Solution Techniques for Constrained Shape Optimization Problems in Shape Spaces

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Table of contents

$$\min_{\Omega} J(\Omega)$$
PDE or VI constraints

- J shape functional depending on a solution of a partial differential equation (PDE) or a variational inequality (VI)

  (Definition: Let $D \subset \mathbb{R}^d$ be non-empty and let $\mathcal{A} \subset \{\Omega \colon \Omega \subset D\}$ denote a set of subsets. A function $J \colon \mathcal{A} \to \mathbb{R}$, $\Omega \mapsto J(\Omega)$ is called a *shape functional*.)

- Ω shape

  $\rightsquigarrow$ *2D shape:* Simply connected, compact subset Ω of $\mathbb{R}^2$ with $\Omega \neq \emptyset$ and $\mathcal{C}^\infty$ boundary $\partial\Omega$

  $\rightsquigarrow$ *How is the set of all shapes defined?*

Space of smooth shapes

2D shape: Simply connected, compact subset Ω of $\mathbb{R}^2$ with $\Omega \neq \varnothing$ and $\mathcal{C}^\infty$ boundary $\partial\Omega$

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- B_e is a manifold (cf. [1])
- *Generalization to higher dimensions:* $B_e(M,N) := \text{Emb}(M,N)/\text{Diff}(M)$, where M is a compact manifold and N denotes a Riemannian manifold with $\dim(M) < \dim(N)$

[1] A. Kriegl and P. Michor. *The Convient Setting of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997.

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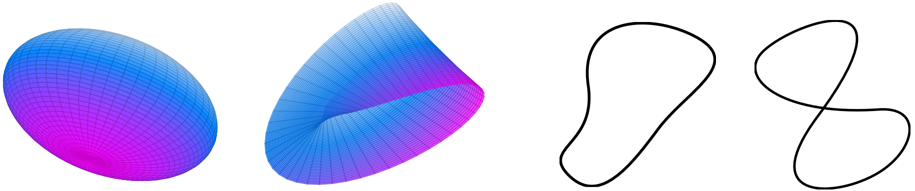

$b_1 \in B_e(S^2, \mathbb{R}^3)$ $b_2 \notin B_e(S^2, \mathbb{R}^3)$ $c_1 \in B_e(S^1, \mathbb{R}^2)$ $c_2 \notin B_e(S^1, \mathbb{R}^2)$

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Optimization on manifolds (cf. [2])

- Line-search methods in $\mathbb{R}^n$: $x_{k+1} = x_k + t_k\xi_k$
- Manifolds are not necessarily linear spaces
  $\Rightarrow$ Select ξ_k as a tangent vector to $\mathcal{M}$ at x_k
  $\Rightarrow$ Points from the tangent space have to be mapped to the manifold

[2] P.-A. Absil, R. Mahony and R. Sepulchre. *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, 2008.

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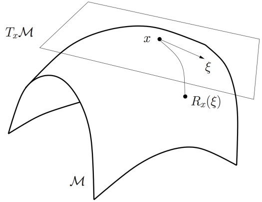

Example: Steepest descent method on B_e

Goal: Find a solution of $\min_{x \in B_e} J(x)$

Input: $x_0 \in B_e$

for $k = 0, 1, \ldots$ **do**

[1] Compute the increment $\xi_k := -\text{grad} J(x^k) \in T_{x_k} B_e$

[2] Set $x_{k+1} := \mathcal{R}_{x_k}(t_k \xi_k)$ for some steplength t_k and a retraction $\mathcal{R}$

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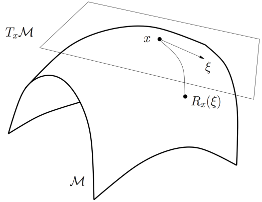

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A *Riemannian metric* on a manifold M is a collection $g = (g_p)_{p \in M}$ of inner products

$$g_p \colon T_pM \times T_pM \to \mathbb{R}, \ (v, w) \mapsto g_p(v, w),$$

one for every $p \in M$, such that the map $M \to \mathbb{R}, \ p \mapsto g_p(X(p), Y(p))$ is smooth for every pair of vector fields X, Y on M.

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Sobolev metric (cf. [3])

$$\boxed{g^1\colon T_cB_e \times T_cB_e \to \mathbb{R},\ (h,k) \mapsto \langle (I - A\triangle_c)\alpha, \beta\rangle_{L^2(c)} \text{ with } A > 0}$$

Here $h = \alpha n$ and $k = \beta n$ denote two elements of the tangent space

$$T_cB_e \cong \{\psi \mid \psi = \alpha n,\ \alpha \in \mathcal{C}^\infty(S^1, \mathbb{R})\}.$$

[3] M. Bauer, P. Harms, and P. Michor. Sobolev metrics on shape space of surfaces. *JGM*, 3(4):389–438, 2011.

Riemannian metrics on B_e

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Riemannian shape gradient w.r.t. g^1

Representation of $DJ(x)$ such that $\quad g^1(\mathrm{grad}J(x), h) = DJ(x)[h] \quad \forall h \in T_xB_e$

[3] M. Bauer, P. Harms, and P. Michor. Sobolev metrics on shape space of surfaces. *JGM*, 3(4):389–438, 2011.

Shape derivative

Definition

Let $D \subset \mathbb{R}^d$ be non-empty and open and let $\Omega \subset D$ be measurable. The *shape derivative* of a shape functional J at Ω in direction $V \in \mathcal{C}_0^k(D, \mathbb{R}^d)$, $k \in \mathbb{N} \cup \{\infty\}$, is defined by the *Eulerian derivative*

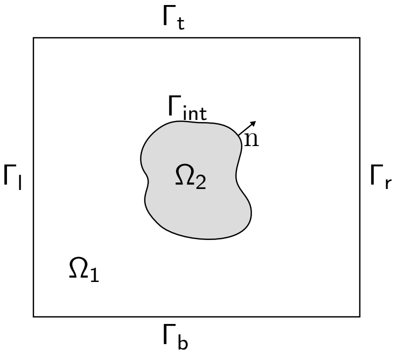
$$DJ(\Omega)[V] = \lim_{t \to 0^+} \frac{J(\Omega_t) - J(\Omega)}{t} \qquad (\Omega_t = \{x + t \cdot V(x) : x \in \Omega\}).$$

- Shape derivative of a shape differentiable functional is a directional derivative in the direction of a differentiable vector field
- ***Hadamard Structure Theorem*** (cf. Theorem 2.17 in [4]):
  - Only the normal part of a vector field on the boundary Γ of a domain Ω has an impact on the value of the shape derivative
  - Existence of a scalar distribution r on Γ
  - If $r \in L^1(\Gamma)$, then $DJ(\Omega)[V] = \int_\Gamma r \langle V, n \rangle \, ds$
- r is often called the *shape gradient* but gradients depend on chosen scalar products defined on the space under consideration

[4] J. Sokolowski and J.-P. Zolésio. *Introduction to Shape Optimization*, volume 16 of *Computational Mathematics* Springer, 1992.

Contents

- Diffusion process
- $\Omega \subset \mathbb{R}^2$ bounded Lipschitz domain
- Two materials $\Omega_1, \Omega_2 \subset \Omega$ with different permeability
- Fixed outer boundary $\partial\Omega = \Gamma_b \,\dot\cup\, \Gamma_l \,\dot\cup\, \Gamma_r \,\dot\cup\, \Gamma_t$
- Variable boundary $\Gamma_{int} \in B_e(S^1, \mathbb{R}^2)$ $\Rightarrow$ Fit to measured concentration
- Homogeneous concentration in Ω for $t = 0$
- Higher concentration on the top in the beginning



Diffusion problem

For **concentration** y and **diffusion coefficient** k:

$$\min_{\Gamma_{\text{int}}} J(y,\Omega) = j(y,\Omega) + j^{\text{reg}}(\Omega) := \frac{1}{2}\int_0^T\int_\Omega (y-\bar{y})^2 dx\, dt + \mu \int_{\Gamma_{\text{int}}} 1\, ds$$

$$\text{s.t. } \frac{\partial y}{\partial t} - \operatorname{div}(k\nabla y) = f \quad \text{in } \Omega \times (0,T]$$

$$y = 1 \quad \text{on } \Gamma_t \times (0,T]$$

$$\frac{\partial y}{\partial n} = 0 \quad \text{on } (\Gamma_b \cup \Gamma_l \cup \Gamma_r) \times (0,T]$$

$$y = y_0 \quad \text{in } \Omega \times \{0\}$$

- Data measurements: $\bar{y} \in L^2\left(0,T; L^2(\Omega)\right)$
- Right hand-side: $f = \text{const.} \quad \text{in } \Omega \times (0,T]$
- Jumping coefficient: $k = \begin{cases} k_1 = \text{const.} & \text{in } \Omega_1 \\ k_2 = \text{const.} & \text{in } \Omega_2 \end{cases}$
- Transmission conditions: $[\![y]\!] = 0, \quad \left[\!\!\left[k\frac{\partial y}{\partial n}\right]\!\!\right] = 0 \quad \text{on } \Gamma_{\text{int}} \times (0,T]$

Shape derivative

Shape derivative of the regularization term j^{reg}

$$Dj^{\text{reg}}(\Omega)[V] = \mu \int_{\Gamma_{\text{int}}} \langle V, n \rangle \, \kappa \, ds$$

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$$\boxed{Dj_\Omega(y,\Omega)[V] = Dj(y,\Omega)[V] = Dj_{\Gamma_{\text{int}}}(y,\Omega)[V]}$$

- $$Dj_\Omega(y,\Omega)[V] = \int_0^T \int_\Omega -k\nabla y^\top \left(\nabla V + \nabla V^\top\right) \nabla p - p \nabla f^\top V + \operatorname{div}(V) \left(\frac{1}{2}(y-\overline{y})^2 + \frac{\partial y}{\partial t} p + k \nabla y^\top \nabla p - fp\right) dx\, dt$$
- $$Dj_{\Gamma_{\text{int}}}(y,\Omega)[V] = \int_0^T \int_{\Gamma_{\text{int}}} [\![k]\!] \, \nabla y_1^\top \nabla p_2 \, \langle V, n \rangle \, ds\, dt$$

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  - Volume formulation: $y \in L^2\left(0, T; H^1(\Omega)\right), p \in W\left(0, T; H^1(\Omega)\right)$
  - Surface formulation: H^2-regularity in space is necessary

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  - Surface formulation: H^2-regularity in space is necessary

Algorithm on (B_e, g^1) (cf. [5])

Evaluate measurements

↓

Solve the state and adjoint equation

↓

Assemble the Dirichlet boundary condition of the linear elasticity equation:

1. Compute the Riemannian shape gradient with respect to g^1 from the surface shape derivative $D_\Gamma \mathcal{L}[V] = \int_\Gamma r \langle V, n \rangle \, ds$ $(r \in L^1(\Gamma))$:

$$\operatorname{grad}^{B_e} \mathcal{L} = qn \text{ mit } (I - A\triangle_\Gamma)q = r \quad (\Gamma \in B_e, A > 0, q \in \mathcal{C}^\infty(\Gamma))$$

2. Compute a Riemannian limited-memory BFGS update

↓

Solve the linear elasticity equation with source term equals zero

↓

Apply the resulting deformation to the finite element mesh

(loop back to: Evaluate measurements)

[5] V. Schulz, M. Siebenborn and K. W. Structured inverse modeling in parabolic diffusion problems. *SICON*, 53(6):3319-3338, 2015.

Contents

Steklov-Poincaré: Motivation and aims

- Derivation of surface shape derivative formulations is a time-consuming process
  $\Rightarrow$ ***Aim:*** *Usage of volume shape derivative expressions*

- Gradient representation and afterwards mesh deformation
  $\Rightarrow$ ***Aim:*** *Gradient representation and mesh deformation all at once*

- $\mathcal{C}^\infty$-shapes
  $\Rightarrow$ ***Aim:*** *Weaken the assumption of $\mathcal{C}^\infty$-shapes*

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Definition of an inner product based on volume formualtions

- Inner product should be derived from the second shape derivative
- Second shape derivative can be related to the Steklov-Poincaré operator (cf. [6])

$\Rightarrow$ Definition of a *Steklov-Poincaré type metric*

[6] S. Schmidt and V. Schulz. Impulse response approximations of discrete shape Hessians with application in CFD. *SICON*, 48(4):2562–2580, 2009.

Definition

$$g^S \colon H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{R},\ (\alpha,\beta) \mapsto \int_\Gamma \alpha \cdot (S^{pr})^{-1} \beta \, ds$$

Here S^{pr} denotes the projected Poincaré-Steklov operator and is given by

$$S^{pr} \colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),\ \xi \mapsto (\gamma_0 U)^\top n,$$

where $U \in H_0^1(\Omega, \mathbb{R}^d)$ solves $a(U,V) = \int_\Gamma \xi \cdot (\gamma_0 V)^\top n \, ds \ \forall V \in H_0^1(\Omega, \mathbb{R}^d)$.

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Shape derivative in surface formulation $DJ_\Gamma[V] = \int_\Gamma r \langle V, n \rangle \, ds$

$\rightsquigarrow$ **Shape gradient with respect to g^S** is given by $h \in T_\Gamma B_e \cong \mathcal{C}^\infty(\Gamma)$ s.t.

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$\Rightarrow h=S^{\mathrm{pr}}r=(\gamma_0 U)^\top n$, where $U\in H^1_0(\Omega,\mathbb{R}^d)$ solves

$$a(U,V)=\int_\Gamma r\cdot(\gamma_0 V)^\top n\, ds\ =DJ_\Gamma[V]=DJ_\Omega[V]\quad\forall V\in H^1_0(\Omega,\mathbb{R}^d)$$

Construction of the g^S-metric yields

$$g^S(u,v) = DJ_\Gamma[V] = DJ_\Omega[V] = a(U,V) \quad \forall V \in H_0^1(\Omega,\mathbb{R}^d) \text{ and } v = (\gamma_0 V)^\top n$$

$\Rightarrow$ *Shape gradient representation* $u = (\gamma_0 U)^\top n$ *and mesh deformation* $U \in H_0^1(\Omega,\mathbb{R}^d)$

Results

Construction of the g^S-metric yields

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⇒ *Shape gradient representation $u = (\gamma_0 U)^\top n$ and mesh deformation $U \in H_0^1(\Omega,\mathbb{R}^d)$*

We have to solve the ***deformation equation***:

$$a(U,V) = b(V) \quad \forall V \in H_0^1(\Omega,\mathbb{R}^d)$$

- $a(\cdot,\cdot)$ symmetric and coercive bilinearform
- $b(V) := DJ_{\text{vol}}(\Omega)[V] + DJ_{\text{surf}}(\Omega)[V]$
  - $J_{\text{vol}}(\Omega)$ parts of J leading to volume shape derivative expressions
  - $J_{\text{surf}}(\Omega)$ parts of J leading to surface shape derivative expressions

⇒ Combination of surface and volume formulation of shape derivatives

Algorithm on (B_e, g^S) (cf. [7])

Evaluate measurements

↓

Solve the state and adjoint equation

↓

Assemble the deformation equation:

- Assemble $DJ_{\text{vol}}(\Omega)[V]$ for V with $\Gamma \cap \text{supp}(V) \neq 0$ as source term
- Assemble $DJ_{\text{surf}}(\Omega)[V]$ as Neumann boundary conditions

↓

Solve the deformation equation

↓

Compute a Riemannian limited-memory BFGS update

↓

Apply the resulting deformation to the finite element mesh

(↺ back to "Evaluate measurements")

[7] V. Schulz, M. Siebenborn and K. W. Efficient PDE constrained optimization based on Steklov-Poincaré type metrics. *SIOPT*, 26(4):2800-2819, 2016.

- Diffusion problem with $T = 20$, $k = \begin{cases} k_1 = 1 \text{ in } \Omega_1 \\ k_2 = 0.001 \text{ in } \Omega_2 \end{cases}$
- $A = 0.001$
- Data $\overline{y}$ are generated from a solution of the state equation for the setting $\Omega_2 = \{x \colon \|x\|_2 < 0.5\}$

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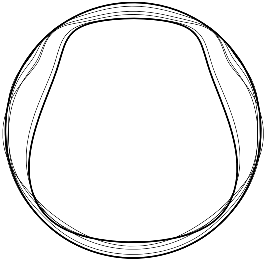

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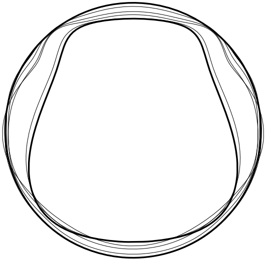

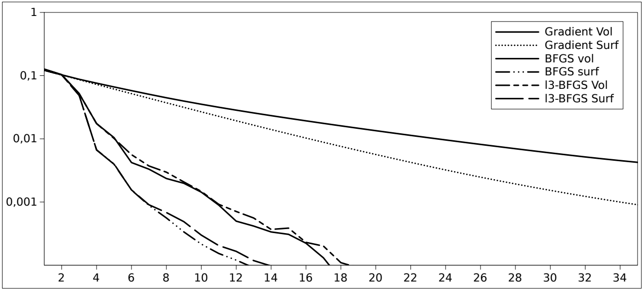

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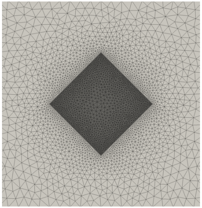

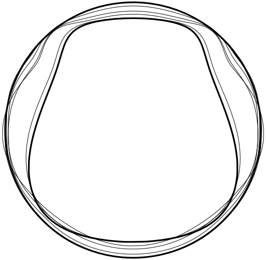

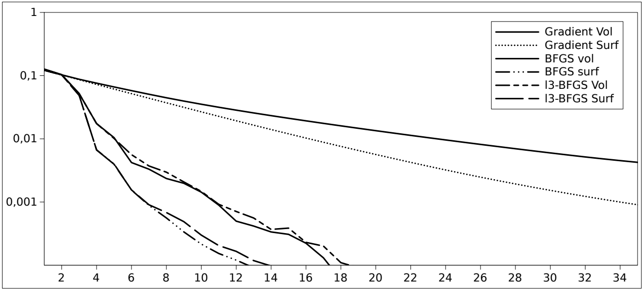

Contents

Space of $H^{1/2}$-shapes

$$\boxed{\mathcal{B}^{1/2}(\Gamma_0, \mathbb{R}^d) := \mathcal{H}^{1/2}(\Gamma_0, \mathbb{R}^d) / \sim}$$

- $\Gamma_0 \subset \mathbb{R}^d$ d-dimensional Lipschitz shape
  (Definition: A d-dimensional *Lipschitz shape* is defined as the boundary $\Gamma_0 = \partial \mathcal{X}_0$ of a compact Lipschitz domain $\mathcal{X}_0 \subset \mathbb{R}^d$ with $\mathcal{X}_0 \neq \emptyset$.)
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Challenges

- Properties of $w \in H^{1/2}(\Gamma_0, \mathbb{R}^d)$ such that $w(\Gamma_0)$ Lipschitz shape
- Independence of the definition of $\mathcal{B}^{1/2}\left(\Gamma_0, \mathbb{R}^d\right)$ from the Lipschitz shape Γ_0
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- Structure of $\mathcal{B}^{1/2}\left(\Gamma_0,\mathbb{R}^d\right)$: Diffeological structure

Diffeological structure of $\mathcal{B}^{1/2}$ (cf. [8])

$$\mathcal{B}^{1/2}(\Gamma_0, \mathbb{R}^d) := \mathcal{H}^{1/2}(\Gamma_0, \mathbb{R}^d) / \sim$$

- $\mathcal{H}^{1/2}(\Gamma_0, \mathbb{R}^d) = \{w : w \in H^{1/2}(\Gamma_0, \mathbb{R}^d)$ injective, continuous; $w(\Gamma_0)$ Lipschitz shape$\}$
- $w_1 \sim w_2 \Leftrightarrow w_1(\Gamma_0) = w_2(\Gamma_0)$, where $w_1, w_2 \in \mathcal{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$

- $\mathcal{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$ is obviously a subset of $H^{1/2}(\Gamma_0, \mathbb{R}^d)$
- $H^{1/2}(\Gamma_0, \mathbb{R}^d)$ is a Banach space and, thus, a manifold
  $\Rightarrow$ We can view $H^{1/2}(\Gamma_0, \mathbb{R}^d)$ with the corresponding diffeology

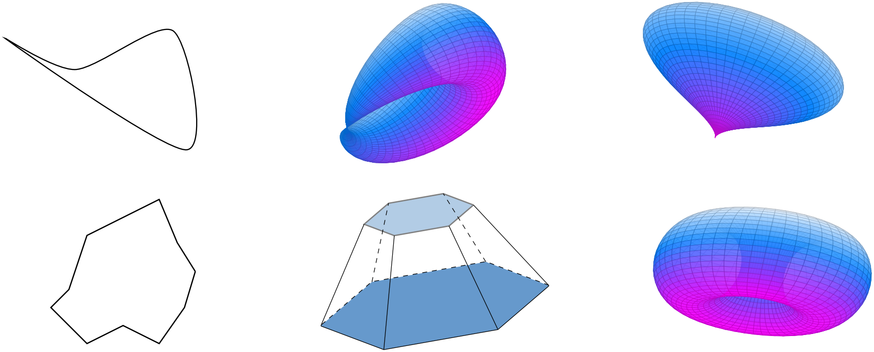
Results in [9]:

- Every subset of a diffeological space carries a natural *subset diffeology*, which is defined by the *pullback* of the ambient diffeology by the *natural inclusion*
- Every quotient of a diffeological space carries a natural *quotient diffeology* defined by the *pushforward* of the diffeology of the source space to the quotient by the *canonical projection*

$\Rightarrow$ We can construct diffeologies on $\mathcal{H}^{1/2}(\Gamma_0, \mathbb{R}^d)$ and $\mathcal{B}^{1/2}(\Gamma_0, \mathbb{R}^d)$ □

[8] K. W. Suitable spaces for shape optimization, 2017. (arXiv:1702.07579)
[9] P. Iglesias-Zemmour. *Diffeology*. Volume 185, American Mathematical Society, 2013.

Diffeological space



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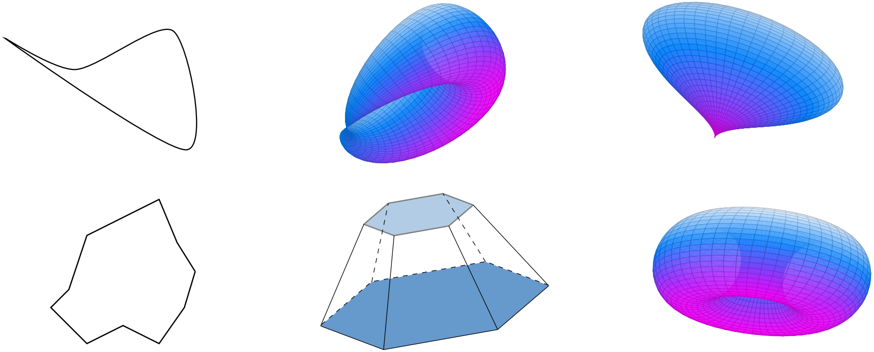

- No theory for shape optimization on diffeological spaces so far
- Diffeological structure suffices for many differential-geometric tools used in shape optimization techniques
- Riemannian structures can be used to measure shape distances and state convergence properties

Contents

VI constrained shape optimization

- No explicit dependence on the domain in classical VIs
- In VI constrained shape optimization problems: Unavoidable source of non-linearity and non-convexity due to the non-linear and non-convex nature of shape spaces

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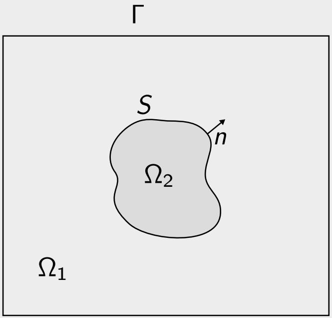
Setting

- $\Omega \subset \mathbb{R}^2$ bounded Lipschitz domain
- Two subdomains $\Omega_1, \Omega_2 \subset \Omega$
- Fixed outer boundary $\partial\Omega = \Gamma_b \,\dot\cup\, \Gamma_l \,\dot\cup\, \Gamma_r \,\dot\cup\, \Gamma_t$
- Variable boundary $S \in B_e(S^1, \mathbb{R}^2)$ or $S \in \mathcal{B}^{1/2}(\Gamma_0, \mathbb{R}^2)$
- Outer normal vector n to Ω_2



VI constrained problem

$$\min_S \; J(y,S) = j(y,S) + j^{\text{reg}}(S) := \frac{1}{2}\int_{\Omega(S)} (y-\bar{y})^2 dx + \mu \int_S 1\, ds$$

$$\begin{aligned}
\text{s.t.} \quad -\Delta y + \lambda &= f && \text{in } \Omega \\
y &\leq \psi && \text{in } \Omega \\
\lambda &\geq 0 && \text{in } \Omega \\
\lambda(y-\psi) &= 0 && \text{in } \Omega \\
y &= 0 && \text{on } \Gamma
\end{aligned}$$

- Obstacle: $\psi \in H^4(\Omega)$ with $0 < \psi \leq M$ for $M > 0$
- Jumping coefficient: $f = \begin{cases} f_{\text{int}} = \text{const.} & \text{in } \Omega_{\text{int}} \\ f_{\text{out}} = \text{const.} & \text{in } \Omega_{\text{out}} \end{cases}$
- Transmission conditions: $[\![y]\!] = 0, \quad \left[\!\!\left[\frac{\partial y}{\partial n}\right]\!\!\right] = 0 \quad \text{on } S$

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$$y = 0 \quad \text{on } \Gamma$$

Regularized version [10]: $\boxed{-\Delta y + \lambda_c = f \quad \text{in } \Omega}$

- $\lambda_c = \max\{0, \overline{\lambda} + c(y-\psi)\}^2$
- $c > 0,\ 0 \ge \overline{\lambda} \in L^4(\Omega)$

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[10] M. Hintermüller and A. Laurain. Optimal shape design subject to elliptic variational inequalities. *SICON*, 49(3):1015-1047, 2011.

Regularized state equation

Idea: Adapt the primal-dual active set (PDAS) algorithm in [11] to our problem

Modified PDAS (mPDAS) algorithm

1. Choose y_0, $k = 0$ and $\lambda_0 = 0$
2. $\mathcal{A}_{k+1} := \{x \colon [\lambda_k + c(y - \psi)](x) > 0\}$ and $\mathcal{I}_{k+1} := \Omega \backslash \mathcal{A}_{k+1}$
3. Compute $y_{k+1} \in H_0^1(\Omega)$ as solution of

$$a(y_{k+1}, v) + \left([\lambda_k + c(y_{k+1} - \psi)]^2, \mathcal{X}_{\mathcal{A}_{k+1}} v \right) = (f, v) \qquad \forall v \in H_0^1(\Omega) \qquad (*)$$

4. $\lambda_{k+1} := \begin{cases} 0 & \text{if } x \in \mathcal{I}_{k+1} \\ \lambda_k + c(y_{k+1} - \psi) & \text{if } x \in \mathcal{A}_{k+1} \end{cases}$
5. Stop or $k := k + 1$ and go to 2.

[11] K. Ito and K. Kunisch. Semi–smooth Newton methods for variational inequalities of the first kind. *ESAIM: Mathematical Modelling and Numerical Analysis*, 37(1):41-62, 2003.

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Linear mPDAS algorithm (cf. [12])

Step 3 in mPDAS algorithm

Compute $y_{k+1} \in H_0^1(\Omega)$ as solution of

$$a(y_{k+1}, v) + \left([\lambda_k + c(y_{k+1} - \psi)]^2, \mathcal{X}_{\mathcal{A}_{\mathbf{k+1}}} v\right) = (f, v) \qquad \forall v \in H_0^1(\Omega) \qquad (*)$$

Problem: $(*)$ is not linear

$\rightsquigarrow$ **Idea:** Compute $\Delta y := y_{k+1} - y_k$ instead of y_{k+1} and use the linearization

$$(\lambda_k + c(y_k + \Delta y - \psi))^2 \doteq (\lambda_k + c(y_k - \psi))^2 + 2c\Delta y(\lambda_k + c(y_k - \psi))$$

Step 3 in linear mPDAS algorithm

a) Compute Δy as solution of

$$a(\Delta y, v) + \left(2c\Delta y\,[\lambda_k + c(y_k - \psi)], \mathcal{X}_{\mathcal{A}_{\mathbf{k+1}}} v\right)$$
$$= (f, v) - a(y_k, v) - \left([\lambda_k + c(y_k - \psi)]^2, \mathcal{X}_{\mathcal{A}_{\mathbf{k+1}}} v\right) \qquad \forall v \in H_0^1(\Omega)$$

b) $y_{k+1} := y_k + \Delta y$

[12] B. Führ, V. Schulz and K. W. Shape optimization for interface identification with obstacle problems. Appears in: *Vietnam Journal of Mathematics*, 2018.

Solution of the regularized state equation

$$\boxed{-\Delta y + \lambda_c = f \quad \text{in } \Omega}$$

- $\lambda_c = \max\{0, \overline{\lambda} + c(y - \psi)\}^2$
- $0 \geq \overline{\lambda} \in L^4(\Omega),\ c = 5,\ f = \begin{cases} f_1 = -10 & \text{in } \Omega_1 \\ f_2 = 100 & \text{in } \Omega_2 \end{cases}$

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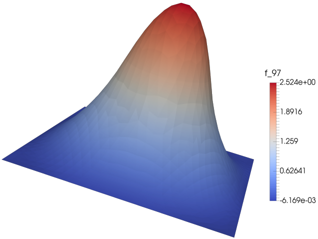

y with $\lambda_c = 0$

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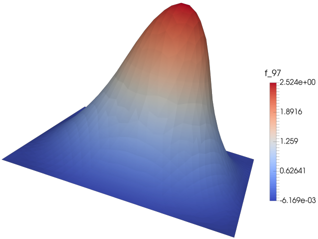

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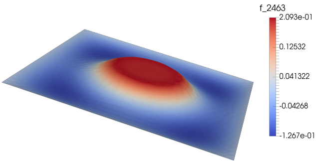
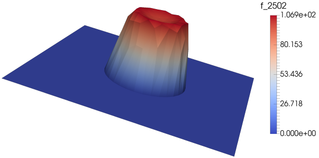


y with $\psi = 0.2$



Corresponding λ_c

Algorithm (based on g^S)

Evaluate objective

↓

Solve the state equation (with the linear mPDAS algorithm)

↓

Solve the adjoint equation

↓

Assemble the deformation equation:

- E.g., choose the weak form of the linear elasticity equation as left hand-side of the deformation equation
- Assemble $DJ_{\text{vol}}(y, X)[V]$ for all test functions V whose support includes S as a source term
- Assemble $DJ_{\text{surf}}(y, X)[V]$ as Neumann boundary conditions

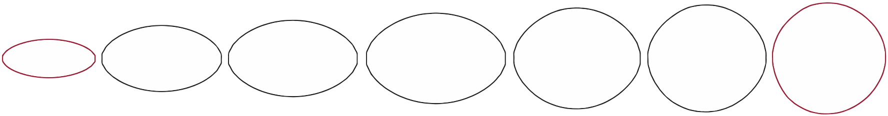
↓

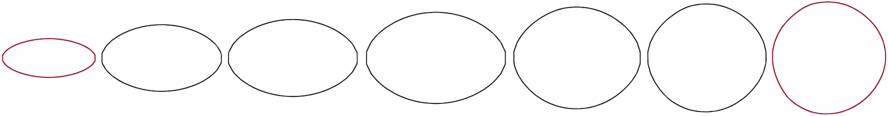
Solve the deformation equation

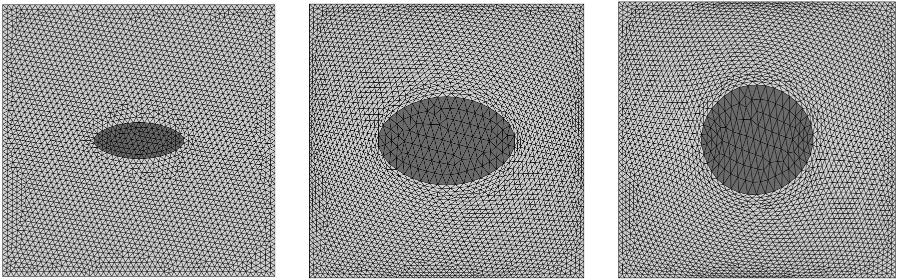
↓

Apply the resulting deformation to the finite element mesh

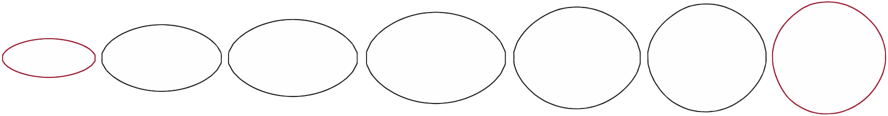
(loop back to: Evaluate objective)

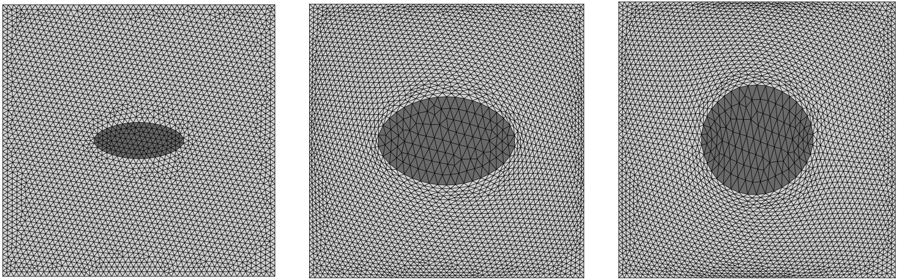






- Computations on unstructured grids with about 1 500 up to 6 000 triangles





- Computations on unstructured grids with about 1 500 up to 6 000 triangles
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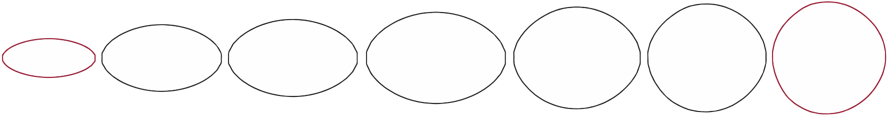
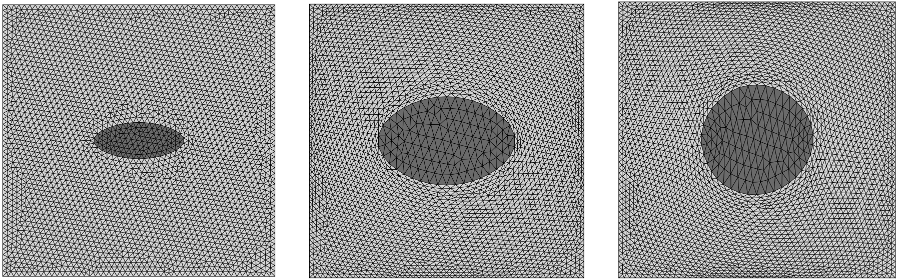
- Largest deformations at the beginning of the iteration process





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- Largest deformations at the beginning of the iteration process
- Algorithmic performance deteriorates if the obstacle problem is strongly binding
- More iterations for tighter obstacles, i.e., small values of ψ (494 iterations for $\psi = 0.5$ vs. 22 iterations for $\psi = 10$)

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Thank you for your attention!