Variations on character varieties

Joint study group of MPI Leipzig, Leipzig University, and Paderborn University

Organisers: Jacques Audibert, Benjamin Delarue, Guenda Palmirotta, Tobias Weich

---

In this study group we give an overview of various definitions of character varieties, the topologies on them, and the most important subspaces. We pursue two main threads. The first and the bulk of the programme, is a detailed study of compactification techniques: the classical Morgan–Shalen compactification [MS84, Ota15] and the more recent real spectrum compactification [Bru88, BIPP21], together with their interpretations as group actions on trees and buildings. The second are applications to spectral geometry: representations of free groups in $\mathrm{SL}(2, \mathbb{R})$ and their relation to resonance phenomena on Schottky surfaces [LMPT24, Wei15]. We also discuss quasi-Fuchsian representations and Thurston's hyperbolization theorem for fibered 3-manifolds [Ota01].

Programme

Block I: Basics

Talk 1 (Oct. 23, Jacques Audibert): Representation varieties

We introduce the representation variety $\mathrm{Hom}(\Gamma, G)$ for a finitely generated group Γ and a topological group G [LM85, Mar25], define the compact-open topology on $\mathrm{Hom}(\Gamma, G)$, and discuss its basic properties. As motivation, we sketch an argument showing that – provided $\mathrm{Hom}(\Gamma, G)$ is irreducible – the set of representations mapping some non-trivial element of Γ to the identity has codimension at least 1, so its complement is open and dense, and in particular contains injective representations.

Talk 2 (Oct. 30, Benjamin Delarue): Character varieties

We study the action of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\Gamma)$ on $\mathrm{Hom}(\Gamma, G)$ [Mar25, Sik12]. Since representations are often only well-behaved up to post-conjugation, the main object of interest is the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. We discuss the role of the outer automorphism group $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ acting on the character variety, together with the analytic and algebraic structures preserved by these actions.

Talk 3 (Nov. 13, Jacques Audibert): Character varieties of surface groups

A surface group is the fundamental group of a closed orientable surface of genus $g \geq 2$. This talk focuses on the connected components of the representation variety $\mathrm{Hom}(\pi_1(S), G)$ and of the character variety $\mathfrak{X}(\pi_1(S), G)$ [Li93, Mar25]. We recall that if G is a connected algebraic group, then $\mathrm{Hom}(\pi_1(S), G)$ has finitely many connected components.

Outlook: Application — Spectral Analysis

Talk 4 (Nov. 27, Tobias Weich): Schottky surfaces and spectral analysis

We present an application of character varieties to spectral geometry. The motivating objects are Schottky surfaces – convex co-compact hyperbolic surfaces whose fundamental group is a free group F_2 – and their spectral invariants: Laplace eigenvalues, Ruelle resonances, and quantum resonances. We discuss the character variety of F_2 in $\mathrm{SL}(2, \mathbb{R})$ with particular attention to the discrete and faithful representations. The main result is a theorem from [Wei15] establishing that, for 3-funneled Schottky surfaces in a geometric limit of large funnel lengths, quantum resonances in a bounded domain equidistribute along explicit lines determined by zeros of a polynomial depending only on the ratio of the funnel lengths. We also mention connections to recent independent work by Li–Matheus–Pan–Tao [LMPT24] and Talbott [Tal25], who obtain substantial generalizations of this result via Berkovich geometry and the Ihara zeta function of an associated graph.

Block II: Compactification

Morgan–Shalen compactification

Talk 5 (Dec. 4, Jacques Audibert): The construction of Morgan–Shalen

The goal of the Morgan–Shalen compactification [MS84, Ota15] is to embed the character variety $\mathfrak{X}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ as a dense subset of a compact Hausdorff space. The construction maps each representation ρ to the tuple of values $(\log(|\mathrm{Tr}(\rho(\gamma_i))| + 2))$ for finitely many generators $\gamma_i \in \Gamma$, and then projectivises. A key lemma establishes that finitely many elements of Γ suffice to determine all other traces via polynomial relations, so the image is relatively compact. We define the resulting compactification $\mathfrak{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \cup \partial_\infty$ and outline its properties.

Talk 6 (Dec. 11, Carsten Peterson): Morgan–Shalen compactification II: valuations and Λ-trees

Starting from an affine algebraic variety X defined over a countable subfield $k \subset \mathbb{C}$, this talk explains how a valuation on the function field of X gives rise to a Λ-tree on which Γ acts [MS84, Ota15]. We make the connection to Thurston's compactification of Teichmüller space explicit: boundary points arising from this construction correspond to projective measured foliations.

Talk 7 (Dec. 18, Carsten Peterson): Morgan–Shalen compactification III: Λ-trees

We develop the theory of Λ-trees, where Λ is a totally ordered abelian group [MS84]. A Λ-metric space is a set equipped with a Λ-valued distance satisfying the usual axioms; the group Λ itself is the prototypical example with $d(x, y) = |x - y|$. A Λ-tree is a Λ-metric space in which any two points are the endpoints of a unique closed segment, and the intersection of two segments sharing a common endpoint is again a segment.

Asymptotic cones

Talk 8 (Jan. 22, Xenia Flamm): Asymptotic cones and degenerations I

We introduce ultrafilters, and use them to define asymptotic cones and ultralimits of metric spaces. We give several examples, and discuss dependencies on the choice of ultrafilter, and observation points.

Talk 9 (Jan. 29, Xenia Flamm): Asymptotic cones and degenerations II

In this talk, we apply asymptotic cones to degenerations of representations. We prove how to obtain, using a suitable scaling sequence, a limiting action on the asymptotic cone of hyperbolic space [Pau88].

Real spectrum compactification

Talk 10 (Feb. 5, Jacques Audibert): Real spectrum: algebraic preliminaries

This talk covers the algebraic foundations of the real spectrum compactification [Bru88]. The central objects are orderings on a field F: a total order compatible with the field operations, determined entirely by its cone of non-negative elements. We introduce the basic theory of orderings on commutative rings and the notion of a prime cone.

Talk 11 (Feb. 12, Xenia Flamm): Real spectrum: topological properties

The real spectrum $\mathrm{Spec}^{\mathrm{R}}(X)$ of a real algebraic variety X with coordinate ring $A = \mathbb{R}[X_1, \ldots, X_n]/I(X)$ consists of pairs $(p, \leq)$ where p is a prime ideal of A and $\leq$ is a total order on $\mathrm{Frac}(A/p)$ [Bru88, BIPP21]. It is topologised by the subbasis $U(a) = \{(p, \leq) : [a]_p > 0\}$. The space is compact but not Hausdorff; we explore its constructible subsets and further topological properties.

Talk 12 (Mar. 5, Xenia Flamm): Real spectrum: compactification of character varieties

We apply the real spectrum to character varieties [BIPP21]. Via the coordinate ring of $\mathfrak{X}(\Gamma, G)$, one forms $\mathrm{Spec}^{\mathrm{R}}(\mathfrak{X}(\Gamma, G))$ and identifies the resulting compact space as a compactification of the classical character variety: a representation ρ embeds as the pair $(\ker(\mathrm{ev}_\rho), \leq_{\mathrm{std}})$.

Talk 13 (Mar. 12, Jacques Audibert): Real spectrum: actions on buildings and asymptotic cones

Boundary points of the real spectrum compactification give rise to group actions on buildings [BIPP21]. Given Γ, a semisimple reductive algebraic group G, and an ordered field K arising from a boundary point, an order-compatible valuation on K yields an action of Γ on the Bruhat–Tits building of G over the residue field. We also discuss the role of asymptotic cones in this picture.

Block III: Quasi-Fuchsian representations

Talk 14 (March 19, 2026, Samuel): Quasi-Fuchsian groups I: Basics

Definition of quasi-Fuchsian groups as discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ whose limit set in the Riemann sphere is a Jordan curve (i.e., homeomorphic to a circle and dividing the sphere into two open discs). Statement of Ahlfors' finiteness theorem and formulation of the Bers map into the product of two copies of Teichmüller space.

Talk 15 (March 26, 2026, Benjamin): Quasi-Fuchsian groups II: quasi-conformal maps

Characterization of quasi-Fuchsian groups as quasi-conformal deformations of Fuchsian groups. We begin with a review of quasi-conformal maps between Riemann surfaces [Can] and define quasi-conformal homeomorphisms using quadrilaterals – embedded closed disks with four marked boundary points. Then we compare this general definition with the more well-known one for diffeomorphisms. We also prepare the proof of Bers's simultaneous uniformization theorem by recalling the measurable Riemann mapping theorem.

Talk 16 (April 16, 2026, Benjamin): Quasi-Fuchsian groups III: quasi-conformal deformations

Proof of the characterization of quasi-Fuchsian groups as quasi-conformal deformations of Fuchsian groups in the cocompact case, based on the reference [Mar74]. We recall several helpful and deep topological statements on three-manufifolds due to Waldhausen and others and construct a quasi-conformal deformation from a given quasi-Fuchsian group through a careful cutting and gluing process.

Talk 17 (April 23, 2026, Samuel): Quasi-Fuchsian space

We recall the definitions of Fuchsian and quasi-Fuchsian groups [Sul85, Can]. A Fuchsian group is a torsion-free discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$, giving a hyperbolic surface $\mathbb{H}^2/\Gamma$. A quasi-Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ obtained from a Fuchsian group by a quasi-conformal deformation of the sphere S^2. For $\Gamma = \pi_1(S)$ with S a surface of genus $g \geq 2$, the quasi-Fuchsian space is defined as an open subset of the character variety $\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbb{C}))$.

Talk 18 (April 30, Jacques): The hyperbolization theorem for fibered 3-manifolds

We present Thurston's hyperbolization theorem (1986) for fibered 3-manifolds [Ota01]. Given a closed orientable surface S and a homeomorphism $\phi \in \mathrm{Homeo}(S)$, the mapping torus

$M_\phi = S \times [0,1]/(x,0) \sim (\phi(x),1)$ admits a complete hyperbolic metric if and only if ϕ is pseudo-Anosov, i.e. no power of ϕ is homotopic to the identity on any essential subsurface.

Block IV: Resonance chains

Talk 19 (Mai 21, 2026, Tobias Weich): Introduction to resonances on hyperbolic surfaces

Talk 20 (Mai 28, 2026, Tobias Weich): Resonance chains and spectral asymptotics

Talk 21 (DATE?, Martin Ulirsch): Berkovich Theory

Talk 22 (DATE?, Martin Ulirsch): Mumford Curves

References

[BIPP21] Marc Burger, Alessandra Iozzi, Anne Parreau, and Maria Beatrice Pozzetti, *The real spectrum compactification of character varieties: characterizations and applications*, C. R. Math. Acad. Sci. Paris **359** (2021), no. 4, 439–463.

[Bru88] G. W. Brumfiel, *The real spectrum compactification of Teichmüller space*, Geometry of group representations (Boulder, CO, 1987), Contemp. Math. **74**, Amer. Math. Soc., Providence, RI, 1988, pp. 51–75.

[Can] R. Canary, *IHP lectures on hyperbolic 3-manifolds*, lecture notes, available at `https://websites.umich.edu/~canary/IHPMay2025.pdf`.

[Li93] Jun Li, *The space of surface group representations*, Manuscr. Math. **78** (1993), no. 3, 223–243.

[LM85] Alexander Lubotzky and Andy R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), no. 336.

[LMPT24] Jialun Li, Carlos Matheus, Wenyu Pan, and Zhongkai Tao, *Selberg, Ihara and Berkovich*, Preprint, arXiv:2412.20754 [math.DS] (2024).

[Mar74] A. Marden, *The Geometry of Finitely Generated Kleinian Groups*, Ann. Math. (2) **99** (1974), 383–462.

[Mar25] A. Maret, *A note on character varieties*, lecture notes, available online, 2025.

[MS84] John W. Morgan and Peter B. Shalen, *Valuations, trees, and degenerations of hyperbolic structures. I*, Ann. Math. (2) **120** (1984), 401–476.

[Ota01] Jean-Pierre Otal, *The hyperbolization theorem for fibered 3-manifolds*, translated from the French by Leslie D. Kay, SMF/AMS Texts Monogr. **7**, Amer. Math. Soc., Providence, RI, 2001.

[Ota15] Jean-Pierre Otal, *Compactification of spaces of representations after Culler, Morgan and Shalen*, in: Berkovich spaces and applications, Lecture Notes in Math. **2119**, Springer, Cham, 2015, pp. 367–413.

[Pau88] Frédéric Paulin, *Topologie de Gromov équivariante, structures hyperboliques et arbres réels*, Invent. Math. 94, No. 1, 53–80 (1988; Zbl 0673.57034)

[Sik12] Adam S. Sikora, *Character varieties*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5173–5208.

[Sul85] Dennis Sullivan, *Quasiconformal homeomorphisms and dynamics. II: Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.

[Tal25] Henry Talbott, *Dynamical zeta functions and resonance chains for infinite-area hyperbolic surfaces with large funnel widths*, Preprint, arXiv:2505.09840 [math.SP] (2025).

[Wei15] Tobias Weich, *Resonance chains and geometric limits on Schottky surfaces*, Commun. Math. Phys. **337** (2015), no. 2, 727–765.