SPECTRAL DECOMPOSITION OF $L^2(\Gamma\backslash G)$

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1. Abstract

In the first series of talks, the proof of Segal's Plancherel theorem was sketched which states that for any locally compact, separable, unimodular, CCR group G, the (left and right) regular representation U of $G \times G$ on $L^2(G)$ disintegrates into a direct integral of irreducibles:

$$(U, L^2(G)) \;\cong\; \left(\int_{\widehat{G}}^{\oplus} \pi \,\widehat{\otimes}\, \pi' \;\mathrm{d}\zeta(\pi),\; \int_{\widehat{G}}^{\oplus} \mathscr{H}_\pi \,\widehat{\otimes}\, \mathscr{H}_\pi' \;\mathrm{d}\zeta(\pi) \right)$$

for the Plancherel measure ζ on the unitary dual $\widehat{G}$ of G equipped with its natural topology. Moreover, it was shown that this general statement translates for groups like $G = \mathbb{R}$ or $G = \mathbb{T}$ into well-known explicit statements from Fourier theory, using that their commutativity makes $\widehat{G}$ into a particularly simple object. For compact groups, one can at least simplify the direct integral into a completed direct sum to obtain the Peter–Weyl theorem as a special case. Moreover, in this case all irreducible unitary representations are shown to be finite-dimensional, so one can also avoid the need to complete the tensor product.

For non-commutative non-compact groups, things become much more complicated: For many such groups, e.g. already for $G = \mathrm{SL}_2(\mathbb{R})$, the spectrum will have both *continuous* and *discrete* parts and the occuring representations will be generically infinite-dimensional. Nonetheless, in this case, it is possible to classify the unitary dual $\widehat{G}$ and use this to obtain an explicit description of the Plancherel theorem of the following form: For all $f \in \mathrm{C}_c^\infty(\mathrm{SL}_2(\mathbb{R}))$, we have

$$f(e) = \frac{1}{2\pi} \left(\sum_{n=1}^{\infty} (n-1)\,\Theta_n(f) + \frac{1}{4i} \int_{i\mathbb{R}} (\Theta_{+,\lambda}(f) \tan(\pi\lambda/2) - \Theta_{-,\lambda}(f) \cot(\pi\lambda/2)) \;\mathrm{d}\lambda \right),$$

where $\Theta_n(f)$ is the global character of the discrete series representations D_n^+ and D_n^- and $\Theta_{\pm,\lambda}$ is the global character of the principal series representation. Here, we use that the global character of any $\pi \in \widehat{G}$, defined a priori as the trace of the trace class operator

$$\pi(f) := \int_G f(g)\pi(g) \;\mathrm{d}g$$

and thus as a distribution on G, is actually given by integration against a locally integrable function Θ_π. This is a consequence of Harish-Chandra's famous regularity theorem on invariant eigendistributions, since the conjugation action by G keeps the trace distribution invariant and the universal enveloping algebra acts by the infinitesimal character of π.

Interestingly, not all representations occuring in the unitary dual of $\mathrm{SL}_2(\mathbb{R})$ seem to contribute to the spectral decomposition. In fact, since G is semisimple, there is a concrete expectation: All the tempered unitary irreducible representations will contribute (principal and discrete series)), all the square-integrable unitary irreducible representations (discrete series) will occur discretely and any non-tempered unitary irreducible representation (complementary series and the trivial

Date: December 4, 2025.

representation) will not show up at all in such disintegration formulae. For non-semisimple groups, this statement is no longer valid and it fails for $G = \mathbb{R}$ already.

For more general (even reductive or semisimple) groups, it is much harder to obtain such an explicit disintegration of the regular action on the space of L-functions. The issue already occurs in the desire to classify the unitary dual (the so-called *unitary dual problem*): Expect for some special cases in low rank, the concrete structure of the unitary dual of real semisimple Lie groups remains largely unknown. The key issue in attaining such a classification is that deciding unitarizability is hard: The *Langlands' classification* yields a complete description of all (tempered) irreducible representations of a given reductive group, and one can easily see which of these representations may be equipped with a non-trivial invariant sesquilinear form; however, it is hard to tell whether there is also such a positive-definite form.

Motivated e.g. from the desire to understand the spectral theory of locally symmetric spaces, one natural generalization is to study the right-regular representation on the space $L^2(\Gamma\backslash G(\mathbb{R}))$, where now G denotes a reductive group over $\mathbb{Q}$ and $\Gamma \subseteq G(\mathbb{Q})$ is a discrete subgroup of cofinite volume. Indeed, the non-cocompact cofinite volume case is most interesting, since then there will be both contributions from the discrete and continuous spectrum. One example of this is the case of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$, which specializes to the upper-half plane for $n = 2$ and whose volume has an interesting expression as follows:

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})) = \zeta(2) \cdot \zeta(3) \cdots \zeta(n).$$

This already suggests some connection to Number Theory. More generally, all congruence subgroups Γ in reductive groups G have this property and we will see that the decomposition of the spectrum will be according to so-called automorphic representations in this case. These automorphic representations, a priori of analytic flavour, admit both conjecturally and provably strong connections to Galois representations and motives in the realm of the Langlands program.

In the two talks, I will first explain the case of $G = \mathrm{SL}_2$, where we will see elliptic modular forms occuring in the spectral decomposition of $L^2(\Gamma\backslash G(\mathbb{R}))$, and use this example to motivate the general definition of an automorphic representation. Subsequently, we will shortly explore the landscape of the Langlands program.

References

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