F}_{>0}(H)$ that is integral on L with $\det(L,F) = 1$. By Table 2.5.4, G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_5^a F)$ for some $a \in \{0,1\}$ and $d \in \{2,10\}$. These groups are:

form	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-10})$
F	${}_{\sqrt{-2}}[(2_+^{1+6}.\mathrm{Alt}_8){:}2]_8$	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2+}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$
$\mathfrak{p}_5 F$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \underset{\sqrt{-2}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5){:}2]$	${}_{\sqrt{-10}}[{}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2-}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$

  Note that this also handles the case where $[G : N] = 4$.

- If $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(9)]_2$, then G contains a normal subgroup N conjugate to ${}_{\infty,2}[\mathrm{SL}_2(3)]_2 \underset{i}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$. Then $\mathrm{End}(\overline{N}) \simeq \mathcal{Q}_{2,3}$ and $\mathrm{Out}(\mathrm{SL}_2(9)) \simeq C_2 \times C_2$ but one outer automorphism interchanges χ_{4a} and χ_{4b}. Therefore $G/N \leq C_2 \times C_2$. Remark 2.2.17 shows that G contains a normal subgroup conjugate to $G_1 := {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \overset{2}{\underset{i}{\boxtimes}} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ or $G_2 := {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \underset{\sqrt{-3}}{\otimes} ({}_{\infty,3}[\mathrm{SL}_2(9)]_2.2)$. (Note that ${}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \underset{\infty,3}{\otimes} {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ is not symplectic and ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ is an irreducible subgroup of $\mathrm{GL}_{32}(\mathbb{Q})$ since $\mathbb{Q}(\sqrt{-2})$ does not split $\mathcal{Q}_{\infty,3}$). The group G_1 is s.i.m.f. and G_2 is only contained in ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3] \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$. □

Lemma 4.9.41

(a) If $F(G) \simeq C_4$ then G is conjugate to ${}_i[C_4]_1 \otimes E_8$, ${}_i[C_4]_1 \otimes M_{8,3}$, ${}_{\infty,3}[\mathrm{SL}_2(7)]_4 \circ C_4$, ${}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2) \overset{2-}{\boxtimes} {}_i[C_4]_1]_8$ or ${}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2) \overset{2+}{\boxtimes} {}_i[C_4]_1]_8$.

(b) $F(G)$ is not conjugate to D_8.

Proof: Suppose $F(G) \simeq C_4$ or D_8. So $E(G)$ embeds into $\mathrm{GL}_8(\mathbb{Q}(i))$. By Table 2.5.1, $E(G)$ is isomorphic to Alt_5, $\mathrm{SL}_2(5)$ (with character $4\chi_{2ab}$), $\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$, $\mathrm{L}_2(7)$, $\mathrm{SL}_2(7)$ (2 representations), $\mathrm{SL}_2(9)$, $2.\mathrm{Alt}_7$ or $2.O_8^+(2)$. (Note that $O_3(\mathcal{B}^o(\mathrm{Sp}_4(3))) \simeq C_3$).
Since $F(G)$ and $\mathrm{Out}(F(G)) \simeq C_2$ are 2-groups, so is $C := C_G(E(G))$. Thus $F(G) = C$ and therefore $G/F^*(G) \leq \mathrm{Out}(E(G))$.

- If $E(G) \simeq \mathrm{Alt}_5$, then $G = \mathcal{B}^o(N) \simeq \pm S_5 \otimes F(G)$ is reducible.

- If $E(G) \simeq {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1$ then $[G : F^*(G)] \leq 2$. Thus $F^*(G) = {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \otimes D_8$ by Lemma 2.2.1. The commuting algebra of $F^*(G)$ is isomorphic to $\mathcal{Q}_{\sqrt{5},\infty}$. The two groups ${}_{\infty,5}[\mathrm{SL}_2(5){\cdot}2]_2 \otimes D_8$ and ${}_{\infty,5}[\mathrm{SL}_2(5) : 2]_2 \otimes D_8$ have $\mathcal{Q}_{\infty,5}$ as commuting algebras. The two groups ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_+}{\boxtimes} D_8$ and ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2_-}{\boxtimes} D_8$ have $\mathcal{Q}_{\infty,2}$ as commuting algebras. These four groups cannot be s.i.m.f. since their commuting algebras contain nontrivial torsion units.

- If $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(9)]_2$ then $[G : F^*(G)] \leq 2$ since one outer automorphism of $\mathrm{SL}_2(9)$ interchanges χ_{4a} and χ_{4b}. So $F^*(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(9)]_2 \otimes D_8$ by Lemma 2.2.1. This group has $\mathcal{Q}_{\infty,3}$ as commuting algebra and is self-centralizing in G. By Remark 2.2.17 one finds that G is a proper subgroup of ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4^2$.

- If $E(G) \simeq {}_{\infty,3}[\mathrm{SL}_2(7)]_4$, then $G \simeq {}_{\infty,3}[\mathrm{SL}_2(7)]_4 \circ C_4$ by Lemma 4.9.3.

In all other cases, let $H \leq F(G)$ be the characteristic subgroup of order 4 and set $N := HE(G)$.

- If $E(G) \simeq \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$ then $B := \mathcal{B}^o(N) \simeq ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2) \otimes {}_i[C_4]_1$. Then B has an irreducible cyclic subgroup of order 60 and $G/B \leq \mathrm{Out}(B)$ shows $\Pi(|G|) = \{2,3,5\}$. It follows from Theorem 4.9.8 that G is conjugate to ${}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2) \overset{2_+}{\boxtimes} {}_i[C_4]_1]_8$ or ${}_{\sqrt{-5}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2) \overset{2_-}{\boxtimes} {}_i[C_4]_1]_8$.

- If $E(G) \simeq \mathrm{L}_2(7)$ then $\mathcal{B}^o(N) \simeq {}_i[C_4]_1 \otimes (\mathrm{L}_2(7).2) = {}_i[C_4]_1 \otimes M_{8,3}$ is already s.i.m.f..

- If $E(G) \simeq \mathrm{SL}_2(7)$ with character $2\chi_{4ab}$, then $N \simeq \mathrm{SL}_2(7) \otimes {}_i[C_4]_1$ has $C := \mathbb{Q}(i, \sqrt{7})$ as commuting algebra with totally real subfield $K \simeq \mathbb{Q}(\sqrt{7})$. Using part (a) of Remark 2.2.8 one finds a N-normal critical lattice L such that $\mathrm{End}_{\mathbb{Z}N}(L) = \mathbb{Z}_C$ and some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L, F) = 1$. Since $\mathbb{Z}_{K,>0}^* = \mathrm{Nr}_{C/K}(\mathbb{Z}_C^*)$, we may restrict ourselves to one class of totally positive units. Further one finds some $x \in N_{\mathrm{GL}_{16}(\mathbb{Q})}(N) \cap \mathrm{GL}_{16}(L)$ such that $\mathfrak{p}_3^x = \mathfrak{p}_3'$ (and x necessarily fixes $\mathfrak{p}_2$ and $\mathfrak{p}_7$). Since $\Pi(|G|) = \{2,3,7\}$, it follows from Table 2.5.4 that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^a \mathfrak{p}_3^b \mathfrak{p}_7^b F)$ for some $a, b \in \{0,1\}$ and $d \in \{1,7\}$. If $b = 1$ these groups equal N. The other four groups are:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-7})$
F	$\lneq {}_i[C_4]_1^8$	$\lneq {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_8^2$
$\mathfrak{p}_2 F$	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	$\lneq {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_8^2$

- If $E(G) \simeq 2.\mathrm{Alt}_7$ then $N \simeq {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4 \otimes {}_i[C_4]_1$ has $C := \mathbb{Q}(i, \sqrt{7})$ as commuting algebra. The group N fixes only one lattice L and there exists some $F \in \mathcal{F}_{>0}(N)$ that is integral on L with $\det(L, F) = 1$. As before, there are eight candidates to check:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-7})$
F	${}_i[C_4]_1 \otimes E_8$	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_8^2$
$\mathfrak{p}_2 F$	${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$	$\lneq {}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_8^2$
$\mathfrak{p}_3\mathfrak{p}_7 F$	N	${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_8 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$
$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_7 F$	N	N

- If $E(G) \simeq 2.O_8^+(2)$ then $\mathcal{B}^o(N) \simeq {}_i[C_4]_1 \otimes E_8$ is already s.i.m.f.. □

Lemma 4.9.42 *$F(G)$ is not conjugate to C_2.*

Proof: Otherwise $G/\pm E(G) \leq \mathrm{Out}(E(G))$. Table 2.5.1 shows that $E(G)$ is isomorphic to Alt_5, $\mathrm{SL}_2(5)$ (with character $4\chi_{2ab}$), $\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{L}_2(7)$ (2 representations), $\mathrm{SL}_2(9)$, $2.\mathrm{Alt}_7$ or $2.O_8^+(2)$.
The case that $E(G)$ is conjugate to $\mathrm{SL}_2(7)$ with character $2\chi_8$ has been handled in Lemma 4.9.3. If $E(G) \simeq \mathrm{L}_2(7)$ then $\mathcal{B}^o(E(G)) \simeq \pm\mathrm{L}_2(7).2$. Since $\mathrm{Out}(E(G)) \simeq C_2$, we get $G = \pm\mathrm{L}_2(7).2$ is reducible. Similarly, if $E(G) \simeq 2.O_8^+(2)$ then $\mathcal{B}^o(E(G)) \simeq E_8$. Since $\mathrm{Out}(E(G)) \simeq C_2$, we get $G = E_8$ is reducible.
Suppose $E(G) \simeq \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$. Then G contains a normal subgroup $B := \mathcal{B}^o(E(G)) \simeq ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2)$ and $G/B \leq \mathrm{Out}(B) \simeq C_2 \times C_2$. Only one class of outer automorphisms can be realized in $\mathrm{GL}_{16}(\mathbb{Q})$ and both extensions (split and non-split) are reducible. Thus this case cannot happen.
In all other cases $E(G)$ is reducible, has a commuting algebra which is not isomorphic to $K^{2\times 2}$ for some totally real number field K and $G/\pm E(G) \leq C_2$. (Note that $\mathrm{Out}(\mathrm{SL}_2(9)) \simeq C_2 \times C_2$ but one outer automorphism interchanges χ_{4a} and χ_{4b}.) This contradicts Lemma 2.2.1. □

4.10 Dimension 18

Theorem 4.10.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_{18}(\mathbb{Q})$ *are*

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
$[2,1,9]$	$_i[C_4]_1^9$	$2^{25}\cdot3^4\cdot5\cdot7$	3	$[1,1,36]$	B_{18}
1	$_i[C_4]_1\otimes A_9$	$2^{10}\cdot3^4\cdot5^2\cdot7$	12	$[1,2,180]$	A_9^2
$[2,2,9]$	$_{\sqrt{-3}}[C_6]_1^9$	$2^{16}\cdot3^{13}\cdot5\cdot7$	1	$[3^9,2,54]$	A_2^9
$[6,1,3]$	$_{\sqrt{-3}}[\pm3_+^{1+2}{:}\mathrm{SL}_2(3)]_3^3$	$2^{13}\cdot3^{13}$	2	$[3^3,2,216]$	E_6^3
2	$_{\sqrt{-3}}[\pm3_+^{1+4}{:}\mathrm{Sp}_4(3)]_9$	$2^8\cdot3^9\cdot5$	2	$[3^5,4,6480]$	H_1
3	$_{\sqrt{-3}}[C_6]_1\otimes A_9$	$2^9\cdot3^5\cdot5^2\cdot7$	4	$[2^2\cdot3^9\cdot5^2,4,270]$	$A_2\otimes A_9$
4	$_{\sqrt{-3}}[\pm3.M_{10}]_9$	$2^5\cdot3^3\cdot5$	2	$[3^9\cdot5^6,6,180]$	$[\pm3.\mathrm{Alt}_6.2^2]_{18}$
$[6,2,3]$	$_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3^3$	$2^{13}\cdot3^4\cdot7^3$	1	$[7^9,4,126]$	$(A_6^{(2)})^3$
5	$_{\sqrt{-7}}[\pm(\mathrm{L}_2(7)\underset{\sqrt{-7}}{\otimes}\mathrm{L}_2(7)){:}2]_9$	$2^8\cdot3^2\cdot7^2$	1	$[7^9,6,336]$	H_2
6	$_{\sqrt{-19}}[\pm\mathrm{L}_2(19)]_9$	$2^3\cdot3^2\cdot5\cdot19$	1	$[19^9,10,342]$	$A_{18}^{(5)}$

where $H_1 := [\pm3_+^{1+4} : \mathrm{Sp}_4(3).2]_{18}$ *and* $H_2 := [\pm(\mathrm{L}_2(7) \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} \mathrm{L}_2(7)) : 2]_{18}$ *are r.i.m.f. subgroups of* $\mathrm{GL}_{18}(\mathbb{Q})$.

Again, before we prove the theorem, we discuss the case that G contains an irreducible cyclic subgroup U. If $|\pm U| = 38$, then Theorem 3.1.1 shows that $G \simeq {}_{\sqrt{-19}}[\pm\mathrm{L}_2(19)]_9$. The remaining case $|\pm U| = 54$ is handled below.

Lemma 4.10.2 *If G contains an irreducible cyclic subgroup U of order* 54 *and* $\Pi(|G|) \subseteq \{2,3,5,7,11,13\}$, *then G is conjugate to* $_{\sqrt{-3}}[C_6]_1^9$ *or* $_{\sqrt{-3}}[\pm3_+^{1+2}{:}\mathrm{SL}_2(3)]_3^3$.

Proof: The centralizing algebra C of U is isomorphic to $\mathbb{Q}(\zeta_{54})$ and has class number 1. Thus G fixes only one lattice L. There exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L with $\det(L,F) = 3$. By Table 2.5.3, G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-3})}(L, \mathfrak{p}_3^k F)$ for some $0 \le k \le 4$. So the result is easily verified. □

Proof (Theorem 4.10.1): Explicit calculations show that the table is correct and it yields s.i.m.f. matrix groups. The symplectic imprimitive matrix groups come from the s.p.i.m.f. subgroups of $\mathrm{Sp}_2(\mathbb{Q})$ and $\mathrm{Sp}_6(\mathbb{Q})$. It follows from Lemma 2.1.21 that $_{\sqrt{-3}}[C_6]_1^9$, $_{\sqrt{-7}}[\pm\mathrm{L}_2(7)]_3^3$ and $_{\sqrt{-3}}[\pm3_+^{1+2}{:}\mathrm{SL}_2(3)]_3^3$ are maximal finite. The group $_i[C_4]_1^9$ fixes 3 lattices and has $\mathbb{Q}(i)$ as commuting algebra. One verifies that this group is also maximal finite.

It remains to classify the s.p.i.m.f. subgroups $G < \mathrm{Sp}_{18}(\mathbb{Q})$. This is done in the following three lemmas by discussing the various Fitting subgroups. In any case we have $F(G) = \prod_{p\in\{2,3,7\}} O_p(G)$ according to Corollary 2.1.16 and Theorem 3.1.1. □

For the remainder of the section suppose $G < \mathrm{Sp}_{18}(\mathbb{Q})$ is s.p.i.m.f..

Lemma 4.10.3 $O_7(G) = 1$.

Proof: By Table 2.5.2 the Fitting subgroup $F(G)$ must be conjugate to $\pm C_7$. Suppose first $E(G) = 1$. By Lemma 2.2.1 G must contain a normal subgroup $N \simeq \pm C_7 : C_3$. But there exists only one such extension and this embeds into $\mathrm{GL}_6(\mathbb{Q})$. Since $[G : N] \leq 2$ this contradicts loc. cit.. So $E(G) \neq 1$ embeds into $\mathrm{GL}_3(\mathbb{Q}(\zeta_7))$. Thus $E(G) \simeq \mathrm{L}_2(7)$ by Table 2.5.1. Then $F^*(G) \simeq \pm C_7 \underset{\sqrt{-7}}{\otimes} \mathrm{L}_2(7)$ is irreducible and fixes only one lattice L. Let $C \simeq \mathbb{Q}(\zeta_{14})$ denote the commuting algebra of $F^*(G)$ and let $K \simeq \mathbb{Q}(\theta_{14})$ be its maximal real subfield. Since $G/F^*(G) \leq C_6 \times C_2$, we only have to consider the prime ideals over $2, 3$ and 7. One finds some $F \in \mathcal{F}_{>0}(F^*(G))$ that is integral on L such that $\det(L, F) = 7^3$. Further C has only one totally complex subfield which is $\mathbb{Q}(\sqrt{-7})$. Hence it follows from Table 2.5.3, that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-7})}(L, \mathfrak{p}_7^a F)$ with $a \in \{0, 1\}$ (for $a \geq 2$ the lattice is not normalized). Both automorphism groups are contained in ${}_{\sqrt{-7}}[\pm(\mathrm{L}_2(7) \underset{\sqrt{-7}}{\otimes} \mathrm{L}_2(7)) : 2]_9$. □

Lemma 4.10.4 *If $O_3(G) \neq 1$ then either*

(a) $O_3(G) \simeq 3_+^{1+4}$ *and G is conjugate to* ${}_{\sqrt{-3}}[\pm 3_+^{1+4} : \mathrm{Sp}_4(3)]_9$.

(b) $O_3(G) \simeq C_3$ *and G is conjugate to* ${}_{\sqrt{-3}}[C_6]_1 \otimes A_9$ *or* ${}_{\sqrt{-3}}[\pm 3.M_{10}]_9$.

Proof: If $O_3(G)$ is irreducible, it is conjugate to 3_+^{1+4} or $3_+^{1+2} \underset{\sqrt{-3}}{\otimes} C_9$ since $O_3(G) \not\simeq C_{27}$ by Lemma 4.10.2. If $O_3(G)$ embeds irreducibly into $\mathrm{GL}_6(\mathbb{Q})$, it is conjugate to C_9 or 3_+^{1+2}. Otherwise $O_3(G) \simeq C_3$ embeds irreducibly into $\mathrm{GL}_2(\mathbb{Q})$.

- If $O_3(G) \simeq 3_+^{1+4}$ then $\mathcal{B}^o(O_3(G)) \simeq {}_{\sqrt{-3}}[\pm 3_+^{1+4} : \mathrm{Sp}_4(3)]_9$ is already s.i.m.f..

- If $O_3(G) \simeq 3_+^{1+2} \underset{\sqrt{-3}}{\otimes} C_9$ then G contains a normal subgroup N conjugate to $\pm(3_+^{1+2} \underset{\sqrt{-3}}{\otimes} C_9) : \mathrm{SL}_2(3)$. The commuting algebra of N is isomorphic to $\mathbb{Q}(\zeta_9)$. Thus $G/N \leq C_6$ and therefore $\Pi(|G|) = \{2, 3\}$. There exists a normal critical set of two lattices $\{L_1, L_2\}$ and $F \in \mathcal{F}_{>0}(N)$ that is integral on the L_i with $\det(L_i, F) \in \{3, 27\}$. By Table 2.5.3 we only have to consider the groups $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-3})}(L_i, \mathfrak{p}_3^a F)$ with $a \in \{0, 1\}$. These groups are subgroups of the s.i.m.f. groups ${}_{\sqrt{-3}}[\pm 3_+^{1+4} : \mathrm{Sp}_4(3)]_9$ or ${}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3^3$.

- If $O_3(G) \simeq C_9$ or 3_+^{1+2} then $C_G(O_3(G))$ embeds into $\mathrm{GL}_3(\mathbb{Q}(\zeta_k))$ with $k \in \{3, 9\}$. In particular $F^*(G) = \pm O_3(G)$. Let $N := \mathcal{B}^o(O_3(G)) \trianglelefteq G$. Then N is conjugate to $\pm C_9$ or $\pm 3_+^{1+2} : \mathrm{SL}_2(3)$. In the first case, let $P \in \mathrm{Syl}_3(G)$. Then $[G : \pm P] \leq 2$ shows $P = N$ and therefore $[G : N] \leq 2$. In the second case, it follows from $\mathrm{Out}(O_3(G)) \simeq \mathrm{GL}_2(3)$ that $[G : N] \leq 2$. So both cases contradict Lemma 2.2.1.

- If $O_3(G) \simeq C_3$, then $C_G(O_3(G))$ embeds into $\mathrm{GL}_9(\mathbb{Q}(\sqrt{-3}))$. So $O_2(G) = \pm I_{18}$ and it follows from Table 2.5.1 that $E(G)$ is conjugate to Alt_{10} or $3.\mathrm{Alt}_6$. In any case $\mathcal{B}^o(F^*(G))$ is already s.i.m.f. and conjugate to one of the groups stated above. □

Lemma 4.10.5 *If $F(G) = O_2(G)$, then G is conjugate to* ${}_{i}[C_4]_1 \otimes A_9$, ${}_{\sqrt{-19}}[\pm \mathrm{L}_2(19)]_9$ *or* ${}_{\sqrt{-7}}[\pm(\mathrm{L}_2(7) \underset{\sqrt{-7}}{\otimes} \mathrm{L}_2(7)):2]_9$.

Proof: Suppose first $F(G) \neq \pm I_{18}$. Then $F(G)$ is isomorphic to C_4 or D_8. Thus G contains a normal subgroup $N \simeq C_4$ and $E(G)$ embeds into $\mathrm{GL}_9(\mathbb{Q}(i))$. By Table 2.5.1 $E(G)$ is conjugate to Alt_{10}. So G contains an irreducible normal subgroup $B := N\mathcal{B}^o(E(G)) \simeq {}_{i}[C_4]_1 \otimes A_9$. The group B fixes 12 lattices and has $\mathbb{Q}(i)$ as commuting algebra. One easily checks that it is s.i.m.f..
If $F(G) = \pm I_{18}$, then $E(G) \neq 1$ embeds irreducibly into $\mathrm{GL}_k(\mathbb{Q})$ for $k \in \{6, 9, 18\}$ by Table 2.5.1.

(a) If $k = 6$ then $E(G) \simeq \mathrm{Alt}_5, \mathrm{Alt}_7, \mathrm{L}_2(7)$ or $U_4(2)$. In any case $[G : \mathcal{B}^o(E(G))] \leq 2$ contradicts Lemma 2.2.1.

(b) If $k = 9$ then $E(G) \simeq \mathrm{Alt}_{10}$ and $\mathcal{B}^o(G) \simeq A_9$ implies $G = \mathcal{B}^o(G)$ is reducible.

(c) If $k = 18$ then $E(G) \simeq \mathrm{L}_2(19)$ or $\mathrm{L}_2(7) \underset{\sqrt{-7}}{\otimes} \mathrm{L}_2(7)$ since $O_3(\mathcal{B}^o(3.\mathrm{Alt}_6)) = C_3$ and $\mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$ fixes no skewsymmetric form. The group ${}_{\sqrt{-19}}[\pm \mathrm{L}_2(19)]_9$ is s.i.m.f. by Theorem 3.2.1. In the second case $\mathcal{B}^o(E(G)) \simeq {}_{\sqrt{-7}}[\pm(\mathrm{L}_2(7) \underset{\sqrt{-7}}{\otimes} \mathrm{L}_2(7)):2]_9$ is irreducible and fixes only one lattice. One checks that it is s.i.m.f.. □

4.11 Dimension 20

Theorem 4.11.1 *The s.i.m.f. subgroups G of $\mathrm{Sp}_{20}(\mathbb{Q})$ are*

#	G	$\|G\|$	$\|\mathcal{Z}(G)\|$	L_{min}	*r.i.m.f. supergroups*
[2, 1, 10]	${}_i[C_4]_1^{10}$	$2^{28}\cdot3^4\cdot5^2\cdot7$	3	$[1,1,40]$	B_{20}
[4, 1, 5]	${}_i[(D_8\otimes C_4).S_3]_2^5$	$2^{28}\cdot3^6\cdot5$	1	$[2^{10},2,120]$	F_4^5
[4, 2, 5]	$({}_i[C_4]_1\otimes A_2)^5$	$2^{18}\cdot3^6\cdot5$	2	$[3^{10},2,60]$	A_2^{10}
1	${}_i[C_4]_1\otimes A_{10}$	$2^{10}\cdot3^4\cdot5^2\cdot7\cdot11$	2	$[11^2,2,220]$	A_{10}^2
2	${}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$	$2^{13}\cdot3^5\cdot5\cdot11$	1	$[2^{10},4,3960]$	$[SU_5(2)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{20}$
[10, 1, 2]	$({}_i[C_4]_1\otimes A_5)^2$	$2^{13}\cdot3^4\cdot5^2$	10	$[3^4,2,120]$	A_5^4
3	${}_i[C_4]_1\otimes[(C_6\times S_4(3)).2]_{10}$	$2^9\cdot3^5\cdot5$	6	$[3^{10},4,540]$	$[(C_6\times S_4(3)).2]_{10}^2$
4	${}_i[(D_8\otimes C_4).S_3]_2\otimes A_5$	$2^9\cdot3^3\cdot5$	6	$[2^{10}\cdot3^4,4,360]$	$F_4\otimes A_5$
5	${}_i[[\pm S_6]_{10} \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_{10}$	$2^7\cdot3^2\cdot5$	6	$[3^8,3,80]$	$[D_8\overset{2(2)}{\otimes}S_6]_{20}$
6	${}_i[C_4]_1\otimes A_{10}^{(2)}$	$2^5\cdot3\cdot5\cdot11$	2	$[11^6,4,440]$	$(A_{10}^{(2)})^2$
7	${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\sqsupset} C_4$	$2^5\cdot3\cdot5\cdot11$	2	$[2^{10}\cdot11^4,6,1320]$	$[\mathrm{SL}_2(11)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{20}$
8	${}_i[C_4]_1\otimes A_{10}^{(3)}$	$2^5\cdot3\cdot5\cdot11$	2	$[11^{10},6,220]$	$(A_{10}^{(3)})^2$
[4, 3, 5]	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2^5$	$2^{23}\cdot3^6\cdot5$	1	$[2^{10},2,120]$	F_4^5
9	${}_{\sqrt{-2}}[{}_{\infty,2}[\pm U_5(2)]_5:2]_{10}$	$2^{12}\cdot3^5\cdot5\cdot11$	1	$[2^{10},4,3960]$	$[SU_5(2)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{20}$
10	${}_{\sqrt{-2}}[2.M_{12}:2]_{10}$	$2^8\cdot3^3\cdot5\cdot11$	1	$[2^{10},4,3960]$	$[2.M_{12}:2]_{20}$
11	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2\otimes A_5$	$2^8\cdot3^3\cdot5$	12	$[2^{10},4,3960]$	$F_4\otimes A_5,\ [SU_5(2)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{20}$
[2, 2, 10]	${}_{\sqrt{-3}}[C_6]_1^{10}$	$2^{18}\cdot3^{14}\cdot5^2\cdot7$	1	$[3^{10},2,60]$	A_2^{10}
[4, 4, 5]	$({}_{\infty,2}[\mathrm{SL}_2(3)]_1\circ C_3)^5$	$2^{18}\cdot3^{11}\cdot5$	2	$[2^{10},2,120]$	F_4^5
[10, 2, 2]	${}_{\sqrt{-3}}[S_4(3)\circ C_3]_5^2$	$2^{15}\cdot3^{10}\cdot5^2$	2	$[2^4\cdot3^{10},4,540]$	$[(C_6\times S_4(3)).2]^2$
12	${}_{\sqrt{-3}}[C_6]_1\otimes A_{10}$	$2^9\cdot3^5\cdot5^2\cdot7\cdot11$	2	$[3^{10}\cdot11^2,4,330]$	$A_2\otimes A_{10}$

13	${}_{\infty,2}[\pm U_5(2)]_5 \circ C_3$	$2^{11}\cdot 3^6\cdot 5\cdot 11$	2	$[2^{10}, 4, 3960]$	$[SU_5(2) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{20}$
14	${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$	$2^9\cdot 3^6\cdot 5$	6	$[2^{10}, 4, 3960]$	$[SU_5(2) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{20}$
15	${}_{\sqrt{-3}}[\pm U_4(2) \circ C_3]_{10}$	$2^7\cdot 3^5\cdot 5$	2	$[2^8\cdot 3^{10}, 6, 1440]$	$[SU_4(2) \overset{2}{\square} C_6]_{20}$
16	${}_{\sqrt{-3}}[\pm \mathrm{L}_2(11) \overset{2(3)}{\otimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$	$2^4\cdot 3^2\cdot 5\cdot 11$	2	$[11^2, 4, 12540]$	$[\pm \mathrm{L}_2(11) \overset{2(3)}{\otimes} D_{12}]_{20}$
17	${}_{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$	$2^4\cdot 3^2\cdot 5\cdot 11$	2	$[11^{10}, 8, 660]$	$[\pm \mathrm{L}_2(11) \overset{2(3)}{\boxtimes} D_{12}]_{20}$
18	${}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(2)}$	$2^4\cdot 3^2\cdot 5\cdot 11$	2	$[3^{10}\cdot 11^6, 8, 660]$	$A_2 \otimes A_{10}^{(2)}$
19	${}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(3)}$	$2^4\cdot 3^2\cdot 5\cdot 11$	2	$[3^{10}\cdot 11^{10}, 12, 330]$	$A_2 \otimes A_{10}^{(3)}$
20	${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3$	$2^3\cdot 3^2\cdot 5\cdot 11$	4	$[2^{10}\cdot 11^4, 6, 1320]$	$[\mathrm{SL}_2(11) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{20}$
21	${}_{\sqrt{-7}}[2.M_{22}:2]_{10}$	$2^9\cdot 3^2\cdot 5\cdot 7\cdot 11$	1	$[7^{10}, 8, 6160]$	$[2.M_{22}:2]_{20}$
$[10, 3, 2]$	${}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5^2$	$2^7\cdot 3^2\cdot 5^2\cdot 11^2$	1	$[11^{10}, 6, 220]$	$(A_{10}^{(3)})^2$
22	${}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$	$2^5\cdot 3^2\cdot 5\cdot 11$	2	$[2^{10}, 4, 3960]$	$[SU_5(2) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{20}$
23	${}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \otimes A_2$	$2^4\cdot 3^2\cdot 5\cdot 11$	4	$[11^{10}, 8, 660]$	$[\mathrm{L}_2(11) \overset{2(3)}{\boxtimes} D_{12}]_{20},\ A_2 \otimes A_{10}^{(3)}$
24	${}_{\sqrt{-19}}[\mathrm{SL}_2(19)]_9$	$2^3\cdot 3^2\cdot 5\cdot 19$	5	$[1, 1, 40]$ $[1, 2, 760]$	B_{20}
$[4, 5, 5]$	${}_{\zeta_{10}}[C_{10}]_1^5$	$2^8\cdot 3\cdot 5^6$	1	$[5^5, 2, 100]$	A_4^5
25	${}_{\zeta_{10}}[\pm 5_+^{1+2}.\mathrm{Sp}_2(5)]_5$	$2^4\cdot 3\cdot 5^4$	2	$[5^3, 4, 12300]$	$[\pm 5_+^{1+2}:\mathrm{GL}_2(5)]_{20}$
26	${}_{\zeta_{10}}[C_{10}]_1 \otimes A_5$	$2^5\cdot 3^2\cdot 5^2$	4	$[2^4\cdot 3^4\cdot 5^5, 4, 300]$	$A_4 \otimes A_5$

Proof: By explicit calculations, one verifies that the above table is correct. The candidates for the symplectic imprimitive matrix groups come from the classification of the s.p.i.m.f. subgroups of $\mathrm{Sp}_2(\mathbb{Q})$, $\mathrm{Sp}_4(\mathbb{Q})$ and $\mathrm{Sp}_{10}(\mathbb{Q})$. According to Lemma 2.1.21 all these groups except ${}_i[C_4]_1^{10}$ and $({}_i[C_4]_1 \otimes A_5)^2$ are s.i.m.f.. These two groups fixes 3 and 10 lattices respectively and they have $\mathbb{Q}(i)$ as commuting algebras. One verifies that these groups are also s.i.m.f.. So it remains to prove the classification of the s.p.i.m.f. subgroups. This is accomplished in Section 4.11.2. □

4.11.1 Irreducible cyclic subgroups

If a s.i.m.f. matrix group $G < \mathrm{Sp}_{20}(\mathbb{Q})$ contains an irreducible cyclic subgroup U, then $|\pm U| \in \{50, 44, 66\}$. These cases are handled below.

Lemma 4.11.2 *Suppose G contains an irreducible cyclic subgroup U of order 50 and $\Pi(|G|) \subseteq \{2,3,5\}$ or there exists some $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L with $\det(L,F) \subseteq \{2,3,5\}$. Then G is conjugate to ${}_{\zeta_{10}}[C_{10}]_1^5$.*

Proof: The commuting algebra of U is isomorphic to $\mathbb{Q}(\zeta_{50})$ and has class number 1. Thus G fixes only one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L with $\det(L,F) = 5$. By Table 2.5.3, G must be conjugate to $\mathrm{Aut}_{\mathbb{Q}(\zeta_{10})}(L, \mathfrak{p}_5^a F)$ for some $0 \le a \le 4$. All these groups are subgroups of ${}_{\zeta_{10}}[C_{10}]_1^5$. □

Lemma 4.11.3 *Suppose G contains an irreducible cyclic subgroup U of order 44 and $\Pi(|G|) \subseteq \{2,3,5,11\}$ or there exists some $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L with $\det(L,F) \subseteq \{2,3,5,11\}$. Then G is conjugate to one of*

$$ {}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5^2,\ {}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1,\ {}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4, $$

$$ {}_{\infty,2}[\pm \mathrm{SL}_2(11)]_5 \overset{2(2)}{\sqsupset} C_4 \text{ or } {}_i[C_4]_1 \otimes H \text{ with } H \in \{A_{10}, A_{10}^{(2)}, A_{10}^{(3)}\}. $$

Proof: The commuting algebra $C := \mathrm{End}(\overline{U}) \simeq \mathbb{Q}(\zeta_{44})$ and has class number 1. Thus G fixes only one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L such that $\det(L,F) = 11^2$. Let $K \simeq \mathbb{Q}(\theta_{44})$ be the maximal totally real subfield of C. There exists some $\sigma \in \mathrm{Gal}(C/K)$ that interchanges $\mathfrak{p}_5$ and $\mathfrak{p}_5'$. Further σ is conjugation by some $x \in N_{\mathrm{GL}_{16}(\mathbb{Q})}(U) \cap \mathrm{GL}(L)$ with $xFx^{\mathrm{tr}} = F$. So Table 2.5.3 shows that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^a \mathfrak{p}_5^b \mathfrak{p}_{11}^{2c-a} F)$ for some $a, b \in \{0,1\}$, $0 \le c \le 2$ and $d \in \{1, 11\}$.

One finds $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-11})}(L, \mathfrak{p}_2^a \mathfrak{p}_5^b \mathfrak{p}_{11}^{2c-a} F) \le \begin{cases} {}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5^2 & \text{if } a = 0 \\ {}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1 & \text{if } a = 1 \end{cases}.$

Further $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-11})}(L, \mathfrak{p}_2^a \mathfrak{p}_5 \mathfrak{p}_{11}^{2c-a} F) = \mathrm{Aut}_{\mathbb{Q}(i)}(L, \mathfrak{p}_2^a \mathfrak{p}_5 \mathfrak{p}_{11}^{2c-a} F)$. The remaining 6 automorphism groups are given below.

form	$\mathbb{Q}(i)$	form	$\mathbb{Q}(i)$
F	${}_i[C_1]_1 \otimes A_{10}$	$\mathfrak{p}_2 \mathfrak{p}_{11}^{-1} F$	${}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$
$\mathfrak{p}_{11}^2 F$	${}_i[C_1]_1 \otimes A_{10}^{(2)}$	$\mathfrak{p}_2 \mathfrak{p}_{11} F$	${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\sqsupset} C_4$
$\mathfrak{p}_{11}^4 F$	${}_i[C_1]_1 \otimes A_{10}^{(3)}$	$\mathfrak{p}_2 \mathfrak{p}_{11}^3 F$	$\lneq {}_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\sqsupset} C_4$

□

Lemma 4.11.4 *Suppose G contains an irreducible cyclic subgroup U of order 66 and $\Pi(|G|) \subseteq \{2,3,5,7,11\}$ or there exists some $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ such that F is integral on L with $\det(L,F) \subseteq \{2,3,5,7,11\}$. Then G is conjugate to one of*

$$_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \otimes A_2,\ _{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1,\ _{\infty,2}[\pm U_5(2)]_5 \circ C_3,$$

$$_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3,\ _{\sqrt{-3}}[\pm \mathrm{L}_2(11) \overset{2(3)}{\otimes} {}_{\sqrt{-3}}[C_6]_1]_{10},\ _{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$$

$$\text{or } _{\sqrt{-3}}[C_6]_1 \otimes H \text{ with } H \in \{A_{10}, A_{10}^{(2)}, A_{10}^{(3)}\}\,.$$

Proof: The commuting algebra C of U is isomorphic to $\mathbb{Q}(\zeta_{66})$ and has class number 1. Thus G fixes only one lattice L. Further there exists some $F \in \mathcal{F}_{>0}(U)$ that is integral on L such that $\det(L,F) = 11^2$. Let $K \simeq \mathbb{Q}(\theta_{66})$ be the maximal totally real subfield of C. There exists some $\sigma \in \mathrm{Gal}(C/K)$ that interchanges $\mathfrak{p}_2$ and $\mathfrak{p}_2'$. Further σ is conjugation by some $x \in N_{\mathrm{GL}_{20}(\mathbb{Q})}(U) \cap \mathrm{GL}(L)$ with $xFx^{\mathrm{tr}} = F$. So Table 2.5.3 shows that G is conjugate to $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-d})}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_{11}^{2c-a}F)$ for some $a,b \in \{0,1\}$, $0 \le c \le 2$ and $d \in \{3,11\}$.

One checks that $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-11})}(L, \mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{p}_{11}^{2c-a}F) \le \begin{cases} _{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \otimes A_2 & \text{if } a = 0 \\ _{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} \mathrm{SL}_2(3) & \text{if } a = 1 \end{cases}$.

Further $\mathrm{Aut}_{\mathbb{Q}(\sqrt{-11})}(L, \mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}^{2c-1}F) = \mathrm{Aut}_{\mathbb{Q}(\sqrt{-3})}(L, \mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}^{2c-1}F)$ for all c. The remaining 9 automorphism groups are given below.

form	$\mathbb{Q}(\sqrt{-3})$	form	$\mathbb{Q}(\sqrt{-3})$
F	$_{\sqrt{-3}}[\pm \mathrm{L}_2(11) \overset{2(3)}{\otimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$	$\mathfrak{p}_3F$	$_{\sqrt{-3}}[C_6]_1 \otimes A_{10}$
$\mathfrak{p}_{11}^2F$	$\lneq {}_{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$	$\mathfrak{p}_3\mathfrak{p}_{11}^2F$	$_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(2)}$
$\mathfrak{p}_{11}^4F$	$_{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$	$\mathfrak{p}_3\mathfrak{p}_{11}^4F$	$_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(3)}$
$\mathfrak{p}_2\mathfrak{p}_{11}^{-1}F$	$_{\infty,2}[\pm U_5(2)]_5 \circ C_3$	$\mathfrak{p}_2\mathfrak{p}_{11}F$	$_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3$
$\mathfrak{p}_2\mathfrak{p}_{11}^3F$	$\lneq {}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$		

□

4.11.2 Proof of Theorem 4.11.1

Suppose that $G < \mathrm{Sp}_{20}(\mathbb{Q})$ is s.p.i.m.f.. Before we discuss all possible Fitting subgroups of G, we describe all groups G that contain an irreducible quasisimple normal subgroup.

Lemma 4.11.5

(a) If G contains a normal subgroup $N \simeq {}_{\infty,2}[\mathrm{SL}_2(11)]_5$ then G is conjugate to $_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3$ or $_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\sqsupset} C_4$.

(b) If G contains a normal subgroup $N \simeq {}_{\infty,2}[\pm U_5(2)]_5$ then G is conjugate to $_{\infty,2}[\pm U_5(2)]_5 \circ C_3$, $_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$ or $_{\sqrt{-2}}[{}_{\infty,2}[\pm U_5(2)]_5{:}2]_{10}$.

Proof: In both cases, N is rationally irreducible and has $\mathcal{Q}_{\infty,2}$ as commuting algebra. By Remark 2.2.17, G contains a subgroup U conjugate to $\pm N:2$ or $\pm N \circ C_i$ with $i \in \{3,4\}$. The commuting algebra of U is $\mathbb{Q}(\sqrt{-d})$ with $d = 2,3,1$ respectively. One easily checks that ${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3$, ${}_{\infty,2}[\pm U_5(2)]_5 \circ C_3$ and ${}_{\sqrt{-2}}[{}_{\infty,2}[\pm U_5(2)]_5 : 2]_{10}$ are s.i.m.f.. Further ${}_{\infty,2}[\pm U_5(2)]_5 \circ C_4$ fixes only one lattice and is only contained in ${}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$. The group ${}_{\infty,2}[\mathrm{SL}_2(11)]_5 : 2$ fixes 4 lattices and is only contained in ${}_{\sqrt{-2}}[2.M_{12} : 2]_{10}$. Finally ${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_4$ fixes 2 lattices and is only contained in ${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\boxdot} C_4$. □

Lemma 4.11.6 *$O_{11}(G)$ is trivial.*

Proof: Suppose $O_{11}(G) \neq 1$. Then $O_{11}(G) \simeq C_{11}$ and it centralizer embeds into $\mathrm{GL}_2(\mathbb{Q}(\zeta_{11}))$. In particular $E(G) = 1$ and therefore $F(G)$ is self centralizing. If $F(G) \neq \pm O_{11}(G)$ then G contains an irreducible cyclic subgroup of order 33 or 44 and $\Pi(|G|) \subseteq \{2,3,5,11\}$. This contradicts Lemmas 4.11.3 and 4.11.4. So $F(G) \simeq C_{22}$ and $[G : F(G)] \in \{5,10\}$ by Lemma 2.2.1. The group $C_{22} : C_5$ has only one faithful rational irreducible representation. It is of degree 10 and has $\mathbb{Q}(\sqrt{-11})$ as commuting algebra. This again contradicts Lemma 2.2.1. □

Lemma 4.11.7 *If $O_5(G)$ is nontrivial then G is conjugate to ${}_{\zeta_{10}}[\pm 5_+^{1+2}.\mathrm{Sp}_2(5)]_5$ or ${}_{\zeta_{10}}[C_{10}]_1 \otimes A_5$.*

Proof: By Table 2.5.2, $O_5(G)$ is conjugate to C_{25}, C_5 or 5_+^{1+2}. The first case contradicts Lemma 4.11.2. The irreducible group $\mathcal{B}^o(5_+^{1+2}) = {}_{\zeta_{10}}[\pm 5_+^{1+2}.\mathrm{Sp}_2(5)]_5$ fixes 2 lattices and has $\mathbb{Q}(\zeta_{10})$ as commuting algebra. One easily checks that it is s.i.m.f.. So we may now assume that $O_5(G) = C_5$. Then $F(G) = \pm C_5$ and $E(G) \neq 1$ by Corollary 2.2.3. Thus $E(G)$ embeds into $\mathrm{GL}_5(\mathbb{Q}(\zeta_{10}))$. It follows from Table 2.5.1 that $E(G)$ is conjugate to Alt_6. Therefore G contains a normal subgroup $N := \mathcal{B}^o(F^*(G)) \simeq {}_{\zeta_{10}}[C_{10}]_1 \otimes A_5$. Since N fixes 4 lattices and has $\mathbb{Q}(\zeta_{10})$ as commuting algebra, one easily checks that N is s.i.m.f.. □

Lemma 4.11.8 *If $O_3(G) \neq 1$ then G is conjugate to one of*

$$ {}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1,\ {}_i[C_4]_1 \otimes [C_6 \times S_4(3).2]_{10},\ {}_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3, $$

$$ {}_{\infty,2}[\pm U_5(2)]_5 \circ C_3,\ {}_{\sqrt{-3}}[\pm U_4(2) \circ C_3]_{10},\ {}_{\sqrt{-3}}[\pm \mathrm{L}_2(11) \overset{2(3)}{\otimes} {}_{\sqrt{-3}}[C_6]_1]_{10},\ {}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \otimes A_2, $$

$$ {}_{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10} \text{ or } {}_{\sqrt{-3}}[C_6]_1 \otimes H \text{ with } H \in \{A_{10}, A_{10}^{(2)}, A_{10}^{(3)}\}. $$

Proof: If $O_3(G) \neq 1$ then $O_3(G) = C_3$. Thus $O_2(G) \in \{C_2, C_4, D_8, Q_8\}$ and $O_p(G) = 1$ for all $p > 3$. Let $N = \mathcal{B}^o(F^*(G))$.

- If $O_2(G) \simeq Q_8$ then $E(G)$ embeds into $\mathrm{GL}_5(\mathbb{Q}(\sqrt{-3}))$. By Table 2.5.1 $E(G)$ is conjugate to Alt_6 or $S_4(3)$.

  In the latter case $N \simeq {}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ fixes 6 lattices and has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra. One checks that it is s.i.m.f.. In the other case $N \simeq A_5 \otimes {}_{\sqrt{-3}}[\mathrm{SL}_2(3) \circ C_3]_2$ has the same endomorphism ring and fixes 18 lattices. One checks that it is only contained in ${}_{\sqrt{-3}}[S_4(3) \circ C_3]_5 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$.

- If $O_2(G) \simeq C_4$ or D_8 then $E(G)$ embeds into $\mathrm{GL}_5(\mathbb{Q}(\zeta_{12}))$. Thus again $E(G)$ is conjugate to A_5 or $S_4(3)$. The group $N_1 := A_5 \otimes C_{12}$ fixes 9 lattices, but using part (a) of Remark 2.2.8 one finds a normal critical lattice L. Let $F \in \mathcal{F}_{>0}(N_1)$, then $\mathrm{Aut}_{\mathbb{Q}(\zeta_{12})}(L,F) = {}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \otimes C_4$. So $E(G) \simeq S_4(3)$ and G must contain a normal subgroup conjugate to $N_2 := {}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \otimes C_4$. The group N_2 fixes 3 lattices but there exists a normal critical one, say L. Further there exists some $F \in \mathcal{F}_{>0}(N)$ that is integral on L such that $\det(L,F) = 2^{10}$ and $\Pi(|G|) = \{2,3,5\}$. The maximal totally real subfield K of $C := \mathrm{End}(\overline{N_2}) \simeq \mathbb{Q}(\zeta_{12})$ is isomorphic to $\mathbb{Q}(\sqrt{3})$. Since $\mathrm{End}_{\mathbb{Z}N_2}(L)$ is the maximal order in C and since $\mathrm{Nr}_{C/K}(\mathbb{Z}_C^*) = \mathbb{Z}_{K,>0}^*$ we may restrict ourselves to one class of totally positive units. Thus Table 2.5.4 shows that there are 4 candidates to check:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-3})$
F	${}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$	${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$
$\mathfrak{p}_2^{-1}\mathfrak{p}_3 F$	${}_i[C_4]_1 \otimes [C_6 \times S_4(3).2]_{10}$	$\lneq {}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5^2$

  Only ${}_i[C_4]_1 \otimes [C_6 \times S_4(3).2]_{10}$ has the correct Fitting subgroup.

- If $O_2(G) = \pm 1$, then $E(G)$ is isomorphic to Alt_6, $S_4(3) = U_4(2)$ (2 representations each), $\mathrm{Alt}_{11}, \mathrm{L}_2(11)$ (3 representations), $\mathrm{SL}_2(11)$ or $U_5(2)$.

  If $E(G) \simeq \mathrm{SL}_2(11)$ or $U_5(2)$ then $G = E(G) \circ C_3$ by Lemma 4.11.5.

  If $E(G) \simeq \mathrm{Alt}_{11}$ then $N \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}$ fixes 2 lattices and is already s.i.m.f..

  If $E(G) \simeq U_4(2)$ with character χ_{10ab} then $N \simeq {}_{\sqrt{-3}}[\pm U_4(2) \circ C_3]_{10}$ is already irreducible and fixes 2 lattices. One checks that it is s.i.m.f..

  If $E(G) \simeq U_4(2)$ with character $2\chi_{5ab}$ then N is conjugate to ${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5$ and $[G : N] = 4$ by Lemma 2.2.1. In particular, G must contain a normal subgroup conjugate to $S_4(3) \otimes A_2$ or $\pm S_4(3) \underset{\sqrt{-3}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$. These two groups fix 15 and 5 lattices respectively. They are only contained in ${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3)]_5^2$.

  If $E(G) \simeq \mathrm{Alt}_6$ with character $4\chi_{5a}$, then $N \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes A_5 \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes S_6$. Since the exceptional outer automorphism of Alt_6 interchanges its 5 dimensional irreducible characters, we have $[G : N] \leq 2$. This contradicts Lemma 2.2.1.

  If $E(G) \simeq \mathrm{Alt}_6$ with character $2\chi_{10}$, then $N \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes [\pm S_6]_{10}$ is already irreducible and fixes 12 lattices. One checks that it is only contained in ${}_{\sqrt{-3}}[\pm U_4(2) \circ C_3]_{10}$.

  If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{10a}$, then $N \simeq {}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(2)}$ has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra and fixes 2 lattices. One checks that it is s.i.m.f..

If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{10b}$, then $N = F^*(G)$ has $\mathbb{Q}(\sqrt{-3})$ as commuting algebra and fixes 6 lattices. One checks that it is only contained in ${}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}$ and ${}_{\sqrt{-3}}[\pm\mathrm{L}_2(11) \overset{2(3)}{\otimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$.
If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{5ab}$, then $F^*(G)$ contains an irreducible cyclic subgroup of order 33 and $|G| = \{2,3,5,11\}$. It follows from Lemma 4.11.4 that G is conjugate to ${}_{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm\mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$, ${}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(3)}$ or ${}_{\sqrt{-11}}[\pm\mathrm{L}_2(11)]_5 \otimes A_2$. □

Lemma 4.11.9 *If $F(G) \simeq D_8 \otimes C_4$ then G is conjugate to ${}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_5$.*

Proof: Since $E(G)$ embeds into $\mathrm{GL}_5(\mathbb{Q}(i))$, it is conjugate to Alt_6. In particular G contains the normal subgroup $N := \mathcal{B}^o(F^*(G)) \simeq {}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_5$. The group N fixes 6 lattices and is easily checked to be s.i.m.f.. □

Lemma 4.11.10 *$F(G) \not\simeq D_8 \otimes D_8$.*

Proof: By Table 2.5.1, $E(G) \simeq \mathrm{Alt}_6$. But then $F^*(G)$ is already irreducible and fixes no skewsymmetric form. □

Lemma 4.11.11 *If $F(G) \simeq Q_8$, then G is conjugate to ${}_{\sqrt{-11}}[\mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ or ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_5$.*

Proof: Since $E(G)$ embeds into $\mathrm{GL}_5(\mathcal{Q}_{\infty,2})$, it is conjugate to Alt_6 or $\mathrm{L}_2(11)$ by Table 2.5.1 (note that $O_3(\mathcal{B}^o(S_4(3))) = C_3$).
In the latter case $\mathcal{B}^o(F^*(G)) \simeq {}_{\sqrt{-11}}[\mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ is already s.i.m.f.. In the first case $N := \mathcal{B}^o(F^*(G)) \simeq {}_{\infty,2}[\mathrm{SL}_2(3)]_1 \otimes A_5$ is irreducible and has $\mathcal{Q}_{\infty,2}$ as commuting algebra. So by Remark 2.2.17, G must contain a subgroup conjugate to $N.2$ or $N \circ C_i$ with $i \in \{3,4\}$. These groups fix $12, 18$ and 6 lattices respectively. One checks that $N:2 \simeq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_5$ is already s.i.m.f.. The other two groups are properly contained in ${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$ or ${}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_5$ respectively. □

Lemma 4.11.12 *$F(G)$ is not conjugate to C_8, D_{16} or QD_{16}.*

Proof: In any case $F(G)$ would contain a cyclic characteristic subgroup of order 8. Thus $E(G)$ embeds into $\mathrm{GL}_5(\mathbb{Q}(\zeta_8))$ and must be conjugate to Alt_6. So G contains a normal subgroup $N \simeq C_8 \otimes A_5$. Since the exceptional outer automorphism of Alt_6 interchanges its absolutely irreducible 5-dimensional characters, we have $G/N \le \mathrm{Out}(C_8) \simeq C_2 \times C_2$. In particular $\Pi(|G|) = \{2,3,5\}$. The group N fixes 20 lattices. By looking at the indices of these lattices in the standard lattice, one sees that $L = \mathbb{Z}^{1\times 20}$ is normal critical and there exists some integral $F \in \mathcal{F}_{>0}(N)$ such that $\det(L, F) = 2^4{\cdot}3^4$. The maximal totally real subfield of $\mathbb{Q}(\zeta_8)$ is isomorphic to $\mathbb{Q}(\sqrt{2})$. So Table 2.5.4 shows that there are 4 groups to check:

form	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{-2})$
F	$({}_i[C_4]_1 \otimes A_5)^2$	$\lneq {}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_5$
$\mathfrak{p}_2 F$	${}_i[(D_8 \otimes C_4).S_3]_2 \otimes A_5$	${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_5$

None of these has the correct Fitting subgroup. □

Lemma 4.11.13

(a) If $F(G) \simeq C_4$, then G is conjugate to ${}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$, ${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\sqsupset} C_4$, ${}_i[[\pm S_6]_{10} \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_{10}$ or ${}_i[C_4]_1 \otimes H$ with $H \in \{A_{10}, A_{10}^{(2)}, A_{10}^{(3)}\}$.

(b) $F(G)$ is not conjugate to D_8.

Proof: In both cases, $F(G)$ contains a characteristic normal cyclic subgroup U of order 4. Thus $E(G)$ embeds into $\mathrm{GL}_{10}(\mathbb{Q}(i))$. By Table 2.5.1, $E(G)$ is conjugate to Alt_6 (2 representations), $\mathrm{L}_2(11)$ (3 representations), Alt_{11}, $\mathrm{SL}_2(11)$ or $U_5(2)$ (note again that $O_3(\mathcal{B}^o(S_4(3))) = C_3$). The groups $U_5(2)$ and $\mathrm{SL}_2(11)$ have been handled in Lemma 4.11.5. Let $N = \mathcal{B}^o(E(G)U)$.

- If $E(G) \simeq \mathrm{Alt}_{11}$, then $N \simeq {}_i[C_4]_1 \otimes A_{10}$ fixes 2 lattices and has $\mathbb{Q}(i)$ as commuting algebra. It is already s.i.m.f..

- If $E(G) \simeq \mathrm{Alt}_6$ with character $2\chi_{10}$, then N has $\mathbb{Q}(i)$ as commuting algebra and fixes 22 lattices. It is only contained in ${}_i[[\pm S_6]_{10} \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_{10}$.

- Suppose $E(G) \simeq \mathrm{Alt}_6$ with character $4\chi_{5a}$. Since the exceptional outer automorphism of Alt_6 interchanges its absolutely irreducible 5-dimensional characters and $\mathcal{B}^o(E(G)) \simeq A_5$, we get that $G/N \leq \mathrm{Out}(F(G))$. In particular $F(G) = D_8$ by Lemma 2.2.1. Let $\triangle(N)$ be a summand of the natural representation of N. Then $\triangle(N) < \mathrm{GL}_{10}(\mathbb{Q})$ is absolutely irreducible and has $\mathbb{Q}$ as commuting algebra. So G embeds into $\mathrm{GL}_{10}(K)$ for some imaginary quadratic number field and $\triangle(N)$ is an irreducible subgroup of $G < \mathrm{GL}_{10}(K)$. One easily checks that only the outer automorphism acting on D_8 and fixing A_5 can be realized in $\mathrm{GL}_{10}(\mathbb{C})$. Only the non-split extension is symplectic, since it can be realized over $\mathbb{Q}(\sqrt{-2})$. It fixes 40 lattices and is only contained in ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_5$.

- If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{10a}$, then $N \simeq {}_i[C_4]_1 \otimes A_{10}^{(2)}$ has $\mathbb{Q}(i)$ as commuting algebra and fixes 2 lattices. One checks that it is s.i.m.f..

- If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{10b}$, then $N = F^*(G)$ has $\mathbb{Q}(i)$ as commuting algebra and fixes 4 lattices. It is only contained in ${}_i[C_4]_1 \otimes A_{10}$.

- If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{5ab}$, then $N = F^*(G)$ contains an irreducible cyclic subgroup of order 44 and $\Pi(|G|) = \{2, 3, 5, 11\}$. Thus G is conjugate to $_i[C_4]_1 \otimes A_{10}^{(3)}$ by Lemma 4.11.3. □

Lemma 4.11.14 *If $F(G) = \pm I_{20}$, then G is conjugate to one of*

$$_{\sqrt{-19}}[\mathrm{SL}_2(19)]_{10},\ _{\sqrt{-2}}[_{\infty,2}[\pm U_5(2)]:2]_{10},\ _{\sqrt{-2}}[2.M_{12}:2]_{10},\ _{\sqrt{-7}}[2.M_{22}:2]_{10}\ .$$

Proof: By Table 2.5.1, $E(G)$ is isomorphic to Alt_6 (2 representations), $\mathrm{L}_2(11)$ (3 representations), Alt_{11}, $\mathrm{SL}_2(11)$, Alt_7, $2.M_{12}$, $2.M_{22}$, $U_5(2)$, $2.L_3(4)$ or $\mathrm{SL}_2(19)$. Note that $O_3(\mathcal{B}^o(U_4(2))) = C_3$ for χ_{5ab} and χ_{10ab}.
The group $_{\sqrt{-19}}[\mathrm{SL}_2(19)]_{10}$ is s.i.m.f. by Theorem 3.2.1. The cases $E(G) \simeq U_5(2)$ or $\mathrm{SL}_2(11)$ are handled in Lemma 4.11.5. Let $N = \mathcal{B}^o(E(G))$.

- If $E(G) \simeq 2.M_{12}$ or $2.M_{22}$ then $N = \pm E(G) : 2$ fixes only one lattice and has $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-7})$ as commuting algebra. One checks that N is s.i.m.f..
- If $E(G) \simeq \mathrm{Alt}_7$, then $N \simeq \pm\mathrm{Alt}_7$ fixes 5 lattices and has $\mathbb{Q}(\sqrt{-7})$ as commuting algebra. One checks that it is only contained in $_{\sqrt{-7}}[2.M_{22}:2]_{10}$.
- If $E(G) \simeq 2.L_3(4)$, then $N \simeq 2.L_3(4).2_2$ fixes 5 lattices and has $\mathbb{Q}(\sqrt{-7})$ as commuting algebra. One checks that it is only contained in $_{\sqrt{-7}}[2.M_{22}:2]_{10}$.
- If $E(G) \simeq \mathrm{Alt}_{11}$, then $G/\pm\mathrm{Alt}_{11} \le \mathrm{Out}(\mathrm{Alt}_{11}) \simeq C_2$ and $N \simeq A_{10} \simeq \pm S_{11}$. Thus $G = N$ is reducible.
- If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{5ab}$ then $N = F(G)$. Hence $G/N \le \mathrm{Out}(N) \simeq C_2$ contradicts Lemma 2.2.1.
- If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{10a}$, then $N \simeq \pm\mathrm{L}_2(11):2$. Hence $G/\pm E(G) \le \mathrm{Out}(\mathrm{L}_2(11)) \simeq C_2$ implies that $G = N$ is reducible.
- If $E(G) \simeq \mathrm{L}_2(11)$ with character $2\chi_{10b}$, then $N = F^*(G)$ and again $[G : N] = 2$. Since χ_{10b} as absolutely irreducible, G embeds into $\mathrm{GL}_{10}(K)$ for some imaginary quadratic number field K such that $N < G$ is irreducible. One easily constructs $N\cdot 2 < \mathrm{GL}_{10}(\mathbb{Q}(\sqrt{-3}))$. But $N\cdot 2$ cannot be s.i.m.f. since $\mathbb{Q}(\sqrt{-3})$ contains a third root of unity.
- If $E(G) \simeq \mathrm{Alt}_6$ with character $4\chi_{5a}$ then $N \simeq A_5 \simeq \pm S_6$. Since the exceptional outer automorphism of Alt_6 interchanges χ_{5a} and χ_{5b}, we see that $G = N$ is reducible.
- If $E(G) \simeq \mathrm{Alt}_6$ with character $2\chi_{10}$ then $N \simeq \pm S_6$ and $[G : N] \le 2$. Again $G < \mathrm{GL}_{10}(K)$ for some imaginary quadratic number field K such that $N < G$ is irreducible. One easily constructs $N:2 < \mathrm{GL}_{10}(\mathbb{Q}(\sqrt{-2}))$. But $N:2$ is only contained in $_{\sqrt{-2}}[2.M_{12}:2]_{10}$. □

4.12 Dimension 22

Theorem 4.12.1 *The s.i.m.f. subgroups G of* $\mathrm{Sp}_{22}(\mathbb{Q})$ *are*

#	G	$\lvert G\rvert$	$\lvert\mathcal{Z}(G)\rvert$	L_{min}	*r.i.m.f. supergroups*
$[2,1,11]$	${}_i[C_4]_1^{11}$	$2^{30}\cdot3^4\cdot5^2\cdot7\cdot11$	3	$[1,1,44]$	B_{22}
1	${}_i[C_4]_1\otimes A_{11}$	$2^{12}\cdot3^5\cdot5^2\cdot7\cdot11$	10	$[3^2,2,264]$	A_{11}^2
$[2,2,11]$	${}_{\sqrt{-3}}[C_6]_1^{11}$	$2^{19}\cdot3^15\cdot5^2\cdot7\cdot11$	1	$[3^{11},2,66]$	A_2^{11}
2	${}_{\sqrt{-3}}[C_6]_1\otimes A_{11}$	$2^{11}\cdot3^6\cdot5^2\cdot7\cdot11$	9	$[3^{11},4,396]$	$A_{11}\otimes A_2$
3	${}_{\sqrt{-3}}[\pm U_5(2)\circ C_3]_{11}$	$2^{11}\cdot3^6\cdot5\cdot11$	4	$[2^2\cdot3,4,49896]$	S
4	${}_{\sqrt{-23}}[\pm\mathrm{L}_2(23)]_{11}$	$2^4\cdot3\cdot11\cdot23$	3	$[23^{11},12,1012]$ $[23^{11},12,506]$	$A_{22}^{(6)}$

where $S := [\pm PSU_6(2).S_3]_{22} < \mathrm{GL}_{22}(\mathbb{Q})$.

Proof: By explicit calculations, one verifies that the above table is correct and yields s.i.m.f. groups. Further the r.i.m.f. supergroups are easily constructed since all s.i.m.f. groups are uniform. It remains to show the completeness of the classification.

The group ${}_{\sqrt{-3}}[C_6]_1^{11}$ is s.i.m.f. by Lemma 2.1.21 and ${}_i[C_4]_1^{11}$ fixes only three lattices. One checks that it is s.i.m.f.. So we may now suppose that G is s.p.i.m.f..

It follows from Corollary 4.1.2 that $E(G)$ is not trivial. Thus by Table 2.5.1, $E(G)$ is conjugate to Alt_{12}, M_{11}, $U_5(2)$ or $\mathrm{L}_2(23)$. In the latter two cases, $E(G)$ is irreducible and $\mathcal{B}^o(E(G))$ is already s.i.m.f..

If $E(G)\simeq\mathrm{Alt}_{12}$ then G contains a normal subgroup $B := \mathcal{B}^o(E(G))\simeq A_{11}\simeq\pm S_{12}$. Clearly B is not self-centralizing. Thus G contains an irreducible subgroup ${}_i[C_4]_1\otimes A_{11}$ or ${}_{\sqrt{-3}}[C_6]_1\otimes A_{11}$. Both groups are s.i.m.f..

If $E(G)\simeq M_{11}$ then $\mathrm{Out}(E(G))$ is trivial. Thus G contains an irreducible subgroup conjugate to ${}_{\sqrt{-3}}[C_6]_1\otimes M_{11}$ or ${}_i[C_4]_1\otimes M_{11}$. The first group fixes 18 lattices and is only contained in ${}_{\sqrt{-3}}[C_6]_1\otimes A_{11}$ and ${}_{\sqrt{-3}}[C_6]_1^{11}$. The second group fixes 15 lattices and is only contained in ${}_i[C_4]_1\otimes A_{11}$ and ${}_i[C_4]_1^{11}$. □

Appendix A

Invariant Forms

For each conjugacy class of s.p.i.m.f. matrix groups in $\mathrm{GL}_{2n}(\mathbb{Q})$ where $1 \leq n \leq 11$ we give a symmetric positive definite form F and a skewsymmetric form S such that

$$\mathrm{Aut}(\mathbb{Z}^{1\times 2n}, \{F,S\}) = \{g \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid gFg^{\mathrm{tr}} = F \text{ and } gSg^{\mathrm{tr}} = S\}$$

is a representative of that class.

Further the pairs (F,S) satisfy the following properties:

(a) $\det(F) = \min\{\det(L,F') \mid (L,F') \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G) \text{ is integral}\}$ where $G := \mathrm{Aut}(\mathbb{Z}^{1\times 2n}, \{F,S\})$.

(b) $S \cdot F^{-1}$ generates the commuting algebra of $\mathrm{Aut}(\mathbb{Z}^{1\times 2n}, \{F,S\})$ and its minimal polynomial $\mu(S\cdot F^{-1}, X)$ is one of

- $X^2 + d$ for some squarefree $d \in \mathbb{N}$.
- $\mu(\zeta_k - \zeta_k^{-1}, X)$ for some even $k \in \mathbb{Z}_{\geq 6}$.
- $\mu(\zeta_{26} + \zeta_{26}^3 + \zeta_{26}^9, X)$.
- $\mu(\sqrt{k}\cdot(\zeta_\ell - \zeta_\ell^{-1}), X)$ where $(k,\ell) \in \{(2,10),\ (3,10),\ (3,16)\}$.
- $\mu(i + \sqrt{-3} + \sqrt{-5}, X)$.

By Algorithm 2.3.3, these pairs (F,S) give an easy way to recover the conjugacy class of any given s.p.i.m.f. subgroup of $\mathrm{GL}_{2n}(\mathbb{Q})$ for $1 \leq n \leq 11$.

In some cases, tensoring such a pair (F,S) with a gram matrix g of some root lattice yields another s.p.i.m.f. automorphism group H. If the pair $(F \otimes g, S \otimes g)$ satisfies property (a) from above, then H is omitted in the list below, since these forms can easily be reconstructed from the name of H.

For example, tensoring the forms of ${}_i[C_4]_1$ with any gram matrix of A_2 yields a pair of forms that satisfies the above properties. Thus the s.p.i.m.f. matrix group ${}_i[C_4]_1 \otimes A_2$ is omitted in the list below.

Similarly, tensoring the forms of ${}_i[C_4]_1$ with any gram matrix of A_5 yields a pair of forms whose automorphism group is conjugate to ${}_i[C_4]_1 \otimes A_5$. But these forms do not satisfy the above properties. Thus the list below contains some other forms for ${}_i[C_4]_1 \otimes A_5$.

The pairs (F, S) are given as a single matrix M. The upper triangular part of M is the upper triangular part of S. The other entries of M are the corresponding entries of F.

For example, $\begin{smallmatrix} 2 & 1 & -3 & 3 \\ -1 & 2 & 1 & -3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{smallmatrix}$ describes the pair $\left(\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -3 & 3 \\ -1 & 0 & 1 & -3 \\ 3 & -1 & 0 & 1 \\ -3 & 3 & -1 & 0 \end{pmatrix}\right)$.

A.1 Dimension 2

1: $_i[C_4]_1$ $\quad$ 2: $_{\sqrt{-3}}[C_6]_1$

$$\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \qquad \begin{smallmatrix} 2 & 3 \\ -1 & 2 \end{smallmatrix}$$

A.2 Dimension 4

1: $_i[(D_8 \otimes C_4).S_3]_2$ $\quad$ 3: $_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2$ $\quad$ 4: $_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3$ $\quad$ 5: $_{\zeta_{10}}[C_{10}]_1$

$$\begin{smallmatrix} 2 & 1 & 0 & -2 \\ -1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{smallmatrix} \qquad \begin{smallmatrix} 2 & 0 & 2 & -2 \\ -1 & 2 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{smallmatrix} \qquad \begin{smallmatrix} 2 & -1 & 2 & -2 \\ -1 & 2 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & -1 & 0 & 2 \end{smallmatrix} \qquad \begin{smallmatrix} 2 & 1 & -3 & 3 \\ -1 & 2 & 1 & -3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{smallmatrix}$$

A.3 Dimension 6

1: $_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3$ $\quad$ 2: $_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3$

$$\begin{smallmatrix} 2 & 3 & -2 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & -2 & 1 \\ 0 & 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{smallmatrix} \qquad \begin{smallmatrix} 4 & 7 & 0 & -7 & 7 & 0 \\ -1 & 4 & 7 & 0 & -7 & 7 \\ -2 & -1 & 4 & 7 & 0 & -7 \\ 1 & -2 & -1 & 4 & 7 & 0 \\ 1 & 1 & -2 & -1 & 4 & 7 \\ -2 & 1 & 1 & -2 & -1 & 4 \end{smallmatrix}$$

A.4 Dimension 8

1: ${}_{i}[(2_{+}^{1+4} \otimes C_4).S_6]_4$

2		0	0	0	0	1	-2	0
-1	2		0	0	0	0	1	-1
0	-1	2		0	-1	0	0	2
0	0	-1	2		0	1	0	-1
0	0	0	-1	2		-1	0	0
0	0	0	0	-1	2		0	0
0	0	0	0	0	-1	2		0
0	0	0	0	-1	0	0	2	

3: ${}_{\sqrt{-2}}[{}_{\infty,2}[2_{-}^{1+4}.\mathrm{Alt}_5]_2 : 2]_4$

2		0	0	-2	2	0	-1	0
-1	2		0	1	0	0	0	-2
0	-1	2		0	0	0	1	0
0	0	-1	2		0	-1	0	1
0	0	0	-1	2		0	1	0
0	0	0	0	-1	2		-2	0
0	0	0	0	0	-1	2		0
0	0	0	0	-1	0	0	2	

4: ${}_{\sqrt{-2}}[F_4 : 2]_4$

2		0	0	0	0	0	2	-2
-1	2		0	0	0	2	-2	0
0	-1	2		0	2	-2	0	0
0	-1	0	2		-2	0	0	0
0	0	0	0	2		0	0	0
0	0	0	0	-1	2		0	0
0	0	0	0	0	-1	2		0
0	0	0	0	0	-1	0	2	

5: ${}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4$

2		3	-2	0	0	0	0	0
-1	2		1	0	0	0	0	0
0	-1	2		-1	0	0	0	2
0	0	-1	2		-1	0	0	0
0	0	0	-1	2		-1	2	-1
0	0	0	0	-1	2		-3	0
0	0	0	0	0	-1	2		0
0	0	0	0	-1	0	0	2	

7: ${}_{\infty,2}[\mathrm{SL}_2(5) : 2]_2 \circ C_3$

4		2	3	-3	-1	0	-2	2
-2	4		-1	0	-1	-1	3	-1
1	-1	4		1	3	6	2	0
1	0	-1	4		-1	-1	2	1
-1	-1	1	1	4		3	3	-2
-2	1	-2	1	1	4		2	0
0	1	-2	2	-1	2	4		3
2	-1	0	1	-2	0	1	4	

9: ${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_4]_4$

2		0	0	0	3	-2	-1	-3
-1	2		0	0	-1	1	3	-1
0	-1	2		0	0	-3	1	3
0	0	-1	2		1	1	-2	-1
0	0	0	-1	2		1	0	0
0	0	0	0	-1	2		0	0
0	0	0	0	0	-1	2		0
0	0	0	0	-1	0	0	2	

10: ${}_{\sqrt{-5}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_4]_4$

4		5	0	0	0	0	0	-5
-2	4		0	0	5	5	0	0
1	-1	4		0	0	0	-5	0
1	0	-1	4		0	5	5	0
-1	-1	1	1	4		0	0	0
-2	1	-2	1	1	4		0	0
0	1	-2	2	-1	2	4		0
2	-1	0	1	-2	0	1	4	

11: ${}_{\sqrt{-5}}[C_{20} : C_4]_4$

2		0	0	0	-4	1	1	1
-1	2		0	0	1	1	-4	1
0	-1	2		0	1	1	1	1
0	0	-1	2		1	-4	1	1
0	0	0	0	2		0	0	0
0	0	0	0	-1	2		0	0
0	0	0	0	0	-1	2		0
0	0	0	0	0	0	-1	2	

12: ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2-}{\square} C_3]_4$

2		3	-1	0	-1	0	0	0
-1	2		0	1	-2	2	-1	2
0	-1	2		1	1	-2	3	-3
0	0	-1	2		1	-2	0	2
0	0	0	-1	2		1	-1	1
0	0	0	0	-1	2		-2	-1
0	0	0	0	0	-1	2		-2
0	0	0	0	-1	0	0	2	

13: ${}_{\sqrt{-6}}[{}_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2+}{\square} C_3]_4$

2		0	0	0	0	2	2	-2
-1	2		0	0	-2	0	0	-2
0	-1	2		0	-2	0	0	4
0	-1	0	2		2	2	-4	0
0	0	0	0	2		0	0	0
0	0	0	0	-1	2		0	0
0	0	0	0	0	-1	2		0
0	0	0	0	0	-1	0	2	

14: ${}_{\sqrt{-6}}[D_{16} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$

2		0	0	0	0	3	3	0
-1	2		0	0	-3	0	0	-3
0	0	2		0	-3	0	0	3
0	0	-1	2		0	3	-3	0
0	0	0	0	2		0	0	0
0	0	0	0	-1	2		0	0
0	0	0	0	0	0	2		0
0	0	0	0	0	0	-1	2	

15: ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$

2		-1	0	-2	0	0	0	4
-1	2		3	0	-2	0	0	0
0	-1	2		1	0	2	0	-2
0	0	-1	2		3	-4	0	0
0	0	0	-1	2		3	0	-1
0	0	0	0	-1	2		1	2
0	0	0	0	0	-1	2		0
0	0	0	0	-1	0	0	2	

16: ${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2-}{\square} C_3]_4$

2		1	0	0	2	0	-6	0
-1	2		-5	4	0	-2	4	0
0	-1	2		-3	0	0	0	0
0	0	-1	2		3	0	0	-6
0	0	0	-1	2		1	-2	5
0	0	0	0	-1	2		3	0
0	0	0	0	0	-1	2		0
0	0	0	0	-1	0	0	2	

17: ${}_{\sqrt{-15}}[{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \overset{2+}{\square} C_3]_4$

4		0	-5	5	5	10	10	0
-2	4		-5	0	-5	-5	-5	5
1	-1	4		-5	-5	0	0	0
1	0	-1	4		5	5	0	5
-1	-1	1	1	4		5	-5	0
-2	1	-2	1	1	4		0	0
0	1	-2	2	-1	2	4		-5
2	-1	0	1	-2	0	1	4	

18: ${}_{\sqrt{-15}}[C_{30} : C_4]_4$

4		0	0	0	6	-9	6	6
-2	4		0	0	-9	6	-9	6
0	-2	4		0	6	-9	6	-9
0	0	-2	4		6	6	-9	6
-2	1	0	0	4		0	0	0
1	-2	1	0	-2	4		0	0
0	1	-2	1	0	-2	4		0
0	0	1	-2	0	0	-2	4	

19: ${}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_5$

2		-2	1	0	0	0	0	0
-1	2		-2	1	0	0	0	0
0	-1	2		-1	0	0	0	0
0	0	-1	2		0	-1	-1	2
0	0	0	-1	2		2	-1	-2
0	0	0	0	-1	2		2	1
0	0	0	0	0	-1	2		-1
0	0	0	0	-1	0	0	2	

21: ${}_{\zeta_{16}-\zeta_{16}^{-1}}[QD_{32}]_2$

1		0	0	1	0	1	0	0
0	1		0	0	1	0	1	0
0	0	1		0	0	1	0	1
0	0	0	1		0	0	1	0
0	0	0	0	1		0	0	1
0	0	0	0	0	1		0	0
0	0	0	0	0	0	1		0
0	0	0	0	0	0	0	1	

A.5 Dimension 10

1: ${}_i[C_4]_1 \otimes A_5$

2		0	0	0	0	0	1	−1	1	−1
1	2		0	0	0	0	2	−1	1	0
−1	0	2		0	0	−1	0	0	−1	1
−1	0	1	2		0	0	0	0	−1	0
1	0	−1	−1	2		−1	0	1	0	0
0	0	0	0	0	2		0	0	−1	−1
0	0	0	0	0	0	2		0	−1	−1
0	0	0	0	0	−1	−1	2		1	1
1	1	−1	−1	0	1	1	−1	3		−2
1	1	−1	−1	0	−1	0	1	1	3	

2: ${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5$

4		0	0	0	−3	0	−3	−3	0	0
−2	4		0	0	3	0	0	3	−3	0
−2	2	4		0	3	0	0	3	0	−3
−2	2	2	4		0	3	0	3	0	0
−1	−1	1	0	4		0	0	0	0	−3
2	0	−2	−1	−2	4		−3	0	0	3
−1	2	0	2	0	1	4		0	−3	0
−1	−1	1	−1	2	−2	−2	4		0	−3
−2	1	2	0	0	−2	−1	2	4		0
−2	2	1	0	−1	−1	0	1	2	4	

3: ${}_{\sqrt{-11}}[\mathrm{L}_2(11)]_5$

6		0	0	0	−11	−11	0	0	0	0
2	6		−11	11	0	−11	11	0	0	0
−2	−3	6		0	0	11	0	0	0	11
−2	1	−2	6		0	0	0	11	0	−11
−3	−2	2	2	6		0	0	11	11	0
−3	1	1	0	2	6		11	0	0	0
2	3	−2	2	0	−1	6		11	11	0
2	−2	2	−1	−1	−2	1	6		0	0
2	−2	0	0	1	−2	−1	2	6		−11
0	2	1	1	2	0	2	0	1	6	

A.6 Dimension 12

2: ${}_{\infty,3}[\pm U_3(3)]_3 \circ C_4$

4		−1	−2	−2	0	0	0	0	0	0	0	1
0	4		0	0	0	1	−1	0	0	−1	−1	0
1	0	4		−1	1	0	0	0	0	−1	0	−1
−2	2	1	4		0	0	0	0	0	−1	−2	−1
0	1	−2	−1	4		−1	0	0	−2	2	1	0
−2	1	−2	1	2	4		0	0	0	0	−1	1
0	−2	0	−2	0	−1	4		1	−1	2	2	−1
1	−2	2	−1	−2	−2	2	4		1	0	1	−1
−2	0	0	2	0	2	−2	−1	4		−2	−2	1
1	0	0	0	−2	−2	0	1	−2	4		1	0
0	2	1	2	−1	−1	−2	0	0	2	4		0
−2	0	−2	0	2	2	0	−1	2	−2	−1	4	

4: ${}_i[{}_{\sqrt{-3}}[\pm 3_+^{1+2}{:}\mathrm{SL}_2(3)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$

4		0	1	1	−2	1	−1	−1	2	1	0	0
2	4		1	0	2	2	1	0	1	1	1	0
−1	1	4		0	2	2	4	0	−1	0	1	0
−1	0	0	4		1	−1	0	1	0	1	0	1
0	0	0	−1	4		−1	0	1	1	0	1	0
−1	0	2	1	1	4		2	1	−2	1	1	−1
−1	−1	0	0	2	2	4		0	−1	0	1	0
1	0	0	1	1	1	0	4		1	−1	−2	−1
2	1	1	0	−1	0	−1	1	4		0	−1	−1
−1	−1	0	1	0	1	0	1	0	4		0	0
0	−1	−1	2	1	1	1	2	1	2	4		−1
0	0	0	−1	0	−1	0	−1	1	2	1	4	

5: ${}_i[\mathrm{L}_2(7) \overset{2(2)}{\otimes} {}_i[C_4]_1]_6$

4		1	0	0	−1	−2	1	0	1	−2	−1	−1
−1	4		1	0	0	1	1	0	1	0	2	0
−2	1	4		−1	0	4	−1	1	0	0	0	1
0	−2	1	4		0	1	0	0	1	0	0	1
1	2	0	−2	4		0	2	2	0	0	1	1
0	−1	0	−1	0	4		−1	−1	0	0	−2	1
1	−1	1	2	0	−1	4		2	1	−1	0	1
0	0	1	0	0	1	0	4		−1	−2	0	−1
−1	1	0	−1	0	0	1	−1	4		0	−1	0
2	0	0	0	2	0	1	0	0	4		−1	0
−1	0	2	2	−1	0	2	0	1	1	4		1
1	0	−1	−1	1	1	1	1	2	2	1	4	

6: ${}_i[C_4]_1 \otimes A_6^{(2)}$

4		0	0	0	0	0	4	−1	−2	1	1	−2
−1	4		0	0	0	0	−1	4	−1	−2	1	1
−2	−1	4		0	0	0	−2	−1	4	−1	−2	1
1	−2	−1	4		0	0	1	−2	−1	4	−1	−2
1	1	−2	−1	4		0	1	1	−2	−1	4	−1
−2	1	1	−2	−1	4		−2	1	1	−2	−1	4
0	0	0	0	0	0	4		0	0	0	0	0
0	0	0	0	0	0	−1	4		0	0	0	0
0	0	0	0	0	0	−2	−1	4		0	0	0
0	0	0	0	0	0	1	−2	−1	4		0	0
0	0	0	0	0	0	1	1	−2	−1	4		0
0	0	0	0	0	0	−2	1	1	−2	−1	4	

7: ${}_i[{}_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_6$

8		0	0	−4	2	1	−1	3	4	−4	−4	0
−2	8		0	2	3	4	1	−4	−1	2	1	−3
−4	4	8		1	−1	1	1	−3	0	1	4	−4
4	−2	−1	8		0	−2	−1	2	4	−8	0	1
−2	−3	1	−2	8		−3	0	2	−2	2	0	−1
−3	4	3	0	1	8		0	−4	−3	0	1	−2
1	3	3	3	−4	2	8		−1	3	−3	1	−2
3	−4	−3	0	0	−4	−3	8		0	0	−2	1
4	−1	0	0	0	−1	1	4	8		0	0	0
−4	2	1	0	0	2	1	−2	−4	8		4	3
4	−1	−4	4	−2	−1	−1	0	0	0	8		4
4	−1	−4	3	−1	−2	−2	3	0	−1	4	8	

8: ${}_{\sqrt{-2}}[\pm \mathrm{L}_2(7) \cdot 2]_6$

2		0	0	0	0	0	0	2	−1	0	1	−1
−1	2		0	0	0	0	−1	0	1	−1	−1	0
0	−1	2		0	0	0	2	−1	0	−1	1	1
0	0	−1	2		0	0	−2	0	1	0	1	−1
0	0	0	−1	2		0	1	0	−2	1	0	−1
0	0	0	0	−1	2		1	−1	2	0	−1	0
0	0	0	0	0	0	2		0	0	0	0	0
0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	−1	2	

9: ${}_{\sqrt{-2}}[{}_{\infty,2}[\mathrm{SL}_2(5)]_3 : 2]_6$

3		1	2	0	−2	−1	1	1	2	−1	0	0
−1	3		−1	0	0	−3	0	1	1	2	−1	1
1	1	3		1	−2	−1	−2	2	0	0	2	−1
0	−1	0	3		2	2	1	1	0	−1	2	1
1	−1	−1	−1	3		0	0	0	1	−1	−1	1
1	−1	−1	1	1	4		3	0	1	−1	−1	1
1	1	1	−1	1	−1	4		1	1	2	−1	1
0	1	1	0	0	−1	0	3		1	1	1	1
1	−1	0	0	0	−1	1	0	3		0	0	−1
−1	0	−1	−1	1	−1	1	−1	0	3		0	−1
0	−1	−1	1	0	2	−2	0	0	−1	3		−1
1	0	1	−1	1	−1	2	1	1	0	−1	3	

10: ${}_{\sqrt{-3}}[6.U_4(3).2]_6$

4		0	3	0	0	0	0	3	0	−3	0	0
0	4		0	0	3	3	0	0	0	0	0	0
1	0	4		3	0	0	0	0	0	0	3	0
−2	2	1	4		3	3	0	−3	0	0	0	0
0	1	−2	−1	4		0	0	0	0	0	−3	0
−2	1	−2	1	2	4		3	0	0	0	−3	0
0	−2	0	−2	0	−1	4		0	0	0	0	0
1	−2	2	−1	−2	−2	2	4		3	−3	0	3
−2	0	0	2	0	2	−2	−1	4		0	0	0
1	0	0	0	−2	−2	0	1	−2	4		0	0
0	2	1	2	−1	−1	−2	0	0	2	4		3
−2	0	−2	0	2	2	0	−1	2	−2	−1	4	

11: ${}_i[{}_{\sqrt{-3}}[\pm 3_+^{1+2} : \mathrm{SL}_2(3)]_3 \otimes {}_{\infty,2}[\mathrm{SL}_2(3)]_1]_6$

4		0	3	1	−2	1	−1	1	6	1	2	2
2	4		3	2	2	2	1	2	3	−1	1	−2
−1	1	4		2	2	2	4	2	−3	2	1	−2
−1	0	0	4		1	1	2	−3	−2	−3	0	−3
0	0	0	−1	4		−3	0	−1	1	2	1	4
−1	0	2	1	1	4		6	−1	−2	−1	1	1
−1	−1	0	0	2	2	4		−2	−1	−2	−1	2
1	0	0	1	1	1	0	4		1	3	6	3
2	1	1	0	−1	0	−1	1	4		2	1	1
−1	−1	0	1	0	1	0	1	0	4		0	0
0	−1	−1	2	1	1	1	2	1	2	4		3
0	0	0	−1	0	−1	0	−1	1	2	1	4	

12: ${}_{\sqrt{-3}}[\pm 3.M_{10}]_6$

8		3	0	0	0	−3	0	−3	−6	0	−3	0
1	8		0	3	−6	−3	3	3	−6	−3	0	−12
2	0	8		−3	−6	0	3	−6	−3	0	0	0
4	−1	−1	8		0	0	3	−3	0	0	−3	0
0	2	2	−2	8		0	−3	6	0	0	0	0
3	−1	4	0	4	8		0	−6	0	0	3	3
−4	−3	−3	−3	−3	−4	8		3	3	6	3	3
−3	3	2	−1	2	−2	−1	8		−3	−3	−3	−6
−2	−2	3	−4	4	2	1	1	8		0	−3	6
−2	1	−4	−2	−2	−4	2	1	−2	8		−6	0
−3	−4	−4	1	0	−1	3	−1	1	2	8		6
2	4	0	−2	4	1	−3	0	2	2	−2	8	

13: ${}_{\sqrt{-3}}[C_6]_1 \otimes M_{6,2}$

10		0	0	0	0	0	15	6	6	6	6	6
4	10		0	0	0	0	6	15	6	−6	−6	6
4	4	10		0	0	0	6	6	15	6	−6	−6
4	−4	4	10		0	0	6	−6	6	15	6	−6
4	−4	−4	4	10		0	6	−6	−6	6	15	6
4	4	−4	−4	4	10		6	6	−6	−6	6	15
−5	−2	−2	−2	−2	−2	10		0	0	0	0	0
−2	−5	−2	2	2	−2	4	10		0	0	0	0
−2	−2	−5	−2	2	2	4	4	10		0	0	0
−2	2	−2	−5	−2	2	4	−4	4	10		0	0
−2	2	2	−2	−5	−2	4	−4	−4	4	10		0
−2	−2	2	2	−2	−5	4	4	−4	−4	4	10	

14: ${}_{\infty,2}[\mathrm{SL}_2(5)]_3 \circ C_3$

4		1	0	1	0	−3	−3	−2	−2	1	−3	1
−1	4		−1	−2	−2	3	2	−1	−2	−1	3	2
2	−1	4		−1	1	−1	1	−1	−3	1	−4	−1
−1	2	−1	4		−1	4	3	1	0	2	3	3
−2	2	−1	1	4		3	2	−1	2	1	2	3
−1	1	1	0	1	4		3	−1	−3	−3	−1	1
1	0	1	−1	0	1	4		1	−1	−1	1	1
0	−1	−1	1	−1	−1	1	4		1	−1	1	0
0	0	1	2	0	1	−1	1	4		0	−3	1
−1	1	1	2	1	1	1	1	2	4		1	0
−1	−1	0	−1	0	1	1	1	1	1	4		−1
1	0	1	1	−1	1	1	0	1	2	1	4	

15: $_{\sqrt{-5}}[_i[C_4]_1 \overset{2_-}{\boxtimes} \mathrm{Alt}_5]_6$

$$\begin{array}{ccccccccccccc}
1 & & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\
0 & 0 & 1 & & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & & 1 & 0 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 &
\end{array}$$

16: $_{\sqrt{-5}}[_i[C_4]_1 \overset{2_+}{\boxtimes} \mathrm{Alt}_5]_6$

$$\begin{array}{ccccccccccccc}
5 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\
2 & 5 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\
2 & 2 & 5 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\
2 & -2 & 2 & 5 & & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & 5 & & 0 & 0 & 0 & 0 & 0 & 0 & -5 \\
2 & 2 & -2 & -2 & 2 & 5 & & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 5 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 2 & 5 & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & -2 & 2 & 5 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 2 & 5 &
\end{array}$$

17: $_{\sqrt{-7}}[\pm \mathrm{L}_2(7)]_3 \underset{\sqrt{-7}}{\otimes} {}_{\infty,3}[\tilde{S}_3]_1$

$$\begin{array}{ccccccccccccc}
4 & & 2 & -3 & 0 & 0 & 0 & 2 & -1 & -4 & 1 & -4 & 0 \\
0 & 4 & & 0 & 2 & -5 & -1 & 2 & 0 & 2 & 0 & 0 & 0 \\
1 & 0 & 4 & & 3 & 0 & 0 & -2 & 0 & 2 & 0 & -1 & -2 \\
-2 & 2 & 1 & 4 & & 1 & 3 & 0 & -1 & 4 & -2 & 0 & 2 \\
0 & 1 & -2 & -1 & 4 & & -2 & -2 & -2 & 0 & 0 & 3 & 2 \\
-2 & 1 & -2 & 1 & 2 & 4 & & 3 & 0 & 0 & 0 & 1 & 2 \\
0 & -2 & 0 & -2 & 0 & -1 & 4 & & 0 & 0 & -2 & 0 & 0 \\
1 & -2 & 2 & -1 & -2 & -2 & 2 & 4 & & -3 & 1 & 0 & -5 \\
-2 & 0 & 0 & 2 & 0 & 2 & -2 & -1 & 4 & & 4 & 2 & 0 \\
1 & 0 & 0 & 0 & -2 & -2 & 0 & 1 & -2 & 4 & & 0 & -2 \\
0 & 2 & 1 & 2 & -1 & -1 & -2 & 0 & 0 & 2 & 4 & & -5 \\
-2 & 0 & -2 & 0 & 2 & 2 & 0 & -1 & 2 & -2 & -1 & 4 &
\end{array}$$

19: $_{\sqrt{-11}}[\mathrm{SL}_2(11)]_6$

$$\begin{array}{ccccccccccccc}
1 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
0 & 0 & 1 & & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 &
\end{array}$$

20: $_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6 \cdot 2_1]_6$

$$\begin{array}{ccccccccccccc}
4 & & 0 & 9 & 6 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & -6 \\
0 & 4 & & 0 & 0 & -3 & 3 & -6 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
-2 & 2 & 1 & 4 & & -3 & 3 & -6 & -3 & 6 & 0 & 0 & 6 \\
0 & 1 & -2 & -1 & 4 & & 6 & 0 & 0 & 0 & 0 & 3 & 0 \\
-2 & 1 & -2 & 1 & 2 & 4 & & 3 & 0 & 0 & 0 & -3 & 0 \\
0 & -2 & 0 & -2 & 0 & -1 & 4 & & 0 & 0 & 0 & 6 & 0 \\
1 & -2 & 2 & -1 & -2 & -2 & 2 & 4 & & 3 & -3 & 0 & -3 \\
-2 & 0 & 0 & 2 & 0 & 2 & -2 & -1 & 4 & & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & -2 & 0 & 1 & -2 & 4 & & -6 & 0 \\
0 & 2 & 1 & 2 & -1 & -1 & -2 & 0 & 0 & 2 & 4 & & -3 \\
-2 & 0 & -2 & 0 & 2 & 2 & 0 & -1 & 2 & -2 & -1 & 4 &
\end{array}$$

21: $_{\sqrt{-15}}[\pm 3.\mathrm{Alt}_6 : 2_1]_6$

$$\begin{array}{ccccccccccccc}
8 & & 15 & 0 & 0 & 0 & -15 & 0 & 15 & 0 & 0 & -15 & 0 \\
1 & 8 & & 0 & -15 & 0 & -15 & 15 & 15 & 0 & 15 & 0 & 0 \\
2 & 0 & 8 & & -15 & 0 & 0 & 15 & 0 & 15 & 0 & 0 & 0 \\
4 & -1 & -1 & 8 & & 0 & 0 & -15 & 15 & 0 & 0 & -15 & 0 \\
0 & 2 & 2 & -2 & 8 & & 0 & 15 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 4 & 0 & 4 & 8 & & 0 & 0 & 0 & 0 & -15 & 15 \\
-4 & -3 & -3 & -3 & -3 & -4 & 8 & & -15 & -15 & 0 & 15 & -15 \\
-3 & 3 & 2 & -1 & 2 & -2 & -1 & 8 & & 15 & 15 & 15 & 0 \\
-2 & -2 & 3 & -4 & 4 & 2 & 1 & 1 & 8 & & 0 & 15 & 0 \\
-2 & 1 & -4 & -2 & -2 & -4 & 2 & 1 & -2 & 8 & & 0 & 0 \\
-3 & -4 & -4 & 1 & 0 & -1 & 3 & -1 & 1 & 2 & 8 & & 0 \\
2 & 4 & 0 & -2 & 4 & 1 & -3 & 0 & 2 & 2 & -2 & 8 &
\end{array}$$

22: $_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$

$$\begin{array}{ccccccccccccc}
4 & & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -3 \\
-2 & 4 & & 1 & 2 & -3 & 0 & 0 & 0 & -1 & 2 & 3 & 3 \\
0 & -1 & 4 & & -3 & 1 & 3 & 1 & -2 & -1 & -4 & -2 & -1 \\
0 & 1 & -2 & 4 & & 1 & -1 & 1 & -1 & 1 & 4 & 2 & 1 \\
-2 & 0 & -1 & 0 & 4 & & -1 & -1 & 2 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & -2 & -1 & 4 & & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & -2 & 1 & 0 & 1 & -2 & 4 & & 0 & -1 & -2 & -4 & -2 \\
2 & 0 & 1 & -1 & -1 & 0 & 0 & 4 & & -3 & -1 & 0 & -1 \\
-1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 4 & & 1 & 2 & 4 \\
1 & -1 & 2 & 0 & 0 & -1 & 1 & 2 & 2 & 4 & & 0 & 0 \\
-1 & 0 & 1 & 1 & 1 & -2 & 2 & 0 & 1 & 2 & 4 & & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 4 &
\end{array}$$

23: $_{\zeta_{26}+\zeta_{26}^3+\zeta_{26}^9}[C_{26} : C_3]_3$

$$\begin{array}{ccccccccccccc}
2 & & 1 & -1 & 3 & -3 & 2 & -1 & 0 & -1 & 3 & -3 & 2 \\
-1 & 2 & & 1 & -2 & 3 & -3 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 2 & & 2 & -2 & 3 & -1 & -2 & 3 & -3 & 1 & 0 \\
0 & 0 & -1 & 2 & & 2 & -2 & 0 & 3 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & & 2 & 1 & -3 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 2 & & 0 & 1 & 0 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & & -1 & 1 & -1 & 3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & & -2 & 2 & -3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & & -2 & 2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & & -2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 &
\end{array}$$

A.7 Dimension 14

1: ${}_i[C_4]_1 \otimes E_7$

2		0	−1	1	1	0	0	1	1	1	0	0	1	0
0	2		0	0	0	0	0	0	1	0	0	−1	0	0
0	1	2		0	0	0	0	0	0	1	1	−2	1	−1
0	1	1	2		0	0	0	0	1	0	0	−1	0	0
0	1	0	1	2		0	0	0	1	−1	0	0	0	0
0	1	1	1	0	2		0	0	0	1	1	−1	0	0
0	1	1	1	1	1	2		0	0	0	1	−1	1	0
0	1	0	1	1	0	0	2		1	−1	−1	0	−1	1
−1	1	0	1	1	1	1	1	3		−1	0	0	−1	1
1	1	0	1	1	1	1	1	1	3		0	0	0	1
1	0	0	0	0	0	0	0	−1	1	2		0	0	0
−1	0	0	0	0	0	0	0	0	−1	−1	2		0	0
1	1	0	1	1	1	1	1	1	2	1	−1	3		1
−1	0	0	0	0	0	0	0	0	−1	−1	1	−1	2	

2: ${}_i[U_3(3) \circ C_4]_7$

3		0	0	0	−1	−1	0	−1	1	0	−1	0	0	0
−1	3		0	−1	0	0	−1	0	0	−1	0	0	0	1
1	1	3		0	−1	−1	0	−1	1	−1	0	1	1	1
1	0	1	3		0	0	1	−3	1	−1	−1	0	0	−1
0	1	0	−1	3		0	−1	1	−1	0	−1	−1	0	1
0	1	0	−1	1	3		−3	1	−1	−1	0	0	−1	0
−1	0	−1	0	0	0	3		0	0	0	−1	−1	0	−1
0	−1	0	0	0	0	−1	3		0	0	0	1	−1	0
0	1	0	0	0	0	1	−1	3		−1	−1	0	0	1
−1	0	0	0	−1	0	1	1	0	3		0	0	0	−1
0	−1	−1	0	0	−1	0	1	0	1	3		0	0	−1
1	−1	0	1	0	−1	0	0	1	−1	1	3		1	0
−1	1	0	−1	1	0	1	0	1	1	1	0	3		0
−1	0	0	0	0	−1	0	1	0	0	0	1	1	3	

A.8 Dimension 16

2: ${}_i[(2_+^{1+6} \otimes C_4).\mathrm{Sp}_6(2)]_8$

4		0	0	1	0	1	0	2	1	−1	0	0	1	0	0	0
2	4		−1	0	−1	0	1	1	0	−1	1	1	1	0	0	−1
2	1	4		0	0	1	0	2	0	0	−1	−1	1	0	0	0
1	2	2	4		−1	0	2	2	0	0	−1	0	0	−1	−2	−1
2	1	2	1	4		2	0	0	−1	1	1	1	0	0	0	0
1	2	1	2	2	4		1	1	0	1	0	1	1	−1	0	−2
−2	−1	0	0	−2	−1	4		0	0	0	−2	0	0	0	0	0
−2	−1	0	0	−2	−1	2	4		0	0	0	0	0	0	−1	−1
−1	−2	0	0	−1	−2	2	2	4		0	−1	0	0	−1	−2	1
−1	1	0	0	−1	1	0	0	−2	4		0	−1	1	1	2	−1
0	1	−1	1	1	2	0	−2	−1	0	4		2	−1	−1	0	−1
0	−1	−1	0	1	1	0	−2	0	−1	2	4		0	−1	0	1
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		0	0	0
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		0	0
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		0
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

3: ${}_i[{}_{\sqrt{-3}}[\mathrm{Sp}_4(3) \circ C_3]_4 \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_8$

6		0	1	1	0	−1	1	−1	0	2	0	1	3	1	1	1
0	6		−1	3	0	1	−1	−1	2	0	2	−1	3	1	1	1
−1	3	6		0	0	−1	1	−2	2	0	2	0	1	−2	0	2
1	−1	−2	6		1	3	0	1	1	1	1	−3	−2	−1	−2	−1
0	0	−2	3	6		0	−3	1	−1	−1	−1	0	0	1	1	−1
1	−3	−3	3	2	6		0	2	−2	0	−2	−1	−1	1	−1	−2
−1	3	3	0	1	0	6		−1	0	0	0	0	1	−1	−2	1
3	−1	−2	−1	−1	2	−1	6		−3	−1	−3	2	2	1	0	−1
−2	−2	−2	1	−1	2	0	1	6		0	−1	−2	−3	0	−3	−3
2	−2	−2	3	3	2	0	−1	0	6		1	0	−1	0	−1	1
−2	−2	−2	1	−1	2	0	−1	3	1	6		−1	−3	0	−3	0
−1	3	0	−1	0	−3	0	−2	−2	−2	−1	6		1	1	1	1
3	3	1	2	0	−1	1	0	−1	1	−1	1	6		−1	−1	2
3	1	−2	1	−1	1	−1	3	0	0	0	1	3	6		−1	1
3	1	0	2	−1	−1	−2	0	−1	1	−1	1	3	3	6		0
−3	1	2	1	1	−2	1	−3	1	−1	0	1	0	−3	0	6	

5: ${}_{i}[C_4]_1 \otimes M_{8,3}$

4		0	0	0	0	0	0	0	4	2	2	−2	−1	2	2	1
2	4		0	0	0	0	0	0	2	4	0	−1	−2	1	2	−1
2	0	4		0	0	0	0	0	2	0	4	−2	−1	2	1	2
−2	−1	−2	4		0	0	0	0	−2	−1	−2	4	2	−1	−2	0
−1	−2	−1	2	4		0	0	0	−1	−2	−1	2	4	0	−1	1
2	1	2	−1	0	4		0	0	2	1	2	−1	0	4	0	0
2	2	1	−2	−1	0	4		0	2	2	1	−2	−1	0	4	1
1	−1	2	0	1	0	1	4		1	−1	2	0	1	0	1	4
0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	4		0	0	0	0	0
0	0	0	0	0	0	0	0	−2	−1	−2	4		0	0	0	0
0	0	0	0	0	0	0	0	−1	−2	−1	2	4		0	0	0
0	0	0	0	0	0	0	0	2	1	2	−1	0	4		0	0
0	0	0	0	0	0	0	0	2	2	1	−2	−1	0	4		0
0	0	0	0	0	0	0	0	1	−1	2	0	1	0	1	4	

6: ${}_{\infty,3}[\mathrm{SL}_2(7)]_4 \circ C_4$

6		1	0	2	1	0	0	−1	2	−1	−1	1	0	2	0	2
1	6		1	0	1	−1	0	−1	0	−2	0	0	0	2	0	−2
−3	−1	6		−2	0	−1	−2	1	−1	0	0	−2	0	−2	−1	−1
2	3	1	6		0	0	−2	−2	1	−2	−1	−1	0	1	−1	0
0	2	2	3	6		1	−2	−1	1	−2	−1	−1	−2	2	−1	0
−2	1	2	2	0	6		−2	−1	1	2	−1	0	2	0	−1	−1
3	1	−2	0	−1	0	6		0	2	−3	−1	0	1	2	0	0
1	0	−2	−1	−2	1	3	6		2	0	−1	1	0	0	0	−1
1	1	−2	1	−2	2	1	0	6		1	−1	3	1	0	−1	0
2	2	−3	0	−2	−2	0	0	0	6		3	−1	2	0	0	0
1	3	0	1	2	−1	2	−1	0	0	6		−1	2	1	2	0
1	−2	−2	−2	−2	−3	−1	−2	0	2	−2	6		−1	−1	0	3
3	3	0	3	1	0	2	1	1	1	2	−2	6		0	−2	−1
−2	1	0	0	−1	3	1	1	3	−1	−1	−1	−1	6		−1	−1
3	−1	−1	0	−2	0	3	0	2	0	1	1	1	−1	6		0
1	2	1	3	0	1	−1	−1	0	1	1	0	1	−1	1	6	

7: ${}_{\sqrt{-2}}[{}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 : 2]_8$

4		0	0	0	0	0	0	0	0	0	0	0	0	2	2	0
2	4		0	0	0	0	0	0	0	0	−2	2	0	0	0	0
2	1	4		0	0	0	0	0	0	0	0	0	0	2	0	0
1	2	2	4		0	0	0	0	0	0	0	2	2	0	0	0
2	1	2	1	4		0	0	0	0	0	0	0	0	0	0	2
1	2	1	2	2	4		0	0	0	0	0	2	0	0	0	0
−2	−1	0	0	−2	−1	4		0	0	2	0	0	0	0	0	−2
−2	−1	0	0	−2	−1	2	4		−2	0	0	0	0	0	−2	0
−1	−2	0	0	−1	−2	2	2	4		0	0	−2	2	0	0	0
−1	1	0	0	−1	1	0	0	−2	4		−2	0	0	0	0	0
0	1	−1	1	1	2	0	−2	−1	0	4		0	2	−2	0	0
0	−1	−1	0	1	1	0	−2	0	−1	2	4		0	0	2	0
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		0	−2	0
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		0	0
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		0
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

8: ${}_{\sqrt{-2}}[2_+^{1+6}.(\mathrm{Alt}_8 : 2)]_8$

2		0	0	0	0	0	0	0	−2	1	0	0	−1	1	−1	1
−1	2		0	0	0	0	0	0	1	−1	1	−1	1	0	1	−2
0	−1	2		0	0	0	0	0	0	2	−1	0	0	−1	0	1
0	0	−1	2		0	0	0	0	1	−2	1	0	0	1	−2	0
0	0	0	−1	2		0	0	0	−1	1	−1	−1	1	0	1	0
0	0	0	0	−1	2		0	0	1	−1	0	1	−1	−1	1	1
0	0	0	0	0	−1	2		0	−1	1	−1	1	1	0	−1	−2
0	0	0	0	−1	0	0	2		0	0	1	1	−2	1	0	0
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

10: ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes_{\sqrt{-2}} {}_{\infty,5}[\mathrm{SL}_2(5) : 2]_2$

4		0	0	0	0	0	0	0	−1	3	1	−1	−4	−1	1	1
−2	4		0	0	0	0	0	0	−1	0	−2	−1	0	0	1	−1
0	0	4		0	0	0	0	0	4	0	−1	−1	1	−2	−1	−2
1	−2	2	4		0	0	0	0	1	0	−1	−1	1	−1	0	2
1	−2	0	2	4		0	0	0	1	0	−1	0	−1	0	0	3
−2	2	−1	−2	0	4		0	0	0	−1	−1	3	2	0	0	−1
−2	2	2	0	−1	0	4		0	2	−1	−4	−3	1	1	−3	0
−1	−1	−2	−1	−1	0	−1	4		0	−4	2	3	2	−1	−2	1
0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	4		0	0	0	0	0
0	0	0	0	0	0	0	0	1	−2	2	4		0	0	0	0
0	0	0	0	0	0	0	0	1	−2	0	2	4		0	0	0
0	0	0	0	0	0	0	0	−2	2	−1	−2	0	4		0	0
0	0	0	0	0	0	0	0	−2	2	2	0	−1	0	4		0
0	0	0	0	0	0	0	0	−1	−1	−2	−1	−1	0	−1	4	

12: ${}_{\infty,2}[2_-^{1+6}.O_6^-(2)]_4 \circ C_3$

4		0	2	−1	2	1	0	0	1	1	0	0	1	0	0	0
2	4		3	0	1	0	−1	1	0	1	−3	−3	1	0	0	1
2	1	4		−2	2	1	0	0	0	0	−1	3	−1	0	−2	0
1	2	2	4		1	0	0	0	0	0	−3	0	0	−1	0	1
2	1	2	1	4		2	0	0	1	1	1	1	0	0	0	−2
1	2	1	2	2	4		−1	1	0	1	−2	−1	3	−3	0	0
−2	−1	0	0	−2	−1	4		2	−2	0	−2	0	0	0	−2	0
−2	−1	0	0	−2	−1	2	4		−2	0	−2	0	0	0	1	3
−1	−2	0	0	−1	−2	2	2	4		2	−1	0	0	−1	0	−1
−1	1	0	0	−1	1	0	0	−2	4		0	−1	1	−1	−2	1
0	1	−1	1	1	2	0	−2	−1	0	4		−2	1	−3	−2	−1
0	−1	−1	0	1	1	0	−2	0	−1	2	4		2	−1	0	−3
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		0	0	2
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		2	−2
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		0
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

13: $_{\sqrt{-3}}[\mathrm{Sp}_4(3)\circ C_3]_4 \underset{\sqrt{-3}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$

6		−6	−3	−3	0	3	−3	3	0	0	0	−3	−9	−3	−3	−3
0	6		−3	3	0	3	−3	3	0	0	0	−3	3	3	3	−3
−1	3	6		0	0	−3	3	0	0	0	0	0	3	0	0	0
1	−1	−2	6		3	3	0	3	−3	3	−3	−3	0	3	0	−3
0	0	−2	3	6		0	−3	3	−3	−3	−3	0	0	3	−3	−3
1	−3	−3	3	2	6		0	0	0	0	0	−3	−3	−3	−3	0
−1	3	3	0	1	0	6		3	0	0	0	0	3	3	0	−3
3	−1	−2	−1	−1	2	−1	6		3	−3	3	0	−6	−3	0	3
−2	−2	−2	1	−1	2	0	1	6		6	3	0	3	0	3	3
2	−2	−2	3	3	2	0	−1	0	6		−3	0	−3	0	−3	−3
−2	−2	−2	1	−1	2	0	−1	3	1	6		3	3	0	3	0
−1	3	0	−1	0	−3	0	−2	−2	−2	−1	6		3	3	3	−3
3	3	1	2	0	−1	1	0	−1	1	−1	1	6		3	3	−6
3	1	−2	1	−1	1	−1	3	0	0	0	1	3	6		3	−3
3	1	0	2	−1	−1	−2	0	−1	1	−1	1	3	3	6		0
−3	1	2	1	1	−2	1	−3	1	−1	0	1	0	−3	0	6	

14: $_{\sqrt{-3}}[C_6]_1 \otimes [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8$

8		0	0	0	0	0	0	0	12	−6	3	3	−3	−6	0	6
−4	8		0	0	0	0	0	0	−6	12	−3	0	−3	3	3	−3
2	−2	8		0	0	0	0	0	3	−3	12	−3	3	−6	−6	0
2	0	−2	8		0	0	0	0	3	0	−3	12	3	3	6	3
−2	−2	2	2	8		0	0	0	−3	−3	3	3	12	3	−3	−6
−4	2	−4	2	2	8		0	0	−6	3	−6	3	3	12	6	0
0	2	−4	4	−2	4	8		0	0	3	−6	6	−3	6	12	3
4	−2	0	2	−4	0	2	8		6	−3	0	3	−6	0	3	12
−4	2	−1	−1	1	2	0	−2	8		0	0	0	0	0	0	0
2	−4	1	0	1	−1	−1	1	−4	8		0	0	0	0	0	0
−1	1	−4	1	−1	2	2	0	2	−2	8		0	0	0	0	0
−1	0	1	−4	−1	−1	−2	−1	2	0	−2	8		0	0	0	0
1	1	−1	−1	−4	−1	1	2	−2	−2	2	2	8		0	0	0
2	−1	2	−1	−1	−4	−2	0	−4	2	−4	2	2	8		0	0
0	−1	2	−2	1	−2	−4	−1	0	2	−4	4	−2	4	8		0
−2	1	0	−1	2	0	−1	−4	4	−2	0	2	−4	0	2	8	

16: $({}_{\infty,2}[\mathrm{SL}_2(3)]_1 \circ C_3) \underset{\sqrt{-3}}{\otimes} {}_{\infty,5}[\mathrm{SL}_2(5):2]_2$

12		−6	−6	18	3	−3	0	−6	0	0	−6	6	0	3	0	−3
6	12		−18	6	0	−3	3	−3	−6	6	0	0	6	0	−6	−6
−6	−6	12		−12	−3	6	3	3	−6	6	6	−6	6	−9	6	3
−6	−6	0	12		3	0	−3	3	6	−6	−6	6	−6	3	−6	−3
−3	−2	1	3	12		6	−6	18	0	−6	3	6	0	6	−6	−6
1	3	0	−2	−6	12		18	−6	−6	9	−3	−6	−3	−9	3	3
2	3	−3	−1	−6	6	12		−12	6	9	0	0	3	−9	−3	−3
0	1	1	−3	−6	6	0	12		−6	3	−6	−12	−3	−3	3	3
−4	−6	6	2	4	−6	−6	−2	12		−12	6	0	3	3	0	3
4	6	−6	−2	0	−3	3	−3	0	12		6	3	3	6	−12	−6
6	4	−2	−6	3	−1	−2	0	2	2	12		3	3	3	−6	−3
−6	−4	2	6	6	−2	−4	0	2	−3	−3	12		−6	0	−3	0
4	6	−6	−2	2	−3	−3	−1	−3	3	3	2	12		0	−3	−6
−3	−6	3	3	0	3	−3	3	3	−6	−1	0	−6	12		6	12
4	2	2	−2	0	3	3	−3	−4	0	2	−3	−1	−2	12		−6
3	0	3	−3	0	3	3	−3	−1	−2	1	0	0	0	6	12	

17: $_{\infty,5}[\mathrm{SL}_2(5) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]_4 \circ C_3$

4		−2	−2	2	−2	2	2	−2	0	−1	1	1	3	0	1	1
−2	4		2	2	0	0	0	0	1	0	0	0	−2	−2	0	−1
−2	2	4		2	−2	−2	−2	2	1	0	2	0	0	−1	3	2
2	−2	−2	4		0	0	2	2	−1	−1	0	0	2	0	0	1
2	0	−2	0	4		2	2	−2	1	1	1	2	0	−1	−1	−2
2	0	−2	0	2	4		−2	−2	2	−1	0	2	0	0	−1	0
2	0	−2	2	2	2	4		2	2	0	1	1	−1	1	−2	−1
−2	0	2	−2	−2	−2	−2	4		1	−1	1	0	−1	1	0	−1
0	−1	1	−1	−1	0	−2	1	4		−1	0	2	3	1	3	1
−1	2	2	−1	−1	−1	0	1	−1	4		2	0	−2	0	1	1
1	−2	0	0	−1	0	−1	1	2	0	4		0	0	2	2	2
1	−2	0	0	0	0	−1	0	2	−2	2	4		2	1	3	0
1	−2	−2	2	0	0	1	−1	−1	0	0	0	4		3	−2	0
−2	2	1	−2	−1	0	−1	1	−1	2	0	−1	−1	4		1	−1
1	−2	−1	2	−1	−1	0	0	−1	−1	0	1	2	−1	4		1
−1	−1	0	1	−2	−2	−1	1	1	−1	0	0	0	−1	1	4	

18: $_{\infty,2}[\mathrm{SL}_2(5) \overset{2(2)}{\boxtimes} D_8]_4 \circ C_3$

8		1	1	0	0	−4	4	−1	0	−4	4	−1	4	4	4	−4
1	8		0	−4	−1	0	2	−5	−4	−2	3	0	−1	−3	−2	−3
3	4	8		0	−5	−3	−2	−4	−4	−3	3	−5	−2	−2	2	−3
4	4	2	8		4	0	3	−5	−4	−3	3	0	3	0	3	−3
4	3	1	4	8		−3	4	0	4	−3	0	−4	0	3	0	0
4	2	1	2	3	8		3	4	3	4	4	0	4	4	4	4
2	4	4	1	4	3	8		0	1	−1	0	−4	−3	−1	0	0
3	1	4	1	2	4	2	8		4	1	2	−6	0	1	1	0
4	2	4	4	4	1	3	4	8		0	3	−5	3	3	3	−3
4	2	3	3	1	4	1	3	2	8		4	−1	4	−4	4	−4
4	3	1	3	2	4	2	2	1	4	8		−4	4	−4	−4	−8
3	2	1	2	4	4	4	2	1	3	4	8		4	1	1	−2
4	1	2	3	2	4	3	2	1	4	4	2	8		−4	4	0
4	1	2	2	1	4	3	3	3	4	4	3	4	8		4	−4
4	4	4	3	2	4	4	3	1	4	4	3	4	4	8		−4
4	1	3	1	2	4	2	4	3	4	0	2	0	4	4	8	

19: $_{\sqrt{-3}}[C_{60}.(C_4 \times C_2)]_8$

8		0	0	0	6	6	−6	−3	0	−3	−3	3	6	0	3	−3
0	8		0	0	6	0	6	3	0	3	3	−3	−6	0	3	−3
2	2	8		0	3	0	3	0	6	−6	0	0	0	0	3	−6
2	2	4	8		3	0	0	0	6	0	6	6	0	−3	−3	0
−2	−2	−1	−3	8		0	0	0	−3	−3	0	−3	0	−6	3	−3
−2	0	−4	−4	2	8		3	0	−12	6	−6	−6	0	−3	0	6
−2	2	1	2	0	−1	8		3	3	3	3	0	−6	−6	3	−6
−3	−1	−4	−2	2	2	1	8		−3	3	0	0	0	0	−3	6
4	0	2	2	−1	−4	−1	−1	8		−6	3	3	3	0	3	−3
1	−1	2	0	−1	−2	−1	−1	−2	8		−3	−3	3	3	0	−3
1	−1	0	0	−2	0	1	0	3	−1	8		0	0	0	3	0
1	3	0	0	1	0	−2	0	3	−1	0	8		0	0	0	0
−2	2	0	−2	4	2	−2	0	−1	−3	−4	4	8		0	0	0
4	4	4	3	−2	−1	2	−4	2	−1	0	0	0	8		3	−6
1	−3	−3	−3	1	2	−1	−1	−1	−2	1	−4	−2	−1	8		0
−1	3	2	2	−3	0	0	−2	−3	3	0	0	0	2	−4	8	

20: $_{\sqrt{-5}}[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\overset{2_-}{\boxtimes}{}_i[C_4]_1]_8$

2		0	0	0	0	0	0	0	3	−1	0	−2	0	1	−1	2
−1	2		0	0	0	0	0	0	0	1	1	1	−1	0	1	−2
0	−1	2		0	0	0	0	0	−2	0	0	−1	1	2	−3	0
0	0	−1	2		0	0	0	0	1	0	−1	2	−1	−1	0	0
0	0	0	−1	2		0	0	0	1	−3	2	0	0	−1	1	1
0	0	0	0	−1	2		0	0	0	1	−2	0	0	2	1	−2
0	0	0	0	0	−1	2		0	−2	1	2	−1	−2	0	0	3
0	0	0	0	−1	0	0	2		−1	3	−1	−2	3	−2	0	−1
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

21: $_{\sqrt{-5}}[((\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2)\overset{2_+}{\boxtimes}{}_i[C_4]_1]_8$

4		0	0	0	0	0	0	0	0	0	0	0	0	5	0	5
−2	4		0	0	0	0	0	0	5	0	0	0	−5	−5	0	0
1	−1	4		0	0	0	0	0	0	5	0	0	0	0	0	0
1	0	−1	4		0	0	0	0	0	0	0	0	0	5	5	0
−1	−1	1	1	4		0	0	0	0	0	0	0	0	0	5	0
−2	1	−2	1	1	4		0	0	0	0	0	5	0	0	5	0
0	1	−2	2	−1	2	4		0	0	0	5	0	0	0	0	0
2	−1	0	1	−2	0	1	4		0	0	0	5	5	5	0	0
0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	−1	4		0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	−1	4		0	0	0	0
0	0	0	0	0	0	0	0	−1	−1	1	1	4		0	0	0
0	0	0	0	0	0	0	0	−2	1	−2	1	1	4		0	0
0	0	0	0	0	0	0	0	0	1	−2	2	−1	2	4		0
0	0	0	0	0	0	0	0	2	−1	0	1	−2	0	1	4	

22: $_{\sqrt{-5}}[_i[(D_8\otimes C_4).S_3]_2\overset{2_+}{\underset{i}{\boxtimes}}{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$

4		2	2	1	−2	1	2	2	−3	1	0	−4	3	−2	−2	2
2	4		1	−2	−3	−2	1	3	−2	3	−3	−5	3	0	0	−1
2	1	4		0	−6	−1	4	4	0	4	−3	−3	3	−4	0	−2
1	2	2	4		−5	−2	2	2	0	4	−3	−4	4	−3	2	−1
2	1	2	1	4		2	2	2	−1	3	−1	−5	0	−2	0	2
1	2	1	2	2	4		1	−1	−2	5	2	−3	1	−1	2	−2
−2	−1	0	0	−2	−1	4		2	2	0	−2	2	2	−2	0	−4
−2	−1	0	0	−2	−1	2	4		0	2	0	4	0	0	3	−3
−1	−2	0	0	−1	−2	2	2	4		0	−1	2	2	−1	2	1
−1	1	0	0	−1	1	0	0	−2	4		0	1	−1	3	2	−3
0	1	−1	1	1	2	0	−2	−1	0	4		−2	1	−3	−2	1
0	−1	−1	0	1	1	0	−2	0	−1	2	4		−2	1	0	3
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		2	0	0
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		0	2
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		4
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

23: $_{\sqrt{-5}}[_i[(D_8\otimes C_4).S_3]_2\overset{2_-}{\underset{i}{\boxtimes}}{}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$

8		−5	−5	0	0	0	0	−5	0	0	0	−5	0	0	0	0
1	8		0	0	5	10	10	5	0	0	5	10	5	5	10	5
3	4	8		0	5	5	10	0	0	5	5	5	10	10	10	5
4	4	2	8		0	0	5	−5	0	−5	−5	0	−5	0	5	5
4	3	1	4	8		5	0	0	0	−5	0	0	0	5	0	0
4	2	1	2	3	8		−5	0	−5	0	0	0	0	0	0	0
2	4	4	1	4	3	8		0	−5	−5	0	0	5	5	0	0
3	1	4	1	2	4	2	8		0	5	0	0	10	5	5	0
4	2	4	4	4	1	3	4	8		0	−5	−5	5	5	5	5
4	2	3	3	1	4	1	3	2	8		0	5	0	0	0	0
4	3	1	3	2	4	2	2	1	4	8		0	0	0	0	0
3	2	1	2	4	4	4	2	1	3	4	8		0	5	−5	0
4	1	2	3	2	4	3	2	1	4	4	2	8		0	0	0
4	1	2	2	1	4	3	3	3	4	4	3	4	8		0	0
4	4	4	3	2	4	4	3	1	4	4	3	4	4	8		0
4	1	3	1	2	4	2	4	3	4	0	2	0	4	4	8	

24: $_{\sqrt{-5}}[_{\infty,5}[\mathrm{SL}_2(5)\overset{2}{\underset{\sqrt{5}}{\boxtimes}}D_{10}]_4:2]_8$

4		4	0	0	4	0	2	0	−3	1	−4	−4	−3	0	−2	−2
−2	4		2	0	−2	−4	0	0	−1	1	−1	1	2	0	0	2
−2	2	4		0	−2	−4	0	2	−1	2	2	1	3	0	3	3
2	−2	−2	4		0	2	2	4	0	0	0	−2	−2	0	−1	−1
2	0	−2	0	4		−4	−4	0	0	−1	−3	−1	−2	0	−1	1
2	0	−2	0	2	4		0	−2	0	1	−3	0	−4	−1	−4	−4
2	0	−2	2	2	2	4		2	0	−2	−2	0	−2	−1	−3	−2
−2	0	2	−2	−2	−2	−2	4		1	0	2	1	1	3	0	0
0	−1	1	−1	−1	0	−2	1	4		2	1	0	2	0	1	−3
−1	2	2	−1	−1	−1	0	1	−1	4		2	0	0	0	−2	1
1	−2	0	0	−1	0	−1	1	2	0	4		−1	−3	0	0	−2
1	−2	0	0	0	0	−1	0	2	−2	2	4		0	1	2	0
1	−2	−2	2	0	0	1	−1	−1	0	0	0	4		1	0	−4
−2	2	1	−2	−1	0	−1	1	−1	2	0	−1	−1	4		−3	2
1	−2	−1	2	−1	−1	0	0	−1	−1	0	1	2	−1	4		−1
−1	−1	0	1	−2	−2	−1	1	1	−1	0	0	0	−1	1	4	

25: $_{\sqrt{-5}}[_i[(D_8\otimes C_4).S_3]_2\overset{2}{\boxtimes}D_{10}]_8$

4		−2	0	4	0	3	0	−6	0	−2	0	4	0	−2	0	4
−2	4		2	−2	−3	0	−3	3	2	0	2	−2	2	0	2	−2
0	−2	4		0	0	3	0	0	0	−2	0	0	0	−2	0	0
0	−2	0	4		6	−3	0	0	−4	2	0	0	−4	2	0	0
−2	1	0	0	4		−2	0	4	0	3	0	−6	0	−2	0	4
1	−2	1	1	−2	4		2	−2	−3	0	−3	3	2	0	2	−2
0	1	−2	0	0	−2	4		0	0	3	0	0	0	−2	0	0
0	1	0	−2	0	−2	0	4		6	−3	0	0	−4	2	0	0
0	0	0	0	−2	1	0	0	4		−2	0	4	0	3	0	−6
0	0	0	0	1	−2	1	1	−2	4		2	−2	−3	0	−3	3
0	0	0	0	0	1	−2	0	0	−2	4		0	0	3	0	0
0	0	0	0	0	1	0	−2	0	−2	0	4		6	−3	0	0
0	0	0	0	0	0	0	0	−2	1	0	0	4		−2	0	4
0	0	0	0	0	0	0	0	1	−2	1	1	−2	4		2	−2
0	0	0	0	0	0	0	0	0	1	−2	0	0	−2	4		0
0	0	0	0	0	0	0	0	0	1	0	−2	0	−2	0	4	

26: $\sqrt{-6}[\infty_{,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \overset{2(3)}{\boxtimes} A_2]_8$

2		0	0	0	0	0	0	0	−4	3	−2	0	1	0	−1	0
−1	2		0	0	0	0	0	0	1	−4	3	0	1	−1	0	−1
0	−1	2		0	0	0	0	0	2	1	−4	3	−1	0	0	0
0	0	−1	2		0	0	0	0	0	0	1	−4	2	0	1	1
0	0	0	−1	2		0	0	0	−1	−1	1	2	−4	3	−1	1
0	0	0	0	−1	2		0	0	0	1	0	0	1	−4	1	2
0	0	0	0	0	−1	2		0	1	0	0	−1	1	3	−4	−2
0	0	0	0	−1	0	0	2		0	1	0	−1	3	−2	2	−4
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

27: $\sqrt{-6}[\infty_{,2}[2_-^{1+4}.\mathrm{Alt}_5]_2 \underset{i}{\overset{2(3)}{\boxtimes}} \infty_{,3}[\tilde{S}_3]_1]_8$

4		−2	2	2	2	2	0	2	0	−2	−2	2	−2	2	−2	0
2	4		−2	0	2	0	−4	−2	0	−4	0	2	−2	4	0	0
2	1	4		4	−2	2	0	2	−2	2	−2	−2	2	−2	−2	0
1	2	2	4		−2	0	−2	2	0	−2	0	−2	2	2	0	0
2	1	2	1	4		−2	0	4	2	0	−4	−4	0	0	0	0
1	2	1	2	2	4		−2	0	0	−2	2	0	0	4	0	−2
−2	−1	0	0	−2	−1	4		0	0	0	2	−2	4	−4	−2	2
−2	−1	0	0	−2	−1	2	4		−4	4	0	−2	4	−4	−2	0
−1	−2	0	0	−1	−2	2	2	4		0	−4	−4	4	−4	0	4
−1	1	0	0	−1	1	0	0	−2	4		4	4	−2	2	2	−4
0	1	−1	1	1	2	0	−2	−1	0	4		−2	0	4	2	2
0	−1	−1	0	1	1	0	−2	0	−1	2	4		−2	0	−2	2
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		0	0	2
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		−2	−2
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		0
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

28: $\sqrt{-6}[\infty_{,2}[\mathrm{SL}_2(3)]_1 \underset{i}{\overset{2}{\boxtimes}} \infty_{,3}[\mathrm{SL}_2(9)]_2]_8$

4		0	0	3	3	0	0	3	3	0	0	−3	3	6	6	0
1	4		0	3	3	0	0	−3	0	0	0	−3	3	0	3	3
2	2	4		3	3	0	0	0	0	3	0	−3	0	3	3	0
1	1	0	4		−3	−3	0	0	0	0	0	−3	0	−3	3	−3
1	1	1	1	4		0	−3	0	3	0	0	0	0	0	0	0
2	1	2	1	1	4		3	3	0	3	6	0	0	0	3	0
1	2	2	1	1	1	4		0	0	0	0	−3	3	0	3	3
1	1	1	2	1	1	1	4		−3	0	0	−6	−3	−3	3	−3
1	−1	1	0	1	2	0	1	4		3	3	0	−3	0	−3	−3
1	1	1	2	0	1	2	2	1	4		0	−6	0	0	3	−3
2	1	2	1	0	2	2	2	1	2	4		−6	0	0	3	−3
1	0	1	1	0	2	1	2	2	2	2	4		0	0	3	−3
1	1	0	2	1	0	1	1	1	1	0	1	4		0	0	0
2	0	1	1	2	2	0	1	2	0	0	0	1	4		0	0
2	1	1	1	2	1	1	1	1	0	1	−1	1	2	4		0
1	1	2	0	2	2	1	0	1	−1	1	1	1	1	1	4	

29: $\sqrt{-6}[F_4 \overset{2}{\boxtimes} \infty_{,3}[\tilde{S}_3]_1]_8$

4		0	0	0	0	0	0	0	0	3	−3	0	3	0	−3	0
0	4		0	0	0	0	0	0	3	0	0	0	0	0	0	−6
−2	0	4		0	0	0	0	0	−3	0	0	0	0	0	0	0
0	−2	−2	4		0	0	0	0	0	0	0	0	0	−3	6	0
0	0	0	−2	4		0	0	0	3	0	0	0	0	0	−6	6
0	0	0	0	−2	4		0	0	0	0	0	−3	0	6	0	−3
0	0	0	0	0	−2	4		0	−3	0	0	6	−6	0	0	0
0	0	0	0	0	0	−2	4		0	−6	0	0	6	−3	0	0
2	0	−1	0	0	0	0	0	4		0	0	0	0	0	0	0
0	2	0	−1	0	0	0	0	0	4		0	0	0	0	0	0
−1	0	2	−1	0	0	0	0	−2	0	4		0	0	0	0	0
0	−1	−1	2	−1	0	0	0	0	−2	−2	4		0	0	0	0
0	0	0	−1	2	−1	0	0	0	0	0	−2	4		0	0	0
0	0	0	0	−1	2	−1	0	0	0	0	0	−2	4		0	0
0	0	0	0	0	−1	2	−1	0	0	0	0	0	−2	4		0
0	0	0	0	0	0	−1	2	0	0	0	0	0	0	−2	4	

30: $\sqrt{-6}[(F_4 \otimes A_2)\!:\!2]_8$

4		0	0	0	0	0	0	0	0	0	0	0	0	0	−6	0
−2	4		0	0	0	0	0	0	0	0	0	0	0	0	0	6
−2	1	4		0	0	0	0	0	0	0	0	0	6	0	0	0
1	−2	−2	4		0	0	0	0	0	0	0	0	0	−6	0	0
0	0	−2	1	4		0	0	0	6	0	0	0	−6	0	0	0
0	0	1	−2	−2	4		0	0	0	−6	0	0	0	6	0	0
0	0	−2	1	0	0	4		0	0	0	−6	0	0	0	6	0
0	0	1	−2	0	0	−2	4		0	0	0	6	0	0	0	−6
0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	−2	1	4		0	0	0	0	0
0	0	0	0	0	0	0	0	1	−2	−2	4		0	0	0	0
0	0	0	0	0	0	0	0	0	0	−2	1	4		0	0	0
0	0	0	0	0	0	0	0	0	0	1	−2	−2	4		0	0
0	0	0	0	0	0	0	0	0	0	−2	1	0	0	4		0
0	0	0	0	0	0	0	0	0	0	1	−2	0	0	−2	4	

32: $\sqrt{-7}[2.\mathrm{Alt}_7]_4 \underset{\sqrt{-7}}{\otimes} \infty_{,3}[\tilde{S}_3]_1$

12		7	0	0	0	7	−7	7	0	0	−7	−7	7	0	0	0
5	12		−7	21	14	0	7	−7	0	0	−7	−7	0	7	0	7
2	3	12		14	7	7	0	0	7	−14	0	−14	0	14	0	7
−6	−3	−4	12		0	0	14	−14	0	14	−7	14	7	0	0	0
−6	−2	1	6	12		7	7	−14	0	0	−7	7	7	0	7	0
3	4	1	−4	1	12		0	0	0	7	−7	0	−7	14	7	0
1	−3	−6	4	1	−4	12		0	−7	7	0	0	14	−14	7	−14
3	3	6	−4	−4	−4	0	12		7	−7	0	−14	0	7	−7	14
−6	0	−5	6	0	−2	−1	−5	12		7	7	7	−7	0	0	0
−6	−6	−4	0	2	−1	1	−5	3	12		14	7	−7	−14	0	−14
1	5	6	−3	1	−1	0	6	−3	−4	12		−14	14	0	7	0
1	−3	−4	−2	1	−2	6	−2	−1	5	−2	12		0	−14	0	−14
5	−2	0	−3	−1	5	−2	−2	−3	1	−6	0	12		0	−7	0
−6	−1	−6	6	0	−2	4	−1	6	2	0	−2	−6	12		7	0
6	4	−4	−4	−3	3	1	−3	−2	−2	1	2	1	−1	12		−7
−6	1	−5	6	6	2	2	−6	6	2	−2	2	−4	6	−1	12	

33: $_{\sqrt{-10}}[_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_+}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$

2		0	0	0	0	0	0	0	4	−3	2	−2	−1	−1	3	3
−1	2		0	0	0	0	0	0	−3	5	−3	−1	3	−2	−1	0
0	−1	2		0	0	0	0	0	0	−2	5	−2	0	1	0	−3
0	0	−1	2		0	0	0	0	3	0	−3	2	−2	3	−4	2
0	0	0	−1	2		0	0	0	−1	−1	−1	−1	3	−2	3	−2
0	0	0	0	−1	2		0	0	−3	1	2	−1	−3	3	−1	3
0	0	0	0	0	−1	2		0	3	−1	−1	1	3	−2	1	−4
0	0	0	0	−1	0	0	2		0	0	1	3	−2	−3	0	2
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

34: $_{\sqrt{-10}}[_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2_-}{\underset{\sqrt{-2}}{\boxtimes}} {}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1]_8$

4		0	0	0	0	0	0	0	5	0	−5	10	0	0	5	5
−2	4		0	0	0	0	0	0	−5	10	0	−5	−5	0	0	−5
1	−1	4		0	0	0	0	0	10	−5	5	0	−5	−10	0	5
1	0	−1	4		0	0	0	0	−5	5	0	5	0	5	5	5
−1	−1	1	1	4		0	0	0	0	−5	10	0	5	−5	−5	0
−2	1	−2	1	1	4		0	0	−5	0	0	0	0	5	0	0
0	1	−2	2	−1	2	4		0	−5	5	−5	5	0	10	10	0
2	−1	0	1	−2	0	1	4		5	0	−5	5	−5	5	5	10
0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	−1	4		0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	−1	4		0	0	0	0
0	0	0	0	0	0	0	0	−1	−1	1	1	4		0	0	0
0	0	0	0	0	0	0	0	−2	1	−2	1	1	4		0	0
0	0	0	0	0	0	0	0	0	1	−2	2	−1	2	4		0
0	0	0	0	0	0	0	0	2	−1	0	1	−2	0	1	4	

35: $_{\sqrt{-10}}[_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \overset{2}{\boxtimes} D_{10}]_8$

4		0	0	4	0	0	0	2	0	0	0	−8	0	0	0	4
−2	4		−4	0	0	0	−2	0	0	0	8	0	0	0	−4	0
0	−2	4		−4	0	2	0	−2	0	−8	0	8	0	4	0	−4
0	−2	0	4		−2	0	2	0	8	0	−8	0	−4	0	4	0
−2	1	0	0	4		0	0	−4	0	0	0	6	0	0	0	−8
1	−2	1	1	−2	4		4	0	0	0	−6	0	0	0	8	0
0	1	−2	0	0	−2	4		4	0	6	0	−6	0	−8	0	8
0	1	0	−2	0	−2	0	4		−6	0	6	0	8	0	−8	0
0	0	0	0	−2	1	0	0	4		0	0	−4	0	0	0	2
0	0	0	0	1	−2	1	1	−2	4		4	0	0	0	−2	0
0	0	0	0	0	1	−2	0	0	−2	4		4	0	2	0	−2
0	0	0	0	0	1	0	−2	0	−2	0	4		−2	0	2	0
0	0	0	0	0	0	0	0	−2	1	0	0	4		0	0	4
0	0	0	0	0	0	0	0	1	−2	1	1	−2	4		−4	0
0	0	0	0	0	0	0	0	0	1	−2	0	0	−2	4		−4
0	0	0	0	0	0	0	0	0	1	0	−2	0	−2	0	4	

36: $_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2) \overset{2_-}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8$

4		0	0	0	0	0	0	0	−6	−3	6	0	−6	6	−6	6
−2	4		0	0	0	0	0	0	−3	6	−3	0	0	0	6	−6
0	−2	4		0	0	0	0	0	6	−3	−6	3	0	0	−6	6
0	0	−2	4		0	0	0	0	0	0	3	6	−3	0	0	−6
0	0	0	−2	4		0	0	0	−6	0	0	−3	6	−9	6	3
0	0	0	0	−2	4		0	0	6	0	0	0	−9	6	3	6
0	0	0	0	0	−2	4		0	−6	6	−6	0	6	3	−6	−6
0	0	0	0	−2	0	0	4		6	−6	6	−6	3	6	−6	−6
−2	1	0	0	0	0	0	0	4		0	0	0	0	0	0	0
1	−2	1	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	1	−2	1	0	0	0	0	0	−2	4		0	0	0	0	0
0	0	1	−2	1	0	0	0	0	0	−2	4		0	0	0	0
0	0	0	1	−2	1	0	1	0	0	0	−2	4		0	0	0
0	0	0	0	1	−2	1	0	0	0	0	0	−2	4		0	0
0	0	0	0	0	1	−2	0	0	0	0	0	0	−2	4		0
0	0	0	0	1	0	0	−2	0	0	0	0	−2	0	0	4	

37: $_{\sqrt{-15}}[((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2) \overset{2_+}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_8$

8		0	0	0	0	0	0	0	0	0	15	15	15	0	0	0
−4	8		0	0	0	0	0	0	0	0	−15	0	−15	15	15	15
2	−2	8		0	0	0	0	0	15	−15	0	15	15	0	0	0
2	0	−2	8		0	0	0	0	15	0	15	0	−15	−15	0	15
−2	−2	2	2	8		0	0	0	15	−15	15	−15	0	−15	−15	0
−4	2	−4	2	2	8		0	0	0	15	0	−15	−15	0	0	0
0	2	−4	4	−2	4	8		0	0	15	0	0	−15	0	0	15
4	−2	0	2	−4	0	2	8		0	15	0	15	0	0	15	0
−4	2	−1	−1	1	2	0	−2	8		0	0	0	0	0	0	0
2	−4	1	0	1	−1	−1	1	−4	8		0	0	0	0	0	0
−1	1	−4	1	−1	2	2	0	2	−2	8		0	0	0	0	0
−1	0	1	−4	−1	−1	−2	−1	2	0	−2	8		0	0	0	0
1	1	−1	−1	−4	−1	1	2	−2	−2	2	2	8		0	0	0
2	−1	2	−1	−1	−4	−2	0	−4	2	−4	2	2	8		0	0
0	−1	2	−2	1	−2	−4	−1	0	2	−4	4	−2	4	8		0
−2	1	0	−1	2	0	−1	−4	4	−2	0	2	−4	0	2	8	

38: $_{\sqrt{-15}}[({}_{\sqrt{5},\infty}[\mathrm{SL}_2(5)]_1 \circ C_3) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_8$

4		2	−6	−2	2	2	6	2	−6	3	−1	−5	3	4	1	1
−2	4		2	2	4	0	4	−4	−1	−2	−4	2	0	−4	−4	−3
−2	2	4		6	6	2	6	−2	−1	−2	−2	0	−2	−7	−1	−4
2	−2	−2	4		4	−4	−2	−2	−3	−1	0	0	4	0	4	3
2	0	−2	0	4		6	2	2	1	−3	3	2	4	3	−1	0
2	0	−2	0	2	4		6	−2	−4	3	0	−2	6	2	3	4
2	0	−2	2	2	2	4		2	−4	−2	−1	−1	7	1	4	7
−2	0	2	−2	−2	−2	−2	4		1	−1	−3	−4	−5	1	−2	−1
0	−1	1	−1	−1	0	−2	1	4		5	4	−4	1	−1	1	−7
−1	2	2	−1	−1	−1	0	1	−1	4		−2	−2	−2	−2	−5	−1
1	−2	0	0	−1	0	−1	1	2	0	4		−2	0	0	6	−2
1	−2	0	0	0	0	−1	0	2	−2	2	4		2	−5	5	0
1	−2	−2	2	0	0	1	−1	−1	0	0	0	4		3	0	4
−2	2	1	−2	−1	0	−1	1	−1	2	0	−1	−1	4		−3	−1
1	−2	−1	2	−1	−1	0	0	−1	−1	0	1	2	−1	4		3
−1	−1	0	1	−2	−2	−1	1	1	−1	0	0	0	−1	1	4	

39: $_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}}{\otimes} ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2)$

4		0	0	0	0	0	0	0	0	2	2	1	−1	3	−2	−2
−2	4		0	0	0	0	0	0	1	−3	0	0	0	−3	0	1
−2	2	4		0	0	0	0	0	−1	0	−3	−3	3	−2	1	0
2	−2	−2	4		0	0	0	0	−2	2	1	0	0	4	0	−2
2	0	−2	0	4		0	0	0	1	−2	1	3	−3	1	−1	0
2	0	−2	0	2	4		0	0	3	−1	3	2	−3	0	−2	0
2	0	−2	2	2	2	4		0	−1	0	2	1	−2	2	0	−2
−2	0	2	−2	−2	−2	−2	4		−1	1	−2	−3	1	−2	0	2
0	−1	1	−1	−1	0	−2	1	4		2	−1	−1	2	−2	1	2
−1	2	2	−1	−1	−1	0	1	−1	4		−3	−3	0	0	−2	−2
1	−2	0	0	−1	0	−1	1	2	0	4		0	0	0	−1	0
1	−2	0	0	0	0	−1	0	2	−2	2	4		0	1	1	0
1	−2	−2	2	0	0	1	−1	−1	0	0	0	4		2	0	−3
−2	2	1	−2	−1	0	−1	1	−1	2	0	−1	−1	4		−2	1
1	−2	−1	2	−1	−1	0	0	−1	−1	0	1	2	−1	4		−2
−1	−1	0	1	−2	−2	−1	1	1	−1	0	0	0	−1	1	4	

40: $_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes}{}_{\infty,2}[2_-^{1+4}.\mathrm{Alt}_5]_2$

4		1	0	2	2	1	−2	−1	0	0	0	0	0	0	1	1
2	4		−2	0	1	0	−3	0	0	0	−1	0	0	0	1	0
2	1	4		2	2	4	−1	−1	−2	2	2	2	0	1	1	−1
1	2	2	4		0	2	−1	1	−2	2	0	0	0	1	0	−2
2	1	2	1	4		1	−1	2	2	0	0	0	2	1	2	−2
1	2	1	2	2	4		−2	1	0	0	−1	0	2	1	1	−2
−2	−1	0	0	−2	−1	4		−2	−3	2	3	1	0	1	0	−1
−2	−1	0	0	−2	−1	2	4		−1	2	−1	0	−1	0	1	−1
−1	−2	0	0	−1	−2	2	2	4		2	0	0	0	−1	0	−2
−1	1	0	0	−1	1	0	0	−2	4		0	2	0	0	−1	1
0	1	−1	1	1	2	0	−2	−1	0	4		−2	2	1	1	−1
0	−1	−1	0	1	1	0	−2	0	−1	2	4		2	1	0	0
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		−2	0	2
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		2	−1
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		0
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

42: $_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes}{}_{\infty,3}[\mathrm{SL}_2(9)]_2$

4		0	−1	3	−1	3	0	2	2	2	3	2	1	1	1	0
1	4		−1	1	0	1	−3	−1	−2	−2	0	−1	−1	0	0	2
2	2	4		3	0	3	−1	1	1	1	1	1	0	2	1	1
1	1	0	4		1	−1	0	−3	0	−1	−2	−2	1	−1	1	1
1	1	1	1	4		2	−1	−2	1	−3	1	−1	0	1	1	3
2	1	2	1	1	4		−1	0	−1	1	1	1	−2	−3	−2	−1
1	2	2	1	1	1	4		0	0	−1	1	−1	1	1	2	3
1	1	1	2	1	1	1	4		0	−1	−1	−1	1	0	2	1
1	−1	1	0	1	2	0	1	4		1	0	−1	−2	−1	0	−2
1	1	1	2	0	1	2	2	1	4		−1	−3	0	1	3	0
2	1	2	1	0	2	2	2	1	2	4		−1	0	0	0	−1
1	0	1	1	0	2	1	2	2	2	2	4		1	−2	1	−1
1	1	0	2	1	0	1	1	1	1	0	1	4		0	2	1
2	0	1	1	2	2	0	1	2	0	0	0	1	4		−1	1
2	1	1	1	2	1	1	1	1	0	1	−1	1	2	4		1
1	1	2	0	2	2	1	0	1	−1	1	1	1	1	1	4	

44: $_{\zeta_{10}}[C_{10}]_1 \underset{\sqrt{5}'}{\otimes}{}_{\infty,3}[\mathrm{SL}_2(3) \overset{2}{\square} C_3]_2$

6		0	−4	1	0	1	−1	3	0	0	0	1	0	6	0	−6
0	6		0	−1	0	−3	3	−3	0	0	0	3	0	−2	−2	0
−1	3	6		2	−2	0	0	−1	−1	−1	0	0	2	0	2	0
1	−1	−2	6		3	0	−2	0	−2	0	−3	3	−2	0	−3	−1
0	0	−2	3	6		0	1	1	−2	0	−3	0	−3	1	−1	−2
1	−3	−3	3	2	6		−3	0	−1	−1	−3	3	1	3	3	−1
−1	3	3	0	1	0	6		−2	0	0	0	0	−1	−1	0	−1
3	−1	−2	−1	−1	2	−1	6		2	−2	1	1	3	3	0	0
−2	−2	−2	1	−1	2	0	1	6		4	0	0	3	−2	0	3
2	−2	−2	3	3	2	0	−1	0	6		−2	0	−3	2	0	−3
−2	−2	−2	1	−1	2	0	−1	3	1	6		−1	0	−3	0	1
−1	3	0	−1	0	−3	0	−2	−2	−2	−1	6		−2	−3	−3	0
3	3	1	2	0	−1	1	0	−1	1	−1	1	6		0	−3	−3
3	1	−2	1	−1	1	−1	3	0	0	0	1	3	6		−3	0
3	1	0	2	−1	−1	−2	0	−1	1	−1	1	3	3	6		0
−3	1	2	1	1	−2	1	−3	1	−1	0	1	0	−3	0	6	

45: $_{\zeta_{10}}[C_{60}.(C_2 \times C_2)]_4$

8		0	3	3	0	−4	0	4	8	0	0	5	0	0	−7	−5
0	8		5	3	0	0	0	0	0	0	4	3	0	0	−3	3
2	2	8		0	1	0	−4	4	0	0	0	0	0	−8	0	−4
2	2	4	8		6	0	−4	5	0	0	−3	3	3	−5	−3	−1
−2	−2	−1	−3	8		3	0	−3	0	−2	3	−4	0	0	5	0
−2	0	−4	−4	2	8		4	1	0	0	3	−3	−3	3	−1	3
−2	2	1	2	0	−1	8		−3	−1	0	−3	−1	0	0	1	3
−3	−1	−4	−2	2	2	1	8		−4	5	0	0	0	−4	0	4
4	0	2	2	−1	−4	−1	−1	8		0	0	0	0	−3	−4	0
1	−1	2	0	−1	−2	−1	−1	−2	8		−5	5	5	0	−2	−3
1	−1	0	0	−2	0	1	0	3	−1	8		−4	−8	0	0	0
1	3	0	0	1	0	−2	0	3	−1	0	8		4	0	0	0
−2	2	0	−2	4	2	−2	0	−1	−3	−4	4	8		0	5	0
4	4	4	3	−2	−1	2	−4	2	−1	0	0	0	8		−5	−2
1	−3	−3	−3	1	2	−1	−1	−1	−2	1	−4	−2	−1	8		0
−1	3	2	2	−3	0	0	−2	−3	3	0	0	0	2	−4	8	

46: $_{\zeta_{16}-\zeta_{16}^{-1}}[(D_8 \otimes QD_{32}).S_3]_4$

2		0	0	0	0	0	0	0	−1	1	0	1	−1	1	−2	0
−1	2		0	0	0	0	0	0	0	1	0	−1	2	−2	1	−1
0	−1	2		0	0	0	0	0	1	−2	1	0	0	0	0	0
0	0	−1	2		0	0	0	0	1	1	−1	−1	0	1	0	0
0	0	0	−1	2		0	0	0	−2	−1	1	0	1	0	−1	−1
0	0	0	0	−1	2		0	0	1	0	−1	1	−1	0	1	1
0	0	0	0	0	−1	2		0	−1	1	1	−1	−1	1	−1	1
0	0	0	0	−1	0	0	2		1	1	−1	1	−1	−1	1	1
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

48: $_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_-}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$

2		0	0	0	0	0	0	0	2	−2	−1	3	−1	0	−1	0
−1	2		0	0	0	0	0	0	−2	2	−2	−1	2	1	−1	−2
0	−1	2		0	0	0	0	0	0	−2	1	1	−1	−2	3	2
0	0	−1	2		0	0	0	0	2	−1	1	−1	−1	3	−1	−1
0	0	0	−1	2		0	0	0	0	0	1	−2	3	−3	0	0
0	0	0	0	−1	2		0	0	−2	0	1	1	−1	2	−1	−2
0	0	0	0	0	−1	2		0	2	2	−3	2	−1	−1	2	0
0	0	0	0	−1	0	0	2		−1	2	−2	1	−2	1	−1	3
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

49: $_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[_{\sqrt{2},\infty}[\tilde{S}_4]_1 \overset{2_+}{\underset{\sqrt{5}'}{\boxtimes}} {}_{\zeta_{10}}[C_{10}]_1]_4$

4		0	2	0	−2	0	2	4	0	0	−4	−4	2	−4	−2	2
2	4		2	0	2	0	0	2	0	0	−2	−2	0	−2	−4	2
2	1	4		−2	−4	−6	4	2	2	0	−2	−4	2	−4	−2	0
1	2	2	4		0	−2	0	−2	−2	4	−2	−2	−2	−2	−2	2
2	1	2	1	4		−4	2	2	4	−4	−4	−2	0	−2	0	4
1	2	1	2	2	4		0	0	0	0	−2	−2	−2	−2	−4	4
−2	−1	0	0	−2	−1	4		−2	−2	2	4	4	0	0	0	−4
−2	−1	0	0	−2	−1	2	4		0	2	4	2	0	0	0	−4
−1	−2	0	0	−1	−2	2	2	4		2	2	2	0	0	4	−4
−1	1	0	0	−1	1	0	0	−2	4		2	−2	−2	4	−2	0
0	1	−1	1	1	2	0	−2	−1	0	4		2	−4	2	0	4
0	−1	−1	0	1	1	0	−2	0	−1	2	4		0	0	4	4
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		0	0	−2
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		0	4
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		2
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

50: $_{\sqrt{2}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[D_{16} \overset{2}{\boxtimes} {}_{\zeta_{10}}[C_{10}]_1]_4$

2		0	0	0	0	0	0	0	−2	1	1	−2	1	−2	0	2
−1	2		0	0	0	0	0	0	2	−3	0	1	1	2	−2	−1
0	−1	2		0	0	0	0	0	−2	2	−2	1	0	−2	1	−1
0	0	−1	2		0	0	0	0	1	1	0	−2	−2	1	−1	2
0	0	0	0	2		0	0	0	1	−3	2	1	−2	−1	0	1
0	0	0	0	−1	2		0	0	1	0	−2	0	1	−1	2	0
0	0	0	0	0	−1	2		0	0	1	2	−2	1	0	−1	−2
0	0	0	0	0	0	−1	2		−2	0	−1	0	−2	2	1	0
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2	

51: $_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(1,1,1)}$

2		0	0	0	0	0	0	0	4	−3	−1	2	−1	3	−2	−2
−1	2		0	0	0	0	0	0	−1	2	0	−1	2	−5	4	0
0	−1	2		0	0	0	0	0	−1	0	3	−2	−1	4	−4	−1
0	0	−1	2		0	0	0	0	1	0	−1	−2	2	−1	0	2
0	0	0	−1	2		0	0	0	−1	2	−4	4	−3	1	0	1
0	0	0	0	−1	2		0	0	−1	−2	2	−2	3	0	1	−3
0	0	0	0	0	−1	2		0	0	0	0	1	0	−2	−2	2
0	0	0	0	−1	0	0	2		2	−1	3	0	−1	−1	2	−1
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

52: $_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(i,1,1)}$

4		1	−2	−2	2	1	−3	−1	0	2	1	3	3	1	1	−1
2	4		1	2	2	6	2	−1	2	1	5	6	−2	0	0	−4
2	1	4		−4	2	1	−1	1	0	0	−3	3	−1	3	0	−2
1	2	2	4		0	2	3	2	4	2	−2	2	−2	1	3	−5
2	1	2	1	4		−2	2	2	4	−2	0	4	2	−2	−1	−3
1	2	1	2	2	4		1	−1	6	−3	1	6	−4	1	3	−3
−2	−1	0	0	−2	−1	4		0	−1	−2	−2	−1	−2	2	1	1
−2	−1	0	0	−2	−1	2	4		−2	1	−3	−2	−2	0	−2	0
−1	−2	0	0	−1	−2	2	2	4		0	−6	−4	2	−1	−3	1
−1	1	0	0	−1	1	0	0	−2	4		3	1	−5	2	−1	−3
0	1	−1	1	1	2	0	−2	−1	0	4		1	1	−2	3	−1
0	−1	−1	0	1	1	0	−2	0	−1	2	4		2	−1	4	−2
−1	1	−1	0	−2	−1	0	2	0	1	−1	−2	4		−4	−3	−1
2	0	0	−1	2	1	−2	−2	−1	−1	1	1	−2	4		0	2
2	0	0	0	0	0	−2	−1	0	0	0	0	0	2	4		1
2	1	2	1	2	0	−2	−1	−1	−1	−1	−1	0	0	0	4	

53: $_{\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})}[C_{60}.(C_2 \times C_2)]_{4,(1,1,-1)}$

4		0	0	0	0	0	0	0	2	3	4	3	0	4	−2	−2
−2	4		0	0	0	0	0	0	1	−2	−4	0	2	−2	1	4
−2	2	4		0	0	0	0	0	−1	−2	0	2	2	−2	3	1
2	−2	−2	4		0	0	0	0	5	4	4	−3	−1	0	−6	0
2	0	−2	0	4		0	0	0	2	−3	−2	4	−3	0	0	3
2	0	−2	0	2	4		0	0	−2	2	1	0	−2	6	−1	0
2	0	−2	2	2	2	4		0	4	2	−1	−1	−2	1	−2	4
−2	0	2	−2	−2	−2	−2	4		−3	−3	−2	1	0	−1	6	−1
0	−1	1	−1	−1	0	−2	1	4		−1	6	4	−1	5	0	−4
−1	2	2	−1	−1	−1	0	1	−1	4		−4	0	0	−1	−1	0
1	−2	0	0	−1	0	−1	1	2	0	4		−2	0	6	−1	−5
1	−2	0	0	0	0	−1	0	2	−2	2	4		2	3	2	0
1	−2	−2	2	0	0	1	−1	−1	0	0	0	4		−2	−4	−3
−2	2	1	−2	−1	0	−1	1	−1	2	0	−1	−1	4		2	2
1	−2	−1	2	−1	−1	0	0	−1	−1	0	1	2	−1	4		−2
−1	−1	0	1	−2	−2	−1	1	1	−1	0	0	0	−1	1	4	

54: $\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})[C_{60}.(C_2 \times C_2)]_{4,(1,-1,1)}$

4		0	0	0	0	0	0	0	−6	3	−3	0	6	−3	0	−3
−2	4		0	0	0	0	0	0	3	0	3	−3	0	0	−6	3
0	−2	4		0	0	0	0	0	−3	3	−6	6	−6	3	3	3
0	0	−2	4		0	0	0	0	0	−3	6	−6	3	−3	3	0
0	0	0	−2	4		0	0	0	6	0	−6	3	0	0	0	0
0	0	0	0	−2	4		0	0	−3	0	3	−3	0	6	−3	−3
0	0	0	0	0	−2	4		0	0	−6	3	3	0	−3	0	0
0	0	0	0	−2	0	0	4		−3	3	3	0	0	−3	0	0
−2	1	0	0	0	0	0	0	4		0	0	0	0	0	0	0
1	−2	1	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	1	−2	1	0	0	0	0	0	−2	4		0	0	0	0	0
0	0	1	−2	1	0	0	0	0	0	−2	4		0	0	0	0
0	0	0	1	−2	1	0	1	0	0	0	−2	4		0	0	0
0	0	0	0	1	−2	1	0	0	0	0	0	−2	4		0	0
0	0	0	0	0	1	−2	0	0	0	0	0	0	−2	4		0
0	0	0	0	1	0	0	−2	0	0	0	0	−2	0	0	4	

55: $\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})[C_{60}.(C_2 \times C_2)]_{4,(i,1,-1)}$

4		0	0	0	4	2	−4	0	0	0	0	0	−2	−1	2	0
−2	4		0	0	2	−4	2	−2	0	0	0	0	−1	2	−1	1
0	−2	4		0	−4	2	−4	4	0	0	0	0	2	−1	2	−2
0	−2	0	4		0	−2	4	4	0	0	0	0	0	1	−2	−2
−2	1	0	0	4		0	0	0	−2	−1	2	0	0	0	0	0
1	−2	1	1	−2	4		0	0	−1	2	−1	1	0	0	0	0
0	1	−2	0	0	−2	4		0	2	−1	2	−2	0	0	0	0
0	1	0	−2	0	−2	0	4		0	1	−2	−2	0	0	0	0
0	0	0	0	−2	1	0	0	4		0	0	0	4	2	−4	0
0	0	0	0	1	−2	1	1	−2	4		0	0	2	−4	2	−2
0	0	0	0	0	1	−2	0	0	−2	4		0	−4	2	−4	4
0	0	0	0	0	1	0	−2	0	−2	0	4		0	−2	4	4
0	0	0	0	0	0	0	0	−2	1	0	0	4		0	0	0
0	0	0	0	0	0	0	0	1	−2	1	1	−2	4		0	0
0	0	0	0	0	0	0	0	0	1	−2	0	0	−2	4		0
0	0	0	0	0	0	0	0	0	1	0	−2	0	−2	0	4	

56: $\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})[C_{60}.(C_2 \times C_2)]_{4,(i,-1,1)}$

6		0	0	0	0	0	0	−3	0	6	6	0	3	0	0	3
0	6		0	0	0	0	0	−3	6	0	0	0	3	0	0	3
−1	3	6		3	6	3	3	0	0	0	0	−3	3	0	0	3
1	−1	−2	6		−3	−6	−3	0	3	3	0	0	0	0	3	3
0	0	−2	3	6		0	−3	−3	0	0	0	9	0	3	3	0
1	−3	−3	3	2	6		0	−3	0	6	6	0	0	−3	3	0
−1	3	3	0	1	0	6		0	3	−3	0	3	3	−3	0	6
3	−1	−2	−1	−1	2	−1	6		−3	3	0	0	0	−3	0	0
−2	−2	−2	1	−1	2	0	1	6		0	−3	−6	−9	−9	−3	0
2	−2	−2	3	3	2	0	−1	0	6		3	6	−3	−3	−3	6
−2	−2	−2	1	−1	2	0	−1	3	1	6		0	−3	−6	−6	3
−1	3	0	−1	0	−3	0	−2	−2	−2	−1	6		3	6	3	0
3	3	1	2	0	−1	1	0	−1	1	−1	1	6		0	0	6
3	1	−2	1	−1	1	−1	3	0	0	0	1	3	6		−3	3
3	1	0	2	−1	−1	−2	0	−1	1	−1	1	3	3	6		3
−3	1	2	1	1	−2	1	−3	1	−1	0	1	0	−3	0	6	

57: $\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})[C_{60}.(C_2 \times C_2)]_{4,(1,-1,-1)}$

4		0	0	0	0	0	0	0	0	6	0	−3	0	0	0	0
−2	4		0	0	0	0	0	0	−6	0	−3	0	0	0	0	0
0	−2	4		0	0	0	0	0	0	3	0	6	0	0	0	0
0	0	−2	4		0	0	0	0	3	0	−6	0	0	0	0	0
−2	1	0	0	4		0	0	0	0	0	0	0	0	−6	0	3
1	−2	1	0	−2	4		0	0	0	0	0	0	6	0	3	0
0	1	−2	1	0	−2	4		0	0	0	0	0	0	−3	0	−6
0	0	1	−2	0	0	−2	4		0	0	0	0	−3	0	6	0
0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−2	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−2	4		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−2	4		0	0	0	0
0	0	0	0	0	0	0	0	−2	1	0	0	4		0	0	0
0	0	0	0	0	0	0	0	1	−2	1	0	−2	4		0	0
0	0	0	0	0	0	0	0	0	1	−2	1	0	−2	4		0
0	0	0	0	0	0	0	0	0	0	1	−2	0	0	−2	4	

58: $\sqrt{3}\cdot(\zeta_{10}-\zeta_{10}^{-1})[C_{60}.(C_2 \times C_2)]_{4,(i,-1,-1)}$

8		0	−9	−3	0	0	0	3	0	−3	−3	−6	0	0	6	−6
0	8		−9	−9	0	18	0	3	−18	3	−3	−6	0	0	6	6
2	2	8		0	−6	0	−3	0	0	0	−6	6	6	0	−3	0
2	2	4	8		−15	0	0	−3	0	0	3	−3	−3	6	0	9
−2	−2	−1	−3	8		−9	0	3	−3	3	−9	−3	0	0	−6	3
−2	0	−4	−4	2	8		3	3	0	0	9	−9	−9	−6	9	−9
−2	2	1	2	0	−1	8		−6	−6	3	−12	−9	0	0	6	3
−3	−1	−4	−2	2	2	1	8		−3	6	0	−6	−12	0	3	0
4	0	2	2	−1	−4	−1	−1	8		0	−3	3	3	9	−3	3
1	−1	2	0	−1	−2	−1	−1	−2	8		−6	6	6	−3	0	−6
1	−1	0	0	−2	0	1	0	3	−1	8		0	0	6	−3	0
1	3	0	0	1	0	−2	0	3	−1	0	8		0	12	0	0
−2	2	0	−2	4	2	−2	0	−1	−3	−4	4	8		0	−3	0
4	4	4	3	−2	−1	2	−4	2	−1	0	0	0	8		6	0
1	−3	−3	−3	1	2	−1	−1	−1	−2	1	−4	−2	−1	8		0
−1	3	2	2	−3	0	0	−2	−3	3	0	0	0	2	−4	8	

59: $\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})[\theta_{16,\infty}[Q_{32}]_1 \overset{2}{\square} C_3]_4$

2		0	0	0	0	0	0	0	4	−1	−1	−1	2	−1	1	−1
−1	2		0	0	0	0	0	0	−2	3	0	−1	2	−2	−1	−1
0	−1	2		0	0	0	0	0	0	−2	2	0	−1	2	−1	−1
0	0	−1	2		0	0	0	0	−1	1	−1	3	−2	−1	0	2
0	0	0	−1	2		0	0	0	0	−3	1	−2	4	−1	1	−3
0	0	0	0	−1	2		0	0	0	2	−2	1	−3	3	−2	2
0	0	0	0	0	−1	2		0	2	−1	0	1	0	−3	4	0
0	0	0	0	−1	0	0	2		0	3	0	−1	−2	1	0	2
0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	2	

60: $_{\sqrt{3}\cdot(\zeta_{16}-\zeta_{16}^{-1})}[D_{32} \overset{2}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_4$

2		0	0	3	0	0	0	0	0	0	0	0	0	0	3	−3
−1	2		3	−3	0	0	0	0	0	0	0	0	0	0	−3	0
0	0	2		0	0	0	0	3	0	0	0	0	0	0	0	0
0	0	−1	2		0	0	−3	0	0	0	0	0	0	0	0	0
0	0	0	0	2		0	3	0	3	0	0	0	0	0	0	0
0	0	0	0	−1	2		0	−3	0	−3	0	0	0	0	0	0
0	0	0	0	0	0	2		0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	−1	2		0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2		0	0	0	3	−3	0	0
0	0	0	0	0	0	0	0	−1	2		0	0	0	3	0	0
0	0	0	0	0	0	0	0	0	0	2		0	−3	0	3	0
0	0	0	0	0	0	0	0	0	0	−1	2		0	3	0	−3
0	0	0	0	0	0	0	0	0	0	0	0	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	2		0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2	

61: $_{\mathbb{Q}(i,\sqrt{3},\sqrt{5})}[C_{60}.C_2]_2$

4		−1	−1	3	−5	−1	1	3	−5	−3	−2	1	−2	2	1	2
1	4		0	2	−6	−1	−1	2	−2	−1	1	−2	−2	5	4	−1
−1	0	4		0	−6	3	−2	0	−3	−3	5	−1	−2	4	5	−3
1	0	0	4		1	−2	3	−2	3	0	−2	0	2	0	−4	0
−1	0	0	−1	4		2	0	−6	0	4	3	−4	4	3	3	−4
1	1	−1	0	0	4		−1	2	−3	−1	−1	5	−5	−1	0	3
−1	−1	0	−1	0	−1	4		1	1	1	0	−2	3	−1	−1	2
−1	0	0	0	0	0	1	4		3	1	−4	3	0	−5	−3	1
−1	0	1	1	2	−1	1	1	4		1	1	−3	5	1	1	−3
1	1	1	2	0	−1	−1	−1	1	4		2	−5	2	6	2	−4
0	1	1	0	−1	1	−2	−2	−1	0	4		3	−5	0	2	1
−1	0	−1	0	2	1	0	1	1	−1	−1	4		1	−1	−1	0
0	0	0	−2	0	1	1	0	−1	0	−1	−1	4		3	3	−1
0	1	2	0	−1	−1	1	1	1	0	0	−1	−1	4		0	3
1	2	−1	0	−1	2	−1	1	−1	0	0	1	1	0	4		1
−2	−1	−1	−2	0	−1	2	1	−1	−2	−1	0	1	−1	−1	4	

62: $_{\zeta_{32}-\zeta_{32}^{-1}}[QD_{64}]_2$

1		1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	1		1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1		1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1		1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1		1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1		1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1		1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1		1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1		1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1		1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1		1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1		1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1		1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1		1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	

63: $_{\zeta_{34}}[C_{34}]_1$

2		0	0	1	−2	1	−1	1	0	0	0	0	0	0	1	−1
−1	2		0	0	−1	1	1	0	−1	0	0	0	0	0	−1	1
0	−1	2		1	0	−1	0	−1	1	0	0	0	0	0	−1	1
0	0	−1	2		1	0	0	0	0	0	0	0	0	0	1	−1
0	0	0	−1	2		0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	−1	2		0	0	1	−1	−1	1	0	0	0	0
0	0	0	0	0	−1	2		0	−1	1	1	−1	0	−1	1	0
0	0	0	0	0	0	−1	2		0	0	1	−1	0	1	−1	0
0	0	0	0	0	0	0	−1	2		0	−1	1	1	−1	0	0
0	0	0	0	0	0	0	0	−1	2		0	−1	0	1	0	1
0	0	0	0	0	0	0	0	0	−1	2		1	−1	0	0	−1
0	0	0	0	0	0	0	0	0	0	−1	2		−1	1	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	2		−1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	2		0	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2		1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	2	

A.9 Dimension 18

1: ${}_{i}[C_4]_1 \otimes A_9$

2		0	−2	−1	0	−1	0	0	0	0	0	−1	0	−1	−1	−1	−1	−1
1	2		−1	−2	0	−1	0	0	0	0	0	−1	0	−1	−1	−1	−1	−1
0	0	2		0	1	−1	1	1	1	1	1	0	−1	0	0	0	−1	0
0	0	1	2		1	−1	1	1	1	1	1	0	−1	0	0	0	0	0
1	1	0	0	2		−1	0	0	0	0	0	−2	0	−1	−1	−1	0	−1
−1	−1	1	1	−1	3		1	1	1	1	1	1	−1	0	0	0	1	0
1	1	0	0	1	0	2		0	0	0	0	−1	0	−2	−1	−1	0	−1
1	1	0	0	1	0	1	2		0	0	0	−1	0	−1	−2	−1	0	−1
1	1	0	0	1	0	1	1	2		0	0	−1	0	−1	−1	−2	0	−1
1	1	0	0	1	0	1	1	1	2		0	−1	0	−1	−1	−1	0	−2
1	1	0	0	1	0	1	1	1	1	2		−1	0	−1	−1	−1	−1	−1
0	0	1	1	0	1	0	0	0	0	0	2		−1	0	0	0	0	0
−1	−1	0	0	−1	0	−1	−1	−1	−1	−1	0	2		1	1	1	0	1
0	0	1	1	0	1	0	0	0	0	0	1	0	2		0	0	0	0
0	0	1	1	0	1	0	0	0	0	0	1	0	1	2		0	0	0
0	0	1	1	0	1	0	0	0	0	0	1	0	1	1	2		0	0
−1	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	3		0
0	0	1	1	0	1	0	0	0	0	0	1	0	1	1	1	0	2	

2: ${}_{\sqrt{-3}}[\pm 3_{+}^{1+4}{:}\mathrm{Sp}_4(3)]_9$

4		0	0	0	0	0	0	0	0	−3	0	0	0	−2	−2	−2	−2	0
2	4		2	2	2	3	−1	1	0	−3	0	0	2	−1	−2	−2	−1	−2
−2	−2	4		−3	0	−3	1	1	1	3	0	0	0	2	2	2	2	0
−2	0	1	4		0	3	−2	1	1	3	0	0	0	2	−1	−1	2	0
−2	−2	2	2	4		0	1	1	1	3	0	0	0	0	0	0	0	0
0	−1	1	1	2	4		0	0	0	0	0	0	0	1	−2	−2	1	0
0	−1	−1	0	1	0	4		−1	1	0	0	0	−1	−2	0	0	−2	1
0	−1	1	1	1	0	1	4		2	3	0	0	−1	−1	1	1	−1	1
−2	−2	1	1	1	0	1	0	4		3	0	0	−1	1	0	0	1	1
1	−1	−1	−1	−1	0	0	1	1	4		0	0	−3	−2	1	−2	1	3
−2	0	0	0	0	0	0	−2	0	−2	4		−3	0	1	1	1	1	0
−2	0	2	2	2	0	0	0	0	−2	1	4		0	1	1	1	1	0
−2	0	2	2	2	0	−1	1	1	−1	0	2	4		1	1	1	1	−3
−2	−1	2	0	0	−1	−2	−1	1	0	1	1	1	4		3	0	4	1
2	2	−2	−1	−2	−2	0	1	−2	1	−1	−1	−1	−1	4		0	−2	1
−2	0	0	1	0	−2	0	−1	2	0	1	1	1	2	0	4		1	1
2	1	0	−2	−2	−1	0	1	−1	1	−1	−1	−1	0	2	−1	4		1
2	2	−2	−2	−2	0	−1	−1	−1	1	0	−2	−1	−1	1	−1	1	4	

4: ${}_{\sqrt{-3}}[\pm 3.M_{10}]_9$

6		0	0	0	0	0	0	0	0	0	−3	0	−3	0	0	0	0	0
0	6		0	0	0	0	0	0	0	0	0	0	0	0	−3	0	−3	0
0	0	6		0	0	0	0	0	3	3	0	0	0	0	0	−3	0	0
0	0	−2	6		0	0	0	0	0	0	0	0	0	0	0	−3	0	−3
−2	0	−2	0	6		0	0	0	0	0	−3	3	0	3	0	0	0	0
0	0	0	−2	0	6		0	−3	−3	0	0	3	3	0	0	0	0	9
−2	0	0	0	0	−2	6		−3	0	3	0	0	0	−3	0	0	0	−3
0	−2	−2	0	−2	1	1	6		0	0	0	0	0	0	−3	0	0	0
−2	0	1	0	0	1	0	0	6		0	0	−3	9	0	0	0	3	0
0	−2	1	0	0	0	1	0	−2	6		0	0	−3	3	0	0	0	0
1	−2	0	−2	1	−2	0	0	0	−2	6		0	0	0	0	0	0	−3
0	−2	0	0	1	1	0	0	1	0	0	6		0	0	0	0	9	3
1	0	−2	0	0	1	0	0	−3	1	0	−2	6		0	3	0	−3	3
0	0	0	−2	1	0	1	0	0	1	0	−2	0	6		0	0	−3	0
−2	1	0	−2	0	0	0	1	−2	0	0	0	1	−2	6		0	0	0
−2	−2	1	1	0	0	−2	0	0	0	0	−2	0	0	0	6		−3	0
0	1	0	0	−2	−2	0	0	1	0	0	−3	1	1	0	1	6		−3
0	0	0	1	0	−3	1	−2	−2	0	1	1	1	0	0	0	1	6	

5: ${}_{\sqrt{-7}}[\pm(\mathrm{L}_2(7) \underset{\sqrt{-7}}{\otimes} \mathrm{L}_2(7)){:}2]_9$

6		0	0	0	0	0	0	0	7	−7	0	0	−7	7	0	7	0	0
2	6		0	0	−7	0	7	0	0	0	0	0	0	0	0	0	0	−7
2	2	6		0	0	7	0	0	0	0	0	−7	0	7	0	0	0	0
2	2	2	6		0	0	0	7	0	0	0	0	0	0	0	7	0	0
2	1	2	2	6		0	−7	0	0	0	0	0	0	0	0	0	0	0
2	2	1	2	2	6		0	0	0	0	7	0	0	0	−7	7	7	0
0	−1	2	2	1	0	6		0	0	0	−7	−7	0	0	0	0	0	0
2	2	2	1	2	0	0	6		0	0	0	0	0	7	7	0	−7	0
1	0	2	2	2	2	0	2	6		0	0	0	0	7	0	7	0	0
1	2	0	2	2	2	2	0	−2	6		0	0	0	−7	0	0	0	0
2	2	−2	0	0	1	−1	2	0	2	6		7	0	0	7	7	0	0
2	2	1	2	0	−2	1	2	0	0	1	6		0	7	7	0	0	0
1	2	2	0	2	0	2	2	−2	2	0	2	6		0	0	−7	0	0
−1	0	1	0	2	2	2	−1	1	1	0	−1	0	6		0	0	7	0
0	2	2	0	2	1	2	1	0	2	−1	1	2	2	6		−7	0	−7
−1	0	0	1	2	−1	2	2	1	0	1	2	1	2	1	6		0	0
0	2	0	2	2	1	2	1	0	2	2	0	2	1	0	2	6		−7
2	1	0	0	2	0	0	0	−2	2	2	2	2	2	1	2	1	6	

6: ${}_{\sqrt{-19}}[\pm \mathrm{L}_2(19)]_9$

10		0	0	19	0	0	0	0	-19	0	0	-19	0	-19	0	0	-19	0
2	10		0	0	0	0	0	0	0	-19	19	0	0	0	19	-19	0	0
4	2	10		0	0	0	0	0	-19	0	0	-19	0	0	0	0	0	19
3	4	2	10		0	0	0	19	0	0	0	0	0	0	0	0	0	0
2	2	2	0	10		-19	0	0	0	0	0	-19	0	0	0	0	0	0
0	4	2	4	1	10		0	19	0	0	19	0	19	0	0	0	0	0
4	2	4	2	2	-2	10		0	0	0	0	-19	0	0	0	0	-19	0
2	2	2	3	4	1	4	10		-19	0	0	-19	0	0	0	0	0	0
1	2	1	2	2	4	0	3	10		0	0	0	19	0	0	0	0	0
4	3	2	2	2	-2	4	2	2	10		0	0	0	-19	0	0	0	0
2	3	-2	2	4	3	2	4	2	-2	10		-19	0	0	0	0	-19	-19
3	4	1	2	1	2	3	3	4	2	1	10		19	0	19	0	0	19
4	2	4	0	4	3	2	2	3	4	2	1	10		-19	-19	0	-19	0
3	2	2	2	-2	2	2	2	4	3	2	2	1	10		0	0	0	0
4	1	2	2	4	2	2	2	2	-2	4	3	-1	2	10		0	0	0
0	3	2	2	2	2	4	4	2	4	2	-2	2	2	-2	10		0	-19
1	4	2	-2	0	2	3	0	2	2	1	4	3	4	2	2	10		0
2	2	3	0	2	2	2	2	0	2	3	-1	2	4	2	3	4	10	

A.10 Dimension 20

2: ${}_{\infty,2}[\pm U_5(2)]_5 \overset{2(2)}{\circ} C_4$

4		0	0	2	0	2	0	1	1	1	-1	1	-1	-1	1	-1	1	-1	-1	0
0	4		-4	-2	2	0	0	1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-2
0	0	4		2	0	0	0	-1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	0
2	2	2	4		1	1	-1	0	0	0	-2	-1	0	0	2	1	-1	-1	-1	-1
0	0	-2	-1	4		0	0	2	-1	-1	1	0	-1	1	-1	-1	0	0	0	0
0	0	0	1	0	4		-4	-2	-1	1	-1	0	1	1	1	1	0	0	0	0
2	0	0	1	0	0	4		2	1	1	1	2	-1	-1	1	-1	0	0	0	0
1	-1	-1	0	2	2	2	4		-1	1	1	1	0	0	0	-1	0	0	0	0
1	-1	-1	0	1	1	1	1	4		0	0	2	0	0	0	-1	1	1	1	2
1	1	1	2	-1	1	-1	-1	0	4		-4	-2	0	0	2	1	1	-1	-1	0
1	-1	1	0	-1	1	1	1	0	0	4		2	0	-2	0	-1	1	1	-1	0
1	-1	1	1	0	2	0	1	2	2	2	4		1	-1	1	0	1	1	-1	1
-1	1	1	0	-1	-1	-1	-2	-2	0	0	-1	4		0	0	2	0	0	0	-1
-1	-1	-1	-2	1	-1	-1	0	0	-2	0	-1	0	4		-4	-2	0	0	2	1
-1	1	-1	0	1	1	-1	0	0	0	-2	-1	0	0	4		2	0	-2	0	-1
-1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	-2	2	2	2	4		1	-1	1	0
1	-1	1	1	0	0	0	0	1	1	-1	1	-2	0	0	-1	4		0	0	2
1	1	1	1	0	0	0	0	-1	1	1	1	0	-2	0	-1	0	4		-4	-2
-1	-1	1	-1	0	0	0	0	1	-1	1	1	0	0	-2	-1	0	0	4		2
0	0	2	1	0	0	0	0	0	0	0	1	-1	-1	-1	-2	2	2	2	4	

3: ${}_i[C_4]_1 \otimes [(C_6 \times S_4(3)).2]_{10}$

4		0	1	0	0	0	0	1	0	-2	-2	0	0	0	-2	0	0	-1	0	-2
1	4		0	0	0	0	0	2	0	-1	1	1	0	2	0	-2	0	-1	0	1
0	0	4		-4	-1	-2	2	0	0	0	0	0	-1	2	-1	-2	2	1	2	0
1	0	0	4		0	0	0	2	-2	1	-2	-2	0	-2	-1	2	0	-2	1	-2
-2	-1	0	1	4		0	0	-1	0	4	1	-2	0	-2	0	0	0	1	2	0
1	2	0	2	-1	4		0	4	-2	-1	-1	0	0	0	1	0	0	-2	0	0
-2	1	0	-2	1	-1	4		-1	2	1	4	1	0	1	1	-2	0	1	0	2
0	0	2	0	0	0	0	4		1	0	0	1	-1	2	-1	-2	2	2	2	0
2	-1	-2	0	-1	-1	-2	-2	6		-1	-2	-1	2	-1	0	2	0	-1	-1	0
0	0	1	0	0	0	0	-1	0	4		0	1	-2	1	2	1	-1	-1	-1	0
0	0	-2	0	0	0	0	-1	2	1	4		2	0	0	2	1	-1	0	-1	0
2	-1	-2	0	-1	-1	-2	0	4	-2	1	6		0	-1	-2	1	0	0	0	-2
-2	0	0	1	2	1	0	0	-2	0	0	-2	4		-1	2	0	0	-1	0	0
-1	-2	-2	-2	-1	-2	0	0	1	-2	1	2	-1	6		0	0	0	2	0	0
2	1	-1	1	-2	1	-2	1	0	0	1	2	0	0	6		1	-1	-2	-2	-2
1	2	2	2	-1	2	-1	0	-2	0	-2	-2	0	-2	1	6		0	-2	1	-1
-2	-2	0	-2	1	-2	1	0	-1	0	0	-1	1	2	-1	-2	4		2	0	1
-2	-2	-2	-1	1	-2	0	-2	2	1	1	0	0	2	-2	-2	1	6		-1	1
-1	-2	1	-2	1	-2	1	0	1	2	0	-1	0	0	-2	-3	2	2	6		2
-2	0	-2	0	0	0	0	0	-1	0	2	-1	2	2	2	-1	1	2	-1	6	

4: $_i[(D_8 \otimes C_4).S_3]_2 \otimes A_5$

4		0	0	0	0	0	1	-1	-1	1	1	1	1	0	0	1	-1	-1	0	0
-2	4		0	0	0	0	0	0	1	-1	-2	1	-2	1	-1	0	1	2	1	0
-2	2	4		0	0	0	0	0	1	-1	-1	0	0	-1	1	0	1	1	-1	0
-2	2	2	4		0	0	0	0	1	-1	-1	0	-2	0	0	0	1	1	0	0
2	-2	0	-2	6		-3	2	0	1	1	0	-1	3	0	2	0	-1	0	-2	0
-2	0	2	2	-1	6		-2	0	-1	-1	1	0	-1	-2	0	-2	-1	1	-2	0
-1	0	0	0	-2	0	4		0	0	0	0	0	0	0	0	2	1	0	0	2
1	0	0	0	0	-2	0	4		0	0	0	0	0	0	0	2	1	0	0	-2
1	-1	-1	-1	-1	-1	2	0	4		0	1	0	0	0	0	1	2	-1	0	1
-1	1	1	1	-1	-1	0	2	0	4		-1	0	0	0	0	1	2	-1	0	-1
1	-2	-1	-1	2	1	0	0	-1	-1	4		-1	2	0	1	0	-2	0	-1	0
-1	-1	0	0	1	2	0	-2	0	0	1	4		0	-1	0	-1	0	1	-2	1
-1	2	2	0	-1	-1	2	2	0	2	0	-2	6		1	1	2	1	0	1	0
0	1	-1	0	-2	-2	0	0	0	0	-2	-1	-1	4		-2	0	0	0	2	0
0	1	-1	0	-2	-2	0	0	0	0	-1	-2	1	2	4		0	0	-1	4	0
-1	0	0	0	-2	2	2	-2	1	-1	0	1	0	0	0	4		0	0	0	2
1	-1	-1	-1	-1	1	1	-1	2	-2	0	0	-1	0	0	2	4		0	0	1
1	-2	-1	-1	2	-1	0	0	1	-1	0	1	-2	0	-1	0	0	4		-1	0
0	1	-1	0	-2	0	0	0	0	0	-1	0	-1	2	0	0	0	-1	4		0
-2	0	0	0	-2	2	2	-2	1	-1	0	1	0	0	0	2	1	0	0	4	

5: $_i[[\pm S_6]_{10} \overset{2(2)}{\boxtimes} {}_i[C_4]_1]_{10}$

4		0	0	-1	-1	1	0	0	1	-2	0	0	1	1	-2	-2	0	0	0	-1
2	4		0	0	-2	0	0	0	1	0	-1	0	0	0	-1	-2	0	0	2	-1
2	2	4		-1	-1	-1	0	0	1	0	0	1	0	0	-1	0	-1	0	2	-1
0	1	1	4		0	-1	2	1	0	1	0	0	-1	-1	-1	0	-2	-1	1	0
0	0	0	-1	3		0	-1	-1	1	-1	0	0	1	1	0	-1	1	3	1	0
-2	-2	-2	-2	1	6		-2	-2	0	0	1	1	-1	1	2	2	2	1	-2	2
1	0	1	0	1	-2	4		0	2	-1	0	0	0	0	0	0	-2	1	0	-1
-1	0	-1	0	1	2	0	4		0	0	1	1	1	1	1	0	0	1	0	1
-1	0	0	0	-1	2	-2	0	4		2	0	1	-1	0	1	1	0	-1	0	1
0	-1	-1	-1	0	-1	0	0	-2	5		-1	0	1	1	-1	-1	1	0	0	-1
-1	1	1	0	1	1	0	2	1	-2	5		1	0	0	1	0	-1	1	2	0
0	-1	0	-1	0	1	-1	0	1	0	1	3		0	0	0	0	0	0	0	0
1	2	0	-1	1	1	0	2	-1	-1	2	0	5		1	1	-1	1	1	0	0
1	1	1	-1	0	1	-1	0	0	0	1	1	2	3		0	0	1	0	0	0
-2	-1	-1	0	1	1	-1	0	1	1	1	1	-1	0	4		0	0	1	1	0
-2	-2	0	0	1	0	0	0	0	1	1	1	-2	-1	2	4		0	1	2	0
0	1	0	2	-1	0	-2	0	0	-1	0	0	1	1	0	-1	4		-1	0	1
1	2	1	0	0	0	-1	-1	1	-1	0	0	1	1	0	-1	1	3		1	-1
2	2	0	0	-1	-2	0	0	0	1	-1	0	1	0	-1	-2	0	1	4		-1
-1	0	0	1	1	1	-1	1	1	-1	2	1	0	0	2	1	1	0	-1	3	

6: $_i[C_4]_1 \otimes A_{10}^{(2)}$

4		0	0	0	0	0	0	0	0	0	4	1	-2	-2	-2	-2	-2	-1	1	2
1	4		0	0	0	0	0	0	0	0	1	4	0	1	1	1	1	1	2	0
-2	0	4		0	0	0	0	0	0	0	-2	0	4	1	2	2	0	2	1	-2
-2	1	1	4		0	0	0	0	0	0	-2	1	1	4	1	2	1	0	-1	0
-2	1	2	1	4		0	0	0	0	0	-2	1	2	1	4	1	1	1	0	-1
-2	1	2	2	1	4		0	0	0	0	-2	1	2	2	1	4	1	2	0	-1
-2	1	0	1	1	1	4		0	0	0	-2	1	0	1	1	1	4	0	0	-2
-1	1	2	0	1	2	0	4		0	0	-1	1	2	0	1	2	0	4	2	-2
1	2	1	-1	0	0	0	2	4		0	1	2	1	-1	0	0	0	2	4	-1
2	0	-2	0	-1	-1	-2	-2	-1	4		2	0	-2	0	-1	-1	-2	-2	-1	4
0	0	0	0	0	0	0	0	0	0	4		0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	4		0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-2	0	4		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-2	1	1	4		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-2	1	2	1	4		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-2	1	2	2	1	4		0	0	0	0
0	0	0	0	0	0	0	0	0	0	-2	1	0	1	1	1	4		0	0	0
0	0	0	0	0	0	0	0	0	0	-1	1	2	0	1	2	0	4		0	0
0	0	0	0	0	0	0	0	0	0	1	2	1	-1	0	0	0	2	4		0
0	0	0	0	0	0	0	0	0	0	2	0	-2	0	-1	-1	-2	-2	-1	4	

7: $_{\infty,2}[\mathrm{SL}_2(11)]_5 \overset{2(2)}{\supset} C_4$

6		0	1	0	-1	2	2	-2	1	1	1	-2	3	1	-1	1	1	1	-1	-1
-2	6		-2	0	1	0	1	-1	0	1	2	0	-2	0	3	0	-1	-2	0	-2
1	0	6		1	-2	1	1	-2	2	-2	0	1	0	0	-1	0	2	2	2	2
-2	2	1	6		-1	-1	-1	1	-2	0	1	0	-1	0	1	1	1	-2	2	-1
1	-1	0	1	6		-1	1	-1	-1	-2	-1	-1	0	-2	-1	0	0	1	2	1
-2	2	1	1	-3	6		0	1	-1	1	-1	-2	1	0	2	0	0	0	1	-1
-2	3	1	3	1	2	6		-1	0	0	1	-1	0	-3	2	0	1	-1	1	-1
0	-1	-2	-3	1	-3	-3	6		1	-1	0	2	-1	1	-2	-1	-2	1	-1	2
-1	2	-2	0	-1	1	2	-1	6		2	2	-1	-1	-1	2	0	2	-1	-2	-2
1	-1	-2	-2	-2	1	-2	1	0	6		-2	-1	2	0	-1	0	-3	0	0	-2
1	2	-2	1	-1	1	1	-2	2	2	6		-2	1	-1	1	0	0	-2	-2	-6
2	2	1	-2	-1	0	-1	0	1	-1	0	6		-1	1	0	1	1	1	-1	0
-3	0	2	1	-2	3	2	-1	-1	0	-1	-3	6		-1	0	-1	0	0	0	1
1	2	-2	-2	0	0	1	1	1	2	1	1	-1	6		0	0	-3	-2	-1	-1
-1	-1	-3	1	-1	0	0	0	0	3	1	-2	0	0	6		0	-2	-1	0	-1
1	2	2	1	-2	2	0	-1	-2	2	2	1	1	0	0	6		0	1	1	-2
-1	1	-2	1	-2	2	-1	0	2	1	2	-1	0	-1	0	0	6		-2	-1	-2
-1	0	2	2	-1	2	1	-1	1	-2	-2	-1	2	-2	-1	-1	2	6		1	2
1	-2	0	0	0	-3	-1	1	-2	-2	-2	-1	0	-1	0	-1	-1	1	6		2
1	2	0	1	-1	-1	1	0	2	-2	0	2	-1	1	-1	0	0	2	2	6	

8: ${}_{i}[C_4]_1 \otimes A_{10}^{(3)}$

6		0	0	0	0	0	0	0	0	0	6	−1	1	−3	−2	−2	−2	−2	2	−1
−1	6		0	0	0	0	0	0	0	0	−1	6	2	−1	−1	2	−2	2	−3	2
1	2	6		0	0	0	0	0	0	0	1	2	6	−3	0	2	0	−2	0	3
−3	−1	−3	6		0	0	0	0	0	0	−3	−1	−3	6	2	1	0	0	−2	−1
−2	−1	0	2	6		0	0	0	0	0	−2	−1	0	2	6	2	3	−2	−2	−2
−2	2	2	1	2	6		0	0	0	0	−2	2	2	1	2	6	2	−2	−1	0
−2	−2	0	0	3	2	6		0	0	0	−2	−2	0	0	3	2	6	−1	0	−2
−2	2	−2	0	−2	−2	−1	6		0	0	−2	2	−2	0	−2	−2	−1	6	−2	0
2	−3	0	−2	−2	−1	0	−2	6		0	2	−3	0	−2	−2	−1	0	−2	6	0
−1	2	3	−1	−2	0	−2	0	0	6		−1	2	3	−1	−2	0	−2	0	0	6
0	0	0	0	0	0	0	0	0	0	6		0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	6		0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	2	6		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−3	−1	−3	6		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−2	−1	0	2	6		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−2	2	2	1	2	6		0	0	0	0
0	0	0	0	0	0	0	0	0	0	−2	−2	0	0	3	2	6		0	0	0
0	0	0	0	0	0	0	0	0	0	−2	2	−2	0	−2	−2	−1	6		0	0
0	0	0	0	0	0	0	0	0	0	2	−3	0	−2	−2	−1	0	−2	6		0
0	0	0	0	0	0	0	0	0	0	−1	2	3	−1	−2	0	−2	0	0	6	

9: ${}_{\sqrt{-2}}[{}_{\infty,2}[\pm U_5(2)]_5 : 2]_{10}$

4		0	0	0	0	0	0	0	0	0	−2	0	0	0	−2	0	2	−2	0	−2
−1	4		0	0	0	2	−2	2	0	0	2	0	2	0	0	0	0	0	0	0
0	0	4		0	0	2	−2	2	0	0	0	0	0	0	0	0	0	−2	2	−2
0	−2	0	4		−2	0	0	0	2	0	0	2	0	0	2	0	−2	0	0	0
0	2	−2	−1	4		0	0	0	−2	2	0	0	2	−2	0	0	0	0	0	2
0	1	−1	−2	0	4		2	0	0	0	0	0	0	2	−2	0	2	0	0	0
1	−1	−1	1	0	−1	4		0	0	0	0	0	−2	−2	0	0	−2	0	0	0
0	0	1	−1	−1	2	−2	4		0	0	−2	0	0	0	0	0	0	0	2	0
−1	0	0	−1	−1	1	−2	1	4		0	0	0	0	2	0	0	2	2	−2	0
0	0	−1	2	1	−1	1	−2	−2	4		2	0	2	0	2	0	0	0	0	2
1	1	0	−1	0	1	1	0	−1	0	4		0	0	0	0	0	0	0	0	−2
−2	−1	0	1	−1	0	−1	1	2	0	−2	4		0	0	2	0	0	2	0	2
1	−1	0	0	−1	1	1	0	−2	1	2	−1	4		0	0	2	0	−2	2	0
0	0	−2	−1	1	1	1	0	1	−1	1	0	0	4		0	0	0	2	0	0
−1	0	1	−1	−1	0	−1	1	0	−1	0	1	1	−1	4		0	0	0	0	0
−1	−1	2	0	−2	0	0	0	0	0	0	1	1	−2	1	4		0	0	0	0
1	0	−1	−1	1	1	1	0	−1	0	1	−2	1	2	−2	−1	4		0	2	0
1	1	−1	−1	2	−1	1	−1	−1	0	1	−1	0	1	1	−2	0	4		0	0
−1	−1	1	1	−2	0	0	1	−1	0	0	0	1	−2	1	2	−1	−2	4		0
−1	0	1	0	−1	−1	0	−1	−1	1	0	0	1	−2	2	2	−1	0	1	4	

10: ${}_{\sqrt{-2}}[2.M_{12} : 2]_{10}$

4		2	−2	0	−2	−2	0	0	0	2	2	0	0	−2	0	−2	−2	2	0	0
−2	4		0	−2	2	0	0	0	0	0	0	0	0	2	0	2	2	2	0	2
1	−1	4		2	0	0	0	0	0	0	0	0	0	−2	0	−2	−2	0	0	0
0	0	0	4		−2	0	−2	−2	−2	0	0	2	0	0	2	0	0	0	0	0
1	−2	2	−1	4		−2	0	0	0	0	−2	0	2	0	0	−2	0	0	0	0
1	−1	0	0	0	4		0	0	2	0	0	−2	2	0	−2	0	−2	0	2	−2
0	0	2	−1	2	−1	4		0	0	0	0	0	0	−2	0	−2	0	0	0	0
−1	0	2	0	1	−1	1	4		−2	0	0	2	0	0	0	0	0	0	−2	0
0	−1	2	0	2	0	1	2	4		0	−2	2	0	0	2	0	0	0	0	0
−2	2	0	1	−1	−2	0	1	1	4		−2	2	−2	2	2	2	2	0	0	2
−1	2	0	1	−1	0	0	0	−1	1	4		0	0	2	0	2	0	0	0	2
−2	2	−2	1	−2	−1	0	−1	−2	1	0	4		0	0	0	0	2	0	0	0
2	−2	1	0	2	−1	0	0	0	−1	0	−2	4		0	0	0	0	0	0	2
−1	−1	−1	−1	1	−1	0	−1	0	0	−2	1	0	4		0	−2	2	−2	0	0
−1	0	−2	0	−1	0	−1	−1	−1	−1	−1	2	−1	1	4		0	0	0	0	−2
−2	0	0	−1	0	0	−1	1	1	1	−1	0	−1	2	0	4		2	−2	0	0
−1	0	0	−1	0	−2	1	0	−1	0	−1	1	0	2	0	1	4		−2	−2	0
−1	1	−1	−1	0	−2	0	1	−1	1	1	0	1	0	−1	0	1	4		−2	2
0	1	1	0	0	−1	0	1	1	1	0	−1	0	0	−2	1	1	1	4		2
−1	0	1	−2	1	−2	2	1	1	1	−1	0	0	1	0	1	2	1	0	4	

11: ${}_{\sqrt{-2}}[\mathrm{GL}_2(3)]_2 \otimes A_5$

4		2	−2	−2	2	2	−2	0	2	0	0	0	0	2	0	−2	0	0	−2	−2
−1	4		0	0	0	−2	0	0	0	0	−2	2	−2	−2	2	2	−2	0	2	2
0	0	4		−2	0	0	0	0	0	0	0	−2	0	0	−2	0	2	0	0	0
0	−2	0	4		0	0	0	0	−2	0	0	−2	2	−2	0	0	0	0	0	0
0	2	−2	−1	4		−2	0	0	0	0	0	2	0	0	4	0	−2	2	0	2
0	1	−1	−2	0	4		−2	0	0	0	0	0	−2	0	0	0	0	0	0	0
1	−1	−1	1	0	−1	4		2	0	0	2	0	2	0	0	0	0	0	0	−2
0	0	1	−1	−1	2	−2	4		0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	−1	−1	1	−2	1	4		2	−2	2	−2	0	−2	0	0	−2	0	0
0	0	−1	2	1	−1	1	−2	−2	4		0	−2	2	−2	2	0	0	0	2	2
1	1	0	−1	0	1	1	0	−1	0	4		0	−2	−2	0	0	−2	0	0	0
−2	−1	0	1	−1	0	−1	1	2	0	−2	4		2	−2	0	2	0	−2	2	2
1	−1	0	0	−1	1	1	0	−2	1	2	−1	4		0	0	0	0	0	0	0
0	0	−2	−1	1	1	1	0	1	−1	1	0	0	4		0	0	−2	0	0	0
−1	0	1	−1	−1	0	−1	1	0	−1	0	1	1	−1	4		2	0	−2	2	0
−1	−1	2	0	−2	0	0	0	0	0	0	1	1	−2	1	4		2	0	0	0
1	0	−1	−1	1	1	1	0	−1	0	1	−2	1	2	−2	−1	4		2	−2	0
1	1	−1	−1	2	−1	1	−1	−1	0	1	−1	0	1	1	−2	0	4		0	0
−1	−1	1	1	−2	0	0	1	−1	0	0	0	1	−2	1	2	−1	−2	4		−2
−1	0	1	0	−1	−1	0	−1	−1	1	0	0	1	−2	2	2	−1	0	1	4	

13: ${}_{\infty,2}[\pm U_5(2)]_5 \circ C_3$

4		−1	0	−2	0	2	1	2	−1	−2	1	0	3	2	1	1	1	1	1	−1
−1	4		0	0	0	−1	−1	−2	0	0	−1	−1	−1	−2	2	−1	−2	3	−1	2
0	0	4		0	0	−1	1	−1	−2	−1	2	−2	2	0	3	0	−1	1	1	1
0	−2	0	4		1	0	−1	3	−1	0	−1	−1	0	−1	−1	−2	1	−1	1	−2
0	2	−2	−1	4		0	−2	−1	1	−1	−2	−1	−1	−1	−1	0	1	0	0	1
0	1	−1	−2	0	4		−1	0	−1	3	1	2	1	−1	2	2	−1	1	0	3
1	−1	−1	1	0	−1	4		0	0	1	−3	1	−1	1	−3	−2	1	−1	−2	−2
0	0	1	−1	−1	2	−2	4		−1	0	2	−1	0	0	3	2	0	1	1	3
−1	0	0	−1	−1	1	−2	1	4		0	3	0	0	−1	0	2	−1	−1	1	−1
0	0	−1	2	1	−1	1	−2	−2	4		−2	0	−1	−1	−1	−2	0	2	−2	−1
1	1	0	−1	0	1	1	0	−1	0	4		0	0	−1	0	0	−1	3	0	2
−2	−1	0	1	−1	0	−1	1	2	0	−2	4		−3	0	−1	−1	0	−1	0	−2
1	−1	0	0	−1	1	1	0	−2	1	2	−1	4		2	1	−1	−1	2	−1	1
0	0	−2	−1	1	1	1	0	1	−1	1	0	0	4		−3	0	0	−1	0	0
−1	0	1	−1	−1	0	−1	1	0	−1	0	1	1	−1	4		−1	0	1	−1	0
−1	−1	2	0	−2	0	0	0	0	0	0	1	1	−2	1	4		−1	0	0	0
1	0	−1	−1	1	1	1	0	−1	0	1	−2	1	2	−2	−1	4		0	−1	3
1	1	−1	−1	2	−1	1	−1	−1	0	1	−1	0	1	1	−2	0	4		0	0
−1	−1	1	1	−2	0	0	1	−1	0	0	0	1	−2	1	2	−1	−2	4		1
−1	0	1	0	−1	−1	0	−1	−1	1	0	0	1	−2	2	2	−1	0	1	4	

14: ${}_{\sqrt{-3}}[\pm S_4(3) \circ C_3]_5 \bigotimes_{\sqrt{-3}} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$

4		1	0	2	0	−2	−1	−2	1	2	−1	0	−3	−2	−1	−1	−1	−1	−1	1
−1	4		−2	−2	2	−1	1	−2	0	0	1	−1	1	2	2	−1	0	3	−1	2
0	0	4		−4	0	1	1	1	0	−3	0	−2	−2	0	1	0	1	1	1	1
0	−2	0	4		−1	0	−1	1	−1	0	−1	1	0	−3	1	2	−1	−1	1	0
0	2	−2	−1	4		−4	0	−3	1	1	0	1	1	1	1	0	−1	2	−2	1
0	1	−1	−2	0	4		−1	0	−1	1	1	−2	−1	1	−2	−2	1	1	−2	−1
1	−1	−1	1	0	−1	4		0	0	3	−1	1	1	−1	−1	0	1	−1	0	−2
0	0	1	−1	−1	2	−2	4		−3	0	2	−3	0	0	1	−2	0	3	−1	1
−1	0	0	−1	−1	1	−2	1	4		0	1	2	0	1	0	0	−1	−1	1	1
0	0	−1	2	1	−1	1	−2	−2	4		−2	0	1	−3	1	2	0	−2	2	1
1	1	0	−1	0	1	1	0	−1	0	4		−2	0	−3	0	2	−1	−1	2	2
−2	−1	0	1	−1	0	−1	1	2	0	−2	4		3	0	1	1	0	−1	2	0
1	−1	0	0	−1	1	1	0	−2	1	2	−1	4		−2	−1	1	1	−2	1	−1
0	0	−2	−1	1	1	1	0	1	−1	1	0	0	4		−1	0	0	1	0	0
−1	0	1	−1	−1	0	−1	1	0	−1	0	1	1	−1	4		1	0	−1	1	0
−1	−1	2	0	−2	0	0	0	0	0	0	1	1	−2	1	4		1	−2	2	−2
1	0	−1	−1	1	1	1	0	−1	0	1	−2	1	2	−2	−1	4		2	−1	−1
1	1	−1	−1	2	−1	1	−1	−1	0	1	−1	0	1	1	−2	0	4		0	2
−1	−1	1	1	−2	0	0	1	−1	0	0	0	1	−2	1	2	−1	−2	4		−3
−1	0	1	0	−1	−1	0	−1	−1	1	0	0	1	−2	2	2	−1	0	1	4	

15: ${}_{\sqrt{-3}}[\pm U_4(2) \circ C_3]_{10}$

6		0	3	0	−3	−3	0	0	0	−3	0	−3	0	0	0	−3	0	−3	−3	0
2	6		3	0	0	−3	−3	−3	−3	−3	0	−3	0	3	0	0	0	0	0	−3
3	3	6		0	−3	−9	−3	0	−3	−3	0	0	3	0	−3	−3	0	−3	0	−3
2	2	2	6		0	−3	0	−3	−3	0	−3	0	3	0	−3	−3	−3	−3	−3	−3
3	2	3	2	6		−3	0	0	0	−3	0	0	3	0	0	0	3	0	0	0
3	3	3	1	3	6		3	3	3	0	3	3	3	3	0	0	3	3	0	0
2	3	3	2	2	3	6		0	0	0	3	0	3	3	0	0	0	0	0	0
2	1	2	1	2	1	2	6		0	3	0	−3	3	3	3	0	3	−3	3	3
2	3	3	1	2	3	0	2	6		0	0	0	3	0	0	0	3	0	3	0
1	3	1	2	1	2	2	1	2	6		0	−3	0	0	0	3	0	0	0	−3
2	2	2	3	0	1	1	2	2	2	6		0	3	3	0	0	0	−3	0	−3
1	1	2	2	2	1	2	3	2	1	2	6		9	3	3	0	0	0	3	0
2	2	1	1	1	−1	1	3	1	2	1	3	6		0	3	0	−3	−3	0	0
2	1	2	2	2	1	−1	1	2	2	1	1	2	6		−3	0	0	−3	0	−3
2	2	1	1	2	2	2	1	0	0	0	−1	1	1	6		3	0	0	0	3
3	2	1	3	2	2	0	0	2	1	2	0	0	2	1	6		3	0	−3	0
2	2	2	3	1	1	0	1	3	0	2	2	1	2	0	3	6		0	3	0
3	2	3	1	0	3	2	1	2	2	3	2	1	1	0	2	2	6		0	0
1	2	0	3	0	0	0	−1	1	2	2	1	2	2	2	3	3	2	6		0
2	3	1	3	0	2	2	1	2	3	3	0	0	1	1	2	2	2	2	6	

16: ${}_{\sqrt{-3}}[\pm \mathrm{L}_2(11) \overset{2(3)}{\otimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$

4		1	3	6	3	1	−1	2	0	−1	0	1	−2	1	−1	1	2	−1	0	1
−1	4		0	−1	−1	−1	1	−1	0	0	1	−2	0	0	3	−1	0	0	−2	−2
−1	0	4		0	0	−2	2	−1	4	0	−3	−1	0	−1	2	−3	2	1	0	−3
−2	1	2	4		0	−2	2	−1	3	−1	−3	−2	−2	1	2	−2	2	2	−3	−2
−1	−1	2	2	4		0	0	−2	3	0	−3	0	0	1	0	−3	1	2	−3	−2
1	1	−2	0	−2	4		2	2	−1	−1	1	−1	−3	0	0	2	1	−1	−1	1
−1	−1	2	0	2	−2	4		−2	3	2	−1	1	2	−2	−1	−3	−1	0	1	−1
0	−1	1	1	0	0	0	4		1	−1	−1	0	−1	3	0	0	1	−1	1	−2
−2	0	0	1	1	−1	1	−1	4		3	−2	2	0	−1	0	0	−2	3	−1	1
1	0	−2	−1	−2	1	−2	−1	1	4		1	1	−2	1	0	2	0	0	0	2
2	−1	1	−1	1	−1	1	1	−2	−1	4		2	0	0	0	−2	1	−2	0	−1
1	−2	−1	0	0	1	−1	2	0	1	0	4		−2	3	−1	2	0	1	−1	0
2	0	−2	−2	−2	1	−2	−1	0	2	0	0	4		−1	−1	2	1	1	0	2
−1	2	1	1	1	0	0	−1	−1	−1	0	−1	−1	4		2	−2	1	0	−3	−2
−1	−1	2	0	0	−2	1	0	0	0	0	−1	−1	0	4		1	−1	−2	4	1
1	−1	−1	0	−1	2	−1	2	−2	0	0	2	0	0	−1	4		2	−1	1	0
−2	2	2	2	1	−1	1	−1	2	0	−1	−2	−1	1	1	−2	4		1	−1	−2
−1	2	−1	0	−2	1	−2	−1	1	2	−2	−1	1	0	0	−1	1	4		0	1
2	−2	0	−1	1	−1	1	1	−1	0	2	1	0	−1	0	1	−1	−2	4		−1
1	−2	−1	0	0	1	−1	2	−1	0	1	2	0	−2	−1	2	−2	−1	1	4	

17: ${}_{\sqrt{-3}}[{}_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \overset{2(3)}{\boxtimes} {}_{\sqrt{-3}}[C_6]_1]_{10}$

8		−2	−6	−3	2	1	−1	3	−3	4	2	0	−4	−3	−2	−1	−3	−2	−3	2
2	8		−6	−4	2	0	−3	3	0	−2	−2	0	0	−2	−3	−1	−4	−3	−1	−2
−2	2	8		0	2	0	−4	0	−1	−2	−1	4	−1	−3	3	0	0	6	3	2
1	0	0	8		2	0	0	−4	−1	2	2	0	0	−6	−2	−2	2	−3	0	−3
−2	0	0	−2	8		0	3	−2	4	−6	6	3	3	2	−6	−4	0	−4	1	−3
−1	4	2	0	2	8		−6	0	3	0	2	−3	0	0	−6	−2	−1	0	3	−2
−1	−1	0	0	3	0	8		−2	3	0	4	6	0	2	0	0	6	−2	4	1
1	−1	−2	0	2	0	−2	8		3	−3	4	1	0	6	−6	0	−2	−4	6	−4
1	−2	1	1	0	1	−1	1	8		4	0	−4	−6	−3	−4	−6	0	2	3	0
0	−2	2	−2	2	0	0	1	0	8		3	3	0	−1	2	−4	−2	2	−2	3
−2	2	1	0	0	2	0	0	0	−3	8		0	0	2	0	0	−4	0	4	−6
2	0	0	0	1	−1	2	1	0	−1	2	8		−4	−2	0	0	0	0	2	−3
0	2	1	−2	3	2	4	0	2	0	0	0	8		3	−3	−3	0	1	6	3
−1	0	3	2	0	2	2	2	3	−1	2	2	3	8		−3	0	2	0	12	0
0	1	3	−2	2	0	0	2	0	2	−2	2	1	1	8		0	2	0	3	2
1	−1	2	2	0	0	0	2	2	0	0	2	1	4	4	8		0	3	6	3
3	0	2	2	0	1	0	2	2	2	0	2	0	2	2	4	8		1	2	4
2	1	2	−1	0	0	−2	0	2	2	−2	0	1	0	4	1	−1	8		0	0
1	1	3	4	−1	1	0	−2	3	0	0	2	0	4	−1	2	2	0	8		0
2	2	2	−1	1	2	1	0	0	3	−2	−1	3	0	2	1	0	4	0	8	

18: ${}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(2)}$

8		0	0	0	0	0	0	0	0	0	12	3	−6	−6	−6	−6	−6	−3	3	6
2	8		0	0	0	0	0	0	0	0	3	12	0	3	3	3	3	3	6	0
−4	0	8		0	0	0	0	0	0	0	−6	0	12	3	6	6	0	6	3	−6
−4	2	2	8		0	0	0	0	0	0	−6	3	3	12	3	6	3	0	−3	0
−4	2	4	2	8		0	0	0	0	0	−6	3	6	3	12	3	3	3	0	−3
−4	2	4	4	2	8		0	0	0	0	−6	3	6	6	3	12	3	6	0	−3
−4	2	0	2	2	2	8		0	0	0	−6	3	0	3	3	3	12	0	0	−6
−2	2	4	0	2	4	0	8		0	0	−3	3	6	0	3	6	0	12	6	−6
2	4	2	−2	0	0	0	4	8		0	3	6	3	−3	0	0	0	6	12	−3
4	0	−4	0	−2	−2	−4	−4	−2	8		6	0	−6	0	−3	−3	−6	−6	−3	12
−4	−1	2	2	2	2	2	1	−1	−2	8		0	0	0	0	0	0	0	0	0
−1	−4	0	−1	−1	−1	−1	−1	−2	0	2	8		0	0	0	0	0	0	0	0
2	0	−4	−1	−2	−2	0	−2	−1	2	−4	0	8		0	0	0	0	0	0	0
2	−1	−1	−4	−1	−2	−1	0	1	0	−4	2	2	8		0	0	0	0	0	0
2	−1	−2	−1	−4	−1	−1	−1	0	1	−4	2	4	2	8		0	0	0	0	0
2	−1	−2	−2	−1	−4	−1	−2	0	1	−4	2	4	4	2	8		0	0	0	0
2	−1	0	−1	−1	−1	−4	0	0	2	−4	2	0	2	2	2	8		0	0	0
1	−1	−2	0	−1	−2	0	−4	−2	2	−2	2	4	0	2	4	0	8		0	0
−1	−2	−1	1	0	0	0	−2	−4	1	2	4	2	−2	0	0	0	4	8		0
−2	0	2	0	1	1	2	2	1	−4	4	0	−4	0	−2	−2	−4	−4	−2	8	

19: ${}_{\sqrt{-3}}[C_6]_1 \otimes A_{10}^{(3)}$

12		0	0	0	0	0	0	0	0	0	18	−3	3	−9	−6	−6	−6	−6	6	−3
−2	12		0	0	0	0	0	0	0	0	−3	18	6	−3	−3	6	−6	6	−9	6
2	4	12		0	0	0	0	0	0	0	3	6	18	−9	0	6	0	−6	0	9
−6	−2	−6	12		0	0	0	0	0	0	−9	−3	−9	18	6	3	0	0	−6	−3
−4	−2	0	4	12		0	0	0	0	0	−6	−3	0	6	18	6	9	−6	−6	−6
−4	4	4	2	4	12		0	0	0	0	−6	6	6	3	6	18	6	−6	−3	0
−4	−4	0	0	6	4	12		0	0	0	−6	−6	0	0	9	6	18	−3	0	−6
−4	4	−4	0	−4	−4	−2	12		0	0	−6	6	−6	0	−6	−6	−3	18	−6	0
4	−6	0	−4	−4	−2	0	−4	12		0	6	−9	0	−6	−6	−3	0	−6	18	0
−2	4	6	−2	−4	0	−4	0	0	12		−3	6	9	−3	−6	0	−6	0	0	18
−6	1	−1	3	2	2	2	2	−2	1	12		0	0	0	0	0	0	0	0	0
1	−6	−2	1	1	−2	2	−2	3	−2	−2	12		0	0	0	0	0	0	0	0
−1	−2	−6	3	0	−2	0	2	0	−3	2	4	12		0	0	0	0	0	0	0
3	1	3	−6	−2	−1	0	0	2	1	−6	−2	−6	12		0	0	0	0	0	0
2	1	0	−2	−6	−2	−3	2	2	2	−4	−2	0	4	12		0	0	0	0	0
2	−2	−2	−1	−2	−6	−2	2	1	0	−4	4	4	2	4	12		0	0	0	0
2	2	0	0	−3	−2	−6	1	0	2	−4	−4	0	0	6	4	12		0	0	0
2	−2	2	0	2	2	1	−6	2	0	−4	4	−4	0	−4	−4	−2	12		0	0
−2	3	0	2	2	1	0	2	−6	0	4	−6	0	−4	−4	−2	0	−4	12		0
1	−2	−3	1	2	0	2	0	0	−6	−2	4	6	−2	−4	0	−4	0	0	12	

20: ${}_{\infty,2}[\mathrm{SL}_2(11)]_5 \circ C_3$

6		2	−5	0	3	−4	−2	2	3	−1	1	4	−9	1	1	−3	1	−3	−1	1
−2	6		2	0	−3	2	−1	−1	0	−1	−2	−2	4	0	−3	−2	1	2	0	2
1	0	6		−1	6	−3	−1	2	−2	0	0	1	−4	2	−1	−2	−2	−4	−2	−2
−2	2	1	6		3	−1	3	−1	2	−2	1	−2	3	0	−3	−3	−3	0	0	1
1	−1	0	1	6		1	1	3	3	0	1	3	0	4	1	−2	0	−3	−2	1
−2	2	1	1	−3	6		0	−3	1	−1	3	−2	1	−2	−2	0	2	2	−1	1
−2	3	1	3	1	2	6		1	2	−2	−1	1	2	3	−4	−4	1	3	1	3
0	−1	−2	−3	1	−3	−3	6		−3	3	−2	0	1	1	4	5	0	−1	1	0
−1	2	−2	0	−1	1	2	−1	6		−2	−4	−1	3	−3	0	−2	0	3	2	0
1	−1	−2	−2	−2	1	−2	1	0	6		2	−1	−2	−2	1	2	3	2	0	2
1	2	−2	1	−1	1	1	−2	2	2	6		0	−1	1	−3	−2	0	2	2	6
2	2	1	−2	−1	0	−1	0	1	−1	0	6		−1	−1	2	−1	3	−1	−3	−2
−3	0	2	1	−2	3	2	−1	−1	0	−1	−3	6		1	−2	3	−2	0	2	1
1	2	−2	−2	0	0	1	1	1	2	1	1	−1	6		2	−2	5	4	−1	3
−1	−1	−3	1	−1	0	0	0	0	3	1	−2	0	0	6		2	2	3	2	1
1	2	2	1	−2	2	0	−1	−2	2	2	1	1	0	0	6		−2	−1	−1	4
−1	1	−2	1	−2	2	−1	0	2	1	2	−1	0	−1	0	0	6		2	3	2
−1	0	2	2	−1	2	1	−1	1	−2	−2	−1	2	−2	−1	−1	2	6		−1	−2
1	−2	0	0	0	−3	−1	1	−2	−2	−2	−1	0	−1	0	−1	−1	1	6		−2
1	2	0	1	−1	−1	1	0	2	−2	0	2	−1	1	−1	0	0	2	2	6	

21: $_{\sqrt{-7}}[2.M_{22}\!:\!2]_{10}$

8		0	0	0	−7	0	0	0	0	−7	0	0	0	0	0	0	0	0	0	0
4	8		0	0	0	−7	0	0	0	−7	0	0	0	0	7	0	0	0	0	0
−2	0	8		0	0	0	0	0	0	7	0	0	7	0	0	7	0	−7	0	0
2	2	2	8		0	0	0	0	0	0	0	0	0	7	7	7	7	0	0	0
1	2	4	4	8		−7	0	0	0	0	0	0	0	0	0	0	0	0	−7	0
0	3	−2	2	−1	8		0	0	0	0	0	0	0	0	7	0	7	0	0	0
−4	−4	2	−4	−2	−4	8		0	0	7	0	0	0	0	0	0	−7	0	0	0
4	4	0	2	2	0	−4	8		0	−7	0	0	0	0	0	0	0	−7	0	0
4	4	−2	4	2	2	−4	4	8		−7	0	0	0	0	7	0	0	0	−7	0
1	3	3	2	2	2	−3	3	−1	8		−7	−7	−7	−7	0	0	7	−7	0	7
−4	−4	4	0	2	−4	4	−4	−4	−1	8		0	0	0	−7	0	0	0	0	0
−4	−4	0	−4	−2	−2	4	−4	−4	−1	4	8		0	0	−7	−7	0	0	0	0
−4	−4	3	0	0	0	4	−4	−4	1	4	4	8		0	0	0	0	0	0	0
−4	−4	4	−1	0	−2	4	−4	−4	1	4	4	4	8		0	0	0	0	0	0
0	−1	0	−3	0	−3	2	2	−1	0	−1	1	0	2	8		−7	−7	7	0	0
−2	0	−1	−1	−4	4	0	−2	−2	2	−2	1	2	2	1	8		0	7	0	0
0	4	2	1	0	1	−1	2	0	3	0	−2	−2	0	−1	2	8		0	0	0
0	−4	−1	−4	−2	−4	2	−1	−4	−1	2	4	0	2	3	−1	−2	8		7	0
−4	−2	4	2	3	0	0	0	−1	2	2	0	2	4	0	0	0	−1	8		−7
4	4	−2	0	−2	2	−4	4	0	3	−4	−4	−4	−4	0	2	4	0	−3	8	

22: $_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \underset{\sqrt{-11}}{\otimes} {}_{\infty,2}[\mathrm{SL}_2(3)]_1$

4		−1	0	6	0	−6	5	−6	−5	2	−3	−4	−1	−2	−3	−3	1	1	1	−1
−1	4		4	0	0	3	−5	6	0	0	−1	3	−1	−6	2	3	−6	−1	3	−2
0	0	4		4	−4	3	1	3	−2	−1	6	−2	6	4	−1	0	3	−3	5	−3
0	−2	0	4		1	−4	3	−5	3	0	−1	3	−4	7	−1	−2	1	3	−3	2
0	2	−2	−1	4		0	−2	3	1	−1	−6	3	−5	−5	−1	4	−3	−4	0	1
0	1	−1	−2	0	4		−1	0	−1	3	−3	2	1	−5	2	−2	−5	1	0	−1
1	−1	−1	1	0	−1	4		0	0	−3	−3	1	−1	1	1	−2	1	3	−2	2
0	0	1	−1	−1	2	−2	4		3	0	2	−1	0	0	−5	−2	0	−3	1	−5
−1	0	0	−1	−1	1	−2	1	4		0	7	−4	4	−1	4	2	−1	3	−3	3
0	0	−1	2	1	−1	1	−2	−2	4		−6	4	−5	3	−1	−2	0	2	−2	−1
1	1	0	−1	0	1	1	0	−1	0	4		−4	4	−1	−4	−4	3	3	0	−2
−2	−1	0	1	−1	0	−1	1	2	0	−2	4		1	4	3	−1	0	3	−4	2
1	−1	0	0	−1	1	1	0	−2	1	2	−1	4		2	−3	−9	3	2	−1	−3
0	0	−2	−1	1	1	1	0	1	−1	1	0	0	4		1	−4	0	3	−4	0
−1	0	1	−1	−1	0	−1	1	0	−1	0	1	1	−1	4		−1	4	1	3	0
−1	−1	2	0	−2	0	0	0	0	0	0	1	1	−2	1	4		3	0	0	0
1	0	−1	−1	1	1	1	0	−1	0	1	−2	1	2	−2	−1	4		−4	−1	−5
1	1	−1	−1	2	−1	1	−1	−1	0	1	−1	0	1	1	−2	0	4		0	0
−1	−1	1	1	−2	0	0	1	−1	0	0	0	1	−2	1	2	−1	−2	4		−3
−1	0	1	0	−1	−1	0	−1	−1	1	0	0	1	−2	2	2	−1	0	1	4	

23: $_{\sqrt{-11}}[\pm \mathrm{L}_2(11)]_5 \otimes A_2$

8		0	0	11	0	11	−11	11	11	0	0	0	0	11	0	11	11	0	11	0
2	8		0	0	0	0	−11	11	0	0	0	0	0	0	11	11	0	11	−11	0
−2	2	8		0	0	0	0	0	−11	0	−11	0	−11	−11	11	0	0	0	−11	0
1	0	0	8		0	0	0	0	−11	0	0	0	0	0	0	0	0	−11	0	−11
−2	0	0	−2	8		0	11	0	0	0	0	−11	11	0	0	0	0	0	−11	11
−1	4	2	0	2	8		0	0	−11	0	0	−11	0	0	0	0	−11	0	−11	0
−1	−1	0	0	3	0	8		0	−11	0	0	0	0	0	0	0	0	0	0	11
1	−1	−2	0	2	0	−2	8		11	11	0	−11	0	0	0	0	0	0	0	0
1	−2	1	1	0	1	−1	1	8		0	0	0	0	11	0	0	0	0	11	0
0	−2	2	−2	2	0	0	1	0	8		−11	−11	0	−11	0	0	0	0	0	11
−2	2	1	0	0	2	0	0	0	−3	8		0	0	0	0	0	0	0	0	0
2	0	0	0	1	−1	2	1	0	−1	2	8		0	0	0	0	0	0	0	11
0	2	1	−2	3	2	4	0	2	0	0	0	8		11	11	11	0	11	0	11
−1	0	3	2	0	2	2	2	3	−1	2	2	3	8		11	0	0	0	0	0
0	1	3	−2	2	0	0	2	0	2	−2	2	1	1	8		0	0	0	−11	0
1	−1	2	2	0	0	0	2	2	0	0	2	1	4	4	8		0	−11	0	−11
3	0	2	2	0	1	0	2	2	2	0	2	0	2	2	4	8		−11	0	0
2	1	2	−1	0	0	−2	0	2	2	−2	0	1	0	4	1	−1	8		0	0
1	1	3	4	−1	1	0	−2	3	0	0	2	0	4	−1	2	2	0	8		0
2	2	2	−1	1	2	1	0	0	3	−2	−1	3	0	2	1	0	4	0	8	

24: $_{\sqrt{-19}}[\mathrm{SL}_2(19)]_9$

1		1	1	−1	1	−1	1	−1	1	1	−1	−1	1	1	1	−1	1	−1	1	1
0	1		1	−1	−1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1
0	0	1		1	1	1	1	1	1	1	−1	1	1	−1	1	1	1	−1	−1	−1
0	0	0	1		1	1	1	−1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1
0	0	0	0	1		1	−1	1	1	−1	−1	1	1	1	−1	−1	−1	−1	−1	1
0	0	0	0	0	1		1	−1	1	−1	1	1	1	1	−1	1	1	−1	1	−1
0	0	0	0	0	0	1		1	−1	1	1	−1	1	−1	−1	1	−1	−1	1	1
0	0	0	0	0	0	0	1		−1	1	1	−1	1	1	1	1	1	−1	−1	1
0	0	0	0	0	0	0	0	1		−1	1	−1	1	−1	1	−1	1	1	−1	−1
0	0	0	0	0	0	0	0	0	1		1	1	−1	−1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1		1	1	−1	1	1	−1	−1	−1	−1
0	0	0	0	0	0	0	0	0	0	0	1		−1	−1	1	1	−1	−1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1		−1	−1	−1	1	−1	1	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	1		−1	1	−1	1	−1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		1	−1	1	1	−1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		1	1	−1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		−1	−1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		−1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	

25: ${}_{\zeta_{10}}[\pm 5_{+}^{1+2}.\mathrm{Sp}_2(5)]_5$

4		−1	−2	−3	−1	1	2	0	0	1	1	−2	2	2	−2	0	0	1	−1	−2
−1	4		0	1	0	0	−1	0	1	−2	1	0	−1	1	1	−2	0	−1	−2	1
1	1	4		−1	1	3	3	−1	1	−2	−2	−2	−2	−1	0	−1	−2	0	−2	1
−1	0	1	4		1	2	−1	−3	−1	−2	−1	1	0	−4	1	0	−3	1	2	0
−1	2	1	2	4		1	0	−1	0	−2	0	1	−1	0	0	−1	−2	0	0	0
1	0	−1	−2	0	4		0	1	0	0	2	0	1	2	−1	−1	1	0	0	−2
0	1	−1	−2	−1	2	4		1	1	1	2	2	−1	2	0	0	1	−3	−2	−2
−2	1	1	1	0	−2	0	4		0	0	−1	1	−2	−1	1	2	−1	0	−1	2
−2	2	1	1	2	0	1	2	4		−3	−1	2	−2	0	0	−1	−2	−1	−1	1
−1	2	0	0	1	−1	0	0	0	4		0	0	0	1	3	0	2	−3	0	0
1	1	0	0	0	−1	−1	−1	−1	2	4		−2	3	2	2	0	1	0	−1	1
−1	−1	−2	0	−1	1	0	−1	−1	−1	−1	4		1	−3	1	−2	1	2	4	0
−1	−1	1	1	1	−1	−2	1	1	0	0	−1	4		0	2	2	0	0	1	2
−2	−1	−1	1	−1	−2	0	2	1	0	0	0	1	4		1	2	−1	−1	−1	1
−1	2	2	1	2	0	−1	1	2	1	0	−1	1	−1	4		−2	−2	0	0	1
−2	1	0	2	2	−2	−1	1	1	2	0	0	1	1	1	4		−1	−2	2	0
−1	2	2	0	1	0	0	2	2	1	0	−1	1	0	2	0	4		1	−1	2
2	−2	1	0	−1	−1	−1	−1	−2	0	1	−1	1	0	−1	0	−1	4		0	0
1	1	2	−1	1	0	−1	−1	0	1	1	−1	1	−2	1	0	1	1	4		3
1	1	2	1	0	−2	0	1	0	0	1	−2	−1	0	0	0	0	1	0	4	

A.11 Dimension 22

1: ${}_{i}[C_4]_1 \otimes A_{11}$

2		0	0	0	0	0	0	0	0	0	0	2	1	−1	1	1	1	1	1	−1	−1	−1
1	2		0	0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	−1	−1	−1
−1	0	2		0	0	0	0	0	0	0	0	−1	0	2	−1	−1	−1	−1	−1	1	1	1
1	0	−1	2		0	0	0	0	0	0	0	1	0	−1	2	1	1	1	1	0	−1	−1
1	0	−1	1	2		0	0	0	0	0	0	1	0	−1	1	2	1	1	1	−1	0	0
1	0	−1	1	1	2		0	0	0	0	0	1	0	−1	1	1	2	1	1	−1	−1	−1
1	0	−1	1	1	1	2		0	0	0	0	1	0	−1	1	1	1	2	1	−1	0	0
1	0	−1	1	1	1	1	2		0	0	0	1	0	−1	1	1	1	1	2	−1	0	0
−1	−1	1	0	−1	−1	−1	−1	3		0	0	−1	−1	1	0	−1	−1	−1	−1	3	0	0
−1	−1	1	−1	0	−1	0	0	0	3		0	−1	−1	1	−1	0	−1	0	0	0	3	2
−1	−1	1	−1	0	−1	0	0	0	2	3		−1	−1	1	−1	0	−1	0	0	0	2	3
0	0	0	0	0	0	0	0	0	0	0	2		0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	2		0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	0	2		0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	−1	2		0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	−1	1	2		0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	−1	1	1	2		0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	−1	1	1	1	2		0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	−1	1	1	1	1	2		0	0	0
0	0	0	0	0	0	0	0	0	0	0	−1	−1	1	0	−1	−1	−1	−1	3		0	0
0	0	0	0	0	0	0	0	0	0	0	−1	−1	1	−1	0	−1	0	0	0	3		0
0	0	0	0	0	0	0	0	0	0	0	−1	−1	1	−1	0	−1	0	0	0	2	3	

2: ${}_{\sqrt{-3}}[C_6]_1 \otimes A_{11}$

4		6	0	3	0	3	0	3	0	3	3	−3	0	3	0	3	−3	3	3	0	0	0
2	4		−3	3	−3	3	−3	3	−3	3	0	0	−3	3	3	3	0	3	3	3	3	3
2	1	4		6	0	3	0	3	0	3	0	0	0	3	0	3	−3	3	3	0	0	0
−1	1	−2	4		−3	0	−3	0	−3	0	0	3	−3	0	0	0	0	0	0	0	0	3
2	1	2	−1	4		6	0	3	0	3	3	−3	0	3	0	3	−3	3	3	0	0	0
−1	1	−1	2	−2	4		−3	0	−3	0	0	3	−3	0	0	0	0	0	0	0	0	3
2	1	2	−1	2	−1	4		6	0	3	3	0	0	3	0	3	−3	3	3	0	0	0
−1	1	−1	2	−1	2	−2	4		−3	0	−3	3	−3	0	0	0	3	0	0	0	0	3
2	1	2	−1	2	−1	2	−1	4		6	3	0	0	3	0	3	−3	3	3	0	0	0
−1	1	−1	2	−1	2	−1	2	−2	4		0	0	−3	0	0	0	0	0	0	0	0	3
1	2	0	0	−1	2	1	1	−1	2	6		3	−3	0	0	0	0	0	0	0	0	3
−1	−2	−2	1	−1	−1	−2	1	0	0	−3	6		3	−3	0	−3	3	−3	−3	0	0	−3
2	1	2	−1	2	−1	2	−1	2	−1	−1	−1	4		6	0	3	−3	3	3	0	0	0
−1	1	−1	2	−1	2	−1	2	−1	2	2	−1	−2	4		0	0	0	0	0	0	0	3
−2	−1	0	0	0	0	0	0	0	0	−2	0	0	0	4		3	0	0	0	0	0	0
−1	1	−1	2	−1	2	−1	2	−1	2	2	−1	−1	2	−1	4		0	0	0	0	0	3
−1	−2	1	−2	1	−2	−1	−1	1	−2	−4	1	1	−2	2	−2	6		0	0	0	0	0
−1	1	−1	2	−1	2	−1	2	−1	2	2	−1	−1	2	0	2	−2	4		0	−3	0	3
−1	1	−1	2	−1	2	−1	2	−1	2	2	−1	−1	2	0	2	−2	2	4		0	−3	3
−2	−1	0	0	0	0	0	0	0	0	−2	0	0	0	2	0	2	−1	0	4		0	0
−2	−1	0	0	0	0	0	0	0	0	−2	0	0	0	2	0	2	0	−1	2	4		0
−2	−1	−2	1	−2	1	−2	1	−2	1	1	1	−2	1	0	1	0	1	1	0	0	4	

3: $_{\sqrt{-3}}[\pm U_5(2) \circ C_3]_{11}$

4		0	3	3	2	2	0	3	0	3	4	0	2	0	0	0	−1	0	−6	2	−2	0
0	4		3	0	−1	2	0	0	0	0	1	0	−1	3	0	0	2	0	0	2	−2	3
−1	1	4		−6	−3	0	−3	0	−3	3	−2	1	−2	3	2	4	4	3	3	0	−2	−1
1	2	2	4		0	3	0	3	−3	3	2	−1	2	3	1	2	2	3	0	3	−4	1
2	1	1	2	4		0	0	3	0	2	2	−2	2	0	−1	0	2	−1	−2	3	−4	−1
0	2	2	1	0	4		−3	0	0	2	−2	0	−2	3	3	2	1	2	1	0	−2	0
2	2	1	2	2	1	4		0	0	3	2	2	0	2	0	0	2	0	−3	1	−1	1
−1	2	2	1	1	2	0	4		0	0	−2	−2	−1	2	1	2	1	0	3	1	−2	−1
0	2	−1	1	0	0	0	0	4		−3	2	−1	0	2	0	0	1	0	0	2	−2	2
−1	−2	1	−1	0	0	−1	0	−1	4		−2	−1	−1	1	2	1	1	0	3	−1	−1	−3
0	1	2	2	0	2	0	2	0	0	4		−3	−1	4	3	4	0	4	2	2	−3	−1
2	−2	−1	−1	0	0	0	−2	−1	1	−1	4		1	−2	0	0	−1	0	−3	0	0	−1
0	1	2	2	2	0	2	1	0	1	1	−1	4		1	0	1	3	0	1	1	−3	0
−2	−1	1	−1	−2	1	−2	0	0	1	0	0	−1	4		2	2	1	2	3	−3	1	0
−2	−2	0	−1	−1	−1	−2	−1	0	2	−1	0	0	2	4		−2	1	0	3	−3	1	−1
−2	−2	0	−2	−2	0	−2	0	−2	1	0	0	−1	2	2	4		−1	0	3	−3	3	−1
−1	−2	0	−2	−2	−1	−2	−1	−1	1	0	1	−1	1	1	1	4		1	1	−1	3	−2
−2	−2	1	−1	−1	0	−2	0	−2	2	0	0	0	2	2	2	1	4		3	−3	1	−1
−2	0	−1	−2	−2	−1	−1	−1	0	−1	−2	−1	−1	1	1	1	1	1	4		−4	4	3
−2	0	2	1	−1	0	−1	1	0	1	2	−2	1	1	1	1	1	1	0	4		0	−1
2	0	−2	0	0	0	1	−2	0	−1	−1	2	−1	−1	−1	−1	−1	−1	0	−2	4		2
2	1	1	1	1	2	1	1	0	−1	1	1	0	0	−1	−1	0	−1	−1	−1	0	4	

4: $_{\sqrt{-23}}[\pm \mathrm{L}_2(23)]_{11}$

12		0	0	0	0	−23	0	23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	12		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	23	0	0	0	−23
6	6	12		0	0	0	0	0	0	0	0	0	0	0	0	0	0	23	0	−23	0	0
4	6	6	12		0	0	0	0	0	0	0	0	0	0	0	0	0	23	−23	0	0	0
4	6	2	6	12		0	0	0	0	0	0	0	23	0	0	0	0	0	−23	23	0	0
3	6	2	2	6	12		0	0	0	0	0	0	23	0	0	0	0	0	−23	0	23	0
−4	−4	−4	0	−4	0	12		0	0	0	0	0	0	0	0	0	0	0	0	0	23	0
−1	−2	2	2	−4	−6	4	12		0	0	0	0	−23	−23	0	0	−23	23	23	0	0	0
4	2	6	6	2	0	−4	−2	12		0	0	0	0	0	0	0	23	0	−23	0	0	0
−6	−4	−6	0	−2	−2	6	0	−4	12		0	0	0	0	23	0	0	0	0	0	0	0
4	0	2	2	−2	0	4	0	4	−2	12		0	0	0	0	−23	23	0	0	0	0	0
4	6	6	4	0	4	2	0	2	−2	6	12		0	0	0	0	0	23	0	−23	0	−23
2	2	−2	−4	−1	3	2	−1	−6	2	0	0	12		0	0	0	0	0	23	0	0	0
4	−2	4	−2	−4	−2	0	3	0	−2	2	2	2	12		0	0	0	0	23	−23	0	0
−2	−2	−4	2	4	0	2	−2	0	3	2	0	−2	−6	12		0	0	0	−23	23	0	0
−4	−6	−4	0	−4	−6	6	6	−2	4	1	−2	−2	2	0	12		0	0	0	0	0	0
6	6	8	8	2	4	0	1	5	−2	5	8	0	4	0	−4	16		23	0	−23	0	0
−2	−1	1	−3	−4	2	4	1	−2	−2	2	1	0	0	−2	0	−3	12		0	0	23	0
−4	2	0	−1	1	1	−4	−1	−1	0	−4	0	−3	−3	1	−4	−2	0	12		0	0	−23
−2	0	1	−2	1	0	−4	−2	2	−4	−2	−3	0	−3	−1	−4	−1	−2	0	12		0	23
4	−2	4	2	2	−1	−1	2	4	−4	2	0	−2	2	0	2	2	−1	−4	2	12		23
4	1	4	−2	−2	−2	−4	0	0	−2	−2	1	2	4	−4	−4	0	−2	−1	1	1	12	

Nomenclature

Let $G < \mathrm{GL}_m(\mathbb{Q})$ be a matrix group and let N and H be any groups.

r.i.m.f.	rational irreducible maximal finite
s.i.m.f.	symplectic irreducible maximal finite
s.p.i.m.f.	symplectic primitive irreducible maximal finite
$\Pi(k)$	set of all primes dividing the integer k
$\tilde{\Pi}(K,k)$	set of primes needed for the m-parameter argument cf. Definition 2.2.10
$\mathbb{F}_q$	finite field with q elements
$\mathcal{Q}_{P_1,\dots,P_r}$	quaternion algebra with center $\mathbb{Q}$ ramified only at the places $P_1,\dots,P_r$
$\mathcal{Q}_{\alpha,P_1,\dots,P_r}$	quaternion algebra with center $\mathbb{Q}(\alpha)$ ramified only at the places $P_1,\dots,P_r$
$\mathbb{Z}_K$	maximal order of an algebraic number field K
$\mathcal{C}l(R)$	ideal class group of a Dedekind ring R
ζ_n	primitive n-th root of unity
θ_n	$\zeta_n + \zeta_n^{-1}$
I_m	$m \times m$ identity matrix
$\mathcal{F}(G)$	set of G-invariant forms cf. Definition 2.1.1
$\mathcal{F}_{sym}(G)$	subset of symmetric G-invariant forms in $\mathcal{F}(G)$
$\mathcal{F}_{sym}(G)$	subset of skewsymmetric G-invariant forms in $\mathcal{F}(G)$
$\mathcal{F}_{>0}(G)$	subset of positive definite forms in $\mathcal{F}_{sym}(G)$
$\overline{G}$	enveloping algebra of G cf. Definition 2.1.1
$\mathrm{End}(\overline{G})$	endomorphism ring or commuting algebra $C_{\mathbb{Q}^{m\times m}}(G)$ of G cf. Definition 2.1.1
$\mathcal{Z}(\Lambda)$	set of Λ-invariant lattices in $\mathbb{Q}^{1\times m}$ for some $\mathbb{Z}$-order $\Lambda \subset \mathbb{Q}^{m\times m}$ cf. Definition 2.1.4

$\mathcal{Z}(G)$	set of G-invariant lattices in $\mathbb{Q}^{1\times m}$ cf. Definition 2.1.4
$\det(L, F)$	determinant of the lattice L in the Euclidean space $(L \otimes \mathbb{R}, F)$
$\mathrm{Aut}_K(L, \mathcal{F})$	group of K-linear automorphisms of the lattice L wrt. the forms in $\mathcal{F}$ cf. Definition 2.1.4
$\mathcal{B}^o(G)$	generalized Bravais group of G cf. Definition 2.1.22
$\pm G$	the group $\langle G, -I_m \rangle$
$O_p(H)$	largest normal p-subgroup of H
$F(H)$	Fitting subgroup of H, i.e. the largest normal nilpotent subgroup of H
$E(H)$	layer of H, i.e. the central product of all components of H
$F^*(H)$	generalized Fitting subgroup of H (central product of $F(H)$ and $E(H)$)
A_n, F_4, E_k	(automorphism groups of) root lattices
C_n	cyclic group of order n
S_n, Alt_n	symmetric and alternating groups on n letters
D_{2n}	dihedral group of order $2n$
QD_{2^n}	quasidihedral group of order 2^n
Q_{2^n}	(generalized) quaternion group of order 2^n
2_+^{1+2n}	central product of n copies of D_8
2_-^{1+2n}	central product of Q_8 and $n-1$ copies of D_8
p_+^{1+2n}	extraspecial p-group of order p^{1+2n} and exponent p (p odd prime)
p_-^{1+2n}	extraspecial p-group of order p^{1+2n} and exponent p^2 (p odd prime)
$G_1 \otimes G_2$	tensor product of the two matrix groups G_1 and G_2. See Section 2.4 for an explanation of the symbols $G_1 \otimes G_2$, $G_1 \underset{Q}{\otimes} G_2$ and $G_1 \circ G_2$
$G_1 \overset{2(p)}{\otimes} G_2$	extension of $G_1 \otimes G_2$ by C_2. See Section 2.4 for an explanation of the symbols $G_1 \overset{2(p)}{\otimes} G_2$, $G_1 \overset{2(p)}{\underset{Q}{\otimes}} G_2$, $G_1 \overset{2(p)}{\boxtimes} G_2$, $G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2$, $G_1 \overset{2(p)}{\boxtimes\!\!\!\supset} G_2$, $G_1 \overset{2(p)}{\underset{Q}{\boxtimes\!\!\!\supset}} G_2$, $G_1 \overset{2(p)}{\circ} G_2$, $G_1 \overset{2(p)}{\square} G_2$ and $G_1 \overset{2(p)}{\square\!\!\!\supset} G_2$
$N{:}H$	semidirect product, i.e. a split extension of N by H
$N{\cdot}H$	nonsplit extension of N by H
$N.H$	any extension of N by H
$H \wr S_k$	wreath product of H and S_k

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