Study Group “Arithmetic Groups and Buildings”

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Introduction

Buildings are mathemtatical objects that have a rich geometric and combinatorial structure and they appear naturally in many mathematical subdisciplines, ranging from number theory over geometric group theory up to analysis. In this seminar we will assume a basic knowledge of the structure theory of buldings and will examine some more advanced topics. One particular focus will be on chamber complexes that can be obtained as quotients of buildings. Therefore we will in particular investigate the construction of arithemtic subgroups which will allow to construct discrete subgroups of the automorphism group of such buildings.

Part I: Chamber Complexes and their covers

Talk 1 : Chamber Complexes and their covers (Daniel Kahl, 25.04. / 02.05.)

Starting from the notion of a chamber complex, we discuss different types of covers. These are the well-known topological covers and the combinatorial covers introduced by Tits. We will be particularly interested in the criteria that a chamber complex must fulfill in order to ensure that the universal cover is a building.

Literature:

- Tits, 1981 *“A Local Approach to Buildings”*
- Ronan, 1980 *“Coverings and automorphisms of chamber systems”*
- Ronan, 2009 Lectures on Buildings Chapter 4
- Pasini, 2003 *“Extending locally truncated chamber systems by sheaves”* Section 2

Talk 2 : Coxeter groups and lattices (Paul Schneider, 23.05.)

The aim of this talk is to present a method which allows a concrete constructiosn of lattices in $SO(p, 1)$ for $p \leq 9$. The method due to Vinberg, is an extension of Poincaré’s theorem and constructs some explicit polyhedral fundameltal domains for d $d \leq 9$. In this case, the group Γ is a Coxeter group. As a byproduct, we will obtain geometric proofs of some of the basic properties of Coxeter groups. Even though the geometric construction may seem less efficient than the arithmetic one, it is still an important tool.

Literature:

- Benoist et al., 2004 *“Five lectures on lattices in semisimple Lie groups”* Lecture 1

Talk 3 : Arithmetic construction of lattices (Lasse Wolf, 06.06.)

The aim of this talk is to give explicit constructions of lattices in the real Lie groups SL(d, R) and SO(p, q). These examples are particular cases of a general arithmetic construction of lattices in any semisimple group G, due to Borel and Harish-Chandra. In fact, Margulis showed that all "irreducible" lattices of G are obtained in this way when the real rank of G is at least 2.

Literature:

- Benoist et al., 2004 *"Five lectures on lattices in semisimple Lie groups"* Lecture 2

Talk 4 : Lattices and Kazhdan's Property (T) (Carlo Kaul, 13.6.)

The aim of this talk is to show how the properties of the unitary repre- sentations of a Lie group G have an influence on the algebraic structure of any lattice Γ of G. We will deal here with a property due to Kazhdan. Namely, using the decreasing properties of the coefficients of unitary representations of G, when G is simple of rank at least 2, we will show that the abelianization of Γ is finite. We will also see that these properties imply mixing properties for some non-relatively compact flows on G/Γ.

Literature:

- Benoist et al., 2004 *"Five lectures on lattices in semisimple Lie groups"* Lecture 3

Talk 5 : Boundaries and Lattices (Kai-Uwe Bux, 27.06.)

The aim of this lecture is to show how measurable Γ-equivariant maps between "boundaries" can be used to prove some algebraic properties for a lattice Γ in a higher rank simple Lie group G. We will prove a theorem of Margulis which says that Γ is almost simple, i.e. any normal subgroup of Γ is either finite or of finite index. Note that the same tool is at the heart of the proof of the Margulis superrigidity theorem, but we will not discuss this here.

Literature:

- Benoist et al., 2004 *"Five lectures on lattices in semisimple Lie groups"* Lecture 4

Talk 6 : Lattices in groups over local fields (Kahl/Hilgert/Weich, 04.07.)

In this talk we will explain that local fields allow us to understand a larger class of groups than arithmetic groups, the so-called S-arithmetic groups. These groups happen to be lattices in locally compact groups G which are products of real and p-adic Lie groups. Moreover, many theorems for lattices in real Lie groups can be extended to lattices in such groups G with a very similar proof. In fact, the main property of ℝ used in these proofs was "locally compact field" and not "archimedean field". Hence, in this talk we will be able to reuse many methods of the previouse talks. As a by-product of this point of view, we will construct cocompact lattices in SL(d, L), where L is a p-adic field, and we will see that when $d \geq 3$, such lattices have property T and are quasisimple.

Literature:

- Benoist et al., 2004 *"Five lectures on lattices in semisimple Lie groups"* Lecture 5

Talk 7 : Non-uniform lattices acting on trees (Carsten Peterson, 11.07.)

In this talk we will discuss what a cusp means in the context of (regular) graphs. In particular we will show how $\Gamma = \mathrm{SL}(2, \mathbb{F}_q[t^{-1}])$ is a non-uniform lattice in $\mathrm{SL}(2, \mathbb{F}_q((t)))$ and what the associated quotient of the Bruhat-Tits tree by Γ looks like. We also hope to discuss Nagao's theorem about the structure of Γ, the more general method of constructing lattices in $\mathrm{SL}(2, \mathbb{F}_q((t)))$ from smooth projective curves over $\mathbb{F}_q$ (with Γ being the example when the underlying curve is $\mathbb{P}^1(\mathbb{F}_q)$), as well as how the Laplacian/adjacency operator is defined on such cusped graphs.

Literature:

- Serre, 1977 Trees
- Efrat, 1991 *"Automorphic spectra on the tree of* PGL_2*"*

References

[Ben+04] Yves Benoist et al. "Five lectures on lattices in semisimple Lie groups". In: *Géométries à courbure négative ou nulle, groupes discrets et rigidités* 18 (2004), pp. 117–176.

[Efr91] I. Efrat. "Automorphic spectra on the tree of PGL_2". In: *L'Enseignement Mathematique* 37 (1991).

[Pas03] A. Pasini. "Extending locally truncated chamber systems by sheaves". In: *Adv. Geom.* (2003).

[Ron09] Mark A. Ronan. *Lectures on Buildings*. University of Chicago Press, 2009.

[Ron80] Mark A. Ronan. "Coverings and automorphisms of chamber systems". In: *Eur. J. Comb.* (1980).

[Ser77] J. P. Serre. *Trees*. Springer Berlin, Heidelberg, 1977.

[Tit81] J. Tits. "A Local Approach to Buildings". In: *The Geometric Vein*. Springer New York, 1981.