Howe's correspondence and characters for dual pairs over Archimedean and non-Archimedean fields

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The Cauchy determinantal identity, 1812

$$\frac{1}{\prod_{1\le i,j\le n}(1-h_ih'_j)} = \sum_{k_1>k_2>\dots>k_n} \frac{|h^{k_1}\,h^{k_2}\dots h^{k_n}|}{|h^{n-1}\,h^{n-2}\dots h^0|}\cdot\frac{|h'^{k_1}\,h'^{k_2}\dots h'^{k_n}|}{|h'^{n-1}\,h'^{n-2}\dots h'^0|}$$

where $$|h^{k_1}\,h^{k_2}\dots h_n^{k_n}| = \det\begin{pmatrix} h_1^{k_1} & h_1^{k_2} & \dots & h_1^{k_n} \\ h_2^{k_1} & h_2^{k_2} & \dots & h_2^{k_n} \\ \dots & \dots & \dots & \dots \\ h_n^{k_1} & h_n^{k_2} & \dots & h_n^{k_n} \end{pmatrix}$$

An interpretation of Cauchy's identity

The formula

$$\omega(g,g')x = gxg'^{t} \qquad (x \in M_{n,n}(\mathbb{C}),\ (g,g') \in \mathrm{U}_n \times \mathrm{U}_n)$$

defines a representation ω of the group $\mathrm{U}_n \times \mathrm{U}_n$ on space $\mathcal{H}_\omega = \mathrm{Sym}(M_{n,n}(\mathbb{C}))$ of the symmetric tensors of $M_{n,n}(\mathbb{C})$.
Taking the trace of of $\omega(g,g')$, one obtains the character formula

$$\Theta_\omega(g,g') = \sum_{\Pi} \Theta_\Pi(g)\Theta_{\Pi'}(g') \qquad (\Pi = \Pi' \in \widehat{\mathrm{U}}_n)\,.$$

Hence one deduces the decomposition

$$\mathcal{H}_\omega = \bigoplus_{\Pi} \mathcal{H}_\Pi \otimes \mathcal{H}_{\Pi'}\,.$$

We get a correspondence of representations $\Pi \leftrightarrow \Pi'$ and a character formula

$$\Theta_\Pi(g) = \int_{\mathrm{U}_n} \Theta_\omega(g,g')\Theta_{\Pi'}(g'^{-1})\,dg\,.$$

Gaussians and Weil factors on a field

$\mathbb{F} = \mathbb{R}$ or a p-adic field (finite commutative extension of $\mathbb{Q}_p$), $p \neq 2$;
dx the Haar measure on $\mathbb{F}$ normalized so that the volume of the closed unit ball is 1.
If $\mathbb{F} = \mathbb{R}$, then choose $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, and define

$$\begin{aligned} \gamma(a) &= \lim_{b\to 0+} \int_{\mathbb{R}} \chi(\frac{1}{2}(a+ib)x^2)\, dx\,, \\ &= |a|^{-\frac{1}{2}} \gamma_W(a)\,, \quad \gamma_W(a) = e^{\frac{\pi i}{4}\mathrm{sgn}(a)} \qquad (a \in \mathbb{R} \setminus \{0\})\,. \end{aligned}$$

If $\mathbb{F} \neq \mathbb{R}$, then choose a unitary character $\chi : \mathbb{F} \to \mathbb{C}^\times$ of the additive group $\mathbb{F}$, and define

$$\begin{aligned} \gamma(a) &= \lim_{r\to\infty} \int_{x\in\mathbb{F}, |x|<r} \chi(\frac{1}{2}(a)x^2)\, dx\,, \\ &= |a|^{-\frac{1}{2}} \gamma_W(a)\,, \quad \gamma_W(a)^8 = 1 \qquad (a \in \mathbb{F} \setminus \{0\})\,. \end{aligned}$$

γ_W is the Weil factor.

Gaussians and Weil factors on a vector space

U finite dimensional vector space over $\mathbb{F}$ with Haar measure μ_{U};
q a nondegenerate quadratic form on U.
If $\mathbb{F} = \mathbb{R}$, then define

$$\begin{aligned} \gamma(q) &= \lim_{p \to 0} \int_{\mathsf{U}} \chi(\frac{1}{2}(q + ip)(u))\, d\mu_{\mathsf{U}}(u)\,, \\ \gamma_W(q) &= \frac{\gamma(q)}{|\gamma(q)|} = \chi(\frac{1}{4}\mathrm{sgn}(q))\,. \end{aligned}$$

If $\mathbb{F} \neq \mathbb{R}$, then define

$$\begin{aligned} \gamma(q) &= \lim_{r \to \infty} \int_{u \in \mathsf{U}, |u| < r} \chi(\frac{1}{2}q(u))\, d\mu_{\mathsf{U}}(u)\,, \\ \gamma_W(q) &= \frac{\gamma(q)}{|\gamma(q)|}\,, \quad \gamma_W(a)^8 = 1\,. \end{aligned}$$

Determinants

$(\mathsf{W}, \langle\cdot,\cdot\rangle)$; $\mathrm{Sp} \ni g$.
If $\mathbb{F} = \mathbb{R}$, pick $J \in \mathfrak{sp}$, $J^2 = -I$, $B(\cdot\cdot) = \langle J\cdot, \cdot\rangle > 0$. Define

$$\color{red}{\det}(g-1 : \mathsf{W}/\,\mathrm{Ker}\,(g-1) \to (g-1)\mathsf{W}) = \det(\langle (g-1)w_i, w_j\rangle_{1\le i,j\le m}),$$

where $w_1, \ldots, w_m$ is any B-orthonormal basis of $\mathrm{Ker}\,(g-1)^{\perp_B} \subseteq \mathsf{W}$.
If $\mathbb{F} \neq \mathbb{R}$, fix a lattice $\mathcal{L} \subseteq \mathsf{W}$ and the corresponding norm

$$N_{\mathcal{L}}(w) = \inf\{|a|^{-1} :\ a \in \mathbb{F}^\times,\ aw \in \mathcal{L}\} \qquad (w \in \mathsf{W}).$$

Let $\mathfrak{o}_{\mathbb{F}} \subseteq \mathbb{F}$ denote the ring of integers. Define

$$\color{red}{\det}(g-1 : \mathsf{W}/\,\mathrm{Ker}\,(g-1) \to (g-1)\mathsf{W}) \\ = \det(\langle (g-1)w_i, w_j\rangle_{1\le i,j\le m})(\mathfrak{o}_{\mathbb{F}}^\times)^2 \in \mathbb{F}^\times/(\mathfrak{o}_{\mathbb{F}}^\times)^2,$$

where $w_1, \ldots, w_m$ are such that the spaces

$$\mathbb{F}w_1, \ldots, \mathbb{F}w_m, \mathrm{Ker}(g-1)$$

span W and are $N_{\mathcal{L}}$-orthogonal, i.e.

$$N_{\mathcal{L}}(a_1w_1 + \cdots + a_mw_m + w) = \max\{N_{\mathcal{L}}(a_1w_1), \ldots, N_{\mathcal{L}}(a_mw_m), N_{\mathcal{L}}(w)\}.$$

The Metaplectic Group [A.-M. Aubert and T.P., 2014]

For $g, g_1, g_2 \in \mathrm{Sp}$, let

$$\Theta^2(g) = \gamma(1)^{2\dim (g-1)\mathsf{W}-2}\left[\gamma(\det(g-1 : \mathsf{W}/\,\mathrm{Ker}(g-1) \to (g-1)\mathsf{W}))\right]^2$$

$$C(g_1, g_2) = \sqrt{\left|\frac{\Theta^2(g_1 g_2)}{\Theta^2(g_1)\Theta^2(g_2)}\right|}\,\gamma_W(q_{g_1,g_2}),$$

where

$$q_{g_1,g_2}(u', u'') = \frac{1}{2}\langle (g_1+1)(g_1-1)^{-1}u', u''\rangle + \frac{1}{2}\langle (g_2+1)(g_2-1)^{-1}u', u''\rangle \quad (u', u'' \in (g_1-1)\mathsf{W} \cap (g_2-1)\mathsf{W}).$$

The Metaplectic Group

$$\widetilde{\mathrm{Sp}} = \left\{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C},\ \ \xi^2 = \Theta^2(g)\right\}$$

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1\xi_2 C(g_1, g_2)).$$

Normalization of Haar measures on vector spaces

Let $\mathbb{F} = \mathbb{R}$.
For any subspace $\mathsf{U} \subseteq \mathsf{W}$ we normalize the Haar measure μ_{U} on U so that the volume of the unit cube with respect to form B is 1.
If $\mathsf{V} \subseteq \mathsf{U}$, then B induces a positive definite form on the quotient U/V and hence a normalized Haar measure $\mu_{\mathsf{U}/\mathsf{V}}$ so that the volume of the unit cube is 1.

Let $\mathbb{F} \neq \mathbb{R}$.
For any subspace $\mathsf{U} \subseteq \mathsf{W}$ we normalize the Haar measure μ_{U} on U so that the volume of the lattice $\mathcal{L} \cap \mathsf{U}$ is 1.
If $\mathsf{V} \subseteq \mathsf{U}$, then we normalized Haar measure $\mu_{\mathsf{U}/\mathsf{V}}$ so that the volume of the lattice $(\mathcal{L} \cap \mathsf{U} + \mathsf{V})/\mathsf{V}$ is 1.

The Weil Representation

$\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ a complete polarization.

$\mathrm{Op} : \mathcal{S}^*(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x, x')v(x')\, d\mu_{\mathrm{X}}(x').$$

Weyl transform $\mathcal{K} : \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathrm{X} \times \mathrm{X})$

$$\mathcal{K}(f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y)\chi\big(\tfrac{1}{2}\langle y, x + x'\rangle\big)\, d\mu_{\mathrm{Y}}(y).$$

An imaginary Gaussian on $(g - 1)\mathrm{W}$

$$\chi_{c(g)}(u) = \chi\big(\tfrac{1}{4}\langle \underbrace{(g + 1)(g - 1)^{-1}}_{c(g)} u, u\rangle\big) \qquad (u = (g - 1)w,\ \ w \in \mathrm{W}).$$

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathrm{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op} \circ \mathcal{K} \circ T(\tilde{g})\,.$$

$(\omega, \mathrm{L}^2(\mathrm{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ.

Dual Pairs

Subgroups $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathrm{W})$ acting reductively on W.

G' is the centralizer of G in Sp and G is the centralizer of G' in Sp.

The preimages $\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}' \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W})$ are also mutual centralizers in the metaplectic group.

For $\mathbb{F} = \mathbb{R}$:

G, G'	stable range
$\mathrm{GL}_n(\mathbb{D}),\ \mathrm{GL}_m(\mathbb{D})$	$n \geq 2m$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$	$p, q \geq 2n$
$\mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{O}_{p,q}$	$n \geq p + q$
$\mathrm{O}_p(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$	$p \geq 4n$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_p(\mathbb{C})$	$n \geq p$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$	$p, q \geq r + s$
$\mathrm{Sp}_{p,q},\ \mathrm{O}^*_{2n}$	$p, q \geq n$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$	$n \geq 2(p + q)$

Howe's Correspondence

[Howe, Waldspurger, Gan, Gan-Sun]

$\mathcal{R}(\widetilde{\mathrm{G}})$ equivalence classes of irreducible admissible representations.
$\mathcal{R}(\widetilde{\mathrm{G}},\omega) \subseteq \mathcal{R}(\widetilde{\mathrm{G}})$ representations realized as quotients of $\mathcal{S}(X)$ by closed $\widetilde{\mathrm{G}}$-invariant subspaces.

For $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}},\omega)$ let $N_\Pi \subseteq \mathcal{S}(X)$ be the intersection of all the closed $\widetilde{\mathrm{G}}$-invariant subspaces $N \subseteq \mathcal{S}(X)$ such that Π is equivalent to $\mathcal{S}(X)/N$.

Then $\mathcal{S}(X)/N_\Pi$ is a representation of both $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$. It is equivalent to

$$\Pi \otimes \Pi_1',$$

for some representation Π_1' of $\widetilde{\mathrm{G}}'$. The representation Π_1' of $\widetilde{\mathrm{G}}'$ has a unique irreducible quotient $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}',\omega)$.

Conversely, starting with $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}',\omega)$ and applying the above procedure with the roles of G and G' reversed, we arrive at the representation $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}},\omega)$.

The resulting bijection

$$\mathcal{R}(\widetilde{\mathrm{G}},\omega) \ni \Pi \longleftrightarrow \Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}',\omega)$$

is called Howe's correspondence, or local θ correspondence, for the pair G, G'.

The wave front set of a distribution for $\mathbb{F} = \mathbb{R}$

Let V be a finite dimensional vector space over $\mathbb{R}$. Recall the Fourier transform

$$\mathcal{F}(\phi)(v^*) = \int_{\mathsf{V}} \phi(v)\chi(-v^*(v))\, d\mu_{\mathsf{V}}(v) \qquad (\phi \in C_c^\infty(\mathsf{V}), v^* \in \mathsf{V}^*)\,.$$

The wave front set of a distribution u on V at a point $v \in V$, denoted $WF_v(u)$, is the complement of the set of all pairs (v, v^*), $v^* \in \mathsf{V}^*$, for which there is a test function $\phi \in C_c^\infty(V)$ with $\phi(v) \neq 0$ and an open cone $\Gamma \subseteq V^*$ containing v^* such that

$$|\mathcal{F}(\phi u)(v_1^*)| \leq C_N(1 + |v_1^*|)^{-N} \qquad (v_1^* \in \Gamma,\ N = 0, 1, 2, ...)\,.$$

This notion behaves well under diffeomorphisms. So for any distribution u on a manifold M, one may define $WF(u) \subseteq T^*M$ as the union of the wave front sets at the individual points.

The wave front set of a distribution for $\mathbb{F} \neq \mathbb{R}$ [D.B. Heifetz, 1985]

Let V be a finite dimensional vector space over $\mathbb{F}$. Recall the Fourier transform

$$\mathcal{F}(\phi)(v^*) = \int_{\mathrm{V}} \phi(v)\chi(-v^*(v))\, d\mu_{\mathrm{V}}(v) \qquad (\phi \in C_c^\infty(\mathrm{V}), v^* \in \mathrm{V}^*)\,.$$

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$$\mathrm{supp}(\mathcal{F}(\phi u)) \cap \Gamma \quad \text{is bounded}\,.$$

This notion behaves well under analytic isomorphisms. So for any distribution u on a manifold M, one may define $WF(u) \subseteq T^*M$ as the union of the wave front sets at the individual points.

The Cauchy Harish-Chandra Integral,

[T.P., 2000] and [H.Y. Loke and T.P. 2018]

Assume that the rank of G' is smaller or equal than the rank of G.

For a Cartan subgroup $\mathrm{H}' \subseteq \mathrm{G}'$ with split part A' let $\mathrm{A}'' \subseteq \mathrm{Sp}$ be the centralizer of A' and $\mathrm{A}''' \subseteq \mathrm{Sp}$ the centralizer of A''.
Let $d\dot{w}$ measure on $\mathrm{A}'''\backslash \mathsf{W}$ defined by

$$\int_{\mathsf{W}} \phi(w)\, d\mu_{\mathsf{W}}(w) = \int_{\mathrm{A}'''\backslash \mathsf{W}} \int_{\mathrm{A}'''} \phi(aw)\, da\, d\dot{w}\,.$$

Define

$$\mathrm{Chc}(f) = \int_{\mathrm{A}'''\backslash \mathsf{W}} \int_{\widetilde{\mathrm{A}''}} f(g) T(g)(w)\, dg\, d\dot{w}\,. \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}''})).$$

For any $h' \in \mathrm{H}'^{reg}$, the intersection of the wave front set of the distribution Chc with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}} \ni \widetilde{g} \longrightarrow \widetilde{h'}\widetilde{g} \in \widetilde{\mathrm{A}''}$$

is empty. Hence there is a unique restriction of the distribution Chc to $\widetilde{\mathrm{G}}$, denoted $\mathrm{Chc}_{\widetilde{h'}}$.

The distribution $\Theta'_{\Pi'}$

Recall the Weyl - Harish-Chandra integration formula

$$\int_{\widetilde{\mathrm{G}}'} \phi(g)\,dg = \sum_{\mathrm{H}'} c_{\mathrm{H}'} \int_{\widetilde{\mathrm{H}'^{reg}}} D(h) \int_{\widetilde{\mathrm{G}}'/\widetilde{\mathrm{H}}'} \phi(g\widetilde{h}g^{-1})\,d\dot{g}\,d\widetilde{h}\,.$$

Define

$$\Theta'_{\Pi'}(f) = C_{\Pi'} \sum c_{\mathrm{H}'} \int_{\widetilde{\mathrm{H}'^{reg}}} D(h)\, \Theta_{\Pi'}(\widetilde{h}^{-1}) \mathrm{Chc}_{\widetilde{h}}(f)\,d\widetilde{h}\,.$$

> Recall: for $(\mathrm{U}_n, \mathrm{U}_n)$ we had the character formula
>
> $\Theta_{\Pi}(g) = \int_{\mathrm{U}} \Theta_\omega(g, g')\Theta_{\Pi'}(g'^{-1})\,dg \overset{\mathrm{def}}{=} \Theta'_{\Pi'}(g)\,.$

For $\mathbb{F} = \mathbb{R}$, this is an invariant eigen-distribution on $\widetilde{\mathrm{G}}$ with the correct infinitesimal character.

For $\mathbb{F} \neq \mathbb{R}$, in the general case there are still some unsolved problems with the convergence of the integrals over some $\widetilde{\mathrm{H}'^{reg}}$.
These problems are solved in many important cases.

Pairs of type I in the stable range

The pair $(\mathrm{G}, \mathrm{G}')$ is of type I if it acts irreducibly on W and W is a single isotypic component under this action.
In this case, there is:

- a division algebra $\mathbb{D}$ with an involution over $\mathbb{F}$
- two vector spaces V and V′ with with non-degenerate Hermitian forms $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ of opposite type

such that

- $\mathsf{W} = \mathsf{V} \otimes_{\mathbb{F}} \mathsf{V}'$,
- G coincides with the isometry group of $(\mathsf{V}, (\cdot, \cdot))$,
- G' coincides with the isometry group of $(\mathsf{V}', (\cdot, \cdot)')$.

The pair $(\mathrm{G}, \mathrm{G}')$ is in the stable range with G' - the smaller member if the dimension of the maximal isotropic subspace of V is greater or equal to the dimension of V′.

The equality $\Theta'_{\Pi'} = \Theta_\Pi$

Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair of type I in the stable range with G' - the smaller member.

Assume that the representation Π' of $\widetilde{\mathrm{G}}'$ is unitary.

Theorem (T.P. 2018)

Let $\mathbb{F} = \mathbb{R}$. Then $\Theta'_{\Pi'} = \Theta_\Pi$.

Idea of the proof. We show that the two distributions are equal on a Zariski open subset $\widetilde{\mathrm{G}}'' \subseteq \widetilde{\mathrm{G}}$. Since both Θ_Π and $\Theta'_{\Pi'}$ is an invariant eigendistribution, Harish-Chandra Regularity Theorem implies that they are equal everywhere.

Theorem (H.Y. Loke and T.P., preprint)

Let $\mathbb{F} \neq \mathbb{R}$, then the integrals defining $\Theta'_{\Pi'}$ converge and $\Theta'_{\Pi'} = \Theta_\Pi$ on a Zariski open subset $\widetilde{\mathrm{G}}'' \subseteq \widetilde{\mathrm{G}}$.

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Thank you