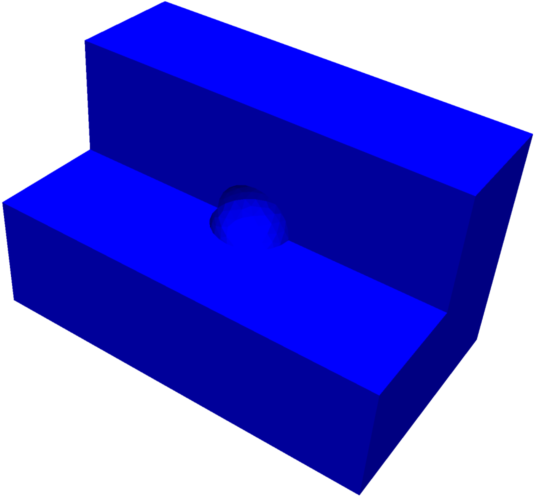
Non-Smooth Geometric Inverse Problems

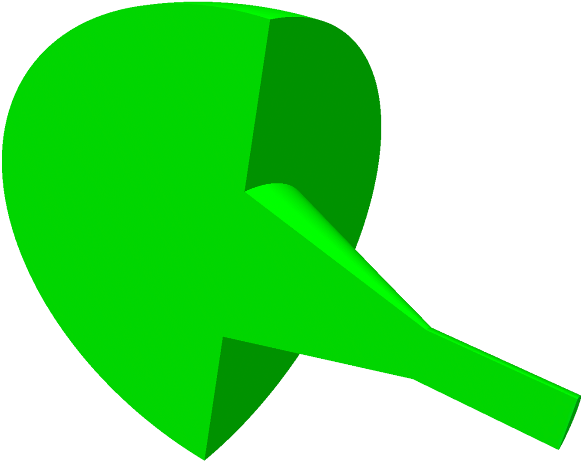
Stephan Schmidt

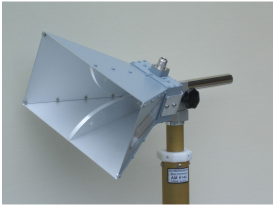
25 June 2018

Maxwell Scattering Problem (joint with M. Schütte, O. Ebel, A. Walther)



Acoustic Horn Design (joint with M. Berggren, E. Wadbro)





- General problem formulation allows treatment of general problems
- Design of acoustic (linear wave) horn antenna, $3.5 \cdot 10^9$ unknowns!

$$\min_{(\varphi,\Gamma_{\text{inc}})} J(\varphi,\Omega) := \frac{1}{2}\int_0^T \int_{\Gamma_{i/o}} \|B(n)(\varphi - \varphi_{\text{meas}})\|_2^2 \, \mathrm{d}\, t \, \mathrm{d}\, s + \delta \int_{\Gamma_{\text{inc}}} 1 \, \mathrm{d}\, s$$

subject to

$$\begin{aligned} \dot{\varphi} + \operatorname{div} F(\varphi) &= 0 \quad \text{in} \quad \Omega \\ \text{BCs} &= g \quad \text{on} \quad \Gamma \end{aligned}$$

Acoustics:

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla p &= 0 \text{ in } \Omega, \\ \frac{\partial p}{\partial t} + c^2 \operatorname{div} u &= 0 \text{ in } \Omega, \\ \frac{1}{2}(p - c\langle u, n\rangle) &= g \text{ on } \Gamma_{\text{i/o}} \end{aligned}$$

Electromagnetism:

$$\begin{aligned} \mu\frac{\partial H}{\partial t} &= -\nabla \times E \text{ in } \Omega, \\ \varepsilon\frac{\partial E}{\partial t} &= \nabla \times H - \sigma E \text{ in } \Omega, \\ \text{BCs} &= g \text{ on } \Gamma_{\text{i/o}} \end{aligned}$$

Aerodynamic Drag Reduction

Minimize Aerodynamic Drag

$$\min_{(U,\Omega)} J(U,\Omega) := \frac{1}{C_\infty} \int_{\Gamma_W} (pn - \tau n) \cdot \psi \,\mathrm{d}\, s$$

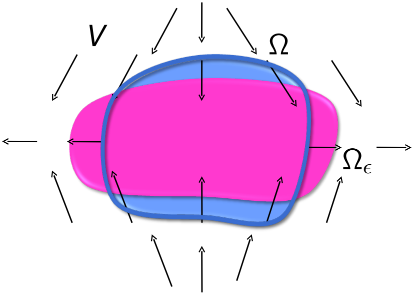
subject to

$$\begin{aligned} 0 = & -(\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}), \nabla \mathbf{v})_\Omega \\ & + (n \cdot (\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u})), \mathbf{v})_\Gamma \quad \forall \mathbf{v} \in \mathcal{H} \end{aligned}$$

Additional essential boundary conditions (Sonntag, S., Gauger, 2015)

- Conserved variables: $\mathbf{u} = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho E)^T$
- Primitive variables: $U = (\rho, u_1, u_2, u_3, p)^T$
- Perfect gas: $p = (\gamma - 1)\rho(E - \frac{1}{2}(u_1^2 + u_2^2 + u_3^2))$

Introduction to Shape Optimization



- Shape is modeled by set Ω
- $\Omega_\epsilon := \{x + \epsilon V(x) : x \in \Omega\}$
- $J : \mathcal{P}(\Omega) \supseteq \mathcal{D} \to \mathbb{R}$: target function
- (Directional) derivative of J with respect to Ω?

- Directional Derivative

$$dJ(\Omega)[V] := \lim_{\epsilon \to 0^+} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon}$$

The Shape Derivative

- Objective function:

$$J_1(\epsilon, \Omega) := \int_{\Omega(\epsilon)} f(\epsilon, x_\epsilon) \, \mathrm{d}\, x_\epsilon \quad \text{or} \quad J_2(\epsilon, \Omega) := \int_{\Gamma(\epsilon)} g(\epsilon, s_\epsilon) \, \mathrm{d}\, s_\epsilon$$

- Take Limit:

$$dJ_1(\Omega)[V] = \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\bigg|_{\epsilon=0} \int_{\Omega(\epsilon)} f(\epsilon, x_\epsilon) \, \mathrm{d}\, x_\epsilon \quad \text{or} \quad dJ_2(\Omega)[V] = \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\bigg|_{\epsilon=0} \int_{\Gamma(\epsilon)} g(\epsilon, s_\epsilon) \, \mathrm{d}\, s_\epsilon$$

- Change of Variables = Change in Domain

$$dJ_1(\Omega)[V] = \int_{\Omega} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Big[f(T_\epsilon(x)) \cdot |\det DT_\epsilon(x)| \Big] + f'(x)[V] \, \mathrm{d}\, x$$

$$dJ_2(\Omega)[V] = \int_{\Gamma} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Big[g(T_\epsilon(s)) \cdot |\det DT_\epsilon(s)| \|(DT_\epsilon(s))^{-T} n(s)\|_2 \Big] + g'(s)[V] \, \mathrm{d}\, s$$

- Local / Shape Derivative: $f'(x)[V] := \frac{\partial}{\partial \epsilon} f(0, x)$

The Shape Derivative (Weak vs Strong)

- Differentiate

$$dJ_1(\Omega)[V] = \int_\Omega \operatorname{div}(fV) + f'[V] \,\mathrm{d}x$$

$$= \int_\Gamma \langle V, n\rangle f \,\mathrm{d}s + \int_\Omega f'[V] \;\mathrm{d}x$$

$$= \int_\Omega f\operatorname{div} V + df[V] \;\mathrm{d}x \quad \text{(Berggren, 2010)}$$

$$dJ_2(\Omega)[V] = \int_\Gamma (\nabla g, V) + g\operatorname{div}_\Gamma V(s) + g'[V] \,\mathrm{d}s = \int_\Gamma \operatorname{div}_\Gamma (gV) + g'[V] \,\mathrm{d}s$$

$$= \int_\Gamma \langle V, n\rangle \left[\frac{\partial g}{\partial n} + \kappa g\right] + g'[V] \,\mathrm{d}s$$

$$= \int_\Gamma g\operatorname{div}_\Gamma V + dg[V] \;\mathrm{d}s$$

- Material Derivative: $df = \langle \nabla f, V\rangle + f'[V]$

Dido's Problem

Find shape of maximum volume for given surface:

$$\max_{\Omega} J(\Omega) := \int_{\Omega} 1 \,\mathrm{d}\, x$$

s.t.

$$\int_{\Gamma} 1 \,\mathrm{d}\, s = A_0$$

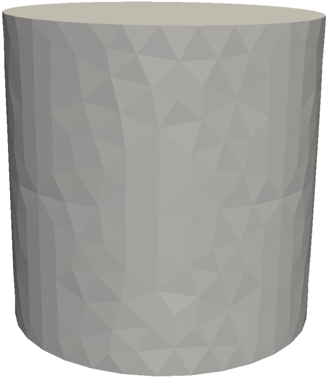
Lagrangian:

$$F(\Omega, \lambda) = \int_{\Omega} -1 \,\mathrm{d}\, x + \lambda \left(\int_{\Gamma} 1 \,\mathrm{d}\, s - A_0 \right)$$

$$dF(\Omega, \lambda)[V] = \int_{\Gamma} \langle V, n \rangle \left[-1 + \lambda\kappa \right] \mathrm{d}\, s \overset{!}{=} 0 \quad \forall V$$

Because $\lambda \in \mathbb{R}$: $\kappa = \frac{1}{\lambda} \in \mathbb{R}$. Thus, curvature constant!
Optimality fulfilled by sphere!!

Dido's Problem (Gradient Descent + Newton)

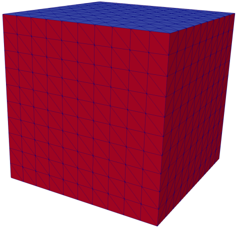


Non-Smoothness and Discretizations

Elongate Unit Cube by Stretching: $V = (0, 0, x_3)^T$

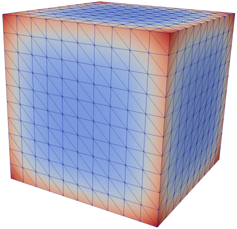
- The Naive Way:

$$A(V) = 4(1(1 + v_3)) + 2 = 6 + 4v_3$$
$$\Rightarrow \frac{\partial A(V)}{\partial v_3} = 4$$



- The Weak Way:

$$dA[V] = \int_\Gamma \operatorname{div}_\Gamma V \, dS = \int_{\bigcup_{i=1}^{4} S_i} 1 \, dS = 4$$



- The Strong Way:

$$dA[V] = \int_\Gamma \langle V, n \rangle \kappa \, dS = 4.133872$$

Shape Linearization of General Conservation Law

Find $\varphi'[V]$ such that

$$
\begin{aligned}
0 = & \int_0^T \int_\Gamma \langle V, n\rangle \left[\langle \lambda, \dot{\varphi}\rangle - \langle F(\varphi), \nabla\lambda\rangle\right] \,\mathrm{d}\,s\,\,\mathrm{d}\,t \\
& + \int_0^T \int_\Gamma \langle V, n\rangle [\langle \nabla(\lambda \cdot F_{\mathrm{b}}^*(\varphi, n)), n\rangle \\
& \quad + \kappa \left(\lambda \cdot F_{\mathrm{b}}^*(\varphi, n) - D_n(\lambda \cdot F_{\mathrm{b}}^*(\varphi, n)) \cdot n\right) + \mathrm{div}_\Gamma \left(D_n^T(\lambda \cdot F_{\mathrm{b}}^*(\varphi, n))\right)] \,\mathrm{d}\,s\,\mathrm{d}\,t \\
& + \int_0^T \int_\Omega \langle \lambda, \dot{\varphi}'[V]\rangle - \langle DF(\varphi)\varphi'[V], \nabla\lambda\rangle \,\mathrm{d}\,x\,\mathrm{d}\,t \\
& + \int_0^T \int_\Gamma \langle \lambda, D_\varphi F_{\mathrm{b}}^*(\varphi, n)\varphi'[V]\rangle \,\mathrm{d}\,s\,\,\mathrm{d}\,t
\end{aligned}
$$

Adjoint Equation

Adjoint equation can be read from the shape-linearized equation: Find λ such that

$$
\begin{aligned}
0 = &\int_0^T \int_\Omega \langle -\dot{\lambda}, \varphi'[V] \rangle - \langle \varphi'[V], D^T F(\varphi) \nabla \lambda \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^T \int_\Gamma \langle \varphi'[V], D_\varphi^T F_\mathrm{b}^*(\varphi, n) \cdot \lambda \rangle \, \mathrm{d}s \, \mathrm{d}t \\
&+ \int_0^T \int_{\Gamma_{\mathrm{i/o}}} \langle B^T(n) B(n) \cdot (\varphi - \varphi_\mathrm{meas}), \varphi'[V] \rangle \, \mathrm{d}s \, \mathrm{d}t
\end{aligned}
$$

$\Rightarrow$ Flux for adjoint can be read: $D_\varphi^T F^*(\varphi, n) \cdot \lambda$

Shape Derivative for Tomography Problems

Maxwell (Existence Results: Cagnol/Eller/Marmorat/Zolésio):

$$dJ(H,E,\Omega)[V]$$

$$= \int_0^T \int_{\Gamma_{\text{inc}}} \langle V, n \rangle \left[\langle \lambda_H, \dot{H} \rangle + \frac{1}{\mu} \langle E, \operatorname{curl} \lambda_H \rangle + \langle \lambda_E, \dot{E} \rangle - \frac{1}{\epsilon} \langle H, \operatorname{curl} \lambda_E \rangle + \frac{\sigma}{\epsilon} \langle \lambda_E, E \rangle \right] \, \mathrm{d}\, s \ \mathrm{d}\, t$$

$$+ \int_0^T \int_{\Gamma_{\text{inc}}} \langle V, n \rangle \operatorname{div} \left(Zc(H \times \lambda_E) \right) \, \mathrm{d}\, s \ \mathrm{d}\, t$$

Horn/Linear Wave:

$$dJ(u,p,\Omega)[V] = \int_0^T \int_{\Gamma_{\text{horn}}} \langle V, n \rangle \left[\langle \lambda_u, \dot{u} \rangle - p \operatorname{div} \lambda_u + \lambda_p \, \dot{p} - c^2 \langle u, \nabla \lambda_p \rangle \right] \, \mathrm{d}\, s \ \mathrm{d}\, t$$

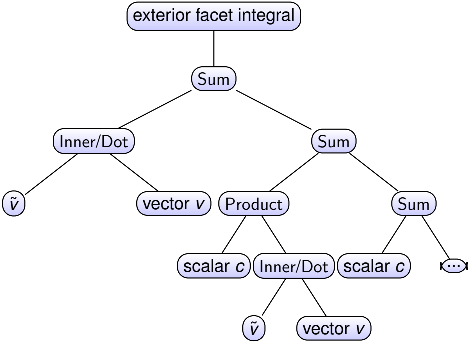
$$+ \int_0^T \int_{\Gamma_{\text{horn}}} \langle V, n \rangle \operatorname{div} \left(c^2 \lambda_p \cdot u \right) \, \mathrm{d}\, s \ \mathrm{d}\, t$$

“Backwards in time” Adjoint Equations for (λ_H, λ_E) and (λ_u, λ_p)

Actual Code in FEniCS/Python

```
U = VectorFunctionSpace(mesh, "CG", 1)
P = FunctionSpace(mesh, "CG", 1)
H = U*P
(u,p) = TrialFunction(H)
(v,q) = TestFunction(H)
u_old = project(Constant((0.0,0.0,0.0)), U)
p_old = project(Constant(0.0), P)
DeltaT = 0.1; c = 345.0
t = 0.0
a = inner(v, (u-u_old)/DeltaT) + p*div(v))*dx
a += inner(q*(p-p_old)/DeltaT - c*c*inner(u, grad(q)))*dx
...
q = Function(H)
while t < 60:
        solve(lhs(a) == rhs(a), q)
        (u,p) = split(q)
        t = t + DeltaT
J=Functional(((p+c*dot(u,n))**2)*0.5*ds(3)*dt)
for (adj,var) in compute_adjoint(J, forget=False):
...
```

Automatic Shape Derivatives in FEniCS/UFL (STS 2018)

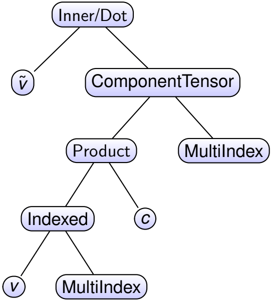


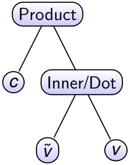
- Change expression into "maximally expanded form"
- Sort all sums closest to integral
- Apply rules to each sub-branch
- Pattern Recognition Problems
- Agument UFL derivatives in dolfin-adjoint/pyadjoint with release 2018.1

Freely Available: `www.bitbucket.org/Epoxid/femorph`

Pattern Recognition Problems

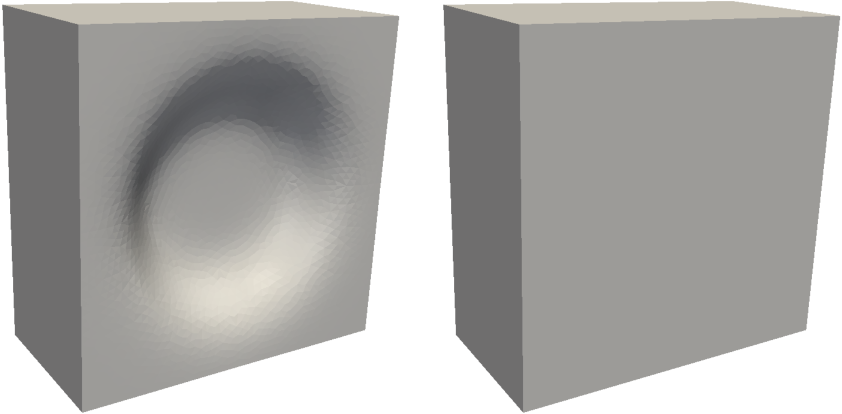
$$\langle c \cdot v, \tilde{v} \rangle = c\langle v, \tilde{v} \rangle,$$
$$\langle v, A\tilde{v} \rangle = \langle A^T v, \tilde{v} \rangle$$





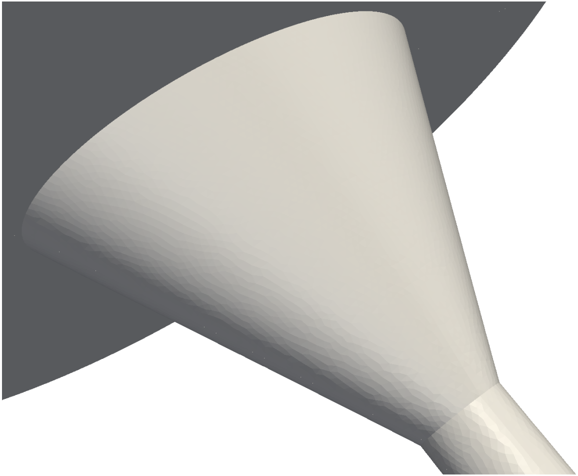
- Two UFL-Representations of the same expression
- Apply Shape Differentiation Rules on either:
  - Left: Ideal for Div-Theorem
  - Right: Human readability, $(DV)^T W = 0$, etc...

Obstacle Without Antenna



- 4.1 – 12.3 Ghz SINC-puls
- 2.4 – 7.3 cm waves, 3.65 cm obstacle
- (time with dolfin-adjoint/pyadjoint integration soon, 2018.1)

Optimal Emitter for Acoustics

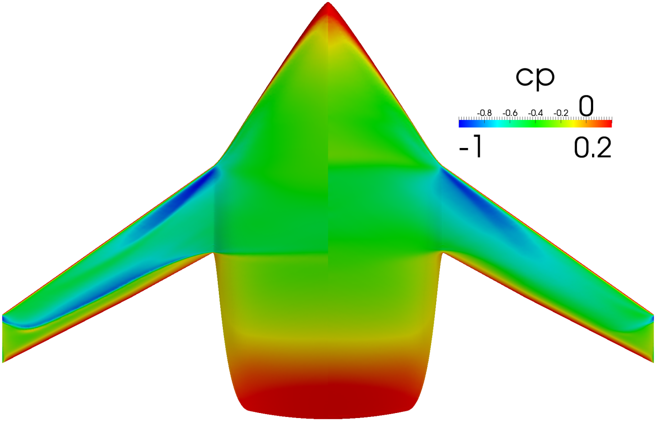


Boundary Data Compression:

$3.5 \cdot 10^9$ unknowns: 26 TB to 3.26 GB, 3 Months on 48 CPUs

(S., Wadbro, Berggren, 2016)

3D Euler Flow: VELA



Shape	C_D	%	C_L	%
460, 517	$3.342 \cdot 10^{-3}$	-30.06%	$1.775 \cdot 10^{-1}$	-0.67%

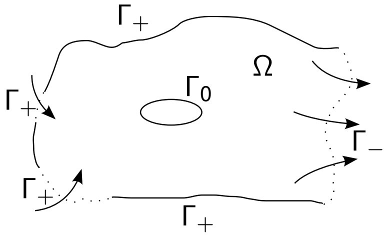
(S., Ilic, Schulz, Gauger 2013)

Model Problem: Incompressible Navier–Stokes

$$\min_{(u,p,\Omega)} E_{NS}(u,p,\Omega) := \frac{1}{2}\int_\Omega \mu \sum_{j,k=1}^{3} \left(\frac{\partial u_k}{\partial x_j}\right)^2 dA$$

subject to

$$\begin{aligned} -\mu\Delta u + \rho u \nabla u + \nabla p &= 0 \quad \text{in} \quad \Omega \\ \operatorname{div} u &= 0 \\ u &= u_+ \quad \text{on} \quad \Gamma_+ \\ u &= 0 \quad \text{on} \quad \Gamma_0 \\ pn - \mu\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \Gamma_- \end{aligned}$$



Material Derivative / Weak Shape Hessian

Weak/Volume/Material form of Shape Derivative:

$$dJ_1[V] = \int_\Omega f \operatorname{div} V + \langle \nabla f, V \rangle + f'[V] \, \mathrm{d}\,x = \int_\Omega f \operatorname{div} V + df[V] \, \mathrm{d}\,x$$

Material derivative $df[V]$ does not commute with differentiation:

- Jacobian / Gradient:

$$d(Df)[V] = Ddf[V] - DfDV$$

- Divergence:

$$d(\operatorname{div} u)[V] = \operatorname{div} du[V] - \operatorname{tr}(Du \cdot DV)$$

- Tangent-Divergence:

$$\operatorname{div}_\Gamma u = \operatorname{div} u - \langle DVn, n \rangle, \quad dn[V] \text{ as above}$$

Weak Shape Hessians

Strategy:
Use adjoint approach to eliminate material derivatives du, not u'

Result:

$$d^2 J_1[V,W] = \int_\Omega f \operatorname{div} V \operatorname{div} W + df[V] \operatorname{div} W + df[W] \operatorname{div} V - f \operatorname{tr}(DVDW) + d^2 f[V,W] \,\mathrm{d}x$$

$$\begin{aligned} &d^2 J_2[V,W] \\ &= \int_\Gamma g \operatorname{div}_\Gamma V \operatorname{div}_\Gamma W + df[V] \operatorname{div}_\Gamma W + df[W] \operatorname{div}_\Gamma V - f \operatorname{tr}(DVDW) + d^2 g[V,W] \\ &\quad + g \Big(\langle (DV)^T n, DWn \rangle + \langle DVn, (DW)^T n \rangle + \langle (DW)^T n, (DV)^T n \rangle - 2\langle DVn, n \rangle \langle DWn, n \rangle \Big) \end{aligned}$$

- $\rightsquigarrow$ Excessively long expressions with normal, curvature or PDEs
- $\rightsquigarrow$ Automatic generation!!

Regularization, Approximate Newton, H^1-Descent

So far: No consideration of the regularization term $R(\Gamma)$.
Standard Choice:

$$R(\Gamma) = \int_\Gamma 1 \; \mathrm{d}\, s$$

Then:

$$dR(\Gamma)[V] = \int_\Gamma \langle V, n \rangle \kappa \; \mathrm{d}\, s$$

$$d^2 R(\Gamma)[V, W] = \int_\Gamma \langle \nabla_\Gamma \langle V, n \rangle, \nabla_\Gamma \langle W, n \rangle \rangle + \langle V, n \rangle \langle W, n \rangle \kappa^2 \; \mathrm{d}\, s$$

Shape-descent in H^1 / Sobolev Gradient Method / Approximate Newton can all be motivated by surface area penalization!

SQP Strategy

Descent Direction:

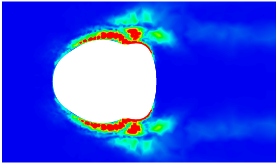
- Volume Hessian has large Kernel!
- Solve during each optimization step: Find W, such that

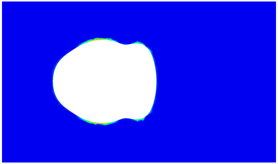
$$KKT(V, W, ...) + \langle V, W \rangle_\Omega + 0.1 \langle \nabla V, \nabla W \rangle_\Omega = dL(V, ...)$$

- UFL-Testfunction: V, UFL-Trialfunction W
- KKT and dL generated automatically

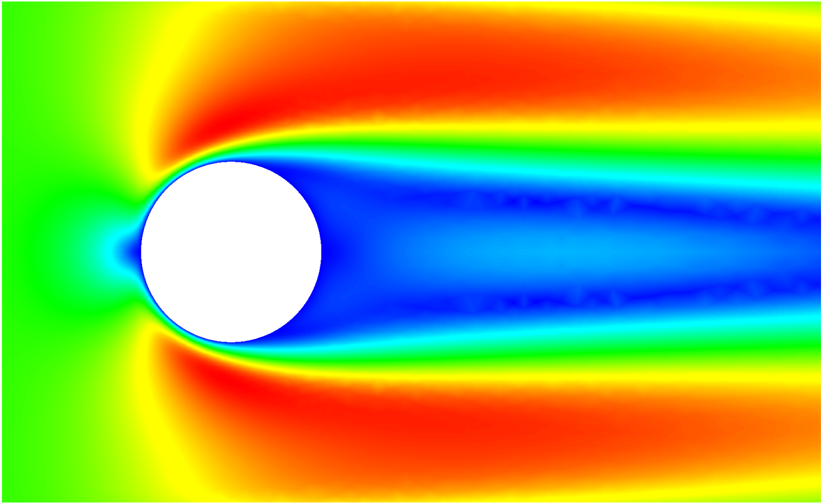
Mesh Defo:

- Boundary trace of W as Dirichlet BC in Laplace mesh deformation
- Inexact PDE $\Rightarrow$ Spurious volume movement (One Shot)





Results



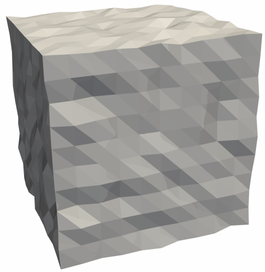
The Regularization Term

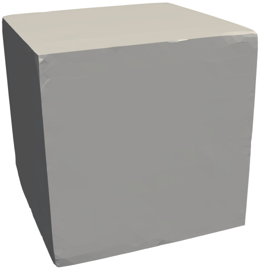
Geometric Inverse Problems:

$$\min_{(\varphi,\Gamma_{\text{inc}})} \frac{1}{2} \int_{\Gamma_{\text{in}}} \int_{t_0}^{t_f} \|(\varphi - f)\|^2 \, \mathrm{d}\, t \, \mathrm{d}\, S + \beta R(\Gamma)$$

subject to

$$\dot{\varphi} + \operatorname{div} F(\varphi) = 0 \quad \text{in} \quad \Omega$$
$$\text{BCs} = g \quad \text{on} \quad \Gamma$$





- Scanning pulse g
- f given measurement data
- Previously:
  R surface area = Laplace Smoothing
- Idea: Regularization to favor kinks
- Idea: Total Variation of Normal!

Primal Minimization Problem (joint with R. Herzog, J.Vidal-Nuñez M. Herrmann, H. Kröner)

- Let's start with functions on surfaces:

$$\min_{u \in BV(S)} \quad \frac{1}{2}\int_S |Ku - f|^2 \; ds + \frac{\alpha}{2}\int_S |u|^2 \; ds + \beta \int_S |\nabla u|$$

- Denoising: $K = Id$, Noisy Texture: f, Denoised Texture: u
- $S \subset \mathbb{R}^3$ is a smooth, compact, orientable and connected surface without boundary
- A function $u \in L^1(S)$ belongs to $BV(S)$ if the TV-seminorm defined by

$$\int_S |\nabla u| = \sup \left\{ \int_S u \operatorname{div} \boldsymbol{\eta} \; ds : \boldsymbol{\eta} \in \boldsymbol{V} \right\}$$

is finite, where $\boldsymbol{V} = \{\boldsymbol{\eta} \in \boldsymbol{C}_c^\infty(\operatorname{int} S, TS) : \; |\boldsymbol{\eta}(p)|_2 \leq 1 \; \forall p \in S\}$ with TS as the *tangent bundle* of S. Note that $BV(S) \hookrightarrow L^2(S)$

Dual Minimization Problem (joint with R. Herzog, J.Vidal-Nuñez M. Herrmann)

- Fenchel Predual Problem:

$$\min_{\boldsymbol{p}\in \boldsymbol{H}(\operatorname{div};S)} \quad \frac{1}{2}\|\operatorname{div} \boldsymbol{p} + K^* f\|^2_{B^{-1}}$$

$$\text{subject to} \quad |\boldsymbol{p}|_2 \leq \beta \quad \text{a.e. on } S$$

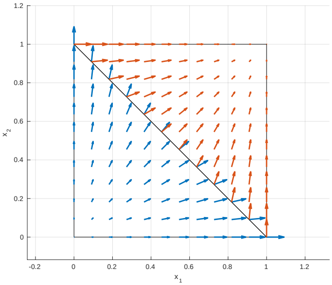
- $\boldsymbol{H}(\operatorname{div};S) :=$
$\left\{\boldsymbol{v} \in \boldsymbol{L}^2(S; T(S)) : \operatorname{div} \boldsymbol{v} \in L^2(S)\right\}$

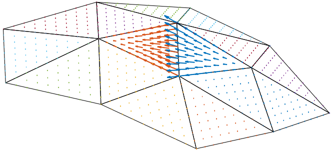
- $\|w\|^2_{B^{-1}} = (w, B^{-1}w)_{L^2(S)} = (w,w)_{B^{-1}}$

with $B := \alpha \operatorname{id} + K^*K \in \mathcal{L}(L^2(S))$

- Connection: $\overline{u} = B^{-1}(\operatorname{div} \overline{\boldsymbol{p}} + K^* f)$

- Transforms non-Differentiability into Box/Ball Constraints





Inner Point Method (joint with R. Herzog, J.Vidal-Nuñez M. Herrmann, H. Kröner)

Logarithmic barrier method to deal with the non-linear inequality constraints:

$$\min_{\boldsymbol{p} \in \boldsymbol{H}(\mathrm{div}\,; S)} \quad \frac{1}{2} \|\mathrm{div}\, \boldsymbol{p} + K^* f\|^2_{B^{-1}} - \mu \int_S \ln \left(\beta^2 - |\boldsymbol{p}|_2^2 \right) \, ds$$

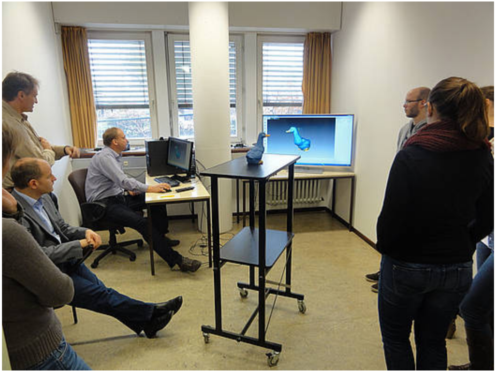
Theorem: Existence and Uniqueness

For every $\mu > 0$, this problem possesses a unique solution $\boldsymbol{p} \in \boldsymbol{H}(\mathrm{div}\,; S)$. It is characterized by

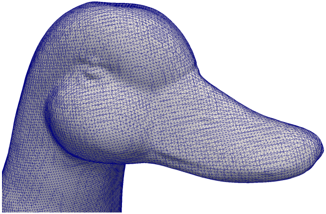
$$\left(\mathrm{div}\, \boldsymbol{p} + K^* f, \ \mathrm{div}\, \delta \boldsymbol{p} \right)_{B^{-1}} + \mu \int_S \frac{2\, (\boldsymbol{p}, \delta \boldsymbol{p})_2}{\beta^2 - |\boldsymbol{p}|_2^2} \, ds = 0$$

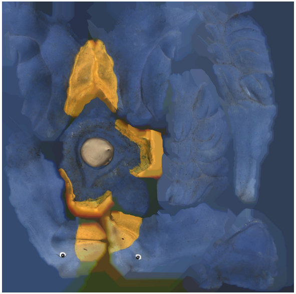
Proof: By construction, extends (Prüfert, Tröltzsch, Weiser, 2008) and (Ulbrich, Ulbrich, 2009) to surfaces

Fully integrated DG-Suite



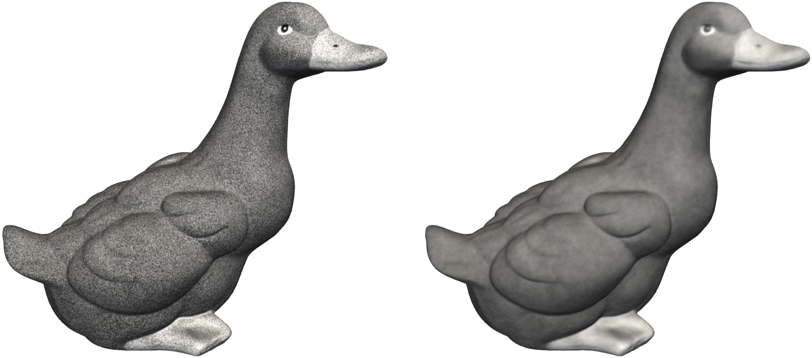
3D Scan ⇒ FEM/DG/Optimization (FEniCS) ⇒ 3D Print





Edge-Preserving TV-Denoising of 3D Scan Data

Interface with 3D Scanner (354, 330 Polygons, 177, 167 Vertices)



- Convert Geometry + Texture (Bitmap) to DG-Space on Surface
- Predual Approach in RT/DG-Space, interior point method

Color Unerazing / Color Inpainting

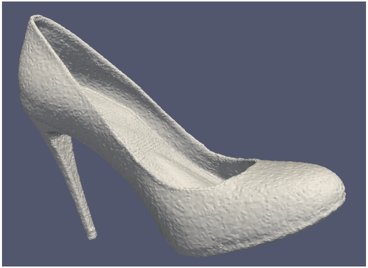
Simulation of missed scan sweep, 3% signal missing

- K: "Identity with Zeros"
- Exactly $100,000$ triangles and $50,002$ vertices
- DG3, RT4

Denoising of Geometries?

How to carry over insights from pictures on surfaces?

- Surface is represented by signed distance function (SDF)
- Normal is gradient of SDF
- Primal approach: gradient of CG1 SDF = Edge Jump
- Split-Bregman Algorithm necessitates “DG0 Skeleton Space” ⇒“HDivTrace”



Non-Smooth Mesh Denoising

Mesh Denoising Formulation

$$\min_{\ell \in \mathcal{CG}_1(D)} \quad \|\ell\|^2_{L^2(\Gamma_0)} + \beta \left|\nabla \ell\right|_{DTV_2(\Gamma(\ell=0))}$$

$$\text{s.t.} \quad \ell(x) < 0 \Leftrightarrow \text{x is 'inside' } \Gamma$$

$$\|\nabla \ell\|_2 = 1 \;\; \forall x \in D$$

Discrete Total Variation

$$\left|\nabla \ell\right|_{DTV_2(\Gamma(\ell=0))} := \sum_{f \in \mathcal{F}_\Gamma} c_{f,\Gamma} \left\| \left(\nabla \ell|^+ - \nabla \ell|^- \right)_f \right\|_2$$

where $c_{f,\Gamma}$ is the length of the intersection between Γ and the facet f

Primal Non-Smooth Optimization Algorithm

Split-Bregman Algorithm: Solve non-smooth problem by introducing new variables: $q = \nabla \ell \in \mathcal{DG}_0^3(D)$ and $p = (q|^+ - q|^-) \in \mathcal{DG}_0^3(\mathcal{F})$

“Augmented Lagrangian Problem”:

$$\min \|\ell\|^2_{L^2(\Gamma_0)} + \beta \sum_{f \in \mathcal{F}_\Gamma} c_{f,\Gamma} \|p_f\|_2 + \frac{\lambda_1}{2} \|q - \nabla \ell - b_T\|^2_{L^2(D)} + \frac{\lambda_2}{2} \|p - (q|^+ - q|^-) - b_F\|^2_{L^2(\mathcal{F}$$

s.t. $\ell(x) < 0 \Leftrightarrow$ x is ‘inside’ Γ

$\|\nabla \ell\|_2 = 1 \ \ \forall x \in D$

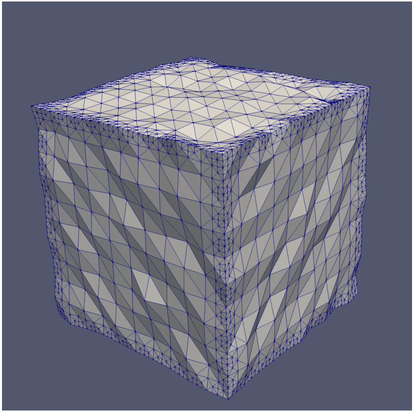
with $b_T \in \mathcal{DG}_0^3(D)$ and $b_F \in \mathcal{DG}_0^3(\mathcal{F})$

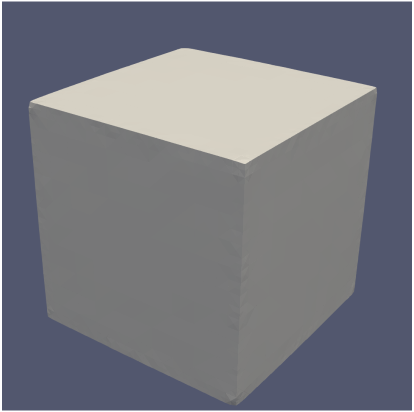
Splitting in Split-Bregman ⇒ "Mesh-Bregman"

Key optimization steps:

- $\hat{\ell}^{(n+1)} := \arg\min_{\ell \in \mathcal{CG}_1(D)} \|\ell\|^2_{L^2(\Gamma_0)} + \frac{\lambda_1}{2}\|q^{(n)} - \nabla \ell - b_T^{(n)}\|^2_{L^2(D)}$
- a) $\ell^{(n+1)} := \hat{\ell}^{(n+1)}$ b) $\ell^{(n+1)} := \ell^{(n)} + \tau_n \left(\hat{\ell}^{(n+1)} - \ell^{(n)}\right)$
- Update: Γ, $\mathcal{F}_\Gamma$, $c_{f,\Gamma}$
- $g^{(n+1)} := \frac{\nabla \ell^{(n+1)}}{\|\nabla \ell^{(n+1)}\|_2}$
- $q^{(n+1)} := \arg\min_{q \in \mathcal{DG}_0^3(D)} \frac{\lambda_1}{2}\|q - g^{(n+1)} - b_T^{(n)}\|^2_{L^2(D)} + \frac{\lambda_2}{2}\|p^{(n)} - (q|^+ - q|^-) - b_F^{(n)}\|^2_{L^2(\mathcal{F})}$
- $j^{(n+1)} := \left(\frac{q^{(n+1)}}{\|q^{(n+1)}\|_2}\Big|^+ - \frac{q^{(n+1)}}{\|q^{(n+1)}\|_2}\Big|^- \right)$
- $p^{(n+1)} :=$ $\arg\min_{p \in \mathcal{DG}_0^3(\mathcal{F})} \beta \sum_{f \in \mathcal{F}_\Gamma} c_{f,\Gamma}\|p_f\|_2 + \frac{\lambda_2}{2}\sum_{f \in \mathcal{F}} c_f \|p_f - j_f^{(n+1)} - (b_F^{(n)})_f\|_2^2$
- $b_T^{(n+1)} := b_T^{(n)} + g^{(n+1)} - q^{(n+1)}$
- $b_F^{(n+1)} := b_F^{(n)} + j^{(n+1)} - p^{(n+1)}$

Results





- Inverse Problems and Surfaces with Kinks!
- Weak and Strong Shape Differentiation
- GPU Computing and Topology Optimization
- Shape SQP-Methods and Automatization