THE GEOMETRY OF CHARACTERS OF COMPACT, CONNECTED LIE GROUPS

Abstract: The theory of Fourier series writes a smooth function f on the circle as an infinite sum of functions of the form $a_n e^{in\theta}$ for $n \in \mathbb{Z}$. The functions $e^{in\theta}$ are the irreducible characters of the circle, and the Fourier coefficient a_n is obtained by integrating f against the conjugate of the corresponding character. More generally, it is natural to consider a compact subset $X \subset \mathbb{R}^n$ with a compact, transitive group of symmetries G. In this generality, there is a theory of Fourier series on X which is founded on an understanding of the irreducible characters of G. In this minicourse, we will discuss the characters of compact, connected Lie groups and their relationship with geometry.

0.1. **Lecture 1: Structure of Lie Groups.** We review the definitions of Lie groups, Lie algebras, and the exponential map. We recall the examples $G = \mathrm{SO}(n)$ and $G = \mathrm{SU}(n)$. We discuss left and right Haar measures, the modular character, and invariant densities on homogeneous spaces.

0.2. **Lecture 2: Characters of Compact, Connected Lie Groups.** We define the character of an irreducible, unitary representation of a compact, connected Lie group. We write down these irreducible characters explicitly for $G = \mathrm{SU}(2)$. We discuss the algebra of invariant differential operators on a compact, connected Lie group. We show that the irreducible characters are eigenfunctions for these differential operators.

0.3. **Lecture 3: Harmonic Analysis on Homogeneous Spaces for Compact, Connected Lie Groups.** Let G be a compact, connected Lie group, and let $H \subset G$ be a closed, unimodular subgroup. We discuss the decomposition of $L^2(G/H)$ into irreducible representations, and we write this decomposition explicitly using characters. We will not have time to give full proofs. We will write everything down explicitly when $G = \mathrm{SO}(3)$, $H = \mathrm{SO}(2)$, $G/H = S^2$. If time permits, we will state the Schur orthogonality relations for irreducible characters.

0.4. **Lecture 4: The Theory of Distributions and the Abelian Fourier Transform.** We define the notion of a distribution on a finite-dimensional, real vector space. We recall the abelian Fourier transform, and we discuss the Fourier transform of a compactly supported distribution. We show that the abelian Fourier transform takes constant coefficient differential operators to polynomials and vice versa.

0.5. **Lecture 5: Characters and Coadjoint Orbits for Compact, Connected Lie Groups.** We introduce the Jacobian of the exponential map. Then we state the Harish-Chandra-Kirillov character formula for irreducible characters of compact, connected Lie groups. While we do not give a complete proof, we use the differential equations from Lecture 2 and the discussion of the Fourier transform and differential operators in Lecture 4 to prove that the abelian Fourier transform of the Lie algebra analogue of the character is supported on a single coadjoint orbit. If time permits, we will discuss applications of this formula to harmonic analysis on homogeneous spaces (Lecture 3).