SPECTRAL DECOMPOSITION OF $L^2(\Gamma\backslash G)$

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1. Abstract

In the first series of talks, the proof of Segal's Plancherel theorem was sketched which states that for any locally compact, separable, unimodular, CCR group G, the (left and right) regular representation U of $G \times G$ on $L^2(G)$ disintegrates into a direct integral of irreducibles:

$$(U, L^2(G)) \;\cong\; \left(\int_{\widehat{G}}^{\oplus} \pi \mathbin{\widehat{\otimes}} \pi' \, \mathrm{d}\zeta(\pi),\; \int_{\widehat{G}}^{\oplus} \mathscr{H}_\pi \mathbin{\widehat{\otimes}} \mathscr{H}_\pi' \, \mathrm{d}\zeta(\pi) \right)$$

for the Plancherel measure ζ on the unitary dual $\widehat{G}$ of G equipped with its natural topology. Moreover, it was shown that this general statement translates for groups like $G = \mathbb{R}$ or $G = \mathbb{T}$ into well-known explicit statements from Fourier theory, using that their commutativity makes $\widehat{G}$ into a particularly simple object. For compact groups, one can at least simplify the direct integral into a completed direct sum to obtain the Peter–Weyl theorem as a special case. Moreover, in this case all irreducible unitary representations are shown to be finite-dimensional, so one can also avoid the need to complete the tensor product.

For non-commutative non-compact groups, things become much more complicated: For many such groups, e.g. already for $G = \mathrm{SL}_2(\mathbb{R})$, the spectrum will have both *continuous* and *discrete* parts and the occuring representations will be generically infinite-dimensional. Nonetheless, in this case, it is possible to classify the unitary dual $\widehat{G}$ and use this to obtain an explicit description of the Plancherel theorem of the following form: For all $f \in \mathrm{C}_c^\infty(\mathrm{SL}_2(\mathbb{R}))$, we have

$$f(e) = \frac{1}{2\pi} \left(\sum_{n=1}^{\infty} (n-1)\, \Theta_n(f) + \frac{1}{4i} \int_{i\mathbb{R}} (\Theta_{+,\lambda}(f) \tan(\pi\lambda/2) - \Theta_{-,\lambda}(f) \cot(\pi\lambda/2)) \, \mathrm{d}\lambda \right),$$

where $\Theta_n(f)$ is the global character of the discrete series representations D_n^+ and D_n^- and $\Theta_{\pm,\lambda}$ is the global character of the principal series representation. Here, we use that the global character of any $\pi \in \widehat{G}$, defined a priori as the trace of the trace class operator

$$\pi(f) := \int_G f(g)\pi(g) \, \mathrm{d}g$$

and thus as a distribution on G, is actually given by integration against a locally integrable function Θ_π. This is a consequence of Harish-Chandra's famous regularity theorem on invariant eigendistributions, since the conjugation action by G keeps the trace distribution invariant and the universal enveloping algebra acts by the infinitesimal character of π.

Interestingly, not all representations occuring in the unitary dual of $\mathrm{SL}_2(\mathbb{R})$ seem to contribute to the spectral decomposition. In fact, since G is semisimple, there is a concrete expectation: All the tempered unitary irreducible representations will contribute (principal and discrete series)), all the square-integrable unitary irreducible representations (discrete series) will occur discretely and any non-tempered unitary irreducible representation (complementary series and the trivial

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representation) will not show up at all in such disintegration formulae. For non-semisimple groups, this statement is no longer valid and it fails for $G = \mathbb{R}$ already.

For more general (even reductive or semisimple) groups, it is much harder to obtain such an explicit disintegration of the regular action on the space of L-functions. The issue already occurs in the desire to classify the unitary dual (the so-called *unitary dual problem*): Expect for some special cases in low rank, the concrete structure of the unitary dual of real semisimple Lie groups remains largely unknown. The key issue in attaining such a classification is that deciding unitarizability is hard: The *Langlands' classification* yields a complete description of all (tempered) irreducible representations of a given reductive group, and one can easily see which of these representations may be equipped with a non-trivial invariant sesquilinear form; however, it is hard to tell whether there is also such a positive-definite form.

Motivated e.g. from the desire to understand the spectral theory of locally symmetric spaces, one natural generalization is to study the right-regular representation on the space $L^2(\Gamma\backslash G(\mathbb{R}))$, where now G denotes a reductive group over $\mathbb{Q}$ and $\Gamma \subseteq G(\mathbb{Q})$ is a discrete subgroup of cofinite volume. Indeed, the non-cocompact cofinite volume case is most interesting, since then there will be both contributions from the discrete and continuous spectrum. One example of this is the case of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$, which specializes to the upper-half plane for $n = 2$ and whose volume has an interesting expression as follows:

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})) = \zeta(2) \cdot \zeta(3) \cdots \zeta(n).$$

This already suggests some connection to Number Theory. More generally, all congruence subgroups Γ in reductive groups G have this property and we will see that the decomposition of the spectrum will be according to so-called automorphic representations in this case. These automorphic representations, a priori of analytic flavour, admit both conjecturally and provably strong connections to Galois representations and motives in the realm of the Langlands program.

In the two talks, I will first explain the case of $G = \mathrm{SL}_2$, where we will see elliptic modular forms occuring in the spectral decomposition of $L^2(\Gamma\backslash G(\mathbb{R}))$, and use this example to motivate the general definition of an automorphic representation. Subsequently, we will shortly explore the landscape of the Langlands program.

2. Rank one automorphic forms

One driving motivation for the theory of automorphic forms is the rank 1 case. A central objective in the realm of this theory is to understand the spectral theory of generalized Laplace operators acting on the L^2-space of certain locally symmetric spaces. The constituents in their spectral decomposition will be so-called L^2-automorphic forms; in the rank 1 case, cusp forms and so-called Maass wave forms will exhaust the discrete part of this automorphic spectrum.

2.1. **Maass wave forms and modular forms as automorphic forms.** First, we recall the definition of a Maaß wave form:

Definition 2.1 (Maaß wave form)**.** A *Maaß (wave) form (of weight* 0 *and level* Γ*)* is a function $f : \mathbb{H} \to \mathbb{C}$ that is Γ-invariant, *of moderate growth at its cusps* and an eigenfunction of the hyperbolic Laplacian $\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

Here, we associate to any cusp c of Γ an element $\gamma_c \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma_c \cdot \infty = c$ and furthermore call a function $f : \Gamma\backslash G \to \mathbb{C}$ *of moderate growth at a cusp* c if $f_\gamma(z) = f(\gamma.z) = f(x + iy)$ is bounded by a polynomial as $y \to \infty$.

We call a Maaß wave form a *Maaß cusp form* if f is even uniformly bounded in y at any cusp, or equivalently, the following integral vanishes for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where we denote by h_γ the width of the cusp associated to γ:

$$\int_0^1 f_\gamma(z + h_\gamma t)\ \mathrm{d}t.$$

If the eigenvalue of f for the Laplace-Beltrami operator Δ is $(1 - s^2)/4$, we call f a Maaß wave form of parameter s. We will demonstrate later why this normalisation makes sense.

(Real-analytic) Eisenstein series constitute the prime example of Maaß wave forms:

Example 2.2. (Real-analytic Eisenstein series)
The function $E : \mathbb{H} \times \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\} \to \mathbb{C}$, defined as

$$E(z, s) = \frac{\pi^{-s}}{2}\Gamma(s) \sum_{(n,m)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\mathrm{Im}(z)^s}{|mz + n|^s},$$

is holomorphic and can be analytically continued to a meromorphic function of s on $\mathbb{C}$. By identifying $G/K = \mathbb{H}$, we get a Maaß form $E(\cdot, s)$ for any $s \in \mathbb{C}$ with respect to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, which we call a *real-analytic (or non-holomorphic) Eisenstein series*. They are not Maaß cusp forms.

We will denote the space of Maaß cusp forms of parameter s with $\mathcal{M}_s(\Gamma)$ and denote the space of holomorphic cusp forms of weight k with $S_k(\Gamma)$. We start by embedding the latter space.

For a cusp form $f \in S_k(\Gamma)$, we write

$$\phi_f : G \to \mathbb{C}, \quad g \mapsto j(g, i)^{-k} f(g.i),$$

where $j(g, z) = az + b$ fulfils the cocycle identity. As G acts transitively on $\mathbb{H}$, the map $\Phi : S_k(\Gamma) \to \mathrm{Maps}(G, \mathbb{C}), f \mapsto \phi_f$ is an injective homomorphism of vector spaces, so we strive to characterise its image. Before doing that, we want to show that the holomorphicity of f forces ϕ_f also to be an eigenfunction of some Laplacian, which later lets us see $S_k(\Gamma)$ and $\mathcal{M}_k(\Gamma)$ as objects of the same type. First, we define how a Laplacian on G (or one should rather say, on $\Gamma\backslash G$) should look like:

Construction 2.3. Let $\mathcal{Z}(\mathfrak{g}) := \mathcal{Z}(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$ be the *center of the universal enveloping algebra* of the complexification of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. By the Harish-Chandra isomorphism, this algebra consists of Weyl group invariant polynomials on the Cartan subalgebra. In our case, the Cartan is 1-dimensional and we have $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[\mathscr{C}]$ for the *Casimir element* $\mathscr{C}$.

The Casimir element of $\mathcal{Z}(\mathfrak{g})$ is concretely given as $\mathscr{C} = \frac{1}{2}(H^2 + T^2 - U^2)$ for the $_2$-triple

$$(H, T, U) = \left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$$

which is a self-dual basis for the trace form on $\mathfrak{g}$ (which is a multiple of the Killing form).

In general, we can identify the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the $C^\infty(G)$-algebra of G-invariant differential operators

$$\mathscr{D}(G)^G := \{D : C^\infty(G) \to C^\infty(G) \text{ linear} \mid \mathrm{supp}(Df) \subseteq \mathrm{supp}(f), L(g)D = DL(g)\ \forall g \in G\},$$

where L denotes the left-regular representation on $C^\infty(G)$ by the isomorphism

$$\mathrm{d}R : \mathcal{U}(\mathfrak{g}) \to \mathscr{D}(G)^G, \quad u \mapsto \mathrm{d}R(u),$$

where we denote by $\mathrm{d}R$ the extension of the map $\mathrm{d}R : \mathfrak{g} \to \mathrm{End}(C^\infty(G))$ (which exists uniquely by the universal property of the functor $\mathcal{U}$) and the latter map is concretely given as

$$(\mathrm{d}R(X)f)(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g \cdot \exp(tX)) = \mathrm{d}f(g) \circ \widetilde{X}_g =: \widetilde{X} f(g),$$

where $\widetilde{X}$ is the invariant vector field associated to X. Therefore, for any $u \in \mathcal{U}(\mathfrak{g})$, $\mathrm{d}R(u)$ and in particular $\mathrm{d}R(\mathscr{C})$ is an invariant differential operator.

With the help of the Iwasawa decomposition

$$G = ANK = \mathbb{R} \times \mathbb{R}_{>0} \times [0, 2\pi), g \mapsto (n_x, a_y, k_\vartheta) \mapsto (x(g), y(g), \vartheta(g)),$$

we can explicitly express this differential operator in coordinates:

$$\mathrm{d}R(\mathscr{C}) = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \, \frac{\partial^2}{\partial x \partial \vartheta},$$

where we suppressed the input. This is the form in which we will call the operator (or sometimes rather the minimal closed extension into an operator on $L^2(\Gamma \backslash G)$ of it) Δ_G.

Remark 2.4. Δ_G is a self-adjoint operator commuting with $R(g)$ for any $g \in G$ and therefore acts as a scalar on any irreducible subrepresentation of $(R, L^2(\Gamma \backslash G))$ by Schur's lemma. We will use this key fact to decompose this representation and connect its decomposition to (Maass) cusp forms and Eisenstein series later.

We know that every holomorphic function is harmonic, i.e. its real part and imaginary part lie in the kernel of the Laplacian on $\mathbb{R}^2$, and one can even characterise holomorphicity by requiring smoothness and the Cauchy-Riemann differential equations to hold. Thus, a characterisation of holomorphicity using our invariant differential operator Δ_G on G seems natural.

Lemma 2.5. For any holomorphic $f : \mathbb{H} \to \mathbb{C}$, we have that ϕ_f is an eigenfunction of the Laplace operator Δ_G with eigenvalue $\frac{k}{2}\left(1 - \frac{k}{2}\right)$.

It now makes sense to state the following proposition regarding the image of ϕ.

Proposition 2.6. The map Φ is an isomorphism of finite-dimensional Hilbert spaces of $S_k(\Gamma)$ onto its image given by the algebra of smooth functions $\phi : G \to \mathbb{C}$ satisfying the following conditions:

(1) $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \Gamma$ (automorphy),

(2) $\phi(g k_\vartheta) = \phi(g) e^{-ik\vartheta}$ for all $\vartheta \in [0, 2\pi)$ (or $k_\vartheta \in K$) (right-K-equivariance),

(3) $\Delta_G \phi = \frac{k}{2}\left(1 - \frac{k}{2}\right) \phi$ (eigenfunction property),

(4) ϕ is bounded at every cusp (regularity),

(5) ϕ is cuspidal at every cusp, i.e. for all $g \in G$ and $\sigma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\int_0^1 \phi\left(\sigma \begin{pmatrix} 1 & th_\sigma \\ 0 & 1 \end{pmatrix} g\right) \, \mathrm{d}t = 0,$$

where we denote by h_σ the width of the cusp $\sigma.\infty$ (cuspidality).

Actually, 5. can be omitted here, as it is implied by 4..

Idea of proof. For well-definedness, one has to do the calculations: 1. and 2. are clear from the definitions, 3. needs some work and uses the harmonicity of holomorphic functions, the fact that they fulfil the Cauchy-Riemann equations and then utilises the explicit parametrisation given as $\phi_f(g) = y(g)^{k/2} e^{-ik\vartheta(g)} f(x(g)+iy(g))$. After using this parametrisation, 4. is the classical result that $y^{k/2} f(x+iy)$ is bounded for every cusp form f and 5. follow from simple manipulations of this integral and the observation that the zeroth Fourier coefficient of an h-periodic function $f : \mathbb{H} \to \mathbb{C}$ can be computed by an integral of the form

$$\int_0^1 f(z+th)\ \mathrm{d}t$$

for any $z \in \mathbb{H}$.

The harder part is the surjectivity: For any ϕ in the algebra of the above functions, we want to define a $f_\phi \in S_k(\Gamma)$ that is a preimage of ϕ under Φ, i.e. $\Phi(f_\phi) = \phi$. We claim that the following definition works: Pick any $g_z \in G$ with $g_z(i) = z$ and set

$$f_\phi(z) = \phi(g_z) j(g_z, i)^k$$

for $z \in \mathbb{H}$. This is independent of the choice of g_z, as ϕ is right-K-equivariant for $K = \mathrm{Stab}(i)$ in a way that makes f_ϕ right-invariant for the action of K, such that the function on the right side factors through $G/K = \mathbb{H}$. It obviously fulfils $\Phi(f_\phi) = \phi$, so it only remains to show that $f_\phi \in S_k(\Gamma)$.

Here, automorphy and cuspidality are just a matter of reversing the explicit computations which served to show the well-definedness of the corestriction of Φ. Holomorphy, however, is harder and needs some structural-theoretic result about the $(\mathfrak{g}, K)$-module of K-finite vectors of discrete series representations; with the help of such a result and the explicit form of the action of the Casimir (or its incarnation as an invariant differential operator), we can show that f_ϕ is a linear combination of lowest weight vectors which are in turn (by definition) killed by the lowering operator $\mathrm{d}R(U)$ of $\mathfrak{g}$ which translates to the following differential operator in our explicit coordinates:

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

This, however, is exactly the Cauchy-Riemann differential operator. For details, consult Section 4 of Chapter 1 of [5].

We remark that this identification is even isometric:

Remark 2.7. Due to the Γ-invariance, boundedness and the fact that Γ is a lattice in G (i.e. $\Gamma\backslash G$ has finite volume), the above function space can be realised inside $L^2(\Gamma\backslash G)$. Thus, we can compare the L^2-inner product with the classically defined Petersson inner product on $S_k(\Gamma)$. Indeed, after normalisation of the Haar measure such that we have the explicit formula

$$\int_G \phi(g)\ \mathrm{d}g = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \phi(x, y, \vartheta)\ \frac{\mathrm{d}x\ \mathrm{d}y}{y^2}\ \mathrm{d}\vartheta$$

and with these inner products, the map Φ is an isometric isomorphism: Denoting by $\mathcal{F}$ a fundamental domain for $\Gamma\backslash G$, we have

$$\int_{\mathcal{F}} |f(z)|^2\ y^k\ \frac{\mathrm{d}x\ \mathrm{d}y}{y^2} = \int_{\Gamma\backslash G} |\phi_f(g)|^2\ \mathrm{d}g.$$

Analogously, for a Maaß cusp form $f \in \mathcal{M}_s(\Gamma)$ (which is actually a modular function for Γ), we write $\Psi : \mathcal{M}_s(\Gamma) \to \mathrm{Maps}(G, \mathbb{C}), f \mapsto \psi_f$, where

$$\psi_f : G \to \mathbb{C}, \quad g \mapsto f(g.i).$$

As G acts transitively on $\mathbb{H}$, ψ is an injective homomorphism of vector spaces, and we can again characterise its image.

Proposition 2.8. The map Ψ is an isomorphism of finite-dimensional Hilbert spaces of $\mathcal{M}_s(\Gamma)$ onto its image given by the algebra of smooth functions $\psi : G \to \mathbb{C}$ satisfying the following conditions:

(1) $\psi(\gamma g) = \psi(g)$ for all $\gamma \in \Gamma$ (automorphy),

(2) $\psi(g k_\vartheta) = \psi(g) e^{-ik\vartheta}$ for all $\vartheta \in [0, 2\pi)$ (or $k_\vartheta \in K$) (right-K-equivariance),

(3) $\Delta_G \psi = \frac{1-s^2}{4}\psi$ (eigenfunction property),

(4) ψ is bounded (and thus cuspidal) (regularity).

Idea of proof. The proof is along very similar lines to the proof of the preceding Proposition and even easier, as we do not have to translate any holomorphicity condition: The definition of a Mass cusp form is already stated as being an eigenfunction of the hyperbolic Laplace-Beltrami operator.

With the help of these two Proposition, we are now able to see Maaß cusp forms and modular cusp forms as two objects of the same kind. We will later define a broad class of objects, called automorphic forms, that encompass these two notions and also the notion of a modular form and general Maaß wave forms. Indeed, Eisenstein series, Maaß cusp forms and modular cusp forms essentially exhaust the space of automorphic forms in rank 1. We will continue this discussion in chapter 2, but first want to recall the general representation theory of G.

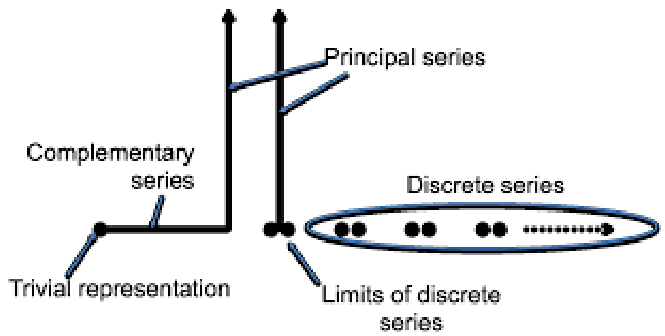
2.2. **Relation to the representation theory of G.** It is a general fact due to Von-Neumann and Mautner which can be found in Dixmier's book on $C^\star$-algebras [4] that, for any type 1 reductive (for example, semisimple) group, one can equip the topological space $\widehat{G}$ consisting of isomorphism classes of irreducible, unitary representations with a Borel (or even Radon) measure $\mu_{\widehat{G}}$, the so-called *Plancherel measure*, such that the right-regular representation on $L^2(G)$ decomposes according to a direct integral of Hilbert spaces indexed by $\widehat{G}$. In other words, there always exists a unitary equivalence

$$L^2(G) \cong \widehat{\bigoplus_{\pi \in \widehat{G}_{\mathrm{disc}}}} (V_\pi \otimes V_\pi^\vee) \oplus \int_{\widehat{G}_{\mathrm{cont}}}^{\oplus} (V_\pi \otimes V_\pi^\vee) \, \mathrm{d}\mu_{\widehat{G}}(\pi)$$

intertwining the right-regular representation R on $L^2(G)$ and $\int_{\widehat{G}}^{\oplus} (1 \otimes \pi^\vee) \, \mathrm{d}\mu_{\widehat{G}}(\pi)$.

Here, $\widehat{G}_{\mathrm{disc}}$ denotes the discrete part of the unitary dual of G consisting of subrepresentations of $(L^2(G), R)$ and $\widehat{G}_{\mathrm{cont}}$ denotes the continuous part of the unitary dual of G contributing to the orthogonal complement of the first space. We call this formula the *Plancherel formula* and it is a general fact that only so-called *tempered* representations occur in this decomposition, where we call a representation tempered if one (or equivalently, any) non-zero matrix coefficient is tempered.

Therefore, to decompose the right-regular representation on $L^2(\Gamma \backslash G)$, it will be crucial to first understand the general representation theory of G. We will therefore recall the classification of the unitary dual of G which was already known to Bargmann [1]. We start with a picture:



This does not only show a set of representatives for $\widehat{G}$, but also reveals topological information on the so-called Fell topology on this set. For example, the trivial representation being a *limit* of complementary series in this representation shows that G does not have Kazhdan's property T.

We start with the principal series, which is a very natural starting point due to the so-called Casselman subrepresentation theorem which states that every irreducible, unitary representation can be realised as a subrepresentation of a principal series.

Construction 2.9 (Principal series representations)**.** Let $G = KAN$ be the Iwasawa decomposition of G and denote $a^\lambda := \exp(\lambda(\log(a)))$ for the logarithm $\log : A \to \mathfrak{a} := \mathrm{Lie}(A)$ and any $\lambda \in \mathbb{C} \cong \mathfrak{a}_{\mathbb{C}}^\star$.

Furthermore, let $P = N_G(N)$ be a minimal-parabolic of G with Langlands decomposition $P = MAN$ and $\varepsilon := \pm 1 : M = \{\pm I_2\} \to \mathbb{C}$ be the character with $\pm 1(I_2) = 1$ and $\pm 1(-I_2) = \pm I_2$. For $\lambda \in \mathbb{C}$ define $\sigma_{\epsilon,\lambda} : P \to \mathbb{C}, p = man \mapsto \varepsilon(m)a^{\lambda+\rho}$ for $\rho \mapsto 1$ under the above isomorphism. (In general, ρ is the half-sum of positiv roots. This so-called *ρ-shift* is a subject of normalisation.)

Then, we define the so-called *normalised induction*

$$\mathrm{Ind}_P^G(\sigma_{\varepsilon,\lambda}) := \{f \in C^\infty(G) : f(pg) = \Delta_P^{1/2}(p)\sigma_{\varepsilon,\lambda}(p)f(g)\},$$

where Δ_P denotes the modular function of P. On this vector space, we have the right-regular action and an inner product given as

$$\langle f, h\rangle = \int_{G/P} f(gP)\overline{h}(gP)\, \mathrm{d}\mu(gP),$$

where μ is a measure invariant for the G-action on G/P (which exists due to our normalisation with respect to the modular function). Completing the space $\mathrm{Ind}_P^G(\sigma_{\epsilon,\lambda})$ equipped with this inner product yields a Hilbert space $H_{\varepsilon,\lambda}$ and the right-regular action on it extends to a Hilbert space representation $\pi_{\varepsilon,\lambda}$ of G on $H_{\varepsilon,\lambda}$.

We call this the *principial series representation of G with parameters ε and λ.*

We collect a few central facts about this family of representations:

Proposition 2.10. Let $\varepsilon = \pm 1$ and $\lambda \in \mathbb{C}$. Then:

(1) $\pi_{\varepsilon,\lambda} \cong \pi_{\varepsilon',\lambda'}$ if and only if $\varepsilon = \varepsilon'$ and $\lambda = \pm\lambda'$.

(2) $\pi_{\varepsilon,\lambda}$ is reducible if and only if:

(a) either $\varepsilon = 1$ and $\lambda \in 2\mathbb{Z}+1$

(b) or $\varepsilon = -1$ and $\lambda \in 2\mathbb{Z}$.

(3) $\pi_{\varepsilon,\lambda}$ is unitarisable if and only if:

(a) $\lambda \in i\mathbb{R}, \varepsilon = \pm 1$

(b) $\lambda \in (-1,1), \varepsilon = +1$.

In the second case, we call $\pi_{\varepsilon,\lambda}$ a *complementary series.*

Thus, a set of representatives of principal series in $\widehat{G}$ is given as $\{\pi_{\varepsilon,\lambda} : (\varepsilon,\lambda) \in X\}$ where

$$X = (i\mathbb{R}_{\geq 0} \times \{1\}) \cup (i\mathbb{R}_{>0} \times \{-1\}) \cup ((0,1) \times \{1\}).$$

The representation $\pi_{0,-1}$ is reducible and splits into two unitarizble subrepresentations $\pi_1^+ \oplus \pi_1^-$, the so-called Steinberg representations. These belong to the discrete series which we will discuss now. The idea is that, as mentioned, for $\varepsilon = 1$ and $\lambda \in 2\mathbb{Z}+1$ or $\varepsilon = -1$ and $\lambda \in 2\mathbb{Z}$ the principal series $\pi_{\varepsilon,\lambda}$ are reducible and we can thus look for a Jordan-Hölder decomposition series or, more specifically, irreducible subrepresentations. We will now show a possibility of constructing them:

Definition 2.11 (Discrete series)**.** Let $k \geq 2$. We know that the automorphism group of $\mathbb{H} = G/K$ is given as $\mathrm{PSL}_2(\mathbb{R})$, so we view an element $g \in G$ as an automorphism of the upper-half plane by linear fractional transformations. Defining $\pi_k^+(g^{-1})f(z) := (bz+d)^{-k} f(g.z)$ we get an action of G on $H_k^+ := \mathcal{O}(\mathbb{H}) \cap L^2(\mathbb{H}, \mu_k)$, where μ_k is the invariant measure on $\mathbb{H}$ given by $\mathrm{d}\mu_k(x+iy) = y^k \cdot \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$. In this manner, we get a Hilbert space representation π_k^+ on H_k^+ which we call the *discrete series of parameter k.*

By interchanging $\mathbb{H}$ with $\overline{\mathbb{H}}$ and y by $|y|$ in the above, we get a Hilbert space representation π_k^-.

We again record the central properties of this family of representations.

Proposition 2.12. (1) π_k^+ and π_k^- are all non-isomorphic to each other, as their $(\mathfrak{g}, K)$-modules are highest weight modules of different highest weights.

(2) Each $\pi_k^{\pm}$ is irreducible.

(3) $\pi_k^{\pm}$ is unitary for each $k \geq 2$.

Furthermore, we have the following growth behaviour:

Definition 2.13. We call a unitary irreducible representation $\pi : G \to \mathrm{GL}(H)$ *square-integrable* (resp. *tempered*) iff one (or equivalently each) non-zero matrix coefficient

$$\pi_{v,w}(g) := \langle \pi(g)v, w \rangle_H$$

is a function in $L^2(G)$ (resp. in $L^{2+\varepsilon}(G)$ for each $\varepsilon > 0$).

Example 2.14. (1) The discrete series representations are square-integrable.

(2) The non-complementary principal series representations are tempered.

(3) The complementary series representations (and the trivial representation) is non-tempered.

Thus, only the non-complementary principal series and discrete series representations contribute to the Plancherel formula of G.

2.3. **The spectral decomposition of $L^2(\Gamma\backslash G)$.** We continue along the lines of Remark 1.4: The minimal closed extension of the differential operator Δ_G into an operator on $L^2(\Gamma\backslash G)$ is self-adjoint and commutes with $R(g)$ for any $g \in G$ (since Δ_G is a central element of the space of invariant differential operators). By Schur's Lemma, it acts on every irreducible subrepresentation of $(R, L^2(\Gamma\backslash G)$ as a scalar.

Furthermore, since K is compact, the restriction of R to K is known to be completely reducible and since K is abelian, every irreducible subrepresentation of $(R|_K, L^2(\Gamma\backslash G))$ is one-dimensional, so K acts as a character on it and we can decompose this representation according to isotypical components indexed by $\widehat{K} \cong \mathbb{Z}$, i.e.:

$$(R|_K, L^2(\Gamma\backslash G)) = \widehat{\bigoplus_{\tau\in\widehat{K}}} L^2(\Gamma\backslash G)[\tau] = \widehat{\bigoplus_{k\in\mathbb{Z}}}\{\phi \in L^2(\Gamma\backslash G) : \phi(gk_\vartheta) = e^{-ik\vartheta}\phi(g)\}.$$

Thus, requiring right-K-equivariance (or right-K-finiteness) is harmless (up to completion). Therefore, decomposing $(R, L^2(\Gamma\backslash G))$ is essentially the same thing as constructing automorphic forms.

We remark that the above is exactly the motivation for introducing $(\mathfrak{g}, K)$-modules: The action of $\mathfrak{g}$ is through invariant differential operators and yields local information and the action of K is through characters and yields global information, and these actions together tell us everything about our representation.

The precise link between automorphic forms and representations will be demonstrated at the end of my second talk (in the third chapter of this manuscript).

As before, we can split up $L^2(\Gamma\backslash G)$ into a *discrete part* $L^2_d(\Gamma\backslash G)$ consisting of the direct sum of all irreducible subrepresentations of R inside $L^2(\Gamma\backslash G)$ and its orthogonal complement $L^2_c(\Gamma\backslash G)$, the so-called *continuous part* (which can be decomposed as a direct Hilbert integral, cf. the Plancherel formula).

However, we can now also split up $L^2(\Gamma\backslash G)$ into a *cuspidal part*, defined as the L^2-closure

$$L^2_0(\Gamma\backslash G) := \overline{\operatorname{span}\{\phi \in C_b(\Gamma\backslash G) : \int_{(\Gamma\cap N)\backslash N} \phi(\sigma n g)\ \mathrm{d}n = 0\ \forall g \in G, \sigma \in \mathrm{SL}_2(\mathbb{Z})\}},$$

which is well-defined since $\Gamma \cap N$ is a lattice inside N and exactly resembles the cuspidality condition from before, the constant functions $\mathbb{C}$ and the so-called *Eisenstein part* $L^2_E(\Gamma\backslash G)$ consisting of the orthogonal complement of $L^2_0(\Gamma\backslash G) \oplus \mathbb{C}$ inside $L^2(\Gamma\backslash G)$.

Therefore, we have two decompositions:

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G) = L^2_0(\Gamma\backslash G) \oplus \mathbb{C} \oplus L^2_E(\Gamma\backslash G).$$

The following Theorem gives a (rather) explicit spectral decomposition of Δ:

Theorem 2.15. We have $L^2_d(\Gamma\backslash G) = L^2_0(\Gamma\backslash G) \oplus \mathbb{C}$ (and thus $L^2_c(\Gamma\backslash G) = L^2_E(\Gamma\backslash G)$ and furthermore:

(1) $L^2_0(\Gamma\backslash G)$ is an invariant subspace of R, the restriction of R to $L^2_0(\Gamma\backslash G)$ is completely reducible and each irreducible component occurs only finitely many times.

(2) Consequently, $L^2_E(\Gamma\backslash G)$ is an invariant subspace of R, and the restriction of R to $L^2_E(\Gamma\backslash G)$ decomposes into a Hilbert space direct integral of spherical principal series

representations. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, this direct integral is of the form

$$R|_{L^2_E(\Gamma\backslash G)} = \int_{\mathbb{R}_+}^{\oplus} \pi_{1,\lambda} \, \mathrm{d}\lambda.$$

Idea of proof. We refer to Lang [6]: He proves that for cusp forms φ, $R(\varphi)$ which is defined by

$$R(\varphi)f(x) := \int_{\Gamma\backslash G} f(xg)\varphi(g) \, \mathrm{d}g,$$

is a compact operator on $L^2_0(\Gamma\backslash G)$ which then implies the completely reducibility and finite-multiplicity-property by general spectral-theoretic facts.

Furthermore, for the second part, he proves that any slowly increasing function ϕ on $\Gamma\backslash\mathbb{H}$ for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ that is orthogonal to all cusp forms can be expressed as an integral average

$$\phi(z) = \int_{i\mathbb{R}_+} \left\langle \phi, E\left(\cdot, \frac{1+\lambda}{2}\right)\right\rangle E\left(z, \frac{1+\lambda}{2}\right) \, \mathrm{d}\lambda$$

for the real-analytic Eisenstein series $E(z, s)$. This is analogous to Parseval's formula and provides a spectral decomposition for the orthogonal complement of cusp forms $L^2_E(\Gamma\backslash G)$.

Remark 2.16. For general congruence subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, the Eisenstein spectrum is a multi-dimensional direct integral, depending on the number of inequivalent cusps of Γ, and includes more general analytic Eisenstein series than the one defined for the full modular group.

For even more general Fuchsian groups of the first type, the Theorem is generally wrong: There are infinite-dimensional representations that can occur discretely but outside the space of cusp forms. These relate to residues at the poles of the Eisenstein series.

The idea of this theorem is that there are four types of automorphic forms for GL_2:

(1) The constant functions, which correspond to the trivial representation of G,

(2) the holomorphic modular forms, which occur as lowest weight vectors for the discrete series representation of G,

(3) Maaß cusp forms, which occur as K-spherical vectors for the principal series representation of G and

(4) (non-holomorphic) Eisenstein series.

The following so-called duality theorem due to Gelfand, Fomin, and Pyatetskii-Shapiro makes this idea more precise:

Theorem 2.17. The discrete series representation π_k^+ of G occurs in the restriction of R to $L^2_0(\Gamma\backslash G)$ with multiplicity $\dim(S_k(\Gamma))$. The spherical principal series representation $\pi_{1,s}$ occurs in the restriction of R to $L^2_0(\Gamma\backslash G)$ with multiplicity $\dim(\mathcal{M}_s(\Gamma))$.

Indeed, this is almost everything we know about the discrete part of $L^2(\Gamma\backslash G)$; except for the following nice result:

Lemma 2.18. In the subspace $L^2_0(\Gamma\backslash G)$, there occurs a subrepresentation corresponding to $\mathcal{M}_s(\Gamma)$ for infinitely many values of $s \in i\mathbb{R}_+$.

Idea of proof. The idea is that these subrepresentations are exactly those having a K-fixed vector, so applying the functor $(-)^K$ to

$$L_0^2(\Gamma\backslash G) = \widehat{\bigoplus_{j\in J}} \pi_j$$

(where J is the index set corresponding to the discrete spectrum), we only get type $\mathcal{M}_s(\Gamma)$ representations for different s on the right-hand side. As, however, every such representation occurs only finitely many times (at most once, actually), but the left-hand side is the space $L_0^2(\Gamma\backslash G/K) = L_0^2(\Gamma\backslash\mathbb{H})$ of modular cusp functions which can be shown to be infinite-dimensional, we get the result.

We close this chapter with a remark about the complementary series:

Remark 2.19. No complementary series representation arises in the spectral decomposition for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is an immediate consequence of the fact that the Selberg-1/4-conjecture is true for the full modular group.

Selberg's conjecture would therefore imply that the complementary series representations never arise in the above spectral decomposition.

We can summarise the contents of this chapter by giving a Plancherel formula for $\Gamma\backslash G$:

Corollary 2.20. For all $f \in L^2(\Gamma\backslash G)$, we have

$$f(z) = \sum_{j\in J} \langle f, \phi_j\rangle \phi_j(z) + \int_{i\mathbb{R}_+} \left\langle f, E\left(\cdot, \frac{1+\lambda}{2}\right)\right\rangle E\left(z, \frac{1+\lambda}{2}\right) \,\mathrm{d}\lambda,$$

where $(\phi_j)_j$ is a set of representatives for the automorphic cusp forms corresponding to the decomposition of the discrete spectrum into irreducible subrepresentations.

3. Onto higher rank

3.1. **Automorphic forms.** Finally, we wish to define the notion of an automorphic form and automorphic representation in a general context. The procedure of upgrading the SL_2-notion is not purely formal, as we shall need to reformulate some SL_2-specific conditions by using the general structure theory of reductive groups. Furthermore, it is convenient (and also customary) to formulate the right-K-finiteness in the language of idempotents. With regards to this, we state the following lemma which first needs a definition:

Definition 3.1 (standard idempotents)**.** Let G be the group of real points of some connected reductive group over $\mathbb{R}$ of Harish-Chandras class. Let K be a maximal compact subgroup of G. Let $\mathrm{d}k$ be the normalised Haar measure on K. Let $\nu \in \widehat{K}$ be a continuous, irreducible representation of K with degree $d(\nu)$ and character χ_ν. View $e_\nu := d(\nu)\chi_\nu \mathrm{d}k$ as a measure on G with support in K. We call an idempotent a standard idempotent if it is the finite sum of idempotents of the form e_ν.

We define $\mathcal{H}(G)$ to be the convolution algebra of distributions on G with support in K. We can therefore view each e_ν as an element of $\mathcal{H}(G)$.

Lemma 3.2. A function $f \in C^\infty(G)$ is right-K-finite if and only if there is a standard idempotent $\xi \in \mathcal{H}(G)$ such that $f \star \xi = f$.

Proof. This is a consequence of the Schur orthogonality relations: If f is of isotypic component μ under K, then $f \star e_\nu = \delta_{\mu,\nu} f$ such that $(-) \star e_\nu : C^\infty(G) \to C^\infty(G)[\nu]$ is a projector onto the ν-isotypical component such that $e_\mu \star e_\nu = 0$ for $\mu \neq \nu$.

Now, every K-finite function f is an element of the direct sum of a finite number of isotypical components such that we can decompose f into its isotypic components $f = \sum_\nu f_\nu$ with $f_\nu \in C^\infty(G)[\nu]$ and define

$$\xi := \sum_{\nu : f \in C^\infty(G)[\nu]} e_\nu$$

to see that $f \star \xi = f$ by bilinearity of $(-) \star (-)$.

The reason why we want to use the language of idempotents instead of just requiring right-K-finiteness is that we also account for the isotypical components which are reached by translation by K. Before defining an automorphic form, we introduce some notation:

Let G be a connected, reductive group over $\mathbb{Q}$. Let $Z \subseteq G$ be the greatest $\mathbb{Q}$-split torus, let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup and let $\mathfrak{g} := \mathrm{Lie}(G(\mathbb{R}))$.

Let $\mathcal{H} := \mathcal{H}(G(\mathbb{R}), K)$ the convolution algebra of distributions on $G(\mathbb{R})$ with support in K which we will call the *Hecke algebra* of the Gelfand pair $(G(\mathbb{R}), K)$.

We will also need to define what a norm function on the real points of any reductive group should be:

Definition 3.3. Let G be a connected, reductive algebraic group over $\mathbb{Q}$. A norm $\|\cdot\|$ on $G(\mathbb{R})$ is a function $\|\cdot\| : G(\mathbb{R}) \to \mathbb{R}$ of the form

$$\|g\| := \sqrt{\mathrm{tr}(\sigma(g)^\star \sigma(g))}$$

for any finite-dimensional, complex representation $\sigma : G(\mathbb{R}) \to \mathrm{GL}(E)$ with finite kernel and closed image inside $\mathrm{End}(E)$, where we take the adjoint with respect to an K-invariant inner product on $G(\mathbb{R})$.

With the help of this, we can define a generalisation of our moderate growth condition from before.

Definition 3.4. We call a function $f : G(\mathbb{R}) \to \mathbb{C}$ *slowly increasing (or of moderate growth)* if there exists a norm $\|\cdot\|$ on $G(\mathbb{R})$ as well as $C > 0$ and $N \in \mathbb{N}$ such that

$$|f(x)| \leq C \cdot \|x\|^N$$

for all $x \in G(\mathbb{R})$. One can check that this is independent of the norm.

Lastly, we need to define what the analogon of a congruence subgroup should be in higher rank. For that, we define the following:

Definition 3.5. Let ι denote an embedding $G \hookrightarrow \mathrm{GL}_n$, i.e. a faithful representation of G. Then, we call a subgroup $\Gamma \subseteq G(\mathbb{Q})$ an *arithmetic subgroup* if there exists such a ι for which $\iota(\Gamma)$ is commensurable with $\Lambda := \iota(G(\mathbb{Q})) \cap \mathrm{GL}_n(\mathbb{Z})$.

Here, we call $\iota(\Gamma)$ and Λ *commensurable* if both $\iota(\Gamma)/(\iota(\Gamma) \cap \Lambda)$ and $\Lambda/(\iota(\Gamma) \cap \Lambda)$ are finite; in other words: if $\iota(\Gamma) \cap \Lambda$ has finite index in both $\iota(\Gamma)$ and Λ.

Now, the following definition of an automorphic form is straight-forward:

Definition 3.6 (Automorphic form in higher rank, classically)**.** With notation as above, let $\Gamma \subseteq G(\mathbb{Q})$ be an arithmetic subgroup. Then, we call a smooth function $f : G(\mathbb{R}) \to \mathbb{C}$ an *automorphic form* (for (Γ, K)) if it satisfies the following conditions:

(1) $f(\gamma.x) = f(x)$ for all $x \in G(\mathbb{R})$ and $\gamma \in \Gamma$,

(2) f is right-K-finite, more concretely: there exists a standard idempotent $\xi \in \mathcal{H}$ such that $f \star \xi = f$,

(3) f is $\mathcal{Z}(\mathfrak{g})$-finite, more concretely: there exists an ideal $I \subseteq \mathcal{Z}(\mathfrak{g})$ of finite codimension such that $Xf = 0$ for all $X \in I$,

(4) f is slowly increasing.

We denote the space of automorphic forms with for (Γ, K) such that $f \star \xi = f$ for $\xi \in \mathcal{H}$ and $f \star I = 0$ for $I \subseteq \mathcal{Z}(\mathfrak{g})$ of finite codimension by $\mathcal{A}(\Gamma, \xi, I, K)$.

Example 3.7. We have seen that for $G = \mathrm{GL}_2, \Gamma = \mathrm{GL}_2(\mathbb{Z}), K = \mathrm{O}_2(\mathbb{R}), \xi = 1, I = (\mathscr{C} - \lambda)$, the space $\mathcal{A}(\Gamma, \xi, I, K)$ is finite-dimensional and consists of Maaß wave forms.

A result of Harish-Chandra shows that this is true generally.

Theorem 3.8 (Harish-Chandra)**.** For any quadrupel (Γ, ξ, I, K) as above, the space $\mathcal{A}(\Gamma, \xi, I, K)$ has finite dimension.

This shows that we can nicely decompose the space of all automorphic forms according to these quadrupels. These finite-dimensional spaces, however, can also be further decomposed according to the action of the center:

Definition 3.9. The center of $G(\mathbb{R})$ acts on f by left translations and by Schur's lemma does so as a scalar on every irreducible component. We denote by $\omega : Z(\mathbb{R}) \cap \Gamma\backslash Z(\mathbb{R}) \to \mathbb{C}^\times$ a central character and by

$$\mathcal{A}(\Gamma, \xi, I, K)_\omega \subseteq \mathcal{A}(\Gamma, \xi, I, K)$$

the subspace of those functions $f \in \mathcal{A}(\Gamma, \xi, I, K)$ such that $f(z.g) = \omega(z)f(g)$ for all $z \in Z(\mathbb{R})$ and $g \in G(\mathbb{R})$.

With the help of structure theory, we can also define what it should mean for an arbitrary automorphic form to be cuspidal:

Definition 3.10 (Cusp forms)**.** An (automorphic) *cusp form* on $G(\mathbb{R})$ (with respect to (Γ, K)) is an automorphic form $f : G(\mathbb{R}) \to \mathbb{C}$ (with respect to (Γ, K)) such that

$$\int_{(\Gamma\cap N(\mathbb{R}))\backslash N(\mathbb{R})} f(n.x) \, \mathrm{d}n = 0$$

for all $x \in G(\mathbb{R})$, where $N \subseteq P$ is the unipotent radical of any proper $\mathbb{Q}$-parabolic subgroup $P \subseteq G$.

We denote by $\mathcal{A}^0(\Gamma, \xi, I, K)$ the subspace of $\mathcal{A}(\Gamma, \xi, I, K)$ consisting of modular cusp forms.

We note that this strictly generalises the definition of cuspidality from before. Also, the following result shows why cuspidality is an interesting condition:

Proposition 3.11. Let f be a smooth function $f : G(\mathbb{R}) \to \mathbb{C}$ satisfying conditions 1. to 3. on an automorphic form, equivariance for the center and cuspidality, but that is not necessarily slowly increasing.

Then, the following are equivalent:

(1) f is of moderate growth, i.e. an automorphic cusp form.

(2) f is bounded.

(3) f is square-integrable modulo $Z(\mathbb{R})\Gamma$.

Therefore, we can view the space of automorphic cusp forms $\mathcal{A}^0(\Gamma, \xi, I, K)_\omega$ as a subspace of the space $L^2(\Gamma\backslash G(\mathbb{R}), \omega)$, where $L^2(\Gamma\backslash G(\mathbb{R}), \omega)$ is defined as the completion of the following set

$$\left\{ f \in C^\infty(\Gamma\backslash G) : \int_{Z(\mathbb{R})\Gamma\backslash G(\mathbb{R})} |f|^2 \; \mathrm{d}g < \infty, \; f(z.g) = \omega(z) f(g) \; \forall z \in Z(\mathbb{R}), g \in G(\mathbb{R}) \right\},$$

which is a Hilbert space when equipped with the L^2-norm on the space $Z(\mathbb{R})\Gamma\backslash G(\mathbb{R})$.

Therefore, constructing automorphic forms for G is, once again, essentially indistinguishable from decomposing the space of K-finite vectors of the right-regular representation on the L^2-space of $\Gamma\backslash G(\mathbb{R})$.

3.2. **Automorphic representations.** Lastly, we will very briefly discuss automorphic representations and their relationship to automorphic forms. To wrap things up, we will make the relationship between representations of $\mathrm{SL}_2(\mathbb{R})$ and Maaß and modular forms precise. For a more thorough account for the theory of automorphic representations, we refer to [2] or [3].

We will concentrate on the L^2-side of things.

Definition 3.12 ($\mathcal{H}$-module structure on L^2)**.** Consider the space $L^2(\Gamma\backslash G(\mathbb{R}))$. This has a natural action R of $\mathcal{H}$ by convolution (which we have already used many times):

$$R : \mathcal{H} \to \mathrm{End}(L^2(\Gamma\backslash G(\mathbb{R}))), \quad R(\phi)f(g) = \int_{G(\mathbb{R})} f(gh)\phi(h) \; \mathrm{d}h.$$

Thus, $L^2(\Gamma\backslash G(\mathbb{R}))$ is naturally an $\mathcal{H}$-module.

On the one hand we know that $G(\mathbb{R})$ acts on $L^2(\Gamma\backslash G(\mathbb{R}))$ naturally by right translations. Note however that we can not give the space of automorphic forms a $G(\mathbb{R})$-module structure, since right translation by $G(\mathbb{R})$ does not respect the K-finiteness condition.

On the other hand, for each $f \in L^2(\Gamma\backslash G(\mathbb{R}))$, the $\mathcal{H}$-module consisting of K-finite vectors inside $\overline{R(G(\mathbb{R}))f} \subseteq L^2(\Gamma\backslash G(\mathbb{R}))$ is admissible (this follows from the finite-dimensionality of spaces of automorphic forms).

Furthermore, we know that the category of admissible $\mathcal{H}_\infty$-modules and admissible $(\mathfrak{g}, K)$-modules are equivalent. This leads to two possibilities of defining what an automorphic representation should be:

Definition 3.13 (Automorphic representation)**.** An *automorphic representation of* $G(\mathbb{R})$ is an admissible representation of $\mathcal{H}$ which is isomorphic to a subquotient of $L^2(\Gamma\backslash G(\mathbb{R}))$.

Equivalently, an automorphic representation of $G(\mathbb{R})$ is an admissible $(\mathfrak{g}, K)$-module which is isomorphic to a subquotient of $L^2(\Gamma\backslash G(\mathbb{R}))$.

Definition 3.14 (Cuspidal, automorphic representation)**.** A *cuspidal automorphic representation* is an automorphic representation which is equivalent to a subrepresentation of $L^2_0(\Gamma\backslash G(\mathbb{R}))$, where $L^2_0(\Gamma\backslash G(\mathbb{R})) \subseteq L^2(\Gamma\backslash G(\mathbb{R}))$ denotes the subspace of L^2-functions satisfying the cuspidality condition.

Note that we can associate to any (cuspidal) automorphic form f a (cuspidal) automorphic representation by the functor $f \mapsto \overline{R(G(\mathbb{R}))f}$.

Example 3.15. Let $f \in \mathcal{M}_s(\Gamma)$ be a Maaß cusp form of parameter s. Then we can associate to f an automorphic form ϕ and look at $\pi := \overline{R(G(\mathbb{R}))\phi}$, we thus get a cuspidal automorphic representation and it turns out that the infinity component π has the same $(\mathfrak{g}, K)$-module structure as the spherical principal series representation of parameter s, i.e. we have $\pi \cong \pi_{1,s}$ as $(\mathfrak{g}, K)$-modules.

By the same procedure, we associate to a modular cusp form $f \in S_k(\Gamma)$ an automorphic representation π. Here, we get that π has the same $(\mathfrak{g}, K)$-module structure as the discrete series representation of weight k, i.e. we have $\pi \cong \pi_k^+$ as $(\mathfrak{g}, K)$-modules.

This establishes a precise link between modular forms and representations of $\mathrm{SL}_2(\mathbb{R})$ and thus finishes our discussion of automorphic representations and automorphic forms.

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