The Capelli eigenvalue problem for Lie superalgebras

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Capelli Operators

- $\mathfrak{g}$: Reductive Lie algebra/Classical Lie superalgebra.
- V : irreducible finite dimensional $\mathfrak{g}$-module such that $\mathscr{P}(V)$ is completely reducible and multiplicity-free.

$$V \cong V_{\bar{0}} \oplus V_{\bar{1}} \quad \rightsquigarrow \quad \mathscr{P}(V) \cong \mathcal{S}(V^*) \cong \mathcal{S}(V_{\bar{0}}^*) \otimes \Lambda(V_{\bar{1}}^*).$$

$$\mathscr{P}(V) \cong \bigoplus_{\lambda \in \mathcal{E}_V} V_\lambda \quad , \quad \mathscr{D}(V) \cong \mathcal{S}(V) \cong \bigoplus_{\lambda \in \Omega} V_\lambda^*.$$

$$\begin{aligned} \mathscr{P}\mathscr{D}(V) &\cong \mathscr{P}(V) \otimes \mathscr{D}(V) \\ &\cong \bigoplus_{\lambda,\mu \in \Omega} V_\lambda \otimes V_\mu^* \quad \cong \bigoplus_{\lambda,\mu \in \Omega} \mathrm{Hom}(V_\mu, V_\lambda) \end{aligned}$$

$$\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda) := \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \\ \{0\} & \text{if } \lambda \neq \mu. \end{cases} \qquad D_\lambda \leftrightarrow 1 \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda)$$

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Example

- $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$, $V := \mathrm{Mat}_{n\times n}(\mathbb{C})$.

$$\mathscr{P}(V) \cong \bigoplus_{\ell(\lambda)\leqslant n} V_\lambda^* \otimes V_\lambda.$$

- For $\lambda = (1) = (1, 0, \ldots)$ we obtain

$$D_{(1)} = \sum_{1\leqslant i,j\leqslant n} x_{i,j} \frac{\partial}{\partial x_{i,j}} \quad \text{(degree operator)}.$$

- For $\lambda := (1^n)$ we obtain

$$D_{(1^n)} = \det(x_{i,j}) \det(\frac{\partial}{\partial x_{i,j}}) \quad \text{(Capelli operator)}.$$

- The basis $\{D_\lambda\}_{\lambda\in\mathcal{E}_V}$ for $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ is called the *Capelli basis*.

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- $D_\lambda : \mathscr{P}(V) \to \mathscr{P}(V)$ is a $\mathfrak{g}$-module homomorphism (D_λ is $\mathfrak{g}$-invariant).
- $\mathscr{P}(V)$ multiplicity-free $\Rightarrow$ $D_\lambda\big|_{V_\mu} = c_\lambda(\mu) I_{V_\mu}$ for all λ, μ.

Problem (Kostant): Compute $c_\lambda(\mu)$.

Example

- $\mathbb{F}$: real division algebra, $d := \dim_{\mathbb{R}}(\mathbb{F}) \in \{1, 2, 4\}$.
- $\mathfrak{g}_{\mathbb{R}} := \mathfrak{gl}_n(\mathbb{F})$, $V_{\mathbb{R}} := \mathrm{Herm}_{n\times n}(\mathbb{F})$, $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $V := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.
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The Capelli Eigenvalue Problem

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Symmetric functions

- $\Lambda_n := \mathbb{C}[x_1, \ldots, x_n]^{S_n}$, $\Lambda := \varprojlim_n \Lambda_n$.
- $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0 \longrightarrow x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.
- Monomial symmetric functions:

$$\mathbf{m}_\lambda := x^\lambda + \cdots$$

- Power symmetric functions:

$$\mathbf{p}_r = \sum_i x_i^r = \mathbf{m}_{(r)} \quad \text{and} \quad \mathbf{p}_\lambda = \mathbf{p}_{\lambda_1} \cdots \mathbf{p}_{\lambda_n}.$$

- Inner product: $\langle \mathbf{p}_\lambda, \mathbf{p}_\mu \rangle_\alpha := \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}$,

  where $z_\lambda := \prod_{r \geqslant 1} r^{\lambda'_r} \lambda'_r!$ for $\lambda' :=$ transpose of λ.
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- $\alpha = 2 \longrightarrow$ Zonal spherical functions.
- General α: Jack symmetric function $\mathbf{J}_\lambda(\cdot, \alpha)$.

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- Inner product: $\langle \mathbf{p}_\lambda, \mathbf{p}_\mu \rangle_\alpha := \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu},$

  where $z_\lambda := \prod_{r \geqslant 1} r^{\lambda'_r} \lambda'_r!$ for $\lambda' :=$ transpose of λ.
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- $\alpha = 2 \quad \longrightarrow \quad$ Zonal spherical functions.
- General α: Jack symmetric function $\mathbf{J}_\lambda(\cdot, \alpha)$.

Symmetric functions

- $\Lambda_n := \mathbb{C}[x_1, \ldots, x_n]^{S_n}$, $\Lambda := \varprojlim_n \Lambda_n$.
- $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0 \longrightarrow x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.
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The spectrum $\mathbf{c}_\lambda$

Sahi '94, Knop–Sahi '96, Okounkov-Olshanski '97, Biedenharn, Louck,...

Fix $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$.

(a) There exists a polynomial $\mathbf{J}^\star_\lambda \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ such that

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where $\boldsymbol{\rho} = \frac{d}{2}(n-1, n-3, \ldots, 3-n, 1-n)$.

(b) $\mathbf{J}^\star_\lambda$ is determined up to a scalar by the following conditions:

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(c) $\mathbf{J}^\star_\lambda(\cdot) = \mathbf{J}_\lambda(\cdot, \frac{2}{d}) +$ lower degree terms.

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Supermathematics

- $\mathbb{Z}/2$-graded vector spaces $V := V_{\bar{0}} \oplus V_{\bar{1}}$.
- $V \otimes W \to V \otimes V$, $v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$.
- $\mathfrak{gl}(V) := \mathrm{End}(V) \cong \mathrm{End}(V)_{\bar{0}} \oplus \mathrm{End}(V)_{\bar{1}}$.
- $V := \mathbb{C}^{m|n}$: $\mathfrak{gl}(\mathbb{C}^{m|n})$.
  - Cartan subalgebra $\mathfrak{h}$: diagonal matrices.
  - Standard basis of $\mathfrak{h}^*$: $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n$.
  - Finite dimensional highest weight modules:

$$\underline{\lambda} = \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \mu_j \delta_j$$

  where $\lambda_i - \lambda_{i+1}, \mu_j - \mu_{j+1} \in \{0, 1, 2, 3, \ldots\}$.
- Other examples: $\mathfrak{osp}(m|2n)$, $\mathfrak{p}(m)$, $\mathfrak{q}(n)$, $\mathfrak{f}(3|1)$, $\mathfrak{g}(2)$, ...

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- $J = J_{\bar{0}} \oplus J_{\bar{1}}$: Jordan superalgebra / $\mathbb{C}$:
  - $a \circ b := (-1)^{|a||b|} b \circ a$.
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TKK Lie superalgebra (the Kantor functor)

$$J \quad \longrightarrow \quad \mathfrak{g}_J := \mathrm{Kan}(J) \ \ \text{Lie superalgebra.}$$

$$\mathfrak{g}_J := \mathfrak{g}_J(-1) \oplus \mathfrak{g}_J(0) \oplus \mathfrak{g}_J(1)$$

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The Lie superbracket of $\mathfrak{g}_J$ is defined by the following relations.

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(ii) $[A, a](x) := A(a, x)$ for $A \in \mathfrak{g}_J(1)$, $a \in \mathfrak{g}_J(-1)$, and $x \in J$.
(iii) $[A, B](x, y) := A(B(x, y)) - (-1)^{|A||B|}B(A(x), y) - (-1)^{|A||B|+|x||y|}B(A(y), x)$ for $A \in \mathfrak{g}_J(0)$, $B \in \mathfrak{g}_J(1)$, and $x, y \in J$.

The TKK Construction

- $J = J_{\bar{0}} \oplus J_{\bar{1}}$: Jordan superalgebra / $\mathbb{C}$:
  - $a \circ b := (-1)^{|a||b|} b \circ a$.
  - $(-1)^{|a||c|}[L_{a\circ b}, L_c] + (-1)^{|b||a|}[L_{b\circ c}, L_a] + (-1)^{|c||b|}[L_{c\circ a}, L_b] = 0.$

$$L_x : J \to J \ , \ L_x(y) := xy.$$

TKK Lie superalgebra (the Kantor functor)

$$J \quad \longrightarrow \quad \mathfrak{g}_J := \mathrm{Kan}(J) \ \text{ Lie superalgebra.}$$

$$\mathfrak{g}_J := \mathfrak{g}_J(-1) \oplus \mathfrak{g}_J(0) \oplus \mathfrak{g}_J(1)$$

- $\mathfrak{g}_J(-1) := J, \quad \mathfrak{g}_J(0) := \mathrm{Span}_{\mathbb{C}}\{L_a, [L_a, L_b] \ : \ a, b \in J\} \subseteq \mathrm{End}_{\mathbb{C}}(J),$

$$\mathfrak{g}_J(1) := \mathrm{Span}_{\mathbb{C}}\{P, [L_a, P] \ : \ a \in J\} \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}^2(J), J),$$

where $P : \mathcal{S}^2(J) \to J$ is the map $P(x, y) := xy$, and

$$[L_a, P](x, y) := a(xy) - (ax)y - (-1)^{|x||y|}(ay)x.$$

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The TKK Construction

Assume J is unital.

- $\mathfrak{g}_J$ is simple if and only if J is simple.
- There is also an associated embedded $\mathfrak{sl}_2$, spanned by

$$e \in \mathfrak{g}_J(-1),\ \ h \in \mathfrak{g}_J(0),\ \ f \in \mathfrak{g}_J(-1).$$

- The homogeneous parts $\mathfrak{g}_J(t)$ are eigenspaces of $\mathrm{ad}_{-\frac{1}{2}h}$.

- We will work with a slight modification $\mathfrak{g}^\flat$ of $\mathfrak{g}_J$.

Unital simple Jordan superalgebras and the corresponding $\mathfrak{g}^\flat$ ($J_{\bar{1}} \neq \{0\}$)

J	$\mathfrak{g}^\flat$
$\mathfrak{gl}(m,n)_+$	$\mathfrak{gl}(2m\|2n)$
$\mathrm{osp}(n,2m)_+$	$\mathfrak{osp}(4n\|2m)$
$(m,2n)_+$	$\mathfrak{osp}(m+3\|2n)$
$\mathrm{D}_t,\ t \neq -1$	$D(2\|1,t)$
F	$F(3\|1)$
$\mathrm{p}(n)_+$	$\mathfrak{p}(2n)$
$\mathrm{q}(n)_+$	$\mathfrak{q}(2n)$
$\mathrm{JP}(0,n)$	$H(n+3)$

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The restricted roots Σ

$$\mathfrak{g} := \mathfrak{g}^\flat(0), \qquad V := \mathfrak{g}^\flat(1) \cong J, \qquad \mathfrak{k} := \mathrm{Stab}_{\mathfrak{g}}(e).$$

	$\mathfrak{g}^\flat$	$\mathfrak{g}$	$\mathfrak{k}$	V
I	$\mathfrak{gl}(2m\|2n)$	$\mathfrak{gl}(m\|n) \oplus \mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m\|n)$	$\mathbb{C}^{m\|n} \otimes (\mathbb{C}^{m\|n})^*$
II	$\mathfrak{osp}(4n\|2m)$	$\mathfrak{gl}(m\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathcal{S}^2(\mathbb{C}^{m\|2n})$
III	$\mathfrak{osp}(m+3\|2n)$	$\mathfrak{gosp}(m+1\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathbb{C}^{m+1\|2n}$
IV	$D(2\|1,t)$	$\mathfrak{gl}(1\|2)$	$\mathfrak{osp}(1\|2)$	$\mathbb{C}_t^{2\|2}$
V	$F(3\|1)$	$\mathfrak{gosp}(2\|4)$	$\mathfrak{osp}(1\|2) \oplus \mathfrak{osp}(1\|2)$	$\mathbb{C}^{6\|4}$
VI	$\mathfrak{p}(2n)$	$\mathfrak{gl}(n\|n)$	$\mathfrak{p}(n)$	$\Pi(\Lambda^2(\mathbb{C}^{n\|n}))$
VII	$\mathfrak{q}(2n)$	$\mathfrak{q}(n) \oplus \mathfrak{q}(n)$	$\mathfrak{q}(n)$	$(\mathbb{C}^{n\|n} \otimes (\mathbb{C}^{n\|n})^*)^{\Pi\otimes\Pi}$

- The symmetric pair $(\mathfrak{g}, \mathfrak{k})$ corresponds to an involution $\Theta : \mathfrak{g} \to \mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \Theta\big|_{\mathfrak{k}} = +1, \quad \Theta\big|_{\mathfrak{p}} = -1.$$

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$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}.$$

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	$\mathfrak{g}^\flat$	$\mathfrak{g}$	$\mathfrak{k}$	V
I	$\mathfrak{gl}(2m\|2n)$	$\mathfrak{gl}(m\|n) \oplus \mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m\|n)$	$\mathbb{C}^{m\|n} \otimes (\mathbb{C}^{m\|n})^*$
II	$\mathfrak{osp}(4n\|2m)$	$\mathfrak{gl}(m\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathcal{S}^2(\mathbb{C}^{m\|2n})$
III	$\mathfrak{osp}(m+3\|2n)$	$\mathfrak{gosp}(m+1\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathbb{C}^{m+1\|2n}$
IV	$D(2\|1,t)$	$\mathfrak{gl}(1\|2)$	$\mathfrak{osp}(1\|2)$	$\mathbb{C}_t^{2\|2}$
V	$F(3\|1)$	$\mathfrak{gosp}(2\|4)$	$\mathfrak{osp}(1\|2) \oplus \mathfrak{osp}(1\|2)$	$\mathbb{C}^{6\|4}$
VI	$\mathfrak{p}(2n)$	$\mathfrak{gl}(n\|n)$	$\mathfrak{p}(n)$	$\Pi(\Lambda^2(\mathbb{C}^{n\|n}))$
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Jordan superalgebras of types A and Q

- Type A – Σ is of type $A(r-1, s-1)$:
  $\Sigma = \Sigma_{\bar{0}} \sqcup \Sigma_{\bar{1}}$ where

$$\Sigma_{\bar{0}} := \left\{\underline{\varepsilon}_i - \underline{\varepsilon}_{i'}\right\}_{1\leqslant i\neq i'\leqslant r} \cup \left\{\underline{\delta}_j - \underline{\delta}_{j'}\right\}_{1\leqslant j\neq j'\leqslant s}$$

  and

$$\Sigma_{\bar{1}} \cup \left\{\pm\left(\underline{\varepsilon}_i - \underline{\delta}_j\right)\right\}_{1\leqslant i\leqslant r, 1\leqslant j\leqslant s},$$

  - $\mathfrak{g}^\flat$ has an even invariant form $\langle\cdot,\cdot\rangle_{\mathfrak{g}^\flat}$. Then $\langle\cdot,\cdot\rangle_{\mathfrak{g}^\flat}\big|_{\mathfrak{a}\times\mathfrak{a}}$ is non-deg., hence induces an isomorphism $\mathfrak{a} \cong \mathfrak{a}^*$ and a bilinear form $\langle\cdot,\cdot\rangle_J : \mathfrak{a}^* \times \mathfrak{a}^* \to \mathbb{C}$.
  - For each $\alpha \in \Sigma$, we define a multiplicity

$$\operatorname{mult}(\alpha) := -\frac{1}{2}\operatorname{sdim}(\mathfrak{g}_\alpha).$$

- Type Q – Σ is of type $Q(r)$:

$$\Sigma := \left\{\underline{\varepsilon}_i - \underline{\varepsilon}_{i'}\right\}_{1\leqslant i\neq i'\leqslant r}$$

  but all roots have graded dimension $(2|2)$.

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Sergeev–Veselov's Deformed root systems $\mathsf{A}_\kappa(r-1, s-1)$

If J is of type A, then there exists some κ such that Σ satisfies

$$\langle \underline{\varepsilon}_i, \underline{\varepsilon}_j \rangle_J = \delta_{i,j} \quad , \quad \langle \underline{\delta}_i, \underline{\delta}_j \rangle_J = \kappa \delta_{i,j},$$

and

$$\mathrm{mult}(\underline{\varepsilon}_i - \underline{\varepsilon}_j) = \kappa, \qquad \mathrm{mult}(\underline{\delta}_i - \underline{\delta}_j) = \kappa^{-1}, \qquad \mathrm{mult}(\pm(\underline{\varepsilon}_i - \underline{\delta}_j)) = 1.$$

Set $\theta_J := -\kappa$.

J	Remarks	r_J	s_J	θ_J	$\pm(\underline{\varepsilon}_i - \underline{\varepsilon}_j)$	$\pm(\underline{\varepsilon}_i - \underline{\delta}_j)$	$\pm(\underline{\delta}_i - \underline{\delta}_j)$
$gl(m,n)_+$	$m, n \geqslant 1$	m	n	1	$2\|0$	$0\|2$	$2\|0$
$osp(n,2m)_+$	$m, n \geqslant 1$	m	n	$\frac{1}{2}$	$1\|0$	$0\|2$	$4\|0$
$(m,2n)_+$	$m, n \geqslant 1$	2	0	$\frac{m-1}{2} - n$	$m-1\|2n$	$-$	$-$
D_t	$t \neq 0, -1$	1	1	$-\frac{1}{t}$	$-$	$0\|2$	$-$
F		2	1	$\frac{3}{2}$	$3\|0$	$0\|2$	$-$

J	Remarks	r_J
$p(n)_+$	$n \geqslant 2$	n
$q(n)_+$	$n \geqslant 2$	n

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$osp(n,2m)_+$	$m, n \geqslant 1$	m	n	$\frac{1}{2}$	$1 \vert 0$	$0 \vert 2$	$4 \vert 0$
$(m,2n)_+$	$m, n \geqslant 1$	2	0	$\frac{m-1}{2} - n$	$m-1 \vert 2n$	$-$	$-$
D_t	$t \neq 0, -1$	1	1	$-\frac{1}{t}$	$-$	$0 \vert 2$	$-$
F		2	1	$\frac{3}{2}$	$3 \vert 0$	$0 \vert 2$	$-$

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$gl(m,n)_+$	$m, n \geqslant 1$	m	n	1	$2\|0$	$0\|2$	$2\|0$
$osp(n,2m)_+$	$m, n \geqslant 1$	m	n	$\frac{1}{2}$	$1\|0$	$0\|2$	$4\|0$
$(m,2n)_+$	$m, n \geqslant 1$	2	0	$\frac{m-1}{2} - n$	$m-1\|2n$	–	–
D_t	$t \neq 0, -1$	1	1	$-\frac{1}{t}$	–	$0\|2$	–
F		2	1	$\frac{3}{2}$	$3\|0$	$0\|2$	–

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$p(n)_+$	$n \geqslant 2$	n
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Sergeev-Veselov polynomials

Fix $\theta \in \mathbb{C}$ (nonzero if $n > 0$).

$\Lambda^{\natural}_{m,n,\theta}$: $\mathbb{C}$-algebra of polynomials $f(x_1, \ldots, x_m, y, \ldots, y_n)$ which are

- separately symmetric in $x := (x_1, \ldots, x_m)$ and in $y := (y_1, \ldots, y_n)$.
- satisfy the relation

$$f\left(x + \tfrac{1}{2}\mathsf{e}_i, y - \tfrac{1}{2}\mathsf{e}_j\right) = f\left(x - \tfrac{1}{2}\mathsf{e}_i, y + \tfrac{1}{2}\mathsf{e}_j\right)$$

on every hyperplane $x_i + \theta y_j = 0$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

$\mathcal{H}(m,n)$: the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_{m+1} \leqslant n$.

For $\lambda \in \mathcal{H}(m,n)$, we set

$$\mathsf{p}_i(\lambda) := \lambda_i - \theta\left(i - \tfrac{1}{2}\right) - \tfrac{1}{2}(n - \theta m) \quad \text{and} \quad \mathsf{q}_j(\lambda) := \langle \lambda'_j - m \rangle - \theta^{-1}\left(j - \tfrac{1}{2}\right) + \tfrac{1}{2}\left(\theta^{-1} n + m\right),$$

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Sergeev-Veselov polynomials

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Sergeev-Veselov polynomials

Theorem (Sergeev–Veselov 2004)

Assume that $\theta \notin \mathcal{S}(m,n)$. Then for each $\lambda \in \mathcal{H}(m,n)$, there exists a unique polynomial $P^*_\lambda \in \Lambda^\natural_{m,n,\theta}$ such that

(i) $\deg(P^*_\lambda) \leqslant |\lambda|$.

(ii) $P^*_\lambda(\mathsf{p}(\mu), \mathsf{q}(\mu), \theta) = 0$ for all $\mu \in \mathcal{H}(m,n)$ such that $|\mu| \leqslant |\lambda|$ and $\mu \neq \lambda$.

(iii) $P^*_\lambda(\mathsf{p}(\lambda), \mathsf{q}(\lambda), \theta) = H_\theta(\lambda)$, where

$$H_\theta(\lambda) := \prod_{1 \leqslant i \leqslant \ell(\lambda)} \prod_{1 \leqslant j \leqslant \lambda_i} (\lambda_i - j + \theta(\lambda'_j - i) + 1).$$

Furthermore, the family of polynomials $\left(P^*_\lambda(x,y,\theta)\right)_{\lambda \in \mathcal{H}(m,n)}$ is a basis of $\Lambda^\natural_{m,n,\theta}$.

$$\mathcal{S}(m,n) := \begin{cases} \left\{-\frac{a}{b} \;:\; a,b \in \mathbb{Z},\; a \geqslant 1, \text{ and } 1 \leqslant b \leqslant m-1\right\} & \text{if } n = 0, \\ \left\{-\frac{a}{b} \;:\; a,b \in \mathbb{Z},\; 0 \leqslant a \leqslant n, \text{ and } b \geqslant 1\right\} & \text{if } m = 0, \\ \mathbb{Q}^{\leqslant 0} & \text{otherwise.} \end{cases}$$

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$\mathscr{P}(V)$ as a $\mathfrak{g}$-module – Type A

Theorem (Sahi-S.-Serganova 2018)

Assume that J is of type A. Then the following assertions hold.

(i) $\mathscr{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module if and only if $\theta_J \notin \mathcal{S}(r_J, s_J)$.

(ii) Whenever (i) holds, $\mathscr{P}(V)$ is a direct sum of irreducible $\mathfrak{g}$-modules whose highest weights are naturally parametrized by $\Omega := \mathcal{H}(r_J, s_J)$:

$$\mathscr{P}(V) \cong \bigoplus_{\lambda \in \Omega} V_\lambda$$

$$\lambda \mapsto \underline{\lambda} \in \mathfrak{h}^* \quad , \quad \mathfrak{a}_\Omega^* := \overline{\{\underline{\lambda} \,:\, \lambda \in \Omega\}}_{\text{Zariski}}$$

$$\tau_J : \mathfrak{a}_\Omega^* \to \mathbb{C}^{r_J + s_J}$$

(iii) Assume that (i) and hence (ii) hold. Then the eigenvalue of the Capelli operator D_μ acting on V_λ is equal to $P_\mu^*(\tau_J(\underline{\lambda}), \theta_J)$, where $\underline{\lambda}$ is the $\mathfrak{b}$-highest weight of μ and τ_J is an affine change of coordinates.

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$\mathscr{P}(V)$ as a $\mathfrak{g}$-module – Type A

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$\mathscr{P}(V)$ as a $\mathfrak{g}$-module – Type A

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Examples

$$\mathcal{H}(m,n) := \{\lambda \,:\, \lambda_{m+1} \leqslant n\}.$$

$J := gl(m,n)_+$

- $\mathfrak{g} := \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \;\; V := \mathbb{C}^{m|n} \otimes (\mathbb{C}^{m|n})^*$
- $\mathscr{P}(V) \cong \bigoplus_{\lambda \in \mathcal{H}(m,n)} V_\lambda^* \otimes V_\lambda$
- $\underline{\lambda} = \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \langle \lambda'_j - m \rangle \delta_j .$

$m = 2,\, n = 4 :$

$$\underline{\lambda} = 6\varepsilon_1 + 5\varepsilon_2 + 3\delta_1 + 2\delta_2 + 2\delta_3 + 0\delta_4 .$$

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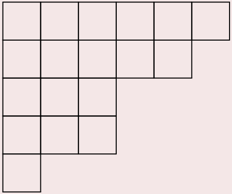

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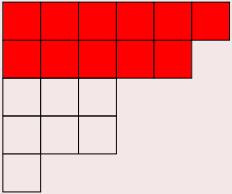

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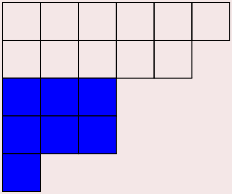

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$m = 2$, $n = 1$, $|\lambda| = 18$.

$$\underline{\lambda} = (8 + 18)\varepsilon_1 + 2(\delta_1 + \delta_2) + 18\zeta. = 26\varepsilon_1 + 2(\delta_1 + \delta_2) + 18\zeta.$$

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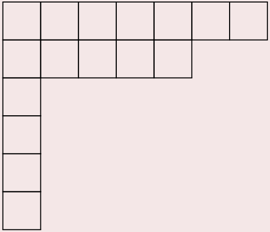

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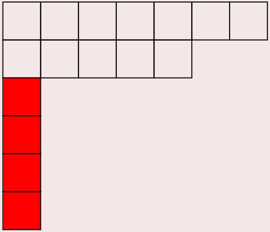

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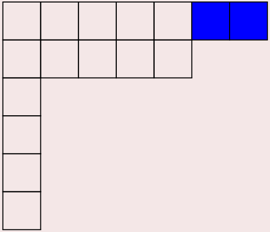

Okounkov-Ivanov polynomials

Γ_n: $\mathbb{C}$-algebra of polynomials symmetric in $x_1, \ldots, x_n$ such that $f(t, -t, x_3, \ldots, x_n)$ is independent of t.

$\mathcal{DP}(n)$: set of partitions of length at most n with distinct parts.

Theorem (Ivanov, 1999)

For every $\lambda \in \mathcal{DP}(n)$, there exists a unique polynomial $Q^*_\lambda \in \Gamma_n$ such that

(i) $\deg(Q^*_\lambda) \leqslant |\lambda|$.

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(iii) $Q^*_\lambda(\lambda) = H(\lambda)$, where $H(\lambda) := \lambda! \prod_{1 \leqslant i < j \leqslant \ell(\lambda)} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}$.

Furthermore, the family of polynomials $\left(Q^*_\lambda\right)_{\lambda \in \mathcal{DP}(n)}$ is a basis of Γ_n.

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Γ_n: $\mathbb{C}$-algebra of polynomials symmetric in $x_1, \ldots, x_n$ such that $f(t, -t, x_3, \ldots, x_n)$ is independent of t.

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Assume that J is of type Q. Then $\mathscr{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module. The highest weights of the irreducible summands of $\mathscr{P}(V)$ are parametrized by $\mathcal{DP}(n)$:

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$\mathscr{P}(V)$ as a $\mathfrak{g}$-module – Type Q

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Strategy of proof

One needs to check the following properties of $c_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$.

- Polynomiality (easy from Harish-Chandra homomorphism).
- Vanishing property (easy representation theoretic argument).
- Symmetry.

Harish-Chandra homomorphism

$$\begin{array}{ccc} \mathbf{Z}(\mathfrak{g}) & \longrightarrow & \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}} \\ {\scriptstyle \mathrm{HC}}\downarrow & & \downarrow \\ \mathscr{P}(\mathfrak{h}^*)^{\mathcal{W}} & \longrightarrow & \mathscr{P}(\mathfrak{a}_\Omega) \end{array}$$

$\mathfrak{a}_\Omega$: Zariski closure in $\mathfrak{h}^*$ of highest weights that occur in Ω.

Proposition (Sahi, S., Serganova 2018)

If $J \ncong F$, then the map

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