EXAMPLES FOR SEGAL'S PLANCHEREL THEOREM

LUKAS LANGEN

Let G be a locally compact, separable, unimodular, CCR group.

The regular representation $U\colon G\times G \curvearrowright L^2(G), (x,y).f(z) \coloneqq f(x^{-1}zy)$ disintegrates into a direct integral of irreducibles. We call this isometric isomorphism the Fourier(-Plancherel) transform

$$\mathcal{F}\colon (U,\, L^2(G)) \xrightarrow{\sim} \left(\int_{\widehat{G}}^{\oplus} \pi \mathbin{\hat{\otimes}} \pi'\; d\zeta(\pi),\; \int_{\widehat{G}}^{\oplus} H_\pi \mathbin{\hat{\otimes}} H'_\pi\; d\zeta(\pi)\right)$$

with Plancherel measure ζ uniquely determined by the isometric property

$$\|f\|_2^2 = \int_{\widehat{G}} \operatorname{tr}(\pi(f)^*\pi(f))\; d\zeta(\pi)$$

for $f \in L^1(G) \cap L^2(G)$.

It is part of the theorem that $\pi(f)$ is a Hilbert-Schmidt operator for $f \in L^1(G) \cap L^2(G)$. Indeed note that $H_\pi \mathbin{\widehat{\otimes}} H'_\pi \cong \mathrm{HS}(H_\pi)$ is isometrically isomorphic with the Hilbert-Schmidt inner product $\langle A, B\rangle_\pi = \operatorname{tr}(A^*B)$ $(A, B \in \mathrm{HS}(H_\pi)$.

Recall that for $f \in L^1(G)$, the Fourier transform is defined via the integrated representation

$$\widehat{f}(\pi) \coloneqq \pi(f) \coloneqq \int_G f(x)\pi(x)\, dx \in \operatorname{End}(H_\pi).$$

Now let $f \in L^1(G) \cap L^2(G)$ and denote $h = \mathcal{F}f \in \mathcal{H} = \int_{\widehat{G}}^{\oplus} H_\pi \widehat{\otimes} H'_\pi\, d\zeta(\pi)$. Then h is (up to identification of nullsets) a square integrable section

$$h\colon \widehat{G} \to \bigcup_{\tau\in\widehat{G}} H_\tau \widehat{\otimes} H'_\tau,\;\; \pi \mapsto h(\pi) \in H_\pi \widehat{\otimes} H'_\pi = \mathrm{HS}(H_\pi).$$

Furthermore, since $\mathcal{F}$ intertwines the actions $U\colon G \times G \curvearrowright L^2(G)$ with $M\colon G \times G \curvearrowright \mathcal{H}$, where

$$M(x,y) = \int_{\widehat{G}}^{\oplus} \pi(x) \mathbin{\widehat{\otimes}} \pi'(y)\, d\zeta(\pi) \in \operatorname{End}(\mathcal{H}),$$

we see[1]

$$\mathcal{F}(U(x,y)f)(\pi) = M(x,y)h(\pi) = \pi(x) \circ h(\pi) \circ \pi(y)^{-1} = \pi(x) \circ \mathcal{F}f(\pi) \circ \pi(y)^{-1}.$$

This corresponds to the classical principle "translation becomes multiplication in the Fourier image".

On the other hand, direct calculation also gives

$$(U(x,y)f)^\wedge(\pi) = \pi(x) \circ \widehat{f}(\pi) \circ \pi(y)^{-1}.$$

Thus $\mathcal{F}$ extends the Fourier transform $\widehat{\cdot}$, i.e. $\mathcal{F}f = \widehat{f}$ for $f \in L^1(G) \cap L^2(G)$.

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[1] at least almost everywhere

1. The case $G = \mathbb{R}$

Since $\mathbb{R}$ is abelian, Schur's Lemma implies that unitary irreducible representations of $\mathbb{R}$ are 1-dimensional and thus correspond to continuous multiplicative characters $\mathbb{R} \to \mathbb{S}_1 \subseteq \mathbb{C}$. Thus

$$\widehat{\mathbb{R}} = \{(\varphi_\xi, V_\xi) : \xi \in \mathbb{R}\} \quad \text{with} \quad \varphi_\xi(t) := e^{-i\xi t},\ V_\xi = \mathbb{C}.$$

Note that $\widehat{\mathbb{R}}$ is again a locally compact topological group (the dual group) isomorphic to $\mathbb{R}$ via $\xi \mapsto (\varphi_\xi, V_\xi)$, as $\mathbb{R}$ is self-dual as a LCA group.

For $f \in L^1(\mathbb{R})$ and $\pi_\xi = (\varphi_\xi, V_\xi)$, we have

$$\pi_\xi(f) = \int_{\mathbb{R}} f(t)\varphi_\xi(t)\, dt,$$

defining an operator $\pi_\xi(f)\colon V_\xi \to V_\xi$. Upon fixing the basis $\{1\} \subseteq \mathbb{C} = V_\xi$, we identify $\pi_\xi(f)$ with a complex number. Thus

$$\operatorname{tr}(\pi_\xi(f)^*\pi_\xi(f)) = |\pi_\xi(f)|^2 = |\widehat{f}(\xi)|^2,$$

where

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)\varphi_\xi(t)\ dt = \int_{\mathbb{R}} f(t)e^{-i\xi t}\ dt$$

denotes the classical Fourier transform.

Thus for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\int_{\mathbb{R}} \operatorname{tr}(\pi_\xi(f)^*\pi_\xi(f))\ d\xi = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2\ d\xi \stackrel{!}{=} \|f\|_2^2$$

shows the Plancherel measure on $\widehat{\mathbb{R}} \cong \mathbb{R}$ to be the (suitably renormalized) Lebesgue measure.

Now

$$(\pi_\xi \otimes \pi'_\xi)(t,s)\colon V_\xi \otimes V'_\xi \to V_\xi \otimes V'_\xi \quad \text{for} \quad (t,s) \in \mathbb{R} \times \mathbb{R}$$

corresponds to the complex number $m_\xi(t,s) := \varphi_\xi(t)\overline{\varphi_\xi(s)} = e^{-i(t-s)\xi}$ acting on $V_\xi \otimes V'_\xi = \mathbb{C}$ by multiplication.

Thus

$$\begin{aligned}(U, L^2(\mathbb{R})) &\cong \left(\int_{\widehat{\mathbb{R}}}^{\oplus} \pi_\xi \otimes \pi'_\xi\, d\xi,\ \int_{\widehat{\mathbb{R}}}^{\oplus} V_\xi \otimes V_\xi^{'}\, d\xi\right) \\ &\cong \left(\int_{\mathbb{R}}^{\oplus} m_\xi\, d\xi, \int_{\mathbb{R}}^{\oplus} \mathbb{C}\, d\xi\right) \\ &\cong (M, L^2(\mathbb{R}))\end{aligned}$$

with $M(t,s)f(z) := e^{-i(t-s)z}f(z)$ recovers the classical Plancherel theorem on $\mathbb{R}$, which gives an isometric isomorphism $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ transforming translations into multiplication operators in the Fourier image, i.e.

$$\mathcal{F}(U(x,y)f)(\xi) = (M(x,y)\mathcal{F}f)(\xi).$$

2. The case $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$

Similarly to before $\widehat{\mathbb{T}}$ consists of the continuous group morphisms $\mathbb{T} \to \mathbb{S}_1$, which can be shown to consist of

$$\widehat{\mathbb{T}} = \{(\varphi_k, V_k) : k \in \mathbb{Z}\} \quad \text{with} \quad \varphi_k(z) := z^{-k}, V_k = \mathbb{C}.$$

Reasoning as before

$$\int_{\mathbb{Z}} |\widehat{f}(k)|^2 \, dk \overset{!}{=} \|f\|_2^2$$

shows the Plancherel measure on $\widehat{\mathbb{T}} \cong \mathbb{Z}$ (as topological groups) to be the counting measure (the suitably normalized Haar measure of $\mathbb{Z}$). Thus

$$\begin{aligned}(U, L^2(\mathbb{T})) &\cong \left(\int_{\widehat{\mathbb{T}}}^{\oplus} \pi_k \otimes \pi_k', \ \int_{\widehat{\mathbb{T}}}^{\oplus} V_k \otimes V_k' \ d\xi\right) \\ &= (\int_{\mathbb{Z}}^{\oplus} m_k \, dk, \ \int_{\mathbb{Z}}^{\oplus} \mathbb{C} \, dk) \\ &\cong (M, \ell^2(\mathbb{Z}))\end{aligned}$$

recovers the classical Plancherel theorem on the torus, giving an isometric isomorphism $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ intertwining translations $U(x,y)$ with the multiplication operators $M(x,y)f(n) := x^{-n}y^n f(n)$, i.e.

$$\mathcal{F}(U(x,y)f)(n) = (M(x,y)\mathcal{F}f)(n).$$

3. The case of LCA groups G

Again, if G is a locally compact separable[2] abelian group, by Schur's Lemma all irreducible unitary representations are given by continuous characters $G \to \mathbb{S}_1$, thus we identify G with the Pontryagin dual group

$$\widehat{G} = \{\alpha\colon G \to \mathbb{S}_1 \text{ cont. group hom.}\}$$

with each character α acting on the one-dimensional space $V_\alpha = \mathbb{C}$ by multiplication with a complex number. Just as before, we reason the Plancherel measure to be the (suitably normalized) Haar measure on the dual group $\widehat{G}$ and conclude

$$\begin{aligned}(U, L^2(G)) &\cong \left(\int_{\widehat{G}}^{\oplus} m_\alpha \, d\alpha, \ \int_{\widehat{G}}^{\oplus} V_\alpha \otimes V_\alpha' \, d\alpha\right) \\ &\cong (M, L^2(\widehat{G}))\end{aligned}$$

with $M(x,y)f(\alpha) := \alpha(x)\overline{\alpha(y)}f(\alpha)$. Thus we recover the Plancherel theorem for locally compact abelian groups establishing an isometric isomorphism

$$\mathcal{F}\colon L^2(G) \to L^2(\widehat{G})$$

such that

$$\mathcal{F}(U(x,y)f)(\alpha) = (M(x,y)\mathcal{F}f)(\alpha).$$

Note that there is a slight inconsistency in the definition of the Fourier transform in the general setting opposed to the abelian setting, as one normally defines

$$\widehat{f}(\pi) = \int_G f(x)\pi(x) \ dx$$

[2]we require separable in the genral setting; this is not important for the LCA theory

in the general setting but

$$\widehat{f}(\alpha) = \int_G f(x)\overline{\alpha(x)}\,dx$$

in the LCA setting, which introduces a minor inconsistency due to the complex conjugation (which we fixed in the previous two examples by using a slightly different enumeration of the unitary dual).

4. The case of compact groups

Suppose now G is a compact separable[3] group. By the Peter-Weyl theorem, $\widehat{G}$ consists of the irreducible finite-dimensional representations and $\widehat{G}$ carries the discrete topology with Plancherel measure just being the counting measure. Thus

$$\begin{aligned}(U, L^2(G)) &\cong \left(\int_{\widehat{G}}^{\oplus} \pi \otimes \pi'\, d\zeta(\pi),\ \int_{\widehat{G}}^{\oplus} V_\pi \otimes V_\pi'\, d\zeta(\pi)\right) \\ &\cong \left(\widehat{\oplus}_{\pi\in\widehat{G}}\, \pi \otimes \pi',\ \widehat{\bigoplus}_{\pi\in\widehat{G}} V_\pi \otimes V_\pi'\right).\end{aligned}$$

References

[1] I. E. Segal, "An Extension of Plancherel's Formula to Separable Unimodular Groups." Ann. Math., vol. 52, no. 2, 1950, pp. 272–292.

[2] N. R. Wallach, *Real reductive groups II*, Pure and Applied Mathematics, vol. 132, Academic Press, Boston, MA, 1992.

[3] J. Dixmier, *C^*-algebras*, North-Holland Mathematical Library, vol. 15, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, Translated from the French by Francis Jellett.

Institut für Mathematik, Universität Paderborn, Warburger Str. 100, D-33098 Paderborn, Germany

Email address: llangen@math.upb.de

[3]we require separability in our general setting; this is not needed for Peter-Weyl. Actually, separability is neither needed in the general setting, see [3]