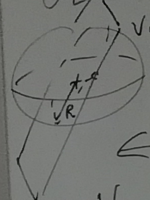
GEOMETRY

ALGEBRAIC VARIETY



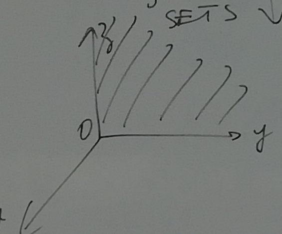
$V_1 = S(C,R) = \{M(x,y,z) \in \mathbb{R}^3 / d^2(C,M) = R^2\}$

$V_2 = P = \{M(x,y,z) \in \mathbb{R}^3 / ax+by+cz+d=0\}$

$V_1 \cap V_2$

$V_1 \subset V_1 \cap V_2$

SMALLER SETS



$\left\{M\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x^2=0\right\} = yOz \quad (x=0)$

$V(2x) = V(x)$

GRAPH OF PL function:

degree 1 semi-algebraic set of codimension 1

BIGGER IDEALS

HILBERT NULLSTELLENSATZ

WEAK: $\mathfrak{m}$ maximal ideal $\mathfrak{m} = \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle$

If $p_1, \dots, p_m \in k[x]$ have no common root

then $I(\{p_1, \dots, p_m\}) = \langle 1 \rangle = k[x]$

$I(V(\langle p_1, \dots, p_m \rangle)) = \sqrt{\langle p_1, \dots, p_m \rangle}$

ALGEBRA

IDEAL

$I_1 \langle (x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 - R^2 \rangle$

$I_2 \langle ax+by+cz+d \rangle$

$\langle (x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 - R^2, ax+by+cz+d \rangle$

$I(V_1 \cap V_2)$

$\langle ax+by+cz+d \rangle = \{p(ax+by+cz+d),\ p \in k[x,y,z]\}$

$I(V_1 \cup V_2) = I(I_1 \cdot I_2)$

$\langle (x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 - R^2 \rangle$

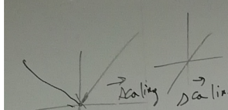
$= \{p \cdot ((x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 - R^2),\ p \in k[x,y,z]\}$

$I(V_1 \cap V_2) =$

$\{p_1 \cdot (ax+by+cz+d) + p_2 \cdot ((x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2),\ (p_1, p_2) \in (k[x,y,z])^2$

$I(V(\langle x^2 \rangle)) = \langle x \rangle$

$\langle x^2 \rangle$



ALGEBRAIC STRUCTURES.

① MAGMA $(M,+)$: SET M, $+ : M \times M \to M$.
$(m_1, m_2) \mapsto m_1 + m_2 \in M$.

② SEMI-GROUP : MAGMA $(M,+)$
ASSOCIATIVITY OF $+$ IN M.
$\forall (m_1, m_2, m_3) \in M^3$, $m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$

③ MONOID: SEMI-GROUP $(M,+)$
$\exists e \in M$, $\forall m \in M$, $m + e = e + m = m$.

④ GROUP: MONOID $(M,+)$
$\forall m \in M$, $\exists m' \in M$, $m + m' = m' + m = e$.

⑥ UNITARY RING: SET R, $+ : R \times R \to R$
$\times : R \times R \to R$
$(R,+)$: ABELIAN GROUP.
$(R,\times)$: MONOID
$\exists 1 \in R$, $\forall r \in R$, $1.r = r.1 = r$.
$\times$ DISTRIBUTIVE W.R.T. $+$:
$\forall (a,b,c) \in R^3$, $a \times (b+c) = (a \times b) + (a \times c)$.

RNG
PSEUDO-RING.
NO 1 IN R

⑦ FIELD $(k$
$(k,+)$: ABEL
$(k^*, \times)$: ABEL
$k^* = k \setminus e_k$
$\times$ DISTRIBU

⑧ IDEAL:
$(I,+)$: ABE
e_R
$(I,\times)$ MONOID
$\times$ DISTRIBUTIVE
$\forall i \in I$, $\forall r \in$

ING
x^n
$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots x_n^{\alpha_n}$
WITHING
$g_1 + g_2 = g_2 + g_1$

...RUCTURES.

$+ : M \times M \to M.$

$(m_1, m_2) \mapsto m_1 + m_2 \in M.$

...+)

... OF + IN M.

$\dots m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$

... $(M, +)$

... $m + e = e + m = m.$

... $,+)$

...M, $m + m' = m' + m = e.$

...T R, $+ : R \times R \to R$

...GROUP.

RNG
PSEUDO-RING.
NO $1 \in R$

$\dots 1.r = r.1 = r.$

... +:

$\dots + c) = (a \times b) + (a \times c).$

(7) FIELD $(k, +, \times)$ $\to$ k set.

$(k, +)$: ABELIAN GROUP.

$(k^*, \times)$: ABELIAN GROUP

$k^* = k \setminus e_k$

$\times$ DISTRIBUTIVE W.R.T. +.

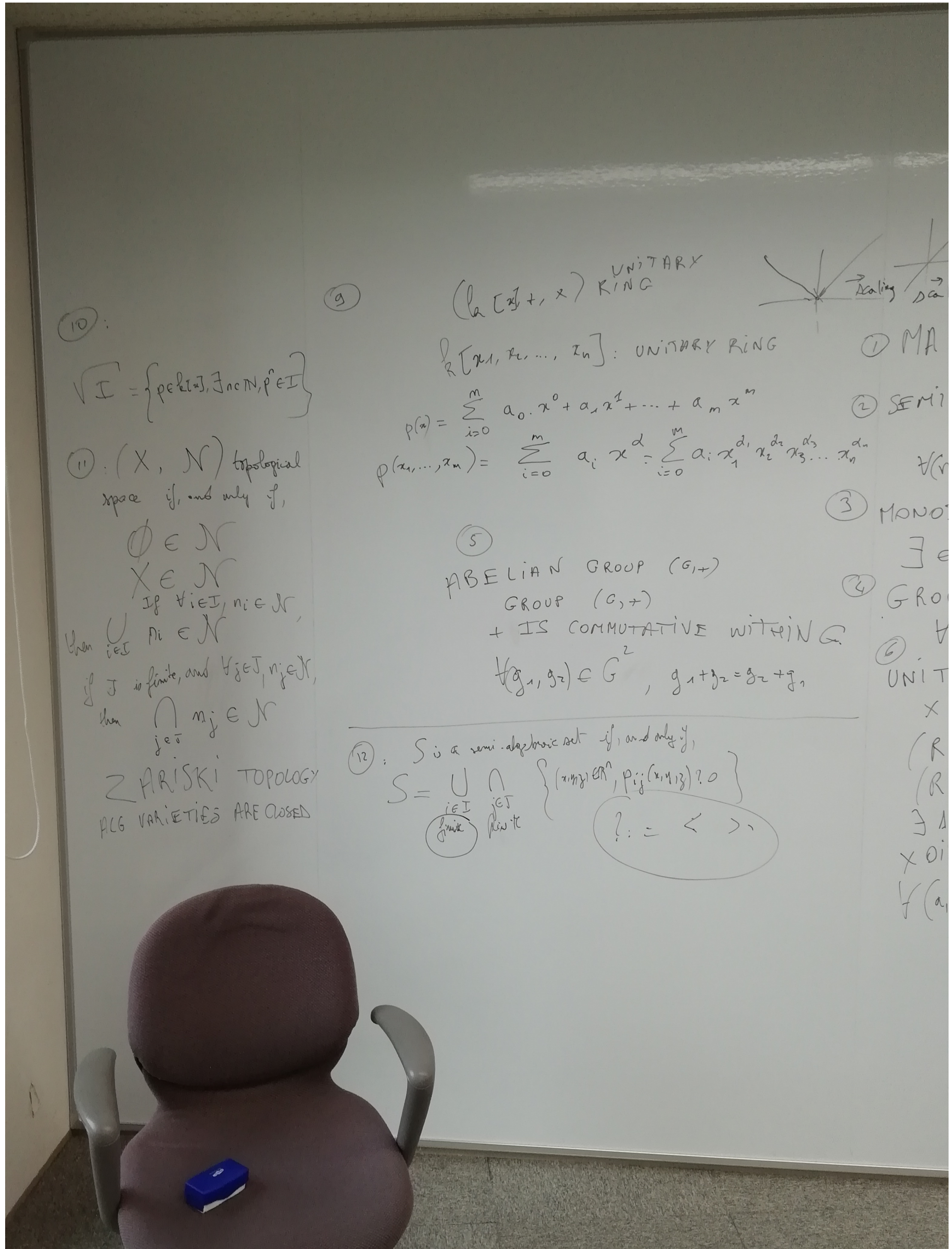
(8) IDEAL: $(I, +, \times)$. $I \subset R$ ring

$(I, +)$: ABELIAN GROUP THAT CONTAINS e_R

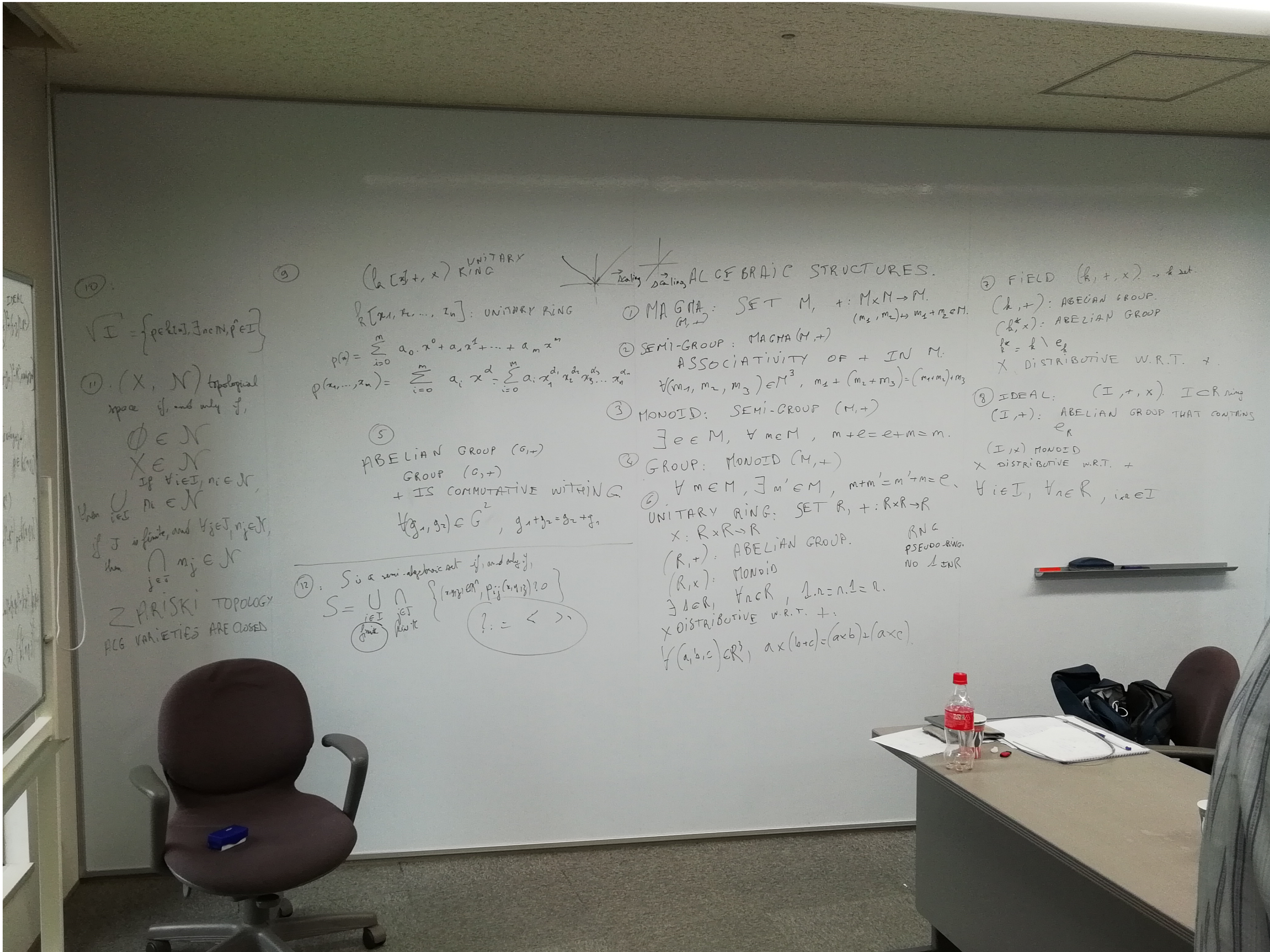
$(I, \times)$ MONOID

$\times$ DISTRIBUTIVE W.R.T. +

$\forall i \in I, \forall r \in R, i \times r \in I$



10 :
$\sqrt{I} = \{ p \in k[x], \exists n \in \mathbb{N}, p^n \in I \}$
11 : $(X, \mathcal{N})$ topological space if, and only if,
$\emptyset \in \mathcal{N}$
$X \in \mathcal{N}$
If $\forall i \in I, n_i \in \mathcal{N}$,
then $\bigcup_{i \in I} n_i \in \mathcal{N}$
if J is finite, and $\forall j \in J, n_j \in \mathcal{N}$,
then $\bigcap_{j \in J} n_j \in \mathcal{N}$
ZARISKI TOPOLOGY
ALG VARIETIES ARE CLOSED
9
$(k[x], +, \times)$ UNITARY RING
$k[x_1, x_2, \dots, x_n]$: UNITARY RING
$p(x) = \sum_{i=0}^{m} a_0 \cdot x^0 + a_1 x^1 + \dots + a_m x^m$
$p(x_1, \dots, x_m) = \sum_{i=0}^{m} a_i x^{\alpha} = \sum_{i=0}^{m} a_i x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots x_n^{\alpha_n}$
Scaling
5
ABELIAN GROUP $(G, +)$
GROUP $(G, +)$
+ IS COMMUTATIVE WITHIN G.
$\forall (g_1, g_2) \in G^2, g_1 + g_2 = g_2 + g_1$
12 : S is a semi-algebraic set if, and only if,
$S = \bigcup_{i \in I} \bigcap_{j \in J} \{ (x,y,z) \in \mathbb{R}^n, p_{ij}(x,y,z) \, ? \, 0 \}$
finite
finite
? : = < > ,



ALGEBRAIC STRUCTURES.
scaling
① MAGMA (M,+): SET M, +: M×M → M.
(m₁, m₂) ↦ m₁ + m₂ ∈ M.
② SEMI-GROUP: MAGMA (M,+)
ASSOCIATIVITY OF + IN M.
∀(m₁, m₂, m₃) ∈ M³, m₁ + (m₂ + m₃) = (m₁ + m₂) + m₃
③ MONOID: SEMI-GROUP (M,+)
∃ e ∈ M, ∀ m ∈ M, m + e = e + m = m.
④ GROUP: MONOID (M,+)
∀ m ∈ M, ∃ m' ∈ M, m + m' = m' + m = e.
⑤ ABELIAN GROUP (G,+)
GROUP (G,+)
+ IS COMMUTATIVE WITHIN G.
∀(g₁, g₂) ∈ G², g₁ + g₂ = g₂ + g₁
⑥ UNITARY RING: SET R, +: R×R → R
×: R×R → R
(R,+): ABELIAN GROUP.
(R,×): MONOID
∃ 1 ∈ R, ∀ r ∈ R, 1.r = r.1 = r.
× DISTRIBUTIVE W.R.T. +:
∀(a,b,c) ∈ R³, a×(b+c) = (a×b) + (a×c).
RNG
PSEUDO-RING.
NO 1 IN R
⑦ FIELD (k, +, ×) → k set.
(k,+): ABELIAN GROUP.
(k*,×): ABELIAN GROUP
k* = k \ e_k
× DISTRIBUTIVE W.R.T. +.
⑧ IDEAL: (I, +, ×). I ⊂ R ring
(I,+): ABELIAN GROUP THAT CONTAINS e_R
(I,×) MONOID
× DISTRIBUTIVE W.R.T. +
∀ i ∈ I, ∀ r ∈ R, i×r ∈ I
⑨ (k[x], +, ×) UNITARY RING
k[x₁, x₂, ..., xₙ]: UNITARY RING
p(x) = Σ_{i=0}^{m} a₀.x⁰ + a₁x¹ + ... + aₘxᵐ
p(x₁, ..., xₙ) = Σ_{i=0}^{m} aᵢ x^α = Σ_{i=0}^{m} aᵢ x₁^α₁ x₂^α₂ x₃^α₃ ... xₙ^αₙ
⑩: √I = {p ∈ k[x], ∃ n ∈ ℕ, pⁿ ∈ I}
⑪. (X, 𝒩) topological space if, and only if,
∅ ∈ 𝒩
X ∈ 𝒩
If ∀ i ∈ I, nᵢ ∈ 𝒩,
then ⋃_{i∈I} nᵢ ∈ 𝒩
if J is finite, and ∀ j ∈ J, nⱼ ∈ 𝒩,
then ⋂_{j∈J} nⱼ ∈ 𝒩
ZARISKI TOPOLOGY
ALG VARIETIES ARE CLOSED
⑫: S is a semi-algebraic set if, and only if,
S = ⋃_{i∈I} ⋂_{j∈J} {(x,y,z) ∈ ℝⁿ, pᵢⱼ(x,y,z) ? 0}
finite finite
? := < >,