INTERSECTION THEOREMS FOR FINITE GENERAL LINEAR GROUPS

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Abstract. A subset Y of the general linear group $\mathrm{GL}(n,q)$ is called t-intersecting if $\mathrm{rk}(x-y) \leqslant n-t$ for all $x,y \in Y$, or equivalently x and y agree pointwise on a t-dimensional subspace of $\mathbb{F}_q^n$ for all $x,y \in Y$. We show that, if n is sufficiently large compared to t, the size of every such t-intersecting set is at most that of the stabiliser of a basis of a t-dimensional subspace of $\mathbb{F}_q^n$. In case of equality, the characteristic vector of Y is a linear combination of the characteristic vectors of the cosets of these stabilisers. We also give similar results for subsets of $\mathrm{GL}(n,q)$ that intersect not necessarily pointwise in t-dimensional subspaces of $\mathbb{F}_q^n$ and for cross-intersecting subsets of $\mathrm{GL}(n,q)$. These results may be viewed as variants of the classical Erdős-Ko-Rado Theorem in extremal set theory and are q-analogs of corresponding results known for the symmetric group. Our methods are based on eigenvalue techniques to estimate the size of the largest independent sets in graphs and crucially involve the representation theory of $\mathrm{GL}(n,q)$.

1. Introduction and results

One of the most famous results in extremal set theory is the Erdős-Ko-Rado Theorem [9]. In its strengthened version [27] it states that, for all fixed k and t and all sufficiently large n, every t-intersecting family of k-subsets of $\{1,2,\dots,n\}$ has size at most $\binom{n-t}{k-t}$ and equality holds if and only if there are t distinct points of $\{1,2,\dots,n\}$ contained in all members of the family.

There are several analogs of the Erdős-Ko-Rado Theorem (see [13], for example). Most notably, following important earlier work [10], [5], [18], [12], a corresponding result for the symmetric group S_n was obtained by Ellis, Friedgut, and Pilpel in a landmark paper [7]. A subset Y of S_n is *t-intersecting* if, for all $x,y \in Y$, there exist distinct $i_1,i_2,\dots,i_t$ in $\{1,2,\dots,n\}$ such that $x(i_k)=y(i_k)$ for all k. It was shown in [7] that, for each fixed t and all sufficiently large n, every t-intersecting set in S_n has size at most $(n-t)!$ and equality holds if and only if Y is a coset of the stabiliser of a t-tuple of distinct points in $\{1,2,\dots,n\}$.

In this paper we consider a q-analog of this problem, namely we study a corresponding problem for the finite general linear groups. Throughout this paper q is

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a fixed prime power and G_n denotes the general linear group of degree n over the finite field $\mathbb{F}_q$, namely the group of invertible $n \times n$ matrices over $\mathbb{F}_q$. We say that two elements $x, y \in G_n$ are *t-intersecting* if there exist linearly independent elements $u_1, u_2, \dots, u_t$ in $\mathbb{F}_q^n$ such that $xu_k = yu_k$ for all k. Equivalently $x, y \in G_n$ are t-intersecting if $\mathrm{rk}(x-y) \leqslant n-t$. A subset Y of G_n is called *t-intersecting* if all pairs in $Y \times Y$ are t-intersecting.

A coset of the stabiliser of a t-tuple of linearly independent elements of $\mathbb{F}_q^n$ has the form

$$\{g \in G_n : gu_1 = v_1, \dots, gu_t = v_t\}$$

for some t-tuples $(u_1, u_2, \dots, u_t)$ and $(v_1, v_2, \dots, v_t)$ of linearly independent elements of $\mathbb{F}_q^n$. We call such a coset a *t-coset*. It is plain that every t-coset is t-intersecting. Note that the size of a t-coset is

$$\prod_{i=t}^{n-1} (q^n - q^i). \tag{1.1}$$

The t-cosets are however not the only t-intersecting sets of this size in G_n, as the transpose of every t-intersecting set is t-intersecting.

We shall often identify a subset Y of G_n with its characteristic vector $1_Y \in \mathbb{C}(G_n)$ (where $\mathbb{C}(G_n)$ is the vector space of functions from G_n to $\mathbb{C}$). It is well known (see [2] or [3], for example) that, since G_n contains a Singer cycle as a regular subgroup, the size of every 1-intersecting set in G_n is at most the expression given in (1.1) for $t = 1$. Meagher and Razafimahatratra [21] have shown that, if Y is a 1-intersecting set of size $q^2 - q$ in G_2, then 1_Y is in the span of the characteristic vectors of the 1-cosets. We prove a corresponding result for all t and n for which n is sufficiently large compared to t.

Theorem 1.1. *Let t be a positive integer and let Y be a t-intersecting set in G_n. If n is sufficiently large compared to t, then*

$$|Y| \leqslant \prod_{i=t}^{n-1} (q^n - q^i)$$

and, in case of equality, 1_Y is spanned by the characteristic vectors of t-cosets.

We also prove a result on cross-intersecting subsets of G_n. Two subsets Y and Z are *t-cross-intersecting* if all pairs in $Y \times Z$ are t-intersecting.

Theorem 1.2. *Let t be a positive integer and let Y and Z be t-cross-intersecting sets in G_n. If n is sufficiently large compared to t, then*

$$\sqrt{|Y| \cdot |Z|} \leqslant \prod_{i=t}^{n-1} (q^n - q^i)$$

and, in case of equality, 1_Y and 1_Z are spanned by the characteristic vectors of t-cosets.

Theorems 1.1 and 1.2 may be seen as q-analogs of [7, Thm. 5 and 6]. It seems plausible that corresponding q-analogs of [7, Thm. 3 and 4] also hold. In the case of t-intersecting sets, this means that the extremal t-intersecting sets in G_n are the t-cosets and their transposes whenever n is sufficiently large compared to t. In fact, Ahanjideh [1] has shown that every 1-intersecting set in G_2 of size q^2-q must be either a 1-coset or the transpose of a 1-coset. We therefore pose the following conjectures.

Conjecture 1.3. *Let Y be a t-intersecting set in G_n whose size meets the bound in Theorem 1.1. If n is sufficiently large compared to t, then Y or Y^T is a t-coset.*

Conjecture 1.4. *Let Y and Z be t-cross-intersecting sets in G_n whose sizes meet the bound in Theorem 1.2. If n is sufficiently large compared to t, then $Y = Z$ and Y or Y^T is a t-coset.*

A subset Y of the symmetric group S_n is *t-set-intersecting* if, for all $x, y \in Y$, there is a subset I of $\{1, 2, \ldots, n\}$ containing t elements such that $x(I) = y(I)$. It was shown in [6] that, for each fixed t and all sufficiently large n, every t-set-intersecting set in S_n has size at most $t!(n-t)!$ and equality holds if and only if Y is a coset of the stabiliser of a subset of $\{1, 2, \ldots, n\}$ containing t elements.

We also obtain a q-analog of this result. We say that two elements $x, y \in G_n$ are *t-space-intersecting* if there exists a t-dimensional subspace U of $\mathbb{F}_q^n$ (or *t-space* for short) such that $xU = yU$. A subset Y of G_n is called *t-space-intersecting* if all pairs in $Y \times Y$ are t-space-intersecting. Of course in this context it would be more natural to replace G_n by the projective linear group $\mathrm{PGL}(n, q)$. However results for G_n and for $\mathrm{PGL}(n, q)$ can be easily translated into each other and for consistency we prefer to work with G_n. A coset of the stabiliser in G_n of a t-space is clearly t-space-intersecting and has order

$$\left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]. \tag{1.2}$$

Note that again the transpose of a t-space-intersecting set is t-space-intersecting. The transpose of the stabiliser of a t-space is in fact the stabiliser of an $(n-t)$-space, so the stabiliser of an $(n-t)$-space is an example of a t-space-intersecting set that has the same size as that of the stabiliser of a t-space.

Using an argument involving a Singer cycle, similarly as that above, Meagher and Spiga [22] have shown that the size of every 1-space-intersecting set in G_n is at most the expression given in (1.2) for $t = 1$. We show that this is true for all t and all sufficiently large n.

Theorem 1.5. *Let t be a positive integer and let Y be a t-space-intersecting set in G_n. If n is sufficiently large compared to t, then*

$$|Y| \leqslant \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, 1_Y *is spanned by the characteristic vectors of cosets of stabilisers of* t*-spaces.*

Again, we have a corresponding result on cross-intersecting subsets of G_n, in which we call two subsets Y and Z of G_n *t-space-cross-intersecting* if all pairs in $Y \times Z$ are t-space-intersecting.

Theorem 1.6. *Let* t *be a positive integer and let* Y *and* Z *be* t*-space-cross-intersecting sets in* G_n*. If* n *is sufficiently large compared to* t*, then*

$$\sqrt{|Y| \cdot |Z|} \leqslant \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, 1_Y *and* 1_Z *are spanned by the characteristic vectors of cosets of stabilisers of* t*-spaces.*

Meagher and Spiga [22] conjectured that the extremal 1-space-intersecting sets in G_n must be cosets of the stabiliser of a 1-space or cosets of the stabiliser of an $(n-1)$-space. This was proved by the same authors for $n = 2$ [22] and $n = 3$ [23] and by Spiga for all $n \geqslant 4$ [25]. We therefore pose the following conjectures.

Conjecture 1.7. *Let* Y *be a* t*-space-intersecting set in* G_n *whose size meets the bound in Theorem 1.5. If* n *is sufficiently large compared to* t*, then* Y *is a coset of the stabiliser of a* t*-space or a coset of the stabiliser of an* $(n-t)$*-space.*

Conjecture 1.8. *Let* Y *and* Z *be* t*-space-cross-intersecting sets in* G_n *whose sizes meet the bound in Theorem 1.6. If* n *is sufficiently large compared to* t*, then* $Y = Z$ *and* Y *is the stabiliser of a* t*-space or the stabiliser of an* $(n-t)$*-space.*

Not surprisingly, as in [7] and [6], our proofs are based on eigenvalue techniques, in particular weighted versions of the Hoffman bound on independent sets in graphs, and crucially involve the representation theory of G_n. We organise this paper as follows. In Section 2 we summarise relevant background on the representation theory of G_n. In Section 3 we recall versions of the Hoffman bound from [7] and explain how they can be applied in our setting. In Section 4 we prepare some key steps of the proofs of our main results and in particular study properties of a matrix related to the character table of G_n. Sections 5 and 6 contain the main arguments of our proofs of Theorems 1.1 and 1.2 and Theorems 1.5 and 1.6, respectively. In Section 7 we prove some auxiliary ingredients used in our proofs.

We close this introduction by noting that, after a first version of this paper was made publically available, Ellis, Kindler, and Lifshitz [8] independently proved a result that is slightly more general than Theorem 1.1 and also proved Conjecture 1.3. Their methods are completely different compared to ours and in particular make no use of the representation theory of G_n.

2. The finite general linear groups

In this section we mostly recall some relevant facts about the conjugacy classes and the character theory of G_n.

2.1. Partitions

An (integer) *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. The set of partitions is denoted by Par. We often omit trailing zeros and write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ if $\lambda_k > 0$ and $\lambda_{k+1} = 0$. The *size* of $(\lambda_1, \lambda_2, \dots)$ is defined to be $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. If $|\lambda| = n$, then we also say that λ is a partition of n. We denote the unique partition of 0 by $\varnothing$.

The *Young diagram* of a partition $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of n is an array of n boxes with left-justified rows and top-justified columns, where row i contains λ_i boxes. To each partition λ belongs a *conjugate partition* λ' whose parts are the number of boxes in the columns of the Young diagram of λ. For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ of the same size, we say that λ is *dominates* μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i \quad \text{for each } k \geqslant 1.$$

This indeed defines a partial order on the set of partitions of a fixed size, which is called the *dominance order*.

2.2. Conjugacy classes

We shall now describe the conjugacy classes of G_n (see [20, Ch. IV,§ 3], for example). Let Φ be the set of monic irreducible polynomials in $\mathbb{F}_q[X]$ distinct from X. For $a \in \mathbb{F}_q^*$ (where $\mathbb{F}_q^*$ is the multiplicative group of $\mathbb{F}_q$), we shall often write a instead of $X - a$ when the meaning is clear from the context. We also write $|f|$ for the degree of $f \in \Phi$. Let Λ be the set of mappings $\underline{\lambda}\colon \Phi \to \mathrm{Par}$ of finite support (with $\varnothing$ being the zero element in Par). We define the *size* of such a mapping to be

$$\|\underline{\lambda}\| = \sum_{f \in \Phi} |\underline{\lambda}(f)| \cdot |f|$$

and put $\Lambda_n = \{\underline{\lambda} \in \Lambda : \|\underline{\lambda}\| = n\}$. The *companion matrix* of $f \in \Phi$ with $f = X^d + f_{d-1}X^{d-1} + \cdots + f_1 X + f_0$ is

$$C(f) = \begin{bmatrix} & & & & -f_0 \\ 1 & & & & -f_1 \\ & 1 & & & -f_2 \\ & & \ddots & & \vdots \\ & & & 1 & -f_{d-1} \end{bmatrix} \in \mathbb{F}_q^{d \times d}$$

(where blanks are filled with zeros). For $f \in \Phi$ of degree d and a positive integer k, we write

$$C(f,k) = \begin{bmatrix} C(f) & I & & & \\ & C(f) & I & & \\ & & \ddots & \ddots & \\ & & & \ddots & I \\ & & & & C(f) \end{bmatrix} \in \mathbb{F}_q^{kd \times kd},$$

where I is an identity matrix of the appropriate size. For $f \in \Phi$ and $\sigma \in \mathrm{Par}$, we define $C(f,\sigma)$ to be the block diagonal matrix of order $|\sigma| \cdot |f|$ with blocks $C(f,\sigma_1), C(f,\sigma_2), \ldots$. Finally, with every $\underline{\sigma} \in \Lambda_n$ we associate the block diagonal matrix $R_{\underline{\sigma}}$ of order n whose blocks are $C(f, \underline{\sigma}(f))$, where f ranges through the support of $\underline{\sigma}$. Then every element g of G_n is conjugate to exactly one matrix $R_{\underline{\sigma}}$ for $\underline{\sigma} \in \Lambda_n$, which is called the *Jordan canonical form* of g. Hence Λ_n indexes the conjugacy classes of G_n; we denote by $C_{\underline{\sigma}}$ the conjugacy class containing $R_{\underline{\sigma}}$. The following result gives an explicit expression for the number of elements in $C_{\underline{\sigma}}$.

Lemma 2.1 ([26, Thm. 1.10.7])**.** *For each $\underline{\sigma} \in \Lambda_n$, we have*

$$\frac{|G_n|}{|C_{\underline{\sigma}}|} = \prod_{f \in \Phi} \prod_{i=1}^{|\underline{\sigma}(f)|} \prod_{j=1}^{m_i(\underline{\sigma}(f))} q^{|f|\, s_i(\underline{\sigma}(f)')} (1 - q^{-|f|\, j}),$$

where $m_i(\sigma) = |\{j \geqslant 1 \colon \sigma_j = i\}|$ and $s_i(\sigma) = \sum_{j=1}^{i} \sigma_j$ for a partition σ.

2.3. Parabolic induction

Recall that, given a finite group G, a subgroup H of G, and a class function ϕ on H, the *induced* class function $\mathrm{Ind}_H^G(\phi)$ on G is given by

$$\mathrm{Ind}_H^G(\phi)(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \phi(xgx^{-1}). \tag{2.1}$$

The character theory of G_n crucially relies on the induction of characters from parabolic subgroups of G_n.

A *composition* is much like a partition, except that the parts do not need to be nonincreasing. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a composition of n. Let P_λ be the parabolic subgroup of G_n consisting of block upper-triangular matrices with block sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$, namely

$$P_\lambda = \left\{ \begin{bmatrix} g_1 & * & \cdots & * \\ & g_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & g_k \end{bmatrix} : g_i \in G_{\lambda_i} \right\}. \tag{2.2}$$

Let $\pi_i : P_\lambda \to G_{\lambda_i}$ be the mapping that projects to the i-th diagonal block, so that

$$\pi_i : \begin{bmatrix} g_1 & * & \cdots & * \\ & g_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & g_k \end{bmatrix} \mapsto g_i. \tag{2.3}$$

Let ϕ_i be a class function on G_{λ_i}. Then

$$\prod_{i=1}^{k} (\phi_i \circ \pi_i)$$

is a class function on P_λ. We define the product $\phi_1 \odot \phi_2 \odot \cdots \odot \phi_k$ to be the induction of this class function to G_n, that is

$$\bigodot_{i=1}^{k} \phi_i = \mathrm{Ind}_{P_\lambda}^{G_n} \left(\prod_{i=1}^{k} (\phi_i \circ \pi_i) \right). \tag{2.4}$$

2.4. Character theory of G_n

The complete set of complex irreducible characters has been obtained by Green [14]. A good treatment of this topic is also contained in [20, Ch. IV]. The complex irreducible representations were obtained by Gelfand [11] and the irreducible representations over fields of nondefining characteristic were obtained by James [17]. The approach of [17] is in fact very similar to the standard combinatorial approach to obtain the complex irreducible representations of the symmetric group (see [24], for example) and we mostly follow [17] to recall some relevant background on the complex characters of G_n.

The irreducible characters of G_n are naturally indexed by Λ_n and, for $\underline{\lambda} \in \Lambda_n$, we denote by $\chi^{\underline{\lambda}}$ the corresponding irreducible character. We shall use the short-hand notation $\chi^{f \mapsto \lambda}$ for $\chi^{\underline{\lambda}}$ if $\underline{\lambda}$ is supported only on $f \in \Phi$ and $\underline{\lambda}(f) = \lambda$. These are typically called the *primary* irreducible characters of G_n. It is well known (see [17, § 8], for example) that the irreducible characters of G_n satisfy

$$\chi^{\underline{\lambda}} = \bigodot_{f \in \Phi} \chi^{f \mapsto \underline{\lambda}(f)}. \tag{2.5}$$

In order to construct the primary irreducible characters, James [17] constructs characters of G_{dm}, denoted by $\xi^{f \mapsto \mu}$, where $f \in \Phi$ has degree d and μ is a partition of m. Writing $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, these characters satisfy [17, (6.2)]

$$\xi^{f \mapsto \mu} = \bigodot_{i=1}^{k} \xi^{f \mapsto (\mu_i)} \tag{2.6}$$

and [17, (7.19)]

$$\xi^{f \mapsto \mu} = \sum_{\lambda} K_{\lambda\mu} \, \chi^{f \mapsto \lambda}, \tag{2.7}$$

where λ ranges over the partitions of $|\mu|$ and $K_{\lambda\mu}$ is a *Kostka number*, which equals the number of semistandard Young tableaux of shape λ and content μ. It is well known (see [24, § 2.11], for example) that the Kostka numbers satisfy

$$K_{\mu\mu} = 1 \text{ and } K_{\lambda\mu} \neq 0 \Rightarrow \lambda \trianglerighteq \mu. \tag{2.8}$$

Conversely it is readily verified that there are integers $H_{\mu\lambda}$ satisfying

$$\chi^{f\mapsto\lambda} = \sum_{\mu} H_{\mu\lambda}\, \xi^{f\mapsto\mu} \tag{2.9}$$

and

$$H_{\lambda\lambda} = 1 \text{ and } H_{\mu\lambda} \neq 0 \Rightarrow \mu \trianglerighteq \lambda \tag{2.10}$$

(see [20, p. 105], for example).

Now, for $\underline{\mu} \in \Lambda_n$, we define the characters

$$\xi^{\underline{\mu}} = \bigodot_{f\in\Phi} \xi^{f\mapsto\underline{\mu}(f)}. \tag{2.11}$$

We denote by $\chi^{\underline{\lambda}}_{\underline{\sigma}}$ and $\xi^{\underline{\mu}}_{\underline{\sigma}}$ the characters $\chi^{\underline{\lambda}}$ and $\xi^{\underline{\mu}}$, respectively, evaluated on the conjugacy class $C_{\underline{\sigma}}$.

We now express $\xi^{\underline{\mu}}$ and $\chi^{\underline{\lambda}}$ in terms of each other. To do so, we define the *shape* of $\underline{\lambda} \in \Lambda_n$ to be the mapping $s : \Phi \to \mathbb{Z}$ given by $s(f) = |\underline{\lambda}(f)|$ for each $f \in \Phi$. We write $\underline{\lambda} \sim \underline{\mu}$ if $\underline{\lambda}, \underline{\mu} \in \Lambda_n$ have the same shape. Then $\sim$ is an equivalence relation on Λ_n. For $\underline{\lambda}, \underline{\mu} \in \Lambda_n$ with $\underline{\lambda} \sim \underline{\mu}$, write

$$K_{\underline{\lambda}\underline{\mu}} = \prod_{f\in\Phi} K_{\underline{\lambda}(f)\underline{\mu}(f)},$$
$$H_{\underline{\mu}\underline{\lambda}} = \prod_{f\in\Phi} H_{\underline{\mu}(f)\underline{\lambda}(f)}.$$

We then find that

$$\xi^{\underline{\mu}} = \sum_{\underline{\lambda}\sim\underline{\mu}} K_{\underline{\lambda}\underline{\mu}}\, \chi^{\underline{\lambda}} \quad \text{for each } \underline{\mu} \in \Lambda_n, \tag{2.12}$$

$$\chi^{\underline{\lambda}} = \sum_{\underline{\mu}\sim\underline{\lambda}} H_{\underline{\mu}\underline{\lambda}}\, \xi^{\underline{\mu}} \quad \text{for each } \underline{\lambda} \in \Lambda_n. \tag{2.13}$$

An explicit expression for the degree $\chi^{\underline{\lambda}}(1)$ (where 1 is the identity of G_n) of $\chi^{\underline{\lambda}}$ is given by the so-called q-analog of the hook-length formula.

Lemma 2.2 ([14, Thm. 14])**.** *We have*

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \prod_{i=1}^{n}(q^i - 1) = \prod_{f\in\Phi} \frac{1}{q^{|f|b(\underline{\lambda}(f))}} \prod_{(i,j)\in\underline{\lambda}(f)} (q^{|f|h_{i,j}(\underline{\lambda}(f))} - 1), \tag{2.14}$$

where, for each partition $\lambda = (\lambda_1, \lambda_2, \dots)$,

$$b(\lambda) = \sum_{i \geqslant 1} (i-1)\lambda_i$$

and $h_{i,j}(\lambda)$ *is the* hook length *of* λ *at* (i,j), *namely*

$$h_{i,j}(\lambda) = \lambda_i + \lambda'_j - i - j + 1$$

and the corresponding product over (i,j) *is over all boxes of the Young diagram of* $\underline{\lambda}(f)$.

It can be readily verified from Lemma 2.2 that the linear (degree-one) irreducible characters of G_n are precisely the primary characters $\chi^{f \mapsto (n)}$, where $|f| = 1$. These are the only characters of G_n that we shall need explicitly. Let α be a generator of the multiplicative group $\mathbb{F}_q^*$ of $\mathbb{F}_q$, let $\omega = \exp(2\pi\sqrt{-1}/(q-1))$ be a complex root of unity, and let $\theta : \mathbb{F}_q^* \to \mathbb{C}$ be the linear character of $\mathbb{F}_q^*$ given by $\theta(\alpha^i) = \omega^i$. The following result is essentially given in [14, pp. 415 and 444].

Lemma 2.3 ([14])**.** *For all* $g \in G_n$, *we have*

$$\chi^{X - \alpha^i \mapsto (n)}(g) = \theta(\det(g)^i).$$

In particular $\chi^{X-1 \mapsto (n)}$ *is the trivial character.*

In what follows we consider certain characters of G_n related to the permutation character of G_n on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$. For $t \leqslant n$, let $H_{n,t}$ be the stabiliser of a fixed t-tuple of linearly independent elements of $\mathbb{F}_q^n$. We define $\zeta^{(t,i)}$ to be the character obtained by inducing the linear character

$$\begin{aligned} H_{n,t} &\to \mathbb{C} \\ g &\mapsto \theta(\det(g)^i) \end{aligned} \tag{2.15}$$

to G_n. Then $\zeta^{(t,0)}$ is the permutation character of G_n on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$. These characters are related to each other in the following way.

Lemma 2.4. *For each* $g \in G_n$, *we have*

$$\zeta^{(t,i)}(g) = \theta(\det(g)^i)\zeta^{(t,0)}(g).$$

Proof. Since similar matrices have the same determinant, we find from (2.1) that

$$\begin{aligned} \zeta^{(t,i)}(g) &= \frac{1}{|H_{n,t}|} \sum_{\substack{x \in G_n \\ xgx^{-1} \in H_{n,t}}} \theta(\det(xgx^{-1})^i) \\ &= \frac{1}{|H_{n,t}|} \sum_{\substack{x \in G_n \\ xgx^{-1} \in H_{n,t}}} \theta(\det(g)^i) \\ &= \theta(\det(g)^i)\zeta^{(t,0)}(g). \end{aligned}$$

□

We shall also need the following information about the decomposition of $\zeta^{(t,i)}$ into irreducible characters of G_n.

Lemma 2.5. *We have*

$$\zeta^{(t,i)} = \sum_{\underline{\lambda}\in\Lambda_n} m_{i,\underline{\lambda}}\,\chi^{\underline{\lambda}},$$

where $m_{i,\underline{\lambda}} \neq 0$ *if and only if* $\underline{\lambda}(\alpha^i)_1 \geqslant n-t$.

Proof. We may choose $H_{n,t}$ to be

$$H_{n,t} = \left\{ \begin{bmatrix} I & * \\ & g \end{bmatrix} : g \in G_{n-t} \right\},$$

so that $H_{n,t}$ is a subgroup of the parabolic subgroup $P_{(t,n-t)}$ given in (2.2). Let π_1 and π_2 be the projections onto the diagonal blocks of orders t and $n-t$, respectively, as given in (2.3). Using Lemma 2.3, the character (2.15) can be written as

$$(1\circ\pi_1)\,(\chi^{X-\alpha^i\mapsto(n-t)}\circ\pi_2). \tag{2.16}$$

where 1 is the trivial character of the trivial subgroup of G_t. By Frobenius reciprocity, 1 induces on G_t to the character

$$\sum_{\underline{\kappa}\in\Lambda_t} \chi^{\underline{\kappa}}(1)\,\chi^{\underline{\kappa}}.$$

Since $P_{(t,n-t)}/H_{n,t} \cong G_t$, it is then readily verified that (2.16) induces on $P_{(t,n-t)}$ to the character

$$\sum_{\underline{\kappa}\in\Lambda_t} \chi^{\underline{\kappa}}(1)\,(\chi^{\underline{\kappa}}\circ\pi_1)\,(\chi^{X-\alpha^i\mapsto(n-t)}\circ\pi_2).$$

Hence, by transitivity of induction, we have

$$\zeta^{(t,i)} = \sum_{\underline{\kappa}\in\Lambda_t} \chi^{\underline{\kappa}}(1)\;(\chi^{\underline{\kappa}}\odot\chi^{X-\alpha^i\mapsto(n-t)}).$$

It is well known [20, Ch. IV, § 4] that, for each fixed $f\in\Phi$, characters $\chi^{f\mapsto\lambda}$ form an algebra with multiplication $\odot$ that is isomorphic to the algebra of symmetric functions and the images of the characters $\chi^{f\mapsto\lambda}$ are the Schur functions. We then find from Pieri's rule (see [20, Ch. I, (5.16)], for example) that

$$\chi^{X-\alpha^i\mapsto\kappa}\odot\chi^{X-\alpha^i\mapsto(n-t)} = \sum_{\lambda}\chi^{X-\alpha^i\mapsto\lambda},$$

where λ runs through all partitions whose Young diagram is obtained from that of κ by adding $n-t$ boxes, no two of which in the same column. Using (2.5) the statement of the lemma is then readily verified. □

3. The Hoffman bound

Henceforth we use the following notation. For a field K and finite sets X and Y, we denote by $K(X,Y)$ the set of $|X| \times |Y|$ matrices A with entries in K, where rows and columns are indexed by X and Y, respectively. For $x \in X$ and $y \in Y$, the (x,y)-entry of A is written as $A(x,y)$. If $|Y| = 1$, then we omit Y, so $K(X)$ is the set of column vectors a indexed by X and, for $x \in X$, the x-entry of a is written as $a(x)$.

The *adjacency matrix* of a graph $\Gamma = (X,E)$ is the matrix $A \in \mathbb{R}(X,X)$ given by

$$A(x,y) = \begin{cases} 1 & \text{for } \{x,y\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Then A is a real symmetric matrix, which of course has an orthonormal system of $|X|$ eigenvectors forming a basis of $\mathbb{R}(X)$. All eigenvalues of A are real and referred to as the eigenvalues of Γ. Note that, if Γ is d-regular, then d is an eigenvalue of Γ and the all-ones vector is a corresponding eigenvector.

Our starting point arises from the following generalised versions of the Hoffman bound [15], stated and proved by Ellis, Friedgut, and Pilpel [7, § 2.4] in the following form.

Proposition 3.1. *Let $\Gamma = (X,E)$ be a graph on n vertices. Suppose that $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$ are regular spanning subgraphs of Γ, all having $\{v_0, v_1, \ldots, v_{n-1}\}$ as an orthonormal system of eigenvectors with v_0 being the all-ones vector. Let $P_i(k)$ be the eigenvalue of v_k in Γ_i. Let $w_0, w_1, \ldots, w_r \in \mathbb{R}$ and write $P(k) = \sum_{i=0}^{r} w_i P_i(k)$.*

(i) If $Y \subseteq X$ is an independent set in Γ, then

$$\frac{|Y|}{|X|} \leqslant \frac{|P_{\min}|}{P(0) + |P_{\min}|},$$

where $P_{\min} = \min_{k \neq 0} P(k)$. In case of equality we have

$$1_Y \in \langle \{v_0\} \cup \{v_k : P(k) = P_{\min}\} \rangle.$$

(ii) If $Y, Z \subseteq X$ are such that there are no edges between Y and Z in Γ, then

$$\sqrt{\frac{|Y|}{|X|}\frac{|Z|}{|X|}} \leqslant \frac{P_{\max}}{P(0) + P_{\max}},$$

where $P_{\max} = \max_{k \neq 0} |P(k)|$. In case of equality we have

$$1_Y, 1_Z \in \langle \{v_0\} \cup \{v_k : |P(k)| = P_{\max}\} \rangle.$$

In order to study graphs induced by G_n and their eigenvalues, we shall bring the theory of association schemes into play. We refer to [4] and [13] for background on association schemes. Every finite group gives rise to an association scheme (see [4, Section 2.7] or [13, Section 3.3] for details). We shall recall relevant background about this association scheme and its symmetrisation for G_n.

For each $\underline{\sigma} \in \Lambda_n$, we define $B_{\underline{\sigma}} = \mathbb{C}(G_n, G_n)$ by

$$B_{\underline{\sigma}}(x,y) = \begin{cases} 1 & \text{for } x^{-1}y \in C_{\underline{\sigma}} \\ 0 & \text{otherwise.} \end{cases}$$

The vector space generated by $\{B_{\underline{\sigma}} : \underline{\sigma} \in \Lambda_n\}$ over the complex numbers turns out to be a commutative matrix algebra $\mathbb{A}$, which contains the identity and the all-ones matrix and is closed under conjugate transposition. The collection of zero-one matrices $B_{\underline{\sigma}}$ therefore defines an association scheme. Since $\mathbb{A}$ is commutative, it can be simultaneously diagonalised and therefore there exists a basis $\{F_{\underline{\lambda}} : \underline{\lambda} \in \Lambda_n\}$ of $\mathbb{A}$ consisting of primitive idempotent matrices. These matrices are given by [4, Theorem II.7.2]

$$F_{\underline{\lambda}} = \frac{\chi^{\underline{\lambda}}(1)}{|G_n|} \sum_{\underline{\sigma} \in \Lambda_n} \chi^{\underline{\lambda}}_{\underline{\sigma}} B_{\underline{\sigma}}. \tag{3.1}$$

Using the orthogonality of characters of the second kind, it is readily verified that

$$B_{\underline{\sigma}} = \sum_{\underline{\lambda} \in \Lambda_n} \frac{|C_{\underline{\sigma}}|}{\chi^{\underline{\lambda}}(1)} \overline{\chi}^{\underline{\lambda}}_{\underline{\sigma}} F_{\underline{\lambda}}, \tag{3.2}$$

where $\overline{\chi}^{\underline{\lambda}}$ is the character of G_n whose values at $g \in G_n$ are the complex conjugates of $\chi^{\underline{\lambda}}(g)$.

For each $f \in \Phi$, let $f^* \in \Phi$ be its reciprocal polynomial, namely the monic polynomial whose roots (in an algebraic closure of $\mathbb{F}_q$) are precisely the inverses of the roots of f. For each $\underline{\lambda} \in \Lambda_n$, define $\underline{\lambda}^*$ to be the element of Λ_n given by $\underline{\lambda}^*(f) = \underline{\lambda}(f^*)$ for all $f \in \Phi$. We record the following lemma, in which we write $C_{\underline{\sigma}}^{-1} = \{g^{-1} : g \in C_{\underline{\sigma}}\}$ for $\underline{\sigma} \in \Lambda_n$.

Lemma 3.2. *Let $\underline{\sigma}, \underline{\lambda} \in \Lambda_n$. Then we have*

(i) $C_{\underline{\sigma}^*} = C_{\underline{\sigma}}^{-1}$,

(ii) $\chi^{\underline{\lambda}^*} = \overline{\chi}^{\underline{\lambda}}$,

(iii) $\chi^{\underline{\lambda}^*}_{\underline{\sigma}} = \chi^{\underline{\lambda}}_{\underline{\sigma}^*}$.

Proof. Statement (i) is a basic fact in linear algebra, (ii) is essentially [17, (7.32)], and (iii) can be deduced from (i) and (ii). □

Let Ω_n be the subset of Λ_n that contains all $\underline{\lambda} \in \Lambda_n$ satisfying $\underline{\lambda} = \underline{\lambda}^*$ and precisely one of $\underline{\lambda}$ or $\underline{\lambda}^*$ for all $\underline{\lambda} \in \Lambda_n$ satisfying $\underline{\lambda} \neq \underline{\lambda}^*$. For $\underline{\lambda} \in \Omega_n$, we define the character

$$\psi^{\underline{\lambda}} = \begin{cases} \chi^{\underline{\lambda}} & \text{for } \underline{\lambda} = \underline{\lambda}^* \\ \chi^{\underline{\lambda}} + \chi^{\underline{\lambda}^*} & \text{otherwise,} \end{cases}$$

and, for $\underline{\sigma} \in \Omega_n$, we define $D_{\underline{\sigma}} = C_{\underline{\sigma}} \cup C_{\underline{\sigma}*}$. Lemma 3.2 implies that $\psi^{\underline{\lambda}}$ is constant on $D_{\underline{\sigma}}$. We write

$$\psi^{\underline{\lambda}}_{\underline{\sigma}} = \psi^{\underline{\lambda}}(g), \quad \text{where } g \text{ is an arbitrary element of } D_{\underline{\sigma}}. \tag{3.3}$$

For $\underline{\sigma}, \underline{\lambda} \in \Omega_n$, write

$$A_{\underline{\sigma}} = \begin{cases} B_{\underline{\sigma}} & \text{for } \underline{\sigma} = \underline{\sigma}^* \\ B_{\underline{\sigma}} + B_{\underline{\sigma}*} & \text{otherwise} \end{cases} \quad \text{and} \quad E_{\underline{\lambda}} = \begin{cases} F_{\underline{\lambda}} & \text{for } \underline{\lambda} = \underline{\lambda}^* \\ F_{\underline{\lambda}} + F_{\underline{\lambda}*} & \text{otherwise.} \end{cases} \tag{3.4}$$

Note that $A_{\underline{\sigma}}$ is symmetric, so all of its eigenvalues are real, and that $E_{\underline{\lambda}}$ has only real entries. Let $V_{\underline{\lambda}}$ be the column span over the reals of $E_{\underline{\lambda}}$ and, for $\underline{\sigma}, \underline{\lambda} \in \Omega_n$, write

$$P(\underline{\lambda}, \underline{\sigma}) = \frac{|D_{\underline{\sigma}}|}{\psi^{\underline{\lambda}}(1)}\, \psi^{\underline{\lambda}}_{\underline{\sigma}}. \tag{3.5}$$

The following lemma, containing essentially standard results, will be crucial in the following.

Lemma 3.3. *We have the following orthogonal direct sum decomposition*

$$\mathbb{R}(G_n) = \bigoplus_{\underline{\lambda} \in \Omega_n} V_{\underline{\lambda}}.$$

Moreover, for all $\underline{\sigma}, \underline{\lambda} \in \Omega_n$, every element of $V_{\underline{\lambda}}$ is an eigenvector of $A_{\underline{\sigma}}$ and the corresponding eigenvalue is $P(\underline{\lambda}, \underline{\sigma})$.

Proof. Since $F_{\underline{\lambda}}$ is a primitive idempotent in $\mathbb{C}(G_n, G_n)$ for each $\underline{\lambda} \in \Lambda_n$, it is readily verified that $E_{\underline{\lambda}}$ is a primitive idempotent in $\mathbb{R}(G_n, G_n)$ for each $\underline{\lambda} \in \Omega_n$. Therefore the $E_{\underline{\lambda}}$ are pairwise orthogonal, namely we have $E_{\underline{\lambda}} E_{\underline{\mu}} = \delta_{\underline{\lambda}\underline{\mu}} E_{\underline{\lambda}}$ for all $\underline{\lambda}, \underline{\mu} \in \Omega_n$. Since $E_{\underline{\lambda}}$ is idempotent, $\mathrm{rk}(E_{\underline{\lambda}})$ is just the trace of $E_{\underline{\lambda}}$. It follows from (3.1) that the trace of $F_{\underline{\lambda}}$ equals $\chi^{\underline{\lambda}}(1)^2$. Hence we have

$$\sum_{\underline{\lambda} \in \Omega_n} \dim V_{\underline{\lambda}} = \sum_{\underline{\lambda} \in \Omega_n} \mathrm{rk}(E_{\underline{\lambda}}) = \sum_{\underline{\lambda} \in \Lambda_n} \chi^{\underline{\lambda}}(1)^2 = |G_n|$$

by standard properties of the degrees of irreducible characters. This proves the first statement. We have $\chi^{\underline{\lambda}}(1) = \chi^{\underline{\lambda}^*}(1)$ by Lemma 2.2, from which together with (3.2) and Lemma 3.2 it is readily verified that

$$A_{\underline{\sigma}} = \sum_{\underline{\lambda} \in \Omega_n} P(\underline{\lambda}, \underline{\sigma})\, E_{\underline{\lambda}}.$$

Since the $E_{\underline{\lambda}}$ are pairwise orthogonal, we obtain the second statement. □

In fact the proof of Lemma 3.3 shows that $\{A_{\underline{\sigma}} : \underline{\sigma} \in \Omega_n\}$ is a symmetric association scheme with primitive idempotents given by $\{E_{\underline{\lambda}} : \underline{\lambda} \in \Omega_n\}$. However we will not exploit this further.

Note that $A_{\underline{\sigma}}$ is the adjacency matrix of a $|D_{\underline{\sigma}}|$-regular graph for each $\underline{\sigma} \in \Omega_n$, except for $\underline{\sigma}$ given by $\underline{\sigma}(1) = (1^n)$, and that $P(\underline{\lambda}, \underline{\sigma}) = |D_{\underline{\sigma}}|$ if $\underline{\lambda} \in \Omega_n$ is given by $X - 1 \mapsto (n)$.

The strategy to prove Theorems 1.1 and 1.2 is as follows (Theorems 1.5 and 1.6 will be proved using slight modifications). We call an element $x \in G$ a *t-derangement* if there is no t-tuple of linearly independent elements of $\mathbb{F}_q^n$ that is fixed by x. Equivalently $x \in G_n$ is a t-derangement if $\mathrm{rk}(x - I) > n - t$. It is readily verified that either all elements of $D_{\underline{\sigma}}$ are t-derangements or none of them. We wish to identify an appropriate subset Σ of Ω_n such that $D_{\underline{\sigma}}$ consists of t-derangements for all $\underline{\sigma} \in \Sigma$ and then apply Proposition 3.1 to the graph Γ with adjacency matrix $\sum_{\underline{\sigma}\in\Sigma} A_{\underline{\sigma}}$ and $|D_{\underline{\sigma}}|$-regular spanning subgraphs $\Gamma_{\underline{\sigma}}$ having adjacency matrix $A_{\underline{\sigma}}$ for $\underline{\sigma} \in \Sigma$. In view of Lemma 3.3, we wish to construct some $w \in \mathbb{R}(\Sigma)$ such that both the minimum value and the negative of the second-largest absolute value over all $\underline{\lambda} \in \Omega_n$ of

$$\sum_{\underline{\sigma}\in\Sigma} w(\underline{\sigma}) P(\underline{\lambda}, \underline{\sigma}) \tag{3.6}$$

equals

$$\eta = -\frac{1}{(q^n - 1)(q^n - q)\cdots(q^n - q^{t-1}) - 1} \tag{3.7}$$

and such that w is normalised in the sense that (3.6) equals 1 if $\psi^{\underline{\lambda}}$ is the trivial character (or equivalently $\underline{\lambda} \in \Omega_n$ is given by $X - 1 \mapsto (n)$). This will ensure that Proposition 3.1 will give the bounds of Theorems 1.1 and 1.2.

4. An invertible matrix

This section contains some key preparations for our main proofs. We first identify relevant conjugacy classes of G_n whose elements are either t-derangements or do not fix a t-space. We then use these conjugacy classes to identify a matrix related to the character table of G_n. A key step is to show that this matrix is invertible.

We call an element of G_n *regular elliptic* if its characteristic polynomial is irreducible. The following lemma shows that regular elliptic elements in G_n play the role of an n-cycle in the symmetric group S_n.

Lemma 4.1 ([19, Prop. 4.4])**.** *Each regular elliptic element of G_n fixes no proper nontrivial subspace of $\mathbb{F}_q^n$.*

Note that, for each $f \in \Phi$ of degree d, its companion matrix satisfies $\det(C_f) = (-1)^d f(0)$. It is well known [16] that, for each $a \in \mathbb{F}_q^*$, there exists an irreducible polynomial $f \in \mathbb{F}_q[x]$ of degree d such that $f(0) = a$. Hence we can always find a polynomial in Φ with prescribed degree and prescribed nonzero determinant of its companion matrix. Also note that, for each $f \in \Phi$, we have $f(0)f^*(0) = 1$ and therefore

$$\det(C_f)\det(C_{f^*}) = 1.$$

We now continue to use α to denote a fixed generator of $\mathbb{F}_q^*$. For all integers ℓ, j satisfying $0 \leqslant \ell < n$ and $0 \leqslant j \leqslant q - 2$, we fix an irreducible polynomial $h_{\ell,j} \in \Phi$

of degree $n-\ell$ such that its companion matrix has determinant α^j and such that $h^*_{\ell,j} = h_{\ell,-j}$. We define

$$\Sigma_{\ell,j} = \{\underline{\sigma} \in \Lambda_n : \underline{\sigma}(h_{\ell,j}) = (1)\}.$$

and

$$\Sigma_\ell = \bigcup_{j=0}^{q-2} \Sigma_{\ell,j} \quad \text{and} \quad \Sigma_{\leqslant t} = \bigcup_{\ell=0}^{t} \Sigma_\ell.$$

Note that, for each $\underline{\sigma} \in \Sigma_{\leqslant t-1}$, the conjugacy class $C_{\underline{\sigma}}$ consists of elements that do not fix a t-space of $\mathbb{F}_q^n$. In addition, for each $\underline{\sigma} \in \Sigma_t$ with the $q-1$ exceptions $\underline{\sigma} \in \Sigma_t$ satisfying $\underline{\sigma}(X-1) = (1^t)$, the conjugacy class $C_{\underline{\sigma}}$ consists of elements that do not fix a t-space pointwise. Next we define

$$\Pi_{k,i} = \{\underline{\lambda} \in \Lambda_n : \underline{\lambda}(\alpha^i)_1 = n-k\}.$$

and

$$\Pi_k = \bigcup_{i=0}^{q-2} \Pi_{k,i} \quad \text{and} \quad \Pi_{\leqslant t} = \bigcup_{k=0}^{t} \Pi_k.$$

Note that, for $k < n/2$, we have $|\Pi_{k,i}| = |\Sigma_{k,i}|$ and $|\Omega_n \cap \Pi_{k,i}| = |\Omega_n \cap \Sigma_{k,i}|$.

We define $Q \in \mathbb{R}(\Omega_n, \Omega_n)$ by

$$Q(\underline{\lambda}, \underline{\sigma}) = \psi^{\underline{\lambda}}_{\underline{\sigma}} \quad \text{for each } \underline{\lambda}, \underline{\sigma} \in \Omega_n$$

and let Q_t be the restriction of Q to $\mathbb{R}(\Omega_n \cap \Pi_{\leqslant t}, \Omega_n \cap \Sigma_{\leqslant t})$. We emphasise that Q_t is a square matrix. A key step in our proof is the following proposition.

Proposition 4.2. *For $n > 2t$, the matrix Q_t has full rank and is independent of n.*

In the remainder of this section we essentially only prove Proposition 4.2. The reader who is interested in maintaining the flow of the proof of our main results may wish to skip to the next section at first reading.

We define $R \in \mathbb{C}(\Lambda_n, \Lambda_n)$ by

$$R(\underline{\lambda}, \underline{\sigma}) = \chi^{\underline{\lambda}}_{\underline{\sigma}} \quad \text{for each } \underline{\lambda}, \underline{\sigma} \in \Lambda_n$$

and let R_t be the restriction of R to $\mathbb{C}(\Pi_{\leqslant t}, \Sigma_{\leqslant t})$. We shall prove a counterpart of Proposition 4.2 for the matrix R_t.

Proposition 4.3. *For $n > 2t$, the matrix R_t has full rank and is independent of n.*

Note that Q_t is obtained from R_t by first applying elementary row operations, then deleting some rows, and then (in view of (3.3)) deleting duplicate columns. Hence Proposition 4.2 follows from Proposition 4.3.

We now prove Proposition 4.3. We let $S \in \mathbb{C}(\Lambda_n, \Lambda_n)$ be the matrix defined by

$$S(\underline{\mu}, \underline{\sigma}) = \xi^{\underline{\mu}}_{\underline{\sigma}} \quad \text{for each } \underline{\mu}, \underline{\sigma} \in \Lambda_n \tag{4.1}$$

and let S_t be the restriction of S to $\mathbb{C}(\Pi_{\leqslant t}, \Sigma_{\leqslant t})$. Now recall the equivalence relation $\sim$ on Λ_n and the numbers $K_{\underline{\lambda}\underline{\mu}}$ from Section 2.4. Define $T \in \mathbb{C}(\Lambda_n, \Lambda_n)$ by

$$T(\underline{\mu}, \underline{\lambda}) = \begin{cases} K_{\underline{\lambda}\underline{\mu}} & \text{for } \underline{\lambda} \sim \underline{\mu} \\ 0 & \text{otherwise} \end{cases}$$

and let T_t be the restriction of T to $\mathbb{C}(\Pi_{\leqslant t}, \Pi_{\leqslant t})$. We first prove the following.

Lemma 4.4.

(i) We have $S = TR$ and T has full rank.

(ii) For $n > 2t$, we have $S_t = T_t R_t$ and T_t has full rank and is independent of n.

Proof. From (2.12) we have $S = TR$ and T is block diagonal, where the blocks are induced by the equivalence classes under $\sim$. Each diagonal block corresponds to one equivalence class. If $s : \Phi \to \mathbb{Z}$ is the shape of such an equivalence class, then the corresponding block can be written as a Kronecker product

$$\bigotimes_{f \in \Phi} K^{(s(f))},$$

where $K^{(m)} \in \mathbb{C}(\mathrm{Par}_m, \mathrm{Par}_m)$ is a Kostka matrix given by $K^{(m)}(\mu, \lambda) = K_{\lambda\mu}$ with the convention $K^{(0)} = (1)$ and Par_m is the set of partitions of m. By (2.8) the Kostka matrices are invertible. Hence T is a block-diagonal matrix whose blocks are Kronecker products of matrices of full rank and so T itself has full rank. This proves (i).

From (2.8) we find that $S_t = T_t R_t$. Note that T_t is still block diagonal with one diagonal block for each equivalence class of Λ_n under $\sim$ whose shape $s : \Phi \to \mathbb{Z}$ satisfies $s(\alpha^i) \geqslant n - t$ for some i. The corresponding block can be written as

$$\tilde{K}^{(s(\alpha^i))} \otimes \bigotimes_{f \in \Phi \setminus \{\alpha^i\}} K^{(s(f))}, \tag{4.2}$$

where $\tilde{K}^{(s(\alpha^i))}$ is the matrix $K^{(s(\alpha^i))}$ restricted to partitions λ of $s(\alpha^i)$ satisfying

$$\lambda \trianglerighteq (n - t, 1^{s(\alpha^i) - (n-t)}).$$

From (2.8) we find that, after a suitable ordering of rows and columns, all matrices occuring in the Kronecker product (4.2) are upper-triangular with ones on the diagonal. Again T_t is a block-diagonal matrix whose blocks are Kronecker products of matrices of full rank and so T_t itself has full rank.

From the proof of [7, Thm. 20] we know that $\tilde{K}^{(s(\alpha^i))}$ is independent of n. Moreover all other matrices occuring in the Kronecker product (4.2) are also independent of n. Hence T_t itself is also independent of n. This proves (ii). □

Next we shall show that the matrix S_t has full rank. Recall that, for a composition λ, we denote by P_λ the parabolic subgroup of $G_{|\lambda|}$ given in (2.2). We start with the following lemma.

Lemma 4.5. *Let m and n be positive integers satisfying $m < n$ and let ϕ and ψ be class functions of G_m and G_n, respectively. Let $\pi_1 : P_{(m,n)} \to G_m$ and $\pi_2 : P_{(m,n)} \to G_n$ be the natural projections onto the corresponding diagonal blocks. Let $g \in P_{(m,n)}$ be such that $\pi_2(g)$ is regular elliptic. Then we have*

$$(\phi \odot \psi)(g) = \phi(\pi_1(g))\, \psi(\pi_2(g)).$$

Proof. From (2.4) we have

$$(\phi \odot \psi)(g) = \frac{1}{|P_{(m,n)}|} \sum_{\substack{x \in G_{m+n} \\ xgx^{-1} \in P_{(m,n)}}} \phi(\pi_1(xgx^{-1}))\psi(\pi_2(xgx^{-1})). \tag{4.3}$$

Since $\pi_2(g)$ is regular elliptic and $m < n$, we find from Lemma 4.1 that g stabilises a unique m-dimensional subspace U of $\mathbb{F}_q^{m+n}$. Hence the number of $x \in G_{m+n}$ such that $xgx^{-1} \in P_{(m,n)}$ is the number of ordered bases $\{u_1, \dots, u_m, w_1, \dots, w_n\}$ of $\mathbb{F}_q^{m+n}$ such that $\{u_1, \dots, u_m\}$ spans U. This number equals $|P_{(m,n)}|$. Since $xgx^{-1} \in P_{(m,n)}$ for each $x \in P_{(m,n)}$, we conclude that

$$\{x \in G_{m+n} : xgx^{-1} \in P_{(m,n)}\} = P_{(m,n)}.$$

Since $\pi_i(xgx^{-1})$ is conjugate to $\pi_i(g)$ for each $i \in \{1, 2\}$ and each $x \in P_{(m,n)}$, the statement of the lemma follows from (4.3). □

We use Lemma 4.5 to prove the following lemma on the structure of the matrix S.

Lemma 4.6. *Let k, ℓ be integers satisfying $0 \leqslant k, \ell < n/2$ and let $\underline{\mu} \in \Pi_{k,i}$ and $\underline{\sigma} \in \Sigma_{\ell,j}$. If $k > \ell$, then we have $\xi^{\underline{\mu}}_{\underline{\sigma}} = 0$. For $k \leqslant \ell$, let ν be the partition obtained from $\underline{\mu}(X - \alpha^i)$ by replacing the part $n - k$ by $\ell - k$ and define $\underline{\nu}, \underline{\tau} \in \Lambda_\ell$ by*

$$\underline{\nu}(f) = \begin{cases} \nu & \text{for } f = X - \alpha^i \\ \underline{\mu}(f) & \text{otherwise} \end{cases} \quad \text{and} \quad \underline{\tau}(f) = \begin{cases} \varnothing & \text{for } f = h_{\ell,j} \\ \underline{\sigma}(f) & \text{otherwise.} \end{cases}$$

If $k \leqslant \ell$, then we have $\xi^{\underline{\mu}}_{\underline{\sigma}} = \xi^{\underline{\nu}}_{\underline{\tau}} \omega^{ij}$.

Proof. Let $g \in C_{\underline{\sigma}}$. Define $\underline{\kappa} \in \Lambda_k$ by

$$\underline{\kappa}(f) = \begin{cases} (\underline{\mu}(\alpha^i)_2, \underline{\mu}(\alpha^i)_3, \dots) & \text{for } f = X - \alpha^i \\ \underline{\mu}(f) & \text{otherwise,} \end{cases}$$

so that by (2.6) and (2.11)

$$\xi^{\underline{\mu}} = \xi^{\underline{\kappa}} \odot \xi^{X - \alpha^i \mapsto (n-k)}, \tag{4.4}$$

For $\xi^{\underline{\mu}}(g)$ to be nonzero, g must be conjugate to an element of $P_{(k,n-k)}$. Each such element fixes a k-dimensional subspace of $\mathbb{F}_q^n$. If $k > \ell$, then by Lemma 4.1, g fixes no k-dimensional subspace of $\mathbb{F}_q^n$ and hence $\xi^{\underline{\mu}}(g) = 0$.

Henceforth assume that $k \leqslant \ell$. We shall frequently use $\xi^{f\mapsto(m)} = \chi^{f\mapsto(m)}$, which follows from (2.7) and (2.8). Since $k \leqslant \ell$ we have

$$\xi^{\underline{\nu}} = \xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i\mapsto(\ell-k)}. \tag{4.5}$$

Write

$$E = \bigcup_{\substack{\underline{\rho}\in\Lambda_{n-k}\\ \underline{\rho}(h_{\ell,j})=(1)}} C_{\underline{\rho}}.$$

We claim that

$$\xi^{X-\alpha^i\mapsto(n-k)}(e) = (\xi^{X-\alpha^i\mapsto(\ell-k)} \odot \xi^{X-\alpha^i\mapsto(n-\ell)})(e) \quad \text{for each } e \in E. \tag{4.6}$$

Indeed, each $e \in E$ is conjugate to an element of $P_{(\ell-k,n-\ell)}$ with blocks $e_1 \in G_{\ell-k}$ and $e_2 \in G_{n-\ell}$ on the main diagonal, where e_2 is regular elliptic. Hence we find from Lemma 2.3 that, for each $e \in E$, the left hand side of (4.6) equals

$$\begin{aligned}\theta(\det(e)^i) &= \theta(\det(e_1)^i) \cdot \theta(\det(e_2)^i)\\ &= \xi^{X-\alpha^i\mapsto(\ell-k)}(e_1) \cdot \xi^{X-\alpha^i\mapsto(n-\ell)}(e_2),\end{aligned}$$

which by Lemma 4.5 equals the right hand side of (4.6). From (4.4) we have

$$\xi^{\underline{\mu}}(g) = \frac{1}{|P_{(k,n-k)}|} \sum_{\substack{x\in G_n\\ xgx^{-1}\in P_{(k,n-k)}}} \xi^{\underline{\kappa}}(\pi_1(xgx^{-1}))\xi^{X-\alpha^i\mapsto(n-k)}(\pi_2(xgx^{-1})),$$

where $\pi_1 : P_{(k,n-k)} \to G_k$ and $\pi_2 : P_{(k,n-k)} \to G_{n-k}$ are the natural projections onto the diagonal blocks. Since $k, \ell < n/2$, Lemma 4.1 implies that each $\pi_2(xgx^{-1})$ occuring in the summation is forced to lie inside E. Hence by subsequent applications of (4.4), (4.6), and (4.5) we then find that

$$\begin{aligned}\xi^{\underline{\mu}}(g) &= (\xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i\mapsto(n-k)})(g)\\ &= (\xi^{\underline{\kappa}} \odot \xi^{X-\alpha^i\mapsto(\ell-k)} \odot \xi^{X-\alpha^i\mapsto(n-\ell)})(g)\\ &= (\xi^{\underline{\nu}} \odot \xi^{X-\alpha^i\mapsto(n-\ell)})(g).\end{aligned}$$

Without loss of generality, we may assume that $g \in P_{(\ell,n-\ell)}$ and that the diagonal blocks of g are g_1 and g_2, where $g_1 \in C_{\underline{\tau}}$ and g_2 is the companion matrix of $h_{\ell,j}$. Since g_2 is regular elliptic, we may apply Lemma 4.5 once more to obtain

$$\xi^{\underline{\mu}}(g) = \xi^{\underline{\nu}}(g_1)\, \xi^{X-\alpha^i\mapsto(n-\ell)}(g_2).$$

Since $g_1 \in C_{\underline{\tau}}$, we have $\xi^{\underline{\nu}}(g_1) = \xi^{\underline{\nu}}_{\underline{\tau}}$, and since g_2 is the companion matrix of $h_{\ell,j}$, we find from Lemma 2.3 that

$$\xi^{X-\alpha^i\mapsto(n-\ell)}(g_2) = \theta(\det(g_2)^i) = \omega^{ij}.$$

Hence we obtain $\xi^{\underline{\mu}}(g) = \xi^{\underline{\nu}}_{\underline{\tau}}\omega^{ij}$, as required. □

We can now prove the required property of the matrix S_t.

Lemma 4.7. *For $n > 2t$, the matrix S_t has full rank and is independent of n.*

Proof. To indicate dependence on n, write $S^{(n)}$ for the matrix S given in (4.1) and $S_t^{(n)}$ for the corresponding restricted matrix S_t. Let $n > 2t$. From Lemma 4.6 we find that all entries in $S_t^{(n)}$ are independent of n, which proves the second statement of the lemma.

To show that $S_t^{(n)}$ is invertible, we view $S_t^{(n)}$ as a block matrix, where the blocks are indexed by Π_k and Σ_ℓ for $k, \ell \in \{0, 1, \ldots, t\}$. Let $B_{k,\ell}$ be the block corresponding to Π_k and Σ_ℓ. Lemma 4.6 implies that $B_{k,\ell}$ is zero for $k > \ell$ and, for $0 \leqslant k \leqslant t$, the block B_{kk} is the Kronecker product of $S^{(k)}$ and the Vandermonde matrix $(\omega^{ij})_{0 \leqslant i,j \leqslant q-2}$. Since the character table of irreducible characters of every finite group is invertible, Lemma 4.4 implies that $S^{(k)}$ is invertible and so B_{kk} is invertible. Hence $S_t^{(n)}$ is block upper-triangular and all diagonal blocks are invertible. Therefore $S_t^{(n)}$ itself is invertible. □

Finally, by combining Lemmas 4.4 and 4.7, we obtain a proof of Proposition 4.3.

5. Proof of Theorems 1.1 and 1.2

Now recall the definition (3.5) of the eigenvalues $P(\underline{\lambda}, \underline{\sigma})$ and the definition (3.7) of the prescribed extremal eigenvalue η. As a first step in constructing the required weight function w occuring in (3.6), we prove the following result.

Proposition 5.1. *Let n and t be positive integers satisfying $n > 2t$. Then there exists $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leqslant t})$ such that $w(\underline{\sigma}) = 0$ for $\underline{\sigma}(1) = (1^t)$ and*

(5.1)
$$\sum_{\underline{\sigma} \in \Omega_n \cap \Sigma_{\leqslant t}} w(\underline{\sigma}) P(\underline{\lambda}, \underline{\sigma}) = \begin{cases} 1 & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{0,0} \\ \eta & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{k,0} \text{ and } 1 \leqslant k \leqslant t \\ 0 & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{k,i} \text{ and } 0 \leqslant k \leqslant t \text{ and } 1 \leqslant i \leqslant q-2 \end{cases}$$

and

$$|w(\underline{\sigma})| \leqslant \frac{\gamma_t}{|D_{\underline{\sigma}}|} \quad \text{for all } \underline{\sigma} \in \Omega_n \cap \Sigma_{\leqslant t} \tag{5.2}$$

for some constant γ_t depending only on t.

Proof. From Proposition 4.2 we know that Q_t has full rank. In view of (3.5) there exists a unique $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leqslant t})$ satisfying (5.1).

We now show that $w(\underline{\sigma}) = 0$ for the $\lfloor q/2 \rfloor + 1$ elements $\underline{\sigma} \in \Omega_n \cap \Sigma_{\leqslant t}$ satisfying $\underline{\sigma}(1) = (1^t)$. Without loss of generality we may assume that Ω_n contains $X - \alpha^i$ and $h_{t,j}$ for all $i, j = 0, 1, \ldots, \lfloor q/2 \rfloor$. Accordingly we define $\underline{\sigma}_j \in \Sigma_{t,j}$ by $\underline{\sigma}_j(1) = (1^t)$ for $j = 0, 1, \ldots, \lfloor q/2 \rfloor$. Recall the definition of the character $\zeta^{(t,i)}$ from Section 2.4 and write $\zeta_{\underline{\sigma}}^{(t,i)}$ for this character evaluated on the conjugacy class $C_{\underline{\sigma}}$. We evaluate the

sum

$$S_i = \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} w(\underline{\sigma})|D_{\underline{\sigma}}|\,(\zeta^{(t,i)}_{\underline{\sigma}} + \zeta^{(t,-i)}_{\underline{\sigma}}) \tag{5.3}$$

in two ways. Since $\zeta^{(t,0)}$ is the permutation character on the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$, we find by Lemma 2.4 that the summand in (5.3) is nonzero only when the elements of $C_{\underline{\sigma}}$ fix a t-tuple of linearly independent elements of $\mathbb{F}_q^n$, hence only when $\underline{\sigma} = \underline{\sigma}_j$ for some j. By the definition of $\underline{\sigma}_j$, each element in $C_{\underline{\sigma}_j}$ has determinant α^j. Hence by applying Lemma 2.4 twice we obtain

$$\zeta^{(t,i)}_{\underline{\sigma}_j} = \omega^{ij}\zeta^{(t,0)}_{\underline{\sigma}_j} = \omega^{ij}\zeta^{(t,0)}_{\underline{\sigma}_0}$$

and therefore

$$S_i = 2\zeta^{(t,0)}_{\underline{\sigma}_0} \sum_{j=0}^{\lfloor q/2\rfloor} w(\underline{\sigma}_j)\,|D_{\underline{\sigma}_j}|\,\cos\left(\frac{2\pi ij}{q-1}\right). \tag{5.4}$$

On the other hand, since $\zeta^{(t,i)} + \zeta^{(t,-i)}$ is a real-valued class function, we find from Lemma 3.2 that it is a linear combination of $\psi^{\underline{\lambda}}$ for $\underline{\lambda} \in \Omega_n$. Hence by Lemma 2.5 there exists numbers $n_{i,\underline{\lambda}}$ such that

$$\zeta^{(t,i)}_{\underline{\sigma}} + \zeta^{(t,-i)}_{\underline{\sigma}} = \sum_{\substack{\underline{\lambda}\in\Omega_n\\ \underline{\lambda}(\alpha^i)_1\geqslant n-t}} n_{i,\underline{\lambda}}\,\psi^{\underline{\lambda}}_{\underline{\sigma}}$$

and hence

$$S_i = \sum_{\substack{\underline{\lambda}\in\Omega_n\\ \underline{\lambda}(\alpha^i)_1\geqslant n-t}} n_{i,\underline{\lambda}} \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} w(\underline{\sigma})|D_{\underline{\sigma}}|\,\psi^{\underline{\lambda}}_{\underline{\sigma}}. \tag{5.5}$$

Since (5.1) holds, we conclude that $S_i = 0$ for each i satisfying $1 \leqslant i \leqslant \lfloor q/2\rfloor$. Since $\zeta^{(t,0)}$ is a permutation character, it contains the trivial character with multiplicity 1 (this can be seen by Frobenius reciprocity, for example). Hence we have $n_{0,\underline{\lambda}} = 2$ for $\underline{\lambda} \in \Omega_n$ satisfying $\underline{\lambda}(1) = (n)$. We therefore find from (5.5) and (5.1) that

$$S_0 = 2 + \eta \sum_{\substack{\underline{\lambda}\in\Omega_n\\ n-t\leqslant\underline{\lambda}(1)_1<n}} n_{0,\underline{\lambda}}\,\psi^{\underline{\lambda}}(1) = 2 + 2\eta(\zeta^{(t,0)}(1) - 1).$$

Since $\zeta^{(t,0)}(1)$ equals the number of t-tuples of linearly independent elements of $\mathbb{F}_q^n$, we have

$$\zeta^{(t,0)}(1) = (q^n - 1)(q^n - q)\cdots(q^n - q^{t-1}). \tag{5.6}$$

Therefore $S_0 = 0$ and so $S_i = 0$ for each i satisfying $0 \leqslant i \leqslant \lfloor q/2\rfloor$. Since each element of $C_{\underline{\sigma}_0}$ fixes a t-tuple of linearly independent elements of $\mathbb{F}_q^n$, we have $\zeta^{(t,0)}_{\underline{\sigma}_0} \neq 0$.

Thus (5.4) implies

$$\sum_{j=0}^{\lfloor q/2\rfloor} w(\underline{\sigma}_j)\,|D_{\underline{\sigma}_j}|\,\cos\left(\frac{2\pi ij}{q-1}\right)=0 \quad \text{for each } i \text{ satisfying } 0\leqslant i\leqslant \lfloor q/2\rfloor$$

and it is readily verified, using that $(\omega^{ij})_{0\leqslant i,j<q-1}$ is a Vandermonde matrix, that this in turn implies that $w(\underline{\sigma}_j)=0$ for all j satisfying $0\leqslant j\leqslant\lfloor q/2\rfloor$, as required.

Now, for each $\underline{\lambda}\in\Omega_n$ satisfying $n-t\leqslant\underline{\lambda}(1)_1<n$, we find from Lemma 2.5 that

$$|\eta|\,\psi^{\underline{\lambda}}(1)\leqslant|\eta|\,(\zeta^{(t,0)}(1)-1)=1,$$

using (5.6). Since $\psi^{\underline{\lambda}}(1)=\chi^{\underline{\lambda}}(1)=1$ for $\underline{\lambda}\in\Pi_{0,0}$, we conclude from (5.1) that

$$\left|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} w(\underline{\sigma})|D_{\underline{\sigma}}|\,\psi^{\underline{\lambda}}_{\underline{\sigma}}\right|\leqslant 1 \quad \text{for each } \underline{\lambda}\in\Omega_n\cap\Pi_{\leqslant t}.$$

By Lemma 4.2 all entries of Q_t (which are precisely the values of $\psi^{\underline{\lambda}}_{\underline{\sigma}}$ occuring in the sum) are independent of n and so are uniformly bounded by some value only depending on t. The same also holds for the inverse of Q_t, which establishes (5.2). □

In what follows we treat the remaining eigenvalues.

Lemma 5.2. *Let n and t be positive integers satisfying $n>2t$ and let $w\in\mathbb{R}(\Omega_n\cap\Sigma_{\leqslant t})$ be such that*

$$|w(\underline{\sigma})|\leqslant\frac{\gamma_t}{|D_{\underline{\sigma}}|} \quad \textit{for all } \underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}$$

for some constant γ_t depending only on t. Then

$$\left|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma})\right|<|\eta| \quad \textit{for all } \underline{\lambda}\in\Omega_n\backslash\Pi_{\leqslant t},$$

provided that n is sufficiently large compared to t.

In the proof of the lemma we use the usual scalar product on class functions of G_n, which is given by

$$\langle\chi,\psi\rangle=\frac{1}{|G_n|}\sum_{g\in G_n}\chi(g)\overline{\psi(g)}, \tag{5.7}$$

where χ,ψ are class functions of G_n.

Proof of Lemma 5.2. By the definition (3.5) of $P(\underline{\lambda},\underline{\sigma})$ and (3.3) we have

$$P(\underline{\lambda},\underline{\sigma})=\frac{|G_n|}{\psi^{\underline{\lambda}}(1)}\langle\psi^{\underline{\lambda}},1_{D_{\underline{\sigma}}}\rangle. \tag{5.8}$$

Since $\chi^{\underline{\lambda}}$ is irreducible, we have $\langle\psi^{\underline{\lambda}},\psi^{\underline{\lambda}}\rangle=1$ or 2 and therefore we obtain, by an application of the Cauchy-Schwarz inequality,

$$|\langle\psi^{\underline{\lambda}},1_{D_{\underline{\sigma}}}\rangle|\leqslant\sqrt{2\langle 1_{D_{\underline{\sigma}}},1_{D_{\underline{\sigma}}}\rangle}=\sqrt{\frac{2|D_{\underline{\sigma}}|}{|G_n|}}.$$

From (5.8) and our hypothesis on w we then find that

$$\begin{aligned}\left|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma})\right| &\leqslant \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} |w(\underline{\sigma})|\,|P(\underline{\lambda},\underline{\sigma})| \\ &\leqslant \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} \frac{\gamma_t}{|D_{\underline{\sigma}}|}\frac{|G_n|}{\psi^{\underline{\lambda}}(1)}\sqrt{\frac{2|D_{\underline{\sigma}}|}{|G_n|}} \\ &\leqslant \frac{\gamma_t\,|\Sigma_{\leqslant t}|}{\psi^{\underline{\lambda}}(1)}\max_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}}\sqrt{\frac{2|G_n|}{|D_{\underline{\sigma}}|}} \\ &\leqslant \frac{\gamma_t\,|\Sigma_{\leqslant t}|}{\chi^{\underline{\lambda}}(1)}\max_{\underline{\sigma}\in\Sigma_{\leqslant t}}\sqrt{\frac{2|G_n|}{|C_{\underline{\sigma}}|}}.\end{aligned}$$

Note that $|\Sigma_{\leqslant t}|$ is independent of n. Using Lemmas 7.1 and 7.2, to be stated and proved in Section 7, we find that there is a constant γ_t', depending only on t, such that

$$\left|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma})\right| \leqslant \frac{\gamma_t'}{q^{n/2}}\,\frac{1}{q^{nt}}$$

for all $\underline{\lambda}\in\Omega_n\backslash\Pi_{\leqslant t}$ and all sufficiently large n. The right hand side is certainly strictly smaller than $1/q^{nt}$ for all sufficiently large n and the proof is completed by noting that $|\eta| > 1/q^{nt}$. □

Recall that $V_{\underline{\lambda}}$ is the column span of $E_{\underline{\lambda}}$. Define

$$U_t = \sum_{\substack{\underline{\lambda}\in\Omega_n \\ \underline{\lambda}(1)_1\geqslant n-t}} V_{\underline{\lambda}}.$$

Now we obtain the following.

Theorem 5.3. *Let t be a positive integer. Then, for all sufficiently large n, the following holds.*

(i) Every t-intersecting set Y in G_n satisfies

$$|Y| \leqslant \prod_{i=t}^{n-1}(q^n - q^i)$$

and, in case of equality, we have $1_Y \in U_t$.

(ii) Every pair of t-cross-intersecting sets Y, Z in G_n satisfies

$$\sqrt{|Y|\cdot|Z|} \leqslant \prod_{i=t}^{n-1}(q^n - q^i)$$

and, in case of equality, we have $1_Y, 1_Z \in U_t$.

Proof. As explained at the end of Section 3, we apply Proposition 3.1 to the graph with adjacency matrix

$$\sum_{\substack{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}\\ \underline{\sigma}(1)\neq(1^t)}} A_{\underline{\sigma}}$$

and the $|D_{\underline{\sigma}}|$-regular spanning subgraphs with adjacency matrix $A_{\underline{\sigma}}$ for those $\underline{\sigma}$ occuring in the above set union. Since none of the elements in $D_{\underline{\sigma}}$ for such $\underline{\sigma}$ fix a t-space pointwise, every t-intersecting set in G_n is an independent set in this graph. Recall from Lemma 3.3 that every element of $V_{\underline{\lambda}}$ is an eigenvector of $A_{\underline{\sigma}}$ with eigenvalue $P(\underline{\lambda},\underline{\sigma})$. Let $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leqslant t})$ be the vector given by Proposition 5.1 and write

$$P(\underline{\lambda}) = \sum_{\substack{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t}\\ \underline{\sigma}(1)\neq(1^t)}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma}).$$

Proposition 5.1 and Lemma 5.2 imply that, for all sufficiently large n, we have

$$P(\underline{\lambda}) = \begin{cases} 1 & \text{for } \underline{\lambda}(1)_1 = n \\ \eta & \text{for } n-t \leqslant \underline{\lambda}(1)_1 < n \end{cases}$$

and $|P(\underline{\lambda})| < |\eta|$ for $\underline{\lambda}(1)_1 < n-t$. Hence, writing $\underline{\lambda}_0$ for $X-1 \mapsto (n)$, we have $P(\underline{\lambda}_0) = 1$ and

$$\eta = \min_{\underline{\lambda}\neq\underline{\lambda}_0} P(\underline{\lambda}) \quad \text{and} \quad |\eta| = \max_{\underline{\lambda}\neq\underline{\lambda}_0} |P(\underline{\lambda})|.$$

Then the required result follows from Proposition 3.1 and the decomposition of $\mathbb{R}(G_n)$ given in Lemma 3.3. □

Our proof of Theorems 1.1 and 1.2 is completed by the following result.

Theorem 5.4. *U_t is spanned by the characteristic vectors of t-cosets.*

Proof. Let $\mathcal{A}_t$ be the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$. Define the incidence matrix $M_t \in \mathbb{C}(G_n, \mathcal{A}_t \times \mathcal{A}_t)$ of elements of G_n versus t-cosets by

$$M_t(x,(u,v)) = \begin{cases} 1 & \text{for } xu = v \\ 0 & \text{otherwise,} \end{cases}$$

so that the columns of M_t are precisely the characteristic vectors of the t-cosets. Let $\zeta^t = \zeta^{(t,0)}$ be the permutation character of the set of t-tuples of linearly independent elements of $\mathbb{F}_q^n$ and define $C_t \in \mathbb{C}(G_n, G_n)$ by

$$C_t(x,y) = \zeta^t(x^{-1}y).$$

Denoting by $\mathbb{1}_{xu=v}$ the indicator of the event that $x \in G_n$ maps u to v, we have

$$\begin{aligned}(M_t M_t^T)(x,y) &= \sum_{u,v} M_t(x,(u,v)) M_t(y,(u,v)) \\ &= \sum_{u,v} \mathbb{1}_{xu=v} \mathbb{1}_{yu=v} \\ &= \sum_{u} \mathbb{1}_{xu=yu} \\ &= \sum_{u} \mathbb{1}_{x^{-1}yu=u} \\ &= \zeta^t(x^{-1}y) = C_t(x,y).\end{aligned}$$

Hence we have $C_t = M_t M_t^T$ and so the column span of C_t equals the column span of M_t or equivalently the span of the characteristic vectors of the t-cosets.

From Lemma 2.5 we have

$$\zeta^t = \sum_{\substack{\underline{\lambda} \in \Lambda_n \\ \underline{\lambda}(1)_1 \geqslant n-t}} m_{\underline{\lambda}} \, \chi^{\underline{\lambda}}$$

for some integers $m_{\underline{\lambda}}$ satisfying $m_{\underline{\lambda}} \neq 0$ for each $\underline{\lambda}$ occuring in the summation. Since ζ^t is real-valued, we find by Lemma 3.2 that $m_{\underline{\lambda}^*} = m_{\underline{\lambda}}$ and therefore have

$$\zeta^t = \sum_{\substack{\underline{\lambda} \in \Omega_n \\ \underline{\lambda}(1)_1 \geqslant n-t}} m_{\underline{\lambda}} \, \psi^{\underline{\lambda}}. \tag{5.9}$$

Lemma 2.2 implies that $\chi^{\underline{\lambda}}(1) = \chi^{\underline{\lambda}^*}(1)$. We therefore obtain from (3.4) and (3.1) that

$$E_{\underline{\lambda}}(x,y) = \frac{\chi^{\underline{\lambda}}(1)}{|G_n|} \psi^{\underline{\lambda}}(x^{-1}y)$$

and thus find from (5.9) that

$$C_t = |G_n| \sum_{\substack{\underline{\lambda} \in \Lambda_n \\ \underline{\lambda}(1)_1 \geqslant n-t}} \frac{m_{\underline{\lambda}}}{\chi^{\underline{\lambda}}(1)} \, E_{\underline{\lambda}}. \tag{5.10}$$

Hence the column span of C_t is contained in U_t. Conversely, let v be a column of $E_{\underline{\kappa}}$ for some $\underline{\kappa} \in \Omega_n$ satisfying $\underline{\kappa}(1)_1 \geqslant n-t$. Since $E_{\underline{\lambda}}$ is idempotent, we have $E_{\underline{\lambda}} v = v$ for $\underline{\kappa} = \underline{\lambda}$ and Lemma 3.3 implies $E_{\underline{\lambda}} v = 0$ for $\underline{\kappa} \neq \underline{\lambda}$. Hence from (5.10) we find that

$$C_t v = |G_n| \frac{m_{\underline{\kappa}}}{\chi^{\underline{\kappa}}(1)} \, v,$$

and, since $m_{\underline{\kappa}} \neq 0$, we conclude that v is in the column span of C_t. This completes the proof. $\square$

6. Proof of Theorems 1.5 and 1.6

Our proofs of Theorems 1.5 and 1.6 follow along similar lines as those in the previous section and therefore our proofs will be less detailed.

Since the parabolic subgroup $P_{(t,n-t)}$ is the stabiliser of a t-space of $\mathbb{F}_q^n$, the character $\xi^{X-1\mapsto(n-t,t)}$ is the permutation character of the set of t-spaces of $\mathbb{F}_q^n$. From (2.7) we obtain its decomposition

$$\xi^{X-1\mapsto(n-t,t)} = \sum_{s=0}^{t} \chi^{X-1\mapsto(n-s,s)}. \tag{6.1}$$

Let $\left[{n \atop k}\right]_q$ denote the q-binomial coefficient, which counts the number of k-spaces of $\mathbb{F}_q^n$. Then we have

$$\xi^{X-1\mapsto(n-t,t)}(1) = \begin{bmatrix} n \\ t \end{bmatrix}_q, \tag{6.2}$$

and so (6.1) implies that

$$\chi^{X-1\mapsto(n-s,s)}(1) = \begin{bmatrix} n \\ s \end{bmatrix}_q - \begin{bmatrix} n \\ s-1 \end{bmatrix}_q. \tag{6.3}$$

Also note that $\psi^{X-1\mapsto\lambda} = \chi^{X-1\mapsto\lambda}$ for all partitions λ. Throughout this section, we define

$$\varepsilon = -\frac{1}{\left[{n \atop t}\right]_q - 1},$$

which will be our prescribed extremal eigenvalue.

We begin with the following counterpart of Proposition 6.1.

Proposition 6.1. *Let n and t be positive integers satisfying $n > 2t$. Then there exists $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leqslant t-1})$ such that*

$$\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma}) = \begin{cases} 1 & \text{for } \underline{\lambda}(1) = (n), \\ \varepsilon & \text{for } \underline{\lambda}(1) = (n-s,s) \text{ with } 1 \leqslant s \leqslant t, \\ 0 & \text{for } \underline{\lambda} \in \Omega_n \cap \Pi_{\leqslant t-1}, \text{ where} \\ & \qquad \underline{\lambda}(1) \neq (n-s,s) \text{ with } 0 \leqslant s \leqslant t-1 \end{cases} \tag{6.4}$$

and

$$|w(\underline{\sigma})| \leqslant \frac{\gamma_t}{|D_{\underline{\sigma}}|} \quad \text{for all } \underline{\sigma} \in \Omega_n \cap \Sigma_{\leqslant t-1}. \tag{6.5}$$

for some constant γ_t depending only on t.

Proof. From Lemma 4.2 we know that Q_{t-1} has full rank. In view of (3.5) there exists a unique $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leqslant t-1})$ satisfying (6.4) except for $\underline{\lambda}$ of the form $\underline{\lambda}(1) = (n-t,t)$.

Next we show that (6.4) also holds when $\underline{\lambda}(1) = (n-t,t)$. By Lemma 4.6 we have $\xi_{\underline{\sigma}}^{X-1\mapsto(n-t,t)} = 0$ for each $\underline{\sigma} \in \Sigma_{\leqslant t-1}$. Hence we have

$$\begin{aligned} 0 &= \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} w(\underline{\sigma})|D_{\underline{\sigma}}|\xi_{\underline{\sigma}}^{X-1\mapsto(n-t,t)} \\ &= \sum_{s=0}^{t}\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} w(\underline{\sigma})|D_{\underline{\sigma}}|\chi_{\underline{\sigma}}^{X-1\mapsto(n-s,s)}, \end{aligned} \tag{6.6}$$

using (6.1). Since (6.4) holds with the only exception $\underline{\lambda}(1) = (n-t,t)$, the inner sum equals 1 for $s=0$ and $\varepsilon\,\chi^{X-1\mapsto(n-s,s)}(1)$ for each s satisfying $1 \leqslant s \leqslant t-1$. Assuming that this is true also for $s=t$ and using (6.3), the right hand side of (6.6) is indeed

$$1 + \varepsilon\sum_{s=1}^{t}\left(\begin{bmatrix} n \\ s \end{bmatrix}_q - \begin{bmatrix} n \\ s-1 \end{bmatrix}_q\right) = 1 + \varepsilon\left(\begin{bmatrix} n \\ t \end{bmatrix}_q - 1\right) = 0.$$

Hence (6.4) also holds when $\underline{\lambda}(1) = (n-t,t)$.

It remains to prove (6.5). For each s satisfying $1 \leqslant s \leqslant t$, we find from (6.1) that

$$|\varepsilon|\,\chi^{X-1\mapsto(n-s,s)}(1) \leqslant |\varepsilon|\,(\xi^{X-1\mapsto(n-t,t)}(1) - 1) = 1,$$

using (6.2). Since $\chi^{X-1\mapsto(n)}(1) = 1$, we conclude from (6.4) that

$$\left|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} w(\underline{\sigma})|D_{\underline{\sigma}}|\,\psi_{\underline{\sigma}}^{\underline{\lambda}}\right| \leqslant 1 \quad \text{for each } \underline{\lambda}\in\Omega_n\cap\Pi_{\leqslant t-1}.$$

By Lemma 4.2 all entries of Q_{t-1} are independent of n and so are uniformly bounded by some value only depending on t. The same also holds for the inverse of Q_t, which establishes (6.5). □

The bound (6.5) and Lemma 5.2 ensure that the right hand side of (6.4) is small in modulus for each $\underline{\lambda} \in \Omega_n\backslash\Pi_t$. It therefore remains to deal with the case that $\underline{\lambda}\in\Omega_n\cap\Pi_t$ except for $\underline{\lambda}\in\Omega_n$ given by $\underline{\lambda}(1) = (n-t,t)$, which is the subject of the following lemma.

Lemma 6.2. *Let $w \in \mathbb{R}(\Omega_n\cap\Sigma_{\leqslant t-1})$ be given in Proposition 6.1 (so that $n > 2t$). Then, for all $\underline{\lambda}\in\Omega_n\cap\Pi_t$ with $\underline{\lambda}(1) \neq (n-t,t)$, we have*

$$\left|\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma})\right| < |\varepsilon|,$$

provided that n is sufficiently large compared to t.

Proof. By slight abuse of notation, we view w as an element of $\mathbb{R}(G_n)$ by setting $w(x) = 0$ if $x \notin \Omega_n\cap\Sigma_{\leqslant t-1}$ and $w(x) = w(\underline{\sigma})$ if $x\in\Omega_n\cap\Sigma_{\leqslant t-1}$ and $x \in D_{\underline{\sigma}}$. Recalling the scalar product on class functions of G_n from (5.7), the statement of the lemma is equivalent to

$$\frac{|G_n|}{\psi^{\underline{\lambda}}(1)}\,|\langle w,\psi^{\underline{\lambda}}\rangle| < |\varepsilon| \tag{6.7}$$

for all $\underline{\lambda} \in \Omega_n \cap \Pi_t$ with $\underline{\lambda}(1) \neq (n-t, t)$, provided that n is sufficiently large compared to t.

Pick $\underline{\lambda} \in \Omega_n \cap \Pi_t$ such that $\underline{\lambda}(1) \neq (n-t, t)$. Then $\underline{\lambda}(\alpha^i)_1 = n-t$ for some i. First assume that $|\underline{\lambda}(1)| \neq n$. Denoting by $\operatorname{Re} x$ the real part of a complex number x, we find from Lemma 3.2 and (2.13) that

$$\tfrac{1}{2}|\langle w, \psi^{\underline{\lambda}}\rangle| \leqslant |\operatorname{Re}\langle w, \chi^{\underline{\lambda}}\rangle| = \Big|\sum_{\underline{\mu}\sim\underline{\lambda}} H_{\underline{\mu}\underline{\lambda}} \operatorname{Re}\langle w, \xi^{\underline{\mu}}\rangle\Big|.$$

Lemma 4.6 implies that $\xi^{\underline{\mu}}_{\underline{\sigma}} = 0$ for each $\underline{\mu} \notin \Pi_{\leqslant t-1}$ and each $\underline{\sigma} \in \Sigma_{\leqslant t-1}$. For $\underline{\mu} \in \Lambda_n$, we have

$$\operatorname{Re}\langle w, \xi^{\underline{\mu}}\rangle = \sum_{\underline{\kappa}\sim\underline{\mu}} K_{\underline{\kappa}\underline{\mu}} \operatorname{Re}\langle w, \chi^{\underline{\kappa}}\rangle.$$

By (2.8), the summation can be taken over all $\underline{\kappa}$ such that $\underline{\kappa}(\alpha^i) \trianglerighteq \underline{\mu}(\alpha^i)$. Hence if $\mu \in \Pi_{\leqslant t-1}$, then $\underline{\kappa} \in \Pi_{\leqslant t-1}$. By the assumed properties of w given in Proposition 6.1, we have $\langle w, \psi^{\underline{\kappa}}\rangle = 0$ for each $\underline{\kappa} \in \Omega_n \cap \Pi_{\leqslant t-1}$ satisfying $|\underline{\kappa}(1)| \neq n$. Since $|\underline{\lambda}(1)| \neq n$ we conclude that $\langle w, \psi^{\underline{\lambda}}\rangle = 0$.

Now assume that $|\underline{\lambda}(1)| = n$ and write $\underline{\lambda}(1) = \lambda$. From (2.9) and (2.10) we have

$$\langle w, \psi^{X-1\mapsto\lambda}\rangle = \sum_{\substack{\mu\trianglerighteq\lambda\\ \mu_1>n-t}} H_{\mu\lambda}\langle w, \xi^{X-1\mapsto\mu}\rangle,$$

since by Lemma 4.6 in the case $\mu_1 = n-t$ we have $\xi^{X-1\mapsto\mu}_{\underline{\sigma}} = 0$ for each $\underline{\sigma} \in \Sigma_{\leqslant t-1}$. From (2.7) and (2.8) we then find that

$$\begin{aligned}\langle w, \psi^{X-1\mapsto\lambda}\rangle &= \sum_{\substack{\mu\trianglerighteq\lambda\\ \mu_1>n-t}} H_{\mu\lambda} \sum_{\kappa\trianglerighteq\mu} K_{\kappa\mu}\langle w, \psi^{X-1\mapsto\kappa}\rangle \\ &= \frac{1}{|G_n|}\sum_{\substack{\mu\trianglerighteq\lambda\\ \mu_1>n-t}} H_{\mu\lambda} + \sum_{\substack{\mu\trianglerighteq\lambda\\ \mu_1>n-t}} H_{\mu\lambda} \sum_{(n)\triangleright\kappa\trianglerighteq\mu} K_{\kappa\mu}\langle w, \psi^{X-1\mapsto\kappa}\rangle, \end{aligned} \tag{6.8}$$

using that $|G_n|\langle w, \psi^{X-1\mapsto(n)}\rangle = 1$ by the assumed properties of w given in Proposition 6.1 and $K_{(n)\mu} = 1$ for each partition μ of n. We first show that the first sum is zero. We have

$$\sum_{\substack{\mu\trianglerighteq\lambda\\ \mu_1>n-t}} H_{\mu\lambda} = \sum_{\mu\trianglerighteq\lambda} K_{(n)\mu}H_{\mu\lambda} - \sum_{\mu\trianglerighteq\lambda} K_{(n-t,t)\mu}H_{\mu\lambda}, \tag{6.9}$$

using that $\lambda_1 = n-t$ and that, for each partition μ of n, we have

$$K_{(n-t,t)\mu} = \begin{cases} 1 & \text{for } \mu_1 = n-t \\ 0 & \text{for } \mu_1 > n-t. \end{cases}$$

It is readily verified that

$$\sum_{\mu \trianglerighteq \lambda} K_{\kappa\mu} H_{\mu\lambda} = \delta_{\kappa\lambda}. \tag{6.10}$$

Since λ is neither (n) nor $(n-t,t)$, we conclude that (6.9) equals zero. Hence (6.8) becomes

$$\langle w, \psi^{X-1\mapsto\lambda}\rangle = \sum_{\substack{\mu \trianglerighteq \lambda \\ \mu_1 > n-t}} H_{\mu\lambda} \sum_{(n)\triangleright\kappa\trianglerighteq\mu} K_{\kappa\mu} \langle w, \psi^{X-1\mapsto\kappa}\rangle. \tag{6.11}$$

By the assumed properties of w given in Proposition 6.1, the inner summand is nonzero only when $\kappa = (n-s,s)$ for some s satisfying $1 \leqslant s \leqslant t-1$. In particular, for κ of this form, Proposition 6.1 and (6.3) give

$$|G_n|\,|\langle w, \psi^{X-1\mapsto\kappa}\rangle| = \frac{\begin{bmatrix} n \\ s \end{bmatrix}_q - \begin{bmatrix} n \\ s-1 \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q - 1} \leqslant \frac{\begin{bmatrix} n \\ t-1 \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q} = \frac{q^t-1}{q^{n-t+1}-1} \leqslant \frac{q^{2t-1}}{q^n}.$$

By Lemma 4.4 the Kostka numbers $K_{\kappa\mu}$ occuring in (6.11) are independent of n and it is readily verified from (6.10) that the numbers $H_{\mu\lambda}$ occuring in (6.11) are also independent of n. Moreover the number of summands in (6.11) is also independent of n. From Lemma 7.2, to be stated and proved in Section 7, we have $\psi^{X-1\mapsto\lambda}(1) \geqslant \delta_{t-1}\, q^{nt}$ for some constant δ_{t-1} only depending on t. Hence there is a constant c_t, depending only on t, such that

$$\frac{|G_n|}{\psi^{X-1\mapsto\lambda}(1)}\,|\langle w, \psi^{X-1\mapsto\lambda}\rangle| \leqslant \frac{c_t}{q^{n(t+1)}}.$$

Since $|\varepsilon| > 1/q^{nt}$, this shows that (6.7) holds provided that n is sufficiently large compared to t. □

Recall that $V_{\underline{\lambda}}$ is the column span of $E_{\underline{\lambda}}$. Define

$$W_t = \sum_{\substack{\underline{\lambda}\in\Omega_n \\ \underline{\lambda}(1)\trianglerighteq(n-t,t)}} V_{\underline{\lambda}}.$$

Now we obtain the following.

Theorem 6.3. *Let t be a positive integer. Then, for all sufficiently large n, the following holds.*

(i) Every t-space-intersecting set Y in G_n satisfies

$$|Y| \leqslant \left[\prod_{i=0}^{t-1}(q^t-q^i)\right]\left[\prod_{i=t}^{n-1}(q^n-q^i)\right]$$

and, in case of equality, we have $1_Y \in W_t$.

(ii) Every pair of t-space-cross-intersecting sets Y, Z in G_n satisfies

$$\sqrt{|Y|\cdot|Z|} \leqslant \left[\prod_{i=0}^{t-1}(q^t - q^i)\right]\left[\prod_{i=t}^{n-1}(q^n - q^i)\right]$$

and, in case of equality, we have $1_Y, 1_Z \in W_t$.

Proof. We apply Proposition 3.1 to the graph with adjacency matrix

$$\sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} A_{\underline{\sigma}}$$

and the $|D_{\underline{\sigma}}|$-regular spanning subgraphs with adjacency matrix $A_{\underline{\sigma}}$ for those $\underline{\sigma}$ occuring in the above set union. Every t-space-intersecting set in G_n is an independent set in this graph. Let $w \in \mathbb{R}(\Omega_n \cap \Sigma_{\leqslant t-1})$ be given by Proposition 6.1 and write

$$P(\underline{\lambda}) = \sum_{\underline{\sigma}\in\Omega_n\cap\Sigma_{\leqslant t-1}} w(\underline{\sigma})P(\underline{\lambda},\underline{\sigma}).$$

Proposition 6.1 and Lemmas 5.2 and 6.2 imply that, for all sufficiently large n, we have

$$P(\underline{\lambda}) = \begin{cases} 1 & \text{for } \underline{\lambda}(1) = (n) \\ \varepsilon & \text{for } \underline{\lambda}(1) = (n-s, s) \text{ with } 1 \leqslant s \leqslant t \end{cases}$$

and $|P(\underline{\lambda})| < |\varepsilon|$ for $\underline{\lambda}(1) \neq (n-s, s)$ with some s satisfying $0 \leqslant s \leqslant t$. Hence, writing $\underline{\lambda}_0$ for $X - 1 \mapsto (n)$, we have $P(\underline{\lambda}_0) = 1$ and

$$\varepsilon = \min_{\underline{\lambda}\neq\underline{\lambda}_0} P(\underline{\lambda}) \quad \text{and} \quad |\varepsilon| = \max_{\underline{\lambda}\neq\underline{\lambda}_0} |P(\underline{\lambda})|.$$

Then the required result follows from Proposition 3.1 and the decomposition of $\mathbb{R}(G_n)$ given in Lemma 3.3. □

Our proof of Theorems 1.5 and 1.6 is completed by the following result.

Theorem 6.4. *W_t is spanned by the characteristic vectors of cosets of stabilisers of t-spaces.*

Proof. The proof is almost identical to that of Theorem 5.4 with $\mathcal{A}_t$ replaced by the set of t-spaces and ζ^t replaced by the permutation character $\xi^{X-1\mapsto(n-t,t)}$ of t-spaces and the decomposition of ζ^t replaced by the decomposition given in (6.1). □

7. Estimates on conjugacy class sizes and character degrees

In this section we provide bounds on the size of certain conjugacy classes and degrees of certain irreducible characters of G_n, which are used in the proof of Lemma 5.2.

Lemma 7.1. *Let n and t be positive integers satisfying $n > 2t$ and let $\underline{\sigma} \in \Sigma_{\leqslant t}$. Then we have*

$$\frac{|G_n|}{|C_{\underline{\sigma}}|} \leqslant q^{t^5} q^n.$$

Proof. From Lemma 2.1 we find that (with the same notation as in Lemma 2.1)

$$\frac{|G_n|}{|C_{\underline{\sigma}}|} \leqslant \prod_{f\in\Phi} \prod_{i=1}^{|\underline{\sigma}(f)|} q^{|f|\, s_i(\underline{\sigma}(f)')\, m_i(\underline{\sigma}(f))}. \tag{7.1}$$

Since $\underline{\sigma} \in \Sigma_{\leqslant t}$ and $t < n/2$, there is exactly one polynomial $h \in \Phi$ of degree at least $n-t$ in the support of $\underline{\sigma}$. This polynomial must satisfy $\underline{\sigma}(h) = (1)$ and the corresponding factor in (7.1) is at most q^n. There are at most t other polynomials in the support of $\underline{\sigma}$. Each such polynomial f has degree at most t and satisfies $|\underline{\sigma}(f)| \leqslant t$ and hence the corresponding factor in (7.1) has a crude upper bound of q^{t^4}. As there are at most t such factors, the proof is completed. □

Lemma 7.2. *Let t be a positive integer. Then there is a constant δ_t such that, for all sufficiently large n and for all $\underline{\lambda} \in \Lambda_n \backslash \Pi_{\leqslant t}$, we have*

$$\chi^{\underline{\lambda}}(1) \geqslant \delta_t\, q^{n(t+1)}.$$

Proof. Let $\underline{\lambda} \in \Lambda_n \backslash \Pi_{\leqslant t}$. Using elementary calculus we find that

$$1 - x \geqslant 4^{-x} \quad \text{for } 0 \leqslant x \leqslant 1/2$$

and therefore

$$\frac{\prod_{i=1}^n (q^i - 1)}{q^{\frac{1}{2}n(n+1)}} = \prod_{i=1}^n \left(1 - \frac{1}{q^i}\right) \geqslant \prod_{i=1}^n \left(1 - \frac{1}{2^i}\right) \geqslant \prod_{i=1}^n 4^{-1/2^i} \geqslant \prod_{i=1}^\infty 4^{-1/2^i} = \frac{1}{4}.$$

Substitute into (2.14) of Lemma 2.2 to give

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \leqslant 4 q^{N(\underline{\lambda}) - M(\underline{\lambda}) - \frac{1}{2}n(n+1)}, \tag{7.2}$$

where

$$N(\underline{\lambda}) = \sum_{f\in\Phi} |f| \sum_{(i,j)\in\underline{\lambda}(f)} h_{i,j}(\underline{\lambda}(f)),$$
$$M(\underline{\lambda}) = \sum_{f\in\Phi} |f|\, b(\underline{\lambda}(f))$$

and b and $h_{i,j}$ are as in Lemma 2.2. Note that for each partition λ, we have

$$\sum_{(i,j)\in\lambda} h_{i,j}(\lambda) \leqslant \sum_{k=1}^{|\lambda|} k = \frac{1}{2}|\lambda|(|\lambda|+1). \tag{7.3}$$

First assume that there exists a polynomial $h \in \Phi$ such that $|h| = 1$ and $\underline{\lambda}(h)'_1 \geqslant n-t$. In this case we have

$$M(\underline{\lambda}) \geqslant b(\underline{\lambda}(h)) \geqslant \sum_{k=1}^{n-t-1} k = \frac{1}{2}(n-t)(n-t-1)$$

and by (7.3)

$$\begin{aligned} N(\underline{\lambda}) &\leqslant \frac{1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|(|\underline{\lambda}(f)|+1) \\ &\leqslant \frac{n+1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)| \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

Therefore (7.2) implies that

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \leqslant 4q^{-\frac{1}{2}(n-t)(n-t-1)},$$

so that we have $\chi^{\underline{\lambda}}(1) \geqslant q^{n(t+1)}$ for all sufficiently large n by very crude estimates.

Hence we can assume that $\underline{\lambda}(f)'_1 \leqslant n-t-1$ and $\underline{\lambda}(f)_1 \leqslant n-t-1$ for all $f \in \Phi$ satisfying $|f| = 1$. Note that the second assumption is implied by the hypothesis $\underline{\lambda} \notin \Pi_{\leqslant t}$. In what follows we use the trivial bound $M(\underline{\lambda}) \geqslant 0$. We distinguish two cases.

In the first case we assume that $|\underline{\lambda}(f)| \leqslant n-t-1$ for all $f \in \Phi$ satisfying $|f| = 1$. Let ℓ be the maximum of $|\underline{\lambda}(f)|$ over all $f \in \Phi$ satisfying $|f| = 1$, hence $\ell \leqslant n-t-1$. By (7.3) we have

$$\begin{aligned} N(\underline{\lambda}) &\leqslant \frac{1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|(|\underline{\lambda}(f)|+1) \\ &= \frac{n}{2} + \frac{1}{2}\sum_{f\in\Phi}|f||\underline{\lambda}(f)|^2. \end{aligned}$$

If $\ell \leqslant n/2$, then we have $|\underline{\lambda}(f)| \leqslant n/2$ for all $f \in \Phi$ and so $N(\underline{\lambda}) \leqslant n^2/4 + n/2$. From (7.2) we then find that $\chi^{\underline{\lambda}}(1) \geqslant q^{n(t+1)}$ for all sufficiently large n, again by very crude estimates. If $\ell > n/2$, then

$$\begin{aligned} N(\underline{\lambda}) &\leqslant \frac{1}{2}(n+\ell^2+(n-\ell)^2) \\ &\leqslant \frac{1}{2}(n+(n-t-1)^2+(t+1)^2) \\ &= \frac{n^2+n}{2} - n(t+1) + (t+1)^2, \end{aligned}$$

where we have used that $x^2 + (n-x)^2$ is increasing for $x \geqslant n/2$. Hence in this case we obtain $1/\chi^{\underline{\lambda}}(1) \leqslant 4q^{-n(t+1)+(t+1)^2}$ by (7.2).

In the remaining case we assume that there exists $h \in \Phi$ such that $|h| = 1$ and $|\underline{\lambda}(h)| \geqslant n-t$. Recall that we also assume that $\underline{\lambda}(h)_1 \leqslant n-t-1$ and $\underline{\lambda}(h)'_1 \leqslant n-t-1$. Since $N(\underline{\lambda})$ depends only on the hook lengths of $\underline{\lambda}(f)$ for $f \in \Phi$, we may replace $\underline{\lambda}(h)$ by its conjugate $\underline{\lambda}(h)'$. Assuming that n is sufficiently large, namely $n \geqslant (t+2)^2$, we have $\underline{\lambda}(h)_1 \geqslant t+2$ or $\underline{\lambda}(h)'_1 \geqslant t+2$ and we assume without loss of generality that

$\underline{\lambda}(h)_1 \geqslant t+2$. Write $\underline{\lambda}(h)_1 = n-r$, so that our assumptions imply $t+1 \leqslant r \leqslant n-t-2$. Then, writing $s = |\underline{\lambda}(h)|$, there exist nonnegative integers c_j satisfying

$$\sum_{j=1}^{n-r} h_{1j}(\underline{\lambda}(h)) = \sum_{j=1}^{n-r}(j+c_j), \quad \text{where} \quad \sum_{j=1}^{n-r} c_j = s-(n-r).$$

Hence

$$\sum_{j=1}^{n-r} h_{1j}(\underline{\lambda}(h)) = \binom{n-r+1}{2} + (s-n+r).$$

Application of (7.3) with $\lambda = (\underline{\lambda}(h)_2, \underline{\lambda}(h)_3, \dots)$ gives

$$\begin{aligned}\sum_{(i,j)\in\underline{\lambda}(h)} h_{i,j}(\underline{\lambda}(h)) &\leqslant \binom{s-n+r+1}{2} + \binom{n-r+1}{2} + (s-n+r)\\ &= \frac{s^2}{2} + \frac{3s}{2} + n^2 - sn - n + r(r-(2n-s-1))\\ &\leqslant \frac{s^2}{2} + \frac{3s}{2} + n^2 - sn - n + (t+1)((t+1)-(2n-s-1)),\end{aligned}$$

since the term depending on r is maximised for $r = t+1$ over the interval $[t+1, n-t-2]$. This last expression equals

$$\frac{s}{2} + \frac{1}{2}s(s-2(n-t-2)) + n^2 - n + (t+1)((t+1)-(2n-1)).$$

The second summand is increasing for $s \geqslant n-t$ and so is at most $\frac{1}{2}n(n-2(n-t-2))$. Hence we obtain

$$\sum_{(i,j)\in\underline{\lambda}(h)} h_{i,j}(\underline{\lambda}(h)) \leqslant \frac{s}{2} + \frac{n^2}{2} - n(t+1) + (t+1)(t+2).$$

Invoking (7.3) once more, we obtain

$$N(\underline{\lambda}) \leqslant \sum_{(i,j)\in\underline{\lambda}(h)} h_{ij} + \frac{1}{2}\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)|(|\underline{\lambda}(f)|+1).$$

We have

$$\frac{s}{2} + \frac{1}{2}\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)| = \frac{1}{2}\sum_{f\in\Phi} |f||\underline{\lambda}(f)| = \frac{n}{2}$$

and

$$\frac{1}{2}\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)|^2 \leqslant \frac{1}{2}\Bigg(\sum_{\substack{f\in\Phi\\ f\neq h}} |f||\underline{\lambda}(f)|\Bigg)^2 \leqslant \frac{t^2}{2}.$$

Collecting all terms, we find that

$$N(\underline{\lambda}) \leqslant \frac{n(n+1)}{2} - n(t+1) + (t+1)(t+2) + \frac{t^2}{2}.$$

From (7.2) we then obtain

$$\frac{1}{\chi^{\underline{\lambda}}(1)} \leqslant 4q^{-n(t+1)+(t+1)(t+2)+\frac{1}{2}t^2},$$

which completes the proof. □

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