Elementary symmetric polynomials and martingales for Heckman-Opdam processes

Margit Rösler and Michael Voit

ABSTRACT. We consider the generators L_k of Heckman-Opdam diffusion processes in the compact and non-compact case in N dimensions for root systems of type A and B, with a multiplicity function of the form $k = \kappa k_0$ with some fixed value k_0 and a varying constant $\kappa \in [0, \infty[$. Using elementary symmetric functions, we present polynomials which are simultaneous eigenfunctions of the L_k for all $\kappa \in]0, \infty[$. This leads to martingales associated with the Heckman-Opdam diffusions $(X_{t,1}, \ldots, X_{t,N})_{t \geq 0}$. As our results extend to the freezing case $\kappa = \infty$ with a deterministic limit after some renormalization, we find formulas for the expectations $\mathbb{E}(\prod_{j=1}^{N}(y - X_{t,j}))$, $y \in \mathbb{C}$.

1. Introduction

In the theory of classical random matrix ensembles there exist several formulas regarding determinants as follows: Let X be a random variable with values in some space of $N \times N$-matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then the expectations $\mathbb{E}(\det(X - yI_N))$ are classical orthogonal polynomials of degree N in $y \in \mathbb{C}$. Such results for the Hermite, Laguerre, and Jacobi ensembles can for example be found in [**DG, FG, A**], where the expectations above can be expressed via classical Hermite, Laguerre and Jacobi polynomials of degree N. These results were extended to the spectra of β-ensembles associated with N-dimensional time-homogeneous diffusion processes in [**KVW, V**] by using some martingales constructed from these diffusions via elementary symmetric polynomials. The generators of the diffusions were Dunkl-Bessel Laplacians and (symmetric) Heckman-Opdam Laplacians in the compact BC case.

In the present paper we study corresponding results for Heckman-Opdam Laplacians in the non-compact BC case as well Heckman-Opdam Laplacians of type A in the compact and noncompact setting. Together with [**KVW, V**], the present paper covers the most important examples related with Calogero-Moser-Sutherland particle models. The basic ideas here are similar to those in [**KVW, V**]; however, while the approach in [**KVW, V**] is mainly based on Ito calculus, we now focus on an algebraic point of view.

The idea is as follows. Let $(X_t)_{t \geq 0}$ be a time-homogeneous diffusion on a suitable closed set $C \subset \mathbb{R}^N$ (such as a Weyl chamber or a fundamental alcove), where the paths are reflected at the boundary ∂C. Then the generator L of the

associated transition semigroup is an elliptic partial differential operator whose domain is contained in the space of functions on $\mathbb{R}^N$ which admit corresponding symmetries on ∂C. We are now interested in functions $f : [0, \infty[\times \mathbb{R}^N \to \mathbb{C}$ for which $(f(t, X_t))_{t \geq 0}$ is a martingale (w.r.t. the canonical filtration), which essentially means that f is L-space-time-harmonic, i.e., $(\frac{\partial}{\partial t} + L)f = 0$; see Section III.10 of [**RW**]. Examples of such harmonic functions can be given in terms of eigenfunctions of L. For a general background in stochastic analysis we also recommend [**P**].

In the framework of Heckman-Opdam theory (see [**HS**], [**HO**]), we fix some root system R (with associated Weyl group W) in $\mathbb{R}^N$ and choose C as an associated Weyl chamber or fundamental alcove. We consider a non-negative multiplicity function $k \geq 0$ on R of the form $k = \kappa \cdot k_0$ with some fixed multiplicity k_0 and a constant $\kappa > 0$. We now study the W-invariant Heckman-Opdam Laplace operators $L_\kappa := L_{\kappa k_0}$ as generators of diffusions (see [**Sch1, Sch2, RR1**]), where the parameter κ is varying. We also study the renormalized generators $\widetilde{L}_\kappa := \frac{1}{\kappa} L_\kappa$ of the renormalized diffusions $(\widetilde{X}_t := X_{t/\kappa})_{t \geq 0}$. In suitable coordinates, these renormalized generators then have the form

$$\widetilde{L}_\kappa f = \frac{1}{\kappa} \Delta f + H f$$

with some second-order differential operator Δ (often a classical Laplacian) and some first-order drift operator H, where both operators are independent of κ. This also works for $\kappa = \infty$ where $(\widetilde{X}_t)_{t \geq 0}$ is the deterministic solution of some ODE associated with H. We now use elementary symmetric polynomials to construct simultanous eigenfunctions f of $\widetilde{L}_\kappa$ for all $\kappa \in]0, \infty]$. In the next step we use martingales associated with these functions f together with information in the deterministic case $\kappa = \infty$ in order to derive formulas for

$$(1.1) \qquad\qquad \mathbb{E}(\prod_{j=1}^{N}(\widetilde{X}_{t,i} - y)) \qquad (y \in \mathbb{C}).$$

For some values of κ, the operators L_κ admit an interpretation as Laplace-Beltrami operators on symmetric spaces. In these cases, (1.1) leads to determinantal formulas for Brownian motions on the symmetric spaces which are closely related, for instance, to some identities in [**R**]. Considering $t \to \infty$ in the compact cases, this also leads to determinantal formulas for the uniform distributions on compact symmetric spaces.

The paper is organized as follows: In Section 2 we recapitulate some well-known facts on Heckman-Opdam hypergeometric functions and polynomials, and the associated Laplacians in the compact and non-compact setting. In the main part, we concentrate on specific root systems. Section 3 is devoted to the compact case of type A, which is related to Calogero-Moser-Sutherland particle models on the torus. Here, for instance, our results lead to determinantal formulas for Brownian motions and the uniform distribution on the unitary groups $U(N)$ and $SU(N)$. In Section 4 we study the non-compact case of type A. Section 5 then contains the non-compact case of type BC.

2. Heckman-Opdam theory

Here we briefly collect some facts from Heckman-Opdam theory; see [**HS, HO**] for a general background.

Let $(\mathfrak{a}, \langle\,.\,,\,.\,\rangle)$ be a Euclidean space of dimension N. We identify $\mathfrak{a}$ with its dual space via the given scalar product. Let R be a crystallographic, possibly not reduced root system in $\mathfrak{a}$ with associated finite reflection group W. We fix a positive subsystem $R_+ \subset R$ and a W-invariant multiplicity $k : R \to [0, \infty[$. The Cherednik operators associated with R_+ and k are defined as

$$(2.1) \quad D_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} \frac{k(\alpha)\langle\alpha, \xi\rangle}{1 - e^{-\langle\alpha, x\rangle}}(f(x) - f(\sigma_\alpha(x))) - \langle\rho(k), \xi\rangle f(x)$$

for $\xi \in \mathbb{R}^n$, with the (weighted) Weyl vector $\rho(k) := \frac{1}{2}\sum_{\alpha \in R_+} k(\alpha)\alpha$. The operators $D_\xi(k)$, $\xi \in \mathfrak{a}$ commute, and for each $\lambda \in \mathfrak{a}_\mathbb{C}$ there is a unique analytic function $G(\lambda, k; .)$ on some W-invariant tubular neighborhood of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$, the so called Opdam-Cherednik kernel, which satisfies

$$(2.2) \quad G(\lambda, k; 0) = 1 \quad \text{and} \quad D_\xi(k)G(\lambda, k; \,.\,) = \langle\lambda, \xi\rangle\, G(\lambda, k; \,.\,) \quad \text{for all } \xi \in \mathfrak{a}.$$

The hypergeometric function associated with R is defined by

$$(2.3) \qquad\qquad F(\lambda, k; z) = \frac{1}{|W|}\sum_{w \in W} G(\lambda, k; w^{-1}z).$$

To introduce the associated Heckman-Opdam polynomials, we need the weight lattice and the set of dominant weights,

$$P = \{\lambda \in \mathfrak{a} : \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z} \;\forall \alpha \in R\}, \quad P_+ = \{\lambda \in P : \langle\lambda, \alpha^\vee\rangle \geq 0 \;\forall \alpha \in R_+\} \supset R_+,$$

where $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$. P_+ carries the usual dominance order. Let

$$\mathcal{T} := \mathrm{span}_\mathbb{C}\{e^{i\lambda},\, \lambda \in P\}$$

be the space of trigonometric polynomials associated with R. The orbit sums

$$M_\lambda = \sum_{\mu \in W\lambda} e^{i\mu}, \quad \lambda \in P_+$$

form a basis of the subspace $\mathcal{T}^W$ of W-invariant polynomials in $\mathcal{T}$. For $Q^\vee := \mathrm{span}_\mathbb{Z}\{\alpha^\vee,\, \alpha \in R\}$, consider the compact torus $T = \mathfrak{a}/2\pi Q^\vee$ with the weight function

$$(2.4) \qquad\qquad \delta_k(x) := \prod_{\alpha \in R_+} \left|\sin\left(\frac{\langle\alpha, x\rangle}{2}\right)\right|^{2k_\alpha}.$$

The Heckman-Opdam polynomials associated with R_+ and k are given by

$$P_\lambda(k; z) = M_\lambda(z) + \sum_{\nu < \lambda} c_{\lambda\nu}(k)M_\nu(z) \quad (\lambda \in P_+,\, z \in \mathfrak{a}_\mathbb{C})$$

where the coefficients $c_{\lambda\nu}(k) \in \mathbb{R}$ are uniquely determined by the condition that $P_\lambda(k; .)$ is orthogonal to M_ν in $L^2(T, \delta_k)$ for all $\nu \in P_+$ with $\nu < \lambda$. It is known that $\{P_\lambda(k, .), \lambda \in P_+\}$ is an orthonormal basis of $L^2(T, \delta_k)^W$ of all W-invariant functions from $L^2(T, \delta_k)$. According to [**HS**], the normalized polynomials

$$R_\lambda(k, z) := P_\lambda(k; z)/P_\lambda(k; 0)$$

can be expressed in terms of the hypergeometic function as

$$(2.5) \qquad\qquad R_\lambda(k, z) = F(\lambda + \rho(k), k; iz).$$

Note that our notion slightly differs from [**HS, HO**], where the polynomials P_λ are defined as exponential polynomials on the torus $i\mathfrak{a}/2\pi iQ^\vee$. We next introduce the Heckman-Opdam Laplacian

$$\Delta_k := \sum_{j=1}^N D_{\xi_j}(k)^2 \ - \ \|\rho(k)\|_2^2,$$

with some orthonormal basis $\xi_1, \ldots, \xi_N$ of $\mathfrak{a}$. The operator Δ_k is independent of the choice of this basis. Denote by L_k the restriction of Δ_k to W-invariant functions. According to [**Sch1**],

$$(2.6) \qquad L_k f(x) = \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \coth\Big(\frac{\langle \alpha, x \rangle}{2}\Big) \cdot \partial_\alpha f(x).$$

The operator L_k is independent of the choice of R_+ and generates a Feller diffusion on some closed Weyl chamber in $\mathfrak{a}$ where the paths are reflected at the boundaries. We notice that by construction, for all $\lambda \in \mathbb{C}$, the hypergeometric functions $\phi_\lambda := F(\lambda, k; \,.\,)$ are eigenfunctions of L_k with eigenvalues

$$\sum_{j=1}^N \langle \lambda, \xi_j \rangle^2 - \|\rho(k)\|_2^2.$$

In the compact case we take the factor i in (2.5) into account as in [**RR1**] and consider the operator

$$(2.7) \qquad \widehat{L}_k f(x) := \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \cot\Big(\frac{\langle \alpha, x \rangle}{2}\Big) \cdot \partial_\alpha f(x).$$

$\widehat{L}_k$ generates a Feller diffusion on a compact fundamental alcove of the affine Weyl group $W_{aff} = W \ltimes 2\pi Q^\vee$ in $\mathfrak{a}$, again with reflecting boundaries. The trigonometric polynomials P_λ ($\lambda \in P_+$) are eigenfunctions of $\widehat{L}_k$ with eigenvalues

$$-\langle \lambda, \lambda + 2\rho(k) \rangle \le 0.$$

They were used in [**RR1**] to construct the transition densities of the diffusions with generators $\widehat{L}_k$. For root systems of type BC, the P_λ are multivariate Jacobi polynomials, and the associated Feller diffusions are multivariate Jacobi processes which were studied in [**Dem**]. For root systems of type A and BC, more details will be worked out in the following sections.

3. The compact case of type A_{N-1}

In this section we study Heckman-Opdam processes of type A_{N-1} in the compact setting. The generators of these processes are the Hamiltonians of interacting particle models of Calogero-Sutherland type with N particles on the torus $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$; see [**LV**] for the background. These processes are diffusions on some fundamental domain of $W = S_N$ in $\mathbb{T}^N$. It will however be convenient to consider also associated diffusions on $\mathbb{R}^N$ with 2π-periodicity such that the diffusions on $\mathbb{T}^N$ appear as images under $x \mapsto e^{ix}$. To introduce the processes on $\mathbb{R}^N$, we consider the root system $R = A_{N-1} = \{\pm(e_i - e_j) : 1 \le i < j \le N\}$ in $\mathbb{R}^N$ with positive subsystem $R_+ = \{e_i - e_j : i > j\}$ and fix a multiplicity parameter $k \in \,]0, \infty[$. Let

$$\omega := (1, \ldots, 1)^T \in \mathbb{R}^N.$$

Then $Q^\vee = \mathbb{Z}^N \cap (\mathbb{R}\omega)^\perp$, and a fundamental domain for the action of $W_{aff} = W \ltimes 2\pi Q^\vee$ in $\mathbb{R}^N$ is given by

$$C_N = \{x \in \mathbb{R}^N : 0 \le \langle \alpha, x \rangle \le 2\pi \; \forall \alpha \in R_+\}$$
$$= \{x \in \mathbb{R}^N : x_1 \le x_2 \le \ldots \le x_N \le x_1 + 2\pi\}.$$

We consider the W-invariant Heckman-Opdam Laplacian

$$(3.1) \qquad \widehat{L}_k f(x) = \Delta f(x) + k \sum_{j=1}^N \sum_{l \ne j} \cot\left(\frac{x_j - x_l}{2}\right) \frac{\partial}{\partial x_j} f(x)$$

with reflecting boundaries, i.e. with domain

$$D(\widehat{L}_k) = \{f|_{C^N} : f \in C^2(\mathbb{R}^N) \text{ invariant under } W_{aff}\}.$$

$\widehat{L}_k$ is the generator of a Feller semigroup of transition operators on C_N, c.f. [**RR1**]. Associated Feller diffusions $(X_{t,k})_{t \ge 0}$ with continuous paths (which are reflected at the boundary of C_N) are called Heckman-Opdam processes of type A_{N-1} on C_N. We also consider the renormalized generators

$$\widetilde{L}_k := \frac{1}{k}\widehat{L}_k = \frac{1}{k}\Delta + \sum_{j=1}^N \sum_{l \ne j} \cot\left(\frac{x_j - x_l}{2}\right) \frac{\partial}{\partial x_j}$$

which degenerate for $k \to \infty$ into

$$\widetilde{L}_\infty = \sum_{j=1}^N \sum_{l \ne j} \cot\left(\frac{x_j - x_l}{2}\right) \frac{\partial}{\partial x_j}.$$

For $k \in]0, \infty[$, the process $\widetilde{X}_k := (\widetilde{X}_{t,k})_{t \ge 0}$ with $\widetilde{X}_{t,k} := X_{t/k,k}$ is a Feller diffusion associated with $\widetilde{L}_k$. It can be also described as solution of the SDE

$$(3.2) \qquad d\widetilde{X}_{t,k,j} = \frac{\sqrt{2}}{\sqrt{k}} dB_{t,j} + \sum_{l \ne j} \cot\left(\frac{\widetilde{X}_{t,k,j} - \widetilde{X}_{t,k,l}}{2}\right) dt \qquad (j = 1, \ldots, N)$$

with some N-dimensional Brownian motion $(B_{t,1}, \ldots, B_{t,N})_{t \ge 0}$. Moreover, for deterministic starting conditions, $\widetilde{L}_\infty$ is the generator of a deterministic process whose paths are the solution of some initial value problem for the ODE

$$(3.3) \qquad \frac{dx_j}{dt}(t) = \sum_{l \ne j} \cot\left(\frac{x_j(t) - x_l(t)}{2}\right) \qquad (j = 1, \ldots, N).$$

As for Dunkl processes in [**AV**], one can show that for initial data in the interior of C_N, the solution $(\widetilde{X}_{t,\infty})_{t \ge 0}$ of this ODE exists for all $t \ge 0$ in the interior of C_N. We also point out that in the Dunkl setting, the ODE analogous to (3.2) has unique solutions for all $t \ge 0$ even for starting points at the boundary of the chamber, see [**VW**]. We expect that such a result is also true in the present setting.

We need the following stationary solutions of (3.3):

LEMMA 3.1. *For each $x_1 \in \mathbb{R}$,*

$$\left(x_1, x_1 + \frac{1}{N}2\pi, x_1 + \frac{2}{N}2\pi, \ldots, x_1 + \frac{N-1}{N}2\pi\right) \in C_N$$

is a stationary solution of (3.3).

PROOF. Assume first that N is odd. As the cotangent is odd and π-periodic, we have for $j = 1, \ldots, N$ that

$$\sum_{l \neq j} \cot\Big(\frac{(j-l)\pi}{N}\Big) = \sum_{l=1}^{N-1} \cot\Big(\frac{l\pi}{N}\Big) = \sum_{l=1}^{(N-1)/2} \Big(\cot\Big(\frac{l\pi}{N}\Big) + \cot\Big(\frac{(N-l)\pi}{N}\Big)\Big) = 0.$$

If N is even, then our computation leads to the additional term $\cot((N\pi/2)/N) = 0$ in the last sum and thus to the same result. This yields the claim. $\square$

The generator $\widetilde{L}_k$ and the associated diffusion $\widetilde{X}_k$ on $\mathbb{R}^N$ can be decomposed into two independent parts, namely the center of gravity and the process of the distances of neighboring particles. This reflects the fact that the usual representation of the symmetric group S_N on $\mathbb{R}^N$ decomposes into two irreducible components. More precisely, consider the center-of-gravity process $\widetilde{X}_k^{cg} := (\widetilde{X}_{t,k}^{cg})_{t\geq 0}$ with

$$\widetilde{X}_{t,k}^{cg} := \frac{1}{N}(\widetilde{X}_{t,k,1} + \ldots + \widetilde{X}_{t,k,N}) \cdot \omega$$

which is the orthogonal projection of $\widetilde{X}_{t,k}$ onto $\mathbb{R}\omega$. Then the diffusion

$$\widetilde{X}_k^{diff} := \widetilde{X}_k - \widetilde{X}_k^{cg}$$

lives on the orthogonal complement $(\mathbb{R}\omega)^\perp \subset \mathbb{R}^N$.

LEMMA 3.2. Let $k \in]0, \infty[$. If $\widetilde{X}_k$ starts in some deterministic point, then the processes $\widetilde{X}_k^{diff}$ and $\widetilde{X}_k^{cg}$ are stochastically independent.

PROOF. By the SDE (3.2) we have

$$(3.4) \qquad d\widetilde{X}_{t,k,j}^{cg} = \frac{\sqrt{2}}{N\sqrt{k}} d\Big(\sum_{l=1}^N B_{t,l}\Big) =: \frac{\sqrt{2}}{\sqrt{Nk}} dB_t$$

with some one-dimensional Brownian motion $(B_t)_{t\geq 0}$ while $\widetilde{X}_k^{diff}$ satisfies

$$d\widetilde{X}_{t,k,j}^{diff} = \frac{\sqrt{2}}{\sqrt{k}} d\Big(B_{t,j} - \frac{1}{N}\sum_{l=1}^N B_{t,l}\Big) + H_j(\widetilde{X}_{t,k}^{diff}) \, dt \quad (j = 1, \ldots, N)$$

where the processes $(B_{t,j} - \frac{1}{N}\sum_{l=1}^N B_{t,l})_{t\geq 0}$ are stochastically independent of $(B_t)_{t\geq 0}$, and the H_j are concrete continuous functions. This implies the claim by the very definition of solutions of SDEs. $\square$

We next transfer all data from $\mathbb{R}^N$ to the torus $\mathbb{T}^N$ via $z_j := e^{ix_j}$ for $j = 1, \ldots, N$, or for short, $z := e^{ix} \in \mathbb{T}^N$. A short computation shows that in z-coordinates, the operator L_k is given by the diffusion operator

$$(3.5) \quad H_k = -\sum_{j=1}^N \Big(z_j \frac{\partial}{\partial z_j}\Big)^2 - k \sum_{j=1}^N \sum_{l \neq j} \frac{z_j + z_l}{z_j - z_l} \cdot z_j \frac{\partial}{\partial z_j}$$

$$= -\sum_{j=1}^N z_j^2 \frac{\partial^2}{\partial z_j^2} - (1 - k(N-1)) \sum_{j=1}^N z_j \frac{\partial}{\partial z_j} - 2k \sum_{j=1}^N \sum_{l \neq j} \frac{z_j^2}{z_j - z_l} \frac{\partial}{\partial z_j},$$

acting on permutation invariant functions from $C^2(\mathbb{T}^N)$. The operator H_k appears in a prominent way in the particle models of Calogero-Sutherland type on $\mathbb{T}$; see Section 2 of [LV]. It is obtained from the Calogero-Sutherland Hamiltonian by

conjugation with the ground state. The operator H_k is the generator of the Feller diffusions $Z_k := (Z_{t,k} := e^{iX_{t,k}})_{t \geq 0}$ on the alcove

$$\mathbb{A}_N := \{e^{ix} : x \in C_N\} \subset \mathbb{T}^N.$$

Further, the operators $\widetilde{H}_k := \frac{1}{k}H_k$ generate the diffusions $\widetilde{Z}_k := (\widetilde{Z}_{t,k} := e^{i\widetilde{X}_{t,k}})_{t \geq 0}$ for $k \in]0, \infty[$. Clearly, this also works for $k = \infty$ where $\widetilde{Z}_\infty$ is deterministic for deterministic initial data.

REMARK 3.3. Besides the generators in (3.5), also the operators

$$(3.6) \qquad D_k := \sum_{j=1}^{N} z_j^2 \frac{\partial^2}{\partial z_j^2} + 2k \sum_{j=1}^{N} \sum_{l \neq j} \frac{z_j^2}{z_j - z_l} \frac{\partial}{\partial z_j}$$

appear in the literature; see e.g. [**F, St, OO**]. Here $-H_k = D_k + E_k$ with the Euler operator

$$(3.7) \qquad E_k := (1 - k(N-1)) \sum_{j=1}^{N} z_j \frac{\partial}{\partial z_j}$$

which commutes with D_k. The $-D_k$ are generators of diffusions with additional drift on $\mathbb{T}$ which rotates the complete system at some fixed speed. Clearly, the subsequent results can be easily translated to $-D_k$. From a stochastic point of view, the diffusions associated with the operators H_k seem to be the most natural ones.

We recall that the (symmetric) eigenfunctions of H_k are Jack polynomials. To become precise, we introduce the following notations. We write

$$\Lambda_N^+ = \{\lambda \in \mathbb{Z}_+^N : \lambda_1 \geq \cdots \geq \lambda_N\}$$

for the set of partitions of length at most N. Denote by C_λ^α, $\lambda \in \Lambda_N^+$, the Jack polynomials of index $\alpha > 0$ in N variables with the normalization

$$(3.8) \qquad (z_1 + \cdots + z_N)^m = \sum_{|\lambda|=m} C_\lambda^\alpha(z) \quad (m \in \mathbb{Z}_+);$$

see [**St, BF**]. The C_λ^α are symmetric and homogeneous of degree $|\lambda| := \lambda_1 + \cdots + \lambda_N$. Moreover, by [**St, BF**], the C_λ^α with index $\alpha = 1/k$ are eigenfunctions of D_k with eigenvalues

$$d_\lambda(k) = \sum_{j=1}^{N} \lambda_j (\lambda_j - 1 + 2k(N-j)).$$

In addition, as C_λ^α is homogeneous of degree $|\lambda|$,

$$E_k C_\lambda^\alpha = (1 - k(N-1))|\lambda| C_\lambda^\alpha.$$

In summary we obtain:

LEMMA 3.4. *For $\lambda \in \Lambda_n^+$, $k \in]0, \infty[$ and $\alpha = 1/k$, C_λ^α is an eigenfunction of $\widetilde{H}_k = \frac{1}{k}H_k$ with eigenvalue*

$$-\sum_{j=1}^{N} \lambda_j \Big(\frac{\lambda_j}{k} + N + 1 - 2j\Big) \leq 0.$$

It is well-known (c.f. [**HO**]) that the polynomials $C_\lambda^{1/k}$, $\lambda \in \Lambda_n^+$ form a complete orthogonal system of $L^2(\mathbb{T}^N, \mu_k)^W$ with the probability measure

$$(3.9) \qquad d\mu_k(z) = \phi_k(z)dz, \ \ \phi_k(z) := c_k \cdot \prod_{j,l:l\neq j} |z_j - z_l|^k \cdot \mathbf{1}_{\mathbb{A}_N}(z),$$

where $c_k > 0$ is a normalization constant and dz denotes the Haar measure on $\mathbb{T}^N$.

We also recapitulate that the elementary symmetric polynomials e_l $(l = 0, \ldots, N)$ in N variables, which are determined by

$$\prod_{j=1}^N (y - x_j) = \sum_{l=0}^N (-1)^l e_l(x)\, y^{N-l} \quad (y \in \mathbb{C}),$$

are Jack polynomials for all $k > 0$ up to normalization. More precisely, by (3.8),

$$(3.10) \qquad e_l(x) = \frac{1}{l!} C_\lambda^\alpha(x) \ \text{ with } \ \lambda = 1^l = (\underbrace{1, \ldots, 1}_{l \text{ times}}, 0, \ldots, 0) \quad (l = 0, \ldots, N).$$

Thus in view of Lemma 3.4, the polynomials e_l are eigenfunctions of $\widetilde{H}_k$ with eigenvalues $-l\left(\frac{1}{k} + N - l\right)$. Moreover, by a continuity argument or by direct computation, the e_l are also eigenfunctions of

$$\widetilde{H}_\infty = (N-1) \sum_{j=1}^N z_j \frac{\partial}{\partial z_j} - 2 \sum_{j=1}^N \sum_{l \neq j} \frac{z_j^2}{z_j - z_l} \frac{\partial}{\partial z_j}$$

with eigenvalues $-l(N - l)$.

We now use these properties to construct martingales from our processes $\widetilde{Z}_k$ for $k \in]0, \infty[$. We recall that by Dynkin's formula (see e.g. Section III.10 of [**RW**]), the following holds for the generator L of a Feller semigroup and an arbitrary associated Feller process $(X_t)_{t\geq 0}$: if f is a bounded eigenfunction of L with eigenvalue $r \in \mathbb{R}$, then the process $(e^{-rt} f(X_t))_{t\geq 0}$ is a martingale w.r.t. the canonical filtration. We thus have:

COROLLARY 3.5. *For $k \in]0, \infty[$ and $z \in \mathbb{A}_N$ consider the diffusion $(\widetilde{Z}_{t,k})_{t\geq 0}$ on $\mathbb{A}_N$ with start in z. Then, for $l = 0, 1, \ldots, N$, the process $(e^{l(1/k+N-l)t} e_l(\widetilde{Z}_{t,k}))_{t\geq 0}$ is a martingale. In particular, for $t \geq 0$,*

$$\mathbb{E}(e_l(\widetilde{Z}_{t,k})) = e^{-l(1/k+N-l)t}\, e_l(z).$$

This statement also holds in the deterministic case $k = \infty$, where we also obtain additional information for $t \to \infty$:

COROLLARY 3.6. *For each starting point z in the interior of $\mathbb{A}_N$, the deterministic process $(\widetilde{Z}_{t,\infty})_{t\geq 0}$ satisfies*

$$(3.11) \qquad e_l(\widetilde{Z}_{t,\infty}) = e^{-l(N-l)t} e_l(z) \qquad (l = 0, 1, \ldots, N).$$

Moreover, the limit $Z := \lim_{t\to\infty} \widetilde{Z}_{t,\infty} \in \mathbb{T}^N$ exists and is given by

$$(3.12) \qquad Z = (Z_1, Z_1 e^{2\pi i/N}, \ldots, Z_1 e^{2\pi i(N-1)/N})$$

where $Z_1 \in \mathbb{T}$ is as follows: If $z = (e^{ix_1}, \ldots, e^{ix_N})$ with $(x_1, \ldots, x_N) \in C_N$, then

$$(3.13) \qquad Z_1 = e^{ix_0} \quad \text{with} \quad x_0 = \frac{x_1 + \ldots + x_N - \pi(N-1)}{N}.$$

PROOF. Eq. (3.11) is obvious. As each $\zeta \in \mathbb{A}_N$ is uniquely determined by the elementary symmetric functions $e_l(\zeta)$, (3.11) and a continuity argument imply that $Z = (Z_1, \ldots, Z_N) := \lim_{t \to \infty} \widetilde{Z}_{t,\infty} \in \mathbb{A}_N$ exists, and that

$$\prod_{j=1}^{N}(y - Z_j) = y^N - e_N(Z) = y^N - z_1 z_2 \cdots z_N.$$

This shows that Z has the form as stated in (3.12) for some $Z_1 \in \mathbb{T}$ with $Z_1^N = z_1 \cdots z_n$. To identify Z_1, we write the initial condition as $z = (e^{ix_1}, \ldots, e^{ix_N})$ with $(x_1, \ldots, x_N) \in C_N$. In the x-coordinates, our process $(\widetilde{X}_{t,\infty} = (x_{t,1}, \ldots, x_{t,N}))_{t \geq 0}$ satisfies the ODE (3.3). This ODE yields that $x_{t,1} + \ldots + x_{t,N}$ is independent of $t \in [0, \infty]$. Therefore, the form of $Z = (e^{ix_{\infty,1}}, \ldots, e^{ix_{\infty,N}})$ in (3.12) yields

$$x_1 + \ldots + x_N = x_{\infty,1} + \ldots + x_{\infty,N} = \sum_{j=0}^{N-1}(x_{\infty,1} + j \cdot 2\pi/N) = Nx_{\infty,1} + (N-1)\pi.$$

This implies (3.13). $\qquad\square$

In the next step we use the decomposition of the processes $\widetilde{X}_k = \widetilde{X}_k^{diff} + \widetilde{X}_k^{cg}$. Then, by Eq. (3.4),

$$\mathbb{E}(e^{i\widetilde{X}_{t,k,j}^{cg}}) = \frac{t}{Nk} \quad (j = 1, \ldots, N).$$

This, Corollary 3.5, and the independence of $\widetilde{X}_k^{diff}, \widetilde{X}_k^{cg}$ yield:

COROLLARY 3.7. *Let $k \in]0, \infty[$ and $x \in C_N$ with $x_1 + \ldots + x_N = 0$. Let $z = e^{ix} \in \mathbb{A}_N$, and consider the diffusion $(\widetilde{Z}_{t,k})_{t \geq 0}$ on $\mathbb{A}_N$ with start in z. Then, for $t \geq 0$ and $l = 1, \ldots, N$,*

$$\mathbb{E}(e_l(e^{i\widetilde{X}_{t,k}^{diff}})) = e^{-l(N-l)(1+1/(Nk))t} e_l(z).$$

In particular, for all l, N and $t \geq 0$ and all starting points z, the expectation

$$\mathbb{E}(e_l(e^{i\widetilde{X}_{t/(1+1/(Nk)),k}^{diff}}))$$

is independent of $k \in]0, \infty]$.

EXAMPLE 3.8. For the starting configuration $x = (0, 2\pi/N, \ldots, (N-1)2\pi/N)$, we have $e_1(z) = \ldots = e_{N-1}(z) = 0$ and $e_N(z) = (-1)^{N-1}$. Hence, by Corollary 3.7,

$$\mathbb{E}(e_l(e^{i\widetilde{X}_{t,k}^{diff}})) = 0 \quad (l = 1, \ldots, N-1), \quad \text{and} \quad \mathbb{E}(e_N(e^{i\widetilde{X}_{t,k}^{diff}})) = (-1)^{N-1}$$

for all $t \geq 0$. We conclude that in this case for all $y \in \mathbb{C}$ and $t \geq 0$,

$$(3.14) \qquad \mathbb{E}\Big(\prod_{j=0}^{N-1}(y - e^{i\widetilde{X}_{t,k,j}^{diff}})\Big) = \prod_{j=1}^{N}(y - e^{j \cdot 2\pi i/N}) = y^N - 1.$$

We next study the limit $t \to \infty$ for $k \in]0, \infty[$ similar to the limit results for $k = \infty$ in Corollary 3.5. We need the following well-known result, which follows easily for instance from the explicit densities of our diffusions in [**RR1**]:

LEMMA 3.9. *Let $k \in]0, \infty[$. Then for each starting point in $\mathbb{A}_N$, the process $(\widetilde{Z}_{t,k})_{t \geq 0}$ converges in distribution for $t \to \infty$ to the probability measure μ_k on $\mathbb{A}_N$ from (3.9).*

This observation and Corollary 3.5 imply:

COROLLARY 3.10. *Let Z be an $\mathbb{A}_N$-valued random variable with distribution μ_k. Then for each $y \in \mathbb{C}$,*

$$\mathbb{E}\big(\prod_{j=1}^{N}(y - Z_j)\big) = y^N.$$

A corresponding result can be also stated under the condition that Z only takes values in the alcove

$$\mathbb{A}_N^1 := \{z \in \mathbb{A}_N : z_1 \cdots z_N = 1\}.$$

For this consider the process $\widetilde{X}_k^{diff}$ in the decomposition $\widetilde{X}_k = \widetilde{X}_k^{diff} + \widetilde{X}_k^{cg}$ above. Then $e^{i\widetilde{X}_k^{diff}}$ is a diffusion on $\mathbb{A}_N^1$ which converges for $t \to \infty$ in distribution to the conditional probability measure $\mu_k^1 \in M^1(\mathbb{A}_N^1)$ of μ_k under the condition $\mathbb{A}_N^1$. (This is, up to normalization, the measure with Haar density (2.4)). Corollary 3.7 now leads to:

COROLLARY 3.11. *Let Z be an $\mathbb{A}_N^1$-valued random variable with distribution μ_k^1. Then for each $y \in \mathbb{C}$,*

$$\mathbb{E}\big(\prod_{j=1}^{N}(y - Z_j)\big) = y^N - 1.$$

Corollaries 3.5, 3.7, 3.10, and 3.11 have applications to Brownian motions and uniform probabilities on compact symmetric spaces of type A.

For a first example, consider the space $C(U(N))$ of all conjugacy classes of $U(N)$, which can be identified with $\mathbb{A}_N$ up to the cyclic group $\mathbb{Z}_N$, i.e., $C(U(N)) \sim \mathbb{A}_N/\mathbb{Z}_N$. In fact, the conjugacy classes are characterized via the ordered spectra of matrices from $U(N)$ where, say, the eigenvalue with the smallest nonnegative argument has the first position. On the other hand, elements in $\mathbb{A}_N$ describe configurations of ordered points on $\mathbb{T}$ where the position of the first entry is arbitrary. It is also well known that the pushforward of the uniform distribution (i.e., the normalized Haar measure) of $U(N)$ under the natural projection $U(N) \mapsto C(U(N)) \sim \mathbb{A}_N/\mathbb{Z}_N$ agrees with the pushforward of μ_k with $k = 1$ under the canonical mapping $\mathbb{A}_N \mapsto \mathbb{A}_N/\mathbb{Z}_N$. Corollary 3.10 thus leads to the following.

COROLLARY 3.12. *Let Z be a uniformly distributed $U(N)$-valued random variable. Then for each $y \in \mathbb{C}$, $\mathbb{E}\big(\det(yI_N - Z)\big) = y^N$.*

The same procedure also works for $SU(N)$, where the space $C(SU(N))$ of conjugacy classes corresponds to $\mathbb{A}_N^1/\mathbb{Z}_N$, and the pushforward of the uniform distribution corresponds to $\mu_k^1 \in M^1(\mathbb{A}_N^1)$ with $k = 1$. Hence, by Corollary 3.11:

COROLLARY 3.13. *Let Z be an $SU(N)$-valued random variable which is uniformly distributed. Then for $y \in \mathbb{C}$, $\mathbb{E}\big(\det(yI_N - Z)\big) = y^N - 1$.*

Corollaries 3.12 and 3.13 are special cases of well-known general formulas for integrals of polynomials on unitary groups w.r.t. uniform distributions in [**CS**]. On the other hand, our approach leads to generalizations of these formulas for Brownian motions on $U(N)$ and $SU(N)$.

For $k = 1/2$ or $k = 2$, our results above are related to the compact symmetric spaces $U(n)/O(n)$ and $U(2n)/Sp(n)$ associated with the root system A_{N-1}.

4. The non-compact case of type A_{N-1}

In this section we start with the W-invariant Heckman-Opdam Laplacians

$$(4.1) \qquad L_k f(x) = \Delta f(x) + k \sum_{j=1}^{N} \sum_{l \neq j} \coth\Big(\frac{x_j - x_l}{2}\Big) \frac{\partial}{\partial x_j} f(x)$$

for $k \in [0, \infty[$ on the Weyl chamber

$$C_N^A := \{x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N\}$$

of type A_{N-1}. As in Section 3, L_k is the generator of a Feller diffusion $(X_{t,k})_{t \geq 0}$ on C_N^A with reflecting boundaries. Again we study the renormalized generators $\widetilde{L}_k := \frac{1}{k} L_k$ which degenerate for $k \to \infty$ into

$$\widetilde{L}_\infty = \sum_{j=1}^{N} \sum_{l \neq j} \coth\Big(\frac{x_j - x_l}{2}\Big) \frac{\partial}{\partial x_j}.$$

The process $\widetilde{X}_k := (\widetilde{X}_{t,k} := X_{t/k,k})_{t \geq 0}$ for $k \in]0, \infty[$ then solves the SDE

$$(4.2) \qquad d\widetilde{X}_{t,k,j} = \frac{\sqrt{2}}{\sqrt{k}} dB_{t,j} + \sum_{l \neq j} \coth\Big(\frac{\widetilde{X}_{t,k,j} - \widetilde{X}_{t,k,l}}{2}\Big) dt \qquad (j = 1, \ldots, N)$$

which degenerates for $k = \infty$ into the ODE

$$(4.3) \qquad \frac{dx_j}{dt}(t) = \sum_{l \neq j} \coth\Big(\frac{x_j(t) - x_l(t)}{2}\Big) \qquad (j = 1, \ldots, N).$$

Again, for initial data in the interior of the chamber C_N^A, the solution $(\widetilde{X}_{t,\infty})_{t \geq 0}$ of these differential equations exists for all $t \geq 0$ in the interior of C_N^A.

Moreover, as in Section 3, the trigonometric elementary symmetric polynomials

$$\widetilde{e}_l(x) := e_l(e^x) \quad (l = 0, \ldots, N, \ x \in C_N^A)$$

are eigenfunctions of $\widetilde{L}_k$ for all k with eigenvalues $l\big(\frac{1}{k} + N - l\big) \geq 0$. This leads to martingales for the diffusions $(\widetilde{X}_{t,k})_{t \geq 0}$ on C_N^A, similar to Corollaries 3.5 and 3.6. However, in the present non-compact case, one has to ensure that the random variables $\widetilde{e}_l(\widetilde{X}_{t,k})$ are integrable. For the starting point $0 \in C_N^A$, this is obvious from Theorem 5.2 of [**Sch1**]. For general starting points, this integrability property can be shown by methods from stochastic analysis in [**Sch2**]. In summary, we get:

COROLLARY 4.1. *For $k \in]0, \infty[$ and $x \in C_N^A$ consider the diffusion $(\widetilde{X}_{t,k})_{t \geq 0}$ on C_N^A with start in x. Then, for $l = 0, 1, \ldots, N$, the process $\big(e^{-l(1/k+N-l)t} \widetilde{e}_l(\widetilde{X}_{t,k})\big)_{t \geq 0}$ is a martingale. In particular, for $t \geq 0$,*

$$\mathbb{E}\big(\widetilde{e}_l(\widetilde{X}_{t,k})\big) = e^{l(1/k+N-l)t} \widetilde{e}_l(x).$$

This result also holds for $k = \infty$. More precisely, the solution $(\widetilde{X}_{t,\infty})_{t \geq 0}$ of the ODE (4.3) with start x in the interior of C_N^A satisfies

$$(4.4) \qquad \widetilde{e}_l(\widetilde{X}_{t,\infty}) = e^{l(N-l)t} \widetilde{e}_l(x) \qquad (l = 0, 1, \ldots, N).$$

As in the preceding section, the diffusions $\widetilde{X}_k$ can be decomposed into the center of gravity $\widetilde{X}_k^{cg} := (\widetilde{X}_{t,k}^{cg})_{t\geq 0}$ with

$$\widetilde{X}_{t,k}^{cg} := \frac{1}{N}(\widetilde{X}_{t,k,1} + \ldots + \widetilde{X}_{t,k,N}) \cdot \omega$$

and the process $\widetilde{X}_k^{diff} := \widetilde{X}_k - \widetilde{X}_k^{cg}$ on the Weyl chamber

$$C_{N,0}^A := \{x \in C_N^A : x_1 + \ldots + x_N = 0\}.$$

As in the proof of Lemma 3.2, the processes $\widetilde{X}_k^{diff}$ and $\widetilde{X}_k^{cg}$ are stochastically independent. Hence, we obtain the following analog of Corollary 3.7.

COROLLARY 4.2. *For $k \in]0, \infty[$ and $x \in C_{N,0}^A$ consider the diffusion $\widetilde{X}_k^{diff}$ on the chamber $C_{N,0}^A$ with start in x. Then, for $t \geq 0$ and $l = 1, \ldots, N$,*

$$\mathbb{E}\big(\widetilde{e}_l(\widetilde{X}_{t,k}^{diff})\big) = e^{l(N-l)(1+1/(Nk))t}\,\widetilde{e}_l(x).$$

This holds also for $k = \infty$ where the expectations can be omitted.

In particular, for all l, N and $t \geq 0$ and all starting points $x \in C_{N,0}^A$, the expectation

$$\mathbb{E}\big(\widetilde{e}_l(\widetilde{X}_{t/(1+1/(Nk)),k}^{diff})\big)$$

is independent of $k \in]0, \infty]$. Corollary 4.2 can be restated as:

COROLLARY 4.3. *For $k \in]0, \infty[$ and $x \in C_{N,0}^A$ consider the diffusion $\widetilde{X}_k^{diff}$ on $C_{N,0}^A$ with start in x. Then, for $t \geq 0$ and $y \in \mathbb{C}$,*

$$\mathbb{E}\big(\prod_{j=1}^N (y - e^{\widetilde{X}_{k,t,j}^{diff}})\big) = P_{(1+1/(Nk))t,N,x}(y)$$

with the polynomial

$$P_{t,N,x}(y) := \sum_{l=0}^N y^{N-l}(-1)^l e^{l(N-l)t}\,\widetilde{e}_l(x).$$

For general starting points $x \in C_{N,0}^A$, the polynomials $P_{t,N,x}$ do not seem to have particularly nice properties. For the starting configuration $x = 0 \in C_{N,0}^A$, we get

$$P_{t,N,0}(y) = \sum_{l=0}^N \binom{N}{l}(-1)^l y^{N-l} e^{l(N-l)t}.$$

We do not have much information about these polynomials. For instance, it can be easily seen that $(P_{t,N,0})_{N\geq 0}$ does not satisfy a three-term-recurrence and thus cannot be orthogonal by Favard's theorem. This is a contrast to the Bessel processes of type A where in a corresponding formula classical Hermite polynomials appear; see [**KVW**].

EXAMPLE 4.4.　　(1) Let $N = 2$. We try to solve the ODE (4.3) with the singular starting point $x = 0 \in C_2^A$. In fact, (4.3) yields that $x_1(t) = -x_2(t)$ and that $y(t) := x_1(t) - x_2(t)$ satisfies $\frac{dy}{dt}(t) = 2\coth(y(t)/2)$. On the other hand, (4.4) for $N = 2$, $l = 1$ suggests that

$$e^{x_1(t)} + e^{-x_1(t)} = 2e^t$$

and thus $x_1(t) = \mathrm{arcosh}(e^t)$. It is now easily checked that in fact,

$$t \mapsto (\mathrm{arcosh}(e^t), -\mathrm{arcosh}(e^t))$$

is continuous on $[0, \infty)$ and solves (4.3) for $t > 0$. Moreover, for $k = \infty$ we have $x^{diff}(t) = x(t)$, and

$$P_{t,2,0}(y) = y^2 - 2e^t y + 1.$$

These polynomials have the zeros $e^{\pm x_1(t)}$ as claimed.

(2) Let $N = 3$. We again try to solve the ODE (4.3) with $x = 0 \in \partial C_3^A$. Here symmetry arguments imply that the solution of the ODE (4.3) must have the form $x(t) = (x_1(t), 0, -x_1(t))$. On the other hand, formula (4.4) with $N = 2$, $l = 1, 2$ suggests that

$$e^{x_1(t)} + e^{-x_1(t)} + 1 = 3e^{2t}$$

and thus $x_1(t) = \mathrm{arcosh}\big(3(e^{2t} - 1)/2\big)$. This indeed gives a solution as in example (1). Moreover, the polynomial

$$P_{t,3,0}(y) = y^3 - 3e^{2t}y^2 + 3e^{2t}y - 1$$

has the zeros 1 and $e^{\pm x_1(t)}$ as claimed.

Unfortunately, we have no closed formulas for $x(t)$ for general dimension N for the starting point $x = 0 \in C_N^A$.

We finally mention that Corollaries 4.2 and 4.3 have applications to Brownian motions on the noncompact symmetric spaces of type A, i.e. on $GL(N, \mathbb{F})/U(N, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

5. The non-compact case of type BC_N

We here start with the nonreduced root system

$$R = BC_N = \{\pm e_i, \pm 2e_i, \pm(e_i \pm e_j);\ 1 \le i < j \le N\} \subset \mathbb{R}^N$$

for $N \ge 2$ with the multiplicity $k = (k_1, k_2, k_3)$, where $k_1, k_2 \ge 0$, $k_3 > 0$ are the values on the roots $e_i, 2e_i, e_i \pm e_j$. As indicated in the introduction, we now reparametrize the multiplicity k. This will be not natural at a first glance, but it will turn out to be useful in the end. We here mainly follow the notations in [**Dem, V**] and define

$$(5.1) \qquad \kappa = k_3, \quad q = N - 1 + \frac{1 + 2k_1 + 2k_2}{2k_3}, \quad p = N - 1 + \frac{1 + 2k_2}{2k_3}.$$

Then $\kappa > 0$, $q \ge p \ge N - 1 + 1/2\kappa$, and

$$(5.2) \qquad k = (k_1, k_2, k_3) = \kappa \cdot \Big(q - p, k_{0,2}, 1\Big) \quad \text{with} \quad k_{0,2} := p - (N - 1) - \frac{1}{2\kappa}.$$

We now regard p, q as fixed parameters and $\kappa > 0$ as a varying parameter, and denote the multiplicity k in (5.2) by k_κ. The associated W-invariant Heckman-Opdam Laplacian (2.6) is

$$L_\kappa = \Delta + \sum_{i=1}^N \kappa\Big((q - p)\coth(\frac{x_i}{2}) + 2k_{0,2}\coth(x_i)$$

$$(5.3) \qquad\qquad + \sum_{j:j\neq i} \big(\coth(\frac{x_i - x_j}{2}) + \coth(\frac{x_i + x_j}{2})\big)\Big)\partial_i$$

on the Weyl chamber

$$C_N^B := \{x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \geq 0\}$$

of type B_N, associated with $R_+ = \{e_i, 2e_i, e_i - e_j : 1 \leq i < j \leq N\}$. As in the preceding sections, the operators L_κ are the generators of diffusions $(X_{t,\kappa})_{t \geq 0}$ on C_N^B where the paths are reflected at ∂C_N^B. The renormalized operators

$$\widetilde{L}_\kappa := \frac{1}{\kappa} L_\kappa$$

then are the generators of the diffusions $(\widetilde{X}_{t,\kappa} := X_{t/\kappa,\kappa})_{t \geq 0}$ which may be regarded as solutions of the SDE

$$d\widetilde{X}_{t,\kappa,j} = \frac{\sqrt{2}}{\sqrt{\kappa}} dB_{t,j} + \sum_{l \neq j} \Big(\coth\Big(\frac{\widetilde{X}_{t,\kappa,j} - \widetilde{X}_{t,\kappa,l}}{2}\Big) + \coth\Big(\frac{\widetilde{X}_{t,\kappa,j} + \widetilde{X}_{t,\kappa,l}}{2}\Big) \Big) dt$$

$$(5.4) \qquad + \Big((q-p) \coth\Big(\frac{\widetilde{X}_{t,\kappa,j}}{2}\Big) + 2k_{0,2} \coth(\widetilde{X}_{t,\kappa,j}) \Big) dt \qquad (j = 1, \ldots, N).$$

For $\kappa \to \infty$, the generator degenerates into

$$\widetilde{L}_\infty = \sum_{i=1}^N \Big((q-p) \coth(\frac{x_i}{2}) + 2(p - (N-1)) \coth(x_i)$$

$$(5.5) \qquad\qquad + \sum_{j:j \neq i} \Big(\coth(\frac{x_i - x_j}{2}) + \coth(\frac{x_i + x_j}{2}) \Big) \Big) \partial_i,$$

and (5.4) becomes the ODE

(5.6)
$$\frac{dx_j}{dt}(t) = \sum_{l \neq j} \Big(\coth\Big(\frac{x_j(t) - x_l(t)}{2}\Big) + \coth\Big(\frac{x_j(t) + x_l(t)}{2}\Big) \Big)$$

$$+ (q-p) \coth\Big(\frac{x_j(t)}{2}\Big) + 2\big(p - (N-1)\big) \coth x_j(t) \qquad (j = 1, \ldots, N).$$

Again, for initial data in the interior of the chamber C_N^B, the solution $(\widetilde{X}_{t,\infty})_{t \geq 0}$ of these differential equations exists for $t \geq 0$ in the interior of C_N^B.

   We now turn to eigenfunctions of the operators $\widetilde{L}_\kappa$ and consider the Heckman-Opdam hypergeometric functions $F_{BC}(\lambda, k_\kappa; x)$ associated with BC_N in the variable $x \in \mathbb{R}^N$, with spectral parameter $\lambda \in \mathbb{C}^N$ and multiplicity k_κ. By Section 2, for $\kappa \in]0, \infty[$, the functions $x \mapsto F_{BC}(\lambda, k_\kappa; x)$ are eigenfunctions of the renormalized Laplacian $\widetilde{L}_\kappa$ with the eigenvalues

$$r_{\lambda,\kappa} := \frac{1}{\kappa} \Big(\sum_{j=1}^N \lambda_j^2 - \|\rho(\kappa)\|_2^2 \Big),$$

where

$$(5.7) \qquad \rho(\kappa) = \frac{1}{2} \sum_{\alpha \in R_+} k_\kappa(\alpha)\alpha \quad \text{with} \quad \rho(\kappa)_j = \frac{\kappa}{2}\big((q-p) + 2k_{0,2} + 2(N-j)\big).$$

   We further consider the associated (normalized) Heckman-Opdam polynomials $R_\lambda(k, .) = R_\lambda^{BC}(k, .)$, as introduced in (2.5), which are indexed by $P_+ = \Lambda_N^+$ (the set of partitions of length at most N). These are multivariate generalizations of

the classical Jacobi polynomials which are well-studied in the literature, see for instance [**BO, L, RR2**]. The polynomials $\widetilde{R}_\lambda$ defined by

$$\widetilde{R}_\lambda(\cos x) := R_\lambda(k; x)$$

form an orthogonal basis of $L^2(\mathbb{A}_N, w_k)$ on

$$\mathbb{A}_N := \{x \in \mathbb{R}^N|\ -1 \le y_1 \le \ldots \le y_N \le 1\}$$

with the weight function

$$(5.8) \qquad w_k(y) := \prod_{i=1}^{N}(1 - y_i)^{k_1+k_2-1/2}(1 + y_i)^{k_2-1/2} \cdot \prod_{i<j}|y_i - y_j|^{2k_3}.$$

Here we are mainly interested in the fact that for $k = k_\kappa$ and $\lambda \in \Lambda_N^+$, the exponential polynomials

$$H_\lambda(x) := \widetilde{R}_\lambda(\cosh x) = F_{BC}(\lambda + \rho(k), k; x)$$

are eigenfunctions of $\widetilde{L}_\kappa$ with the eigenvalues

$$r_\lambda = \frac{1}{\kappa}\langle\lambda, \lambda + 2\rho(k)\rangle$$

$$= \frac{1}{\kappa}\sum_{j=1}^{N}\lambda_j(\lambda_j + k_1 + 2k_2 + 2k_3(N - j))$$

$$(5.9) \qquad = \sum_{j=1}^{N}\lambda_j\Big(\frac{\lambda_j - 1}{\kappa} + p + q + 2 - 2j\Big).$$

We now consider the partitions $\lambda(n) := 1^n \in \Lambda_+^N$ for $n = 0, \ldots, N$. It is known (see Section 5 of [**V**] and in particular Lemma 5.1 there) that the Jacobi polynomials $\widetilde{R}_{\lambda(n)}$ are of the form

$$(5.10) \qquad \widetilde{R}_{\lambda(n)} = \sum_{l=0}^{n} c_{n,l}(p, q) \cdot e_l \quad \text{with } c_{n,n}(p, q) \neq 0,$$

where the e_l are again the elementary symmetric polynomials in N variables and the coefficients $c_{n,l}(p, q) \in \mathbb{R}$ depend on p, q only and not on κ. This observation will be crucial in the following and is the reason for our parametrization of k by p, q, κ above. For more details on the $c_{n,l}(p, q)$ we refer to [**V**]. In summary, the functions $H_{\lambda(n)}$ with $n = 0, \ldots, N$ are independent of κ and simultaneous eigenfunctions of the operators $\widetilde{L}_\kappa$ for all $\kappa \in\ ]0, \infty[$ with the eigenvalues

$$r_n = n(p + q - n + 1).$$

Clearly, this observation also holds for $\kappa = \infty$. This implies:

LEMMA 5.1. *For each starting point* $x \in C_N^B$ *of the processes* $(\widetilde{X}_{t,\kappa})_{t\ge 0}$ *with* $\kappa \in\ ]0, \infty]$, *the processes*

$$(5.11) \qquad \Big(e^{-r_n t} \cdot H_{\lambda(n)}\big(\widetilde{X}_{t,\kappa}\big)\Big)_{t\ge 0}$$

are martingales for $n = 0, \ldots, N$, *where the numbers* r_n *and the functions* $H_{\lambda(n)}$ *do not depend on* κ.

In particular, for x in the interior of C_N^B, the solution $(\widetilde{X}_{t,\infty})_{t\geq 0}$ of the ODE (5.6) with $\widetilde{X}_{0,\infty} = x$ satisfies

$$H_{\lambda(n)}(\widetilde{X}_{t,\infty}) = e^{r_n t} H_{\lambda(n)}(x) \quad for \quad t \geq 0,\, n = 0, \ldots, N.$$

Proof. This follows from our preceding considerations and the fact that, similar as in the non-compact case of type A, the random variables $H_{\lambda(n)}(\widetilde{X}_{t,\kappa})$ are integrable for $\kappa < \infty$ and $t > 0$. $\qquad\square$

We may invert (5.10) and write the elementary symmetric polynomials e_l as linear combinations of the $\widetilde{R}_{\lambda(n)}$ for $l, n = 0, \ldots, N$ with coefficients independent of κ. Lemma 5.1 thus implies:

COROLLARY 5.2. *Fix some deterministic starting point $x \in C_N^B$ as well as the parameters p, q. Consider the associated diffusions $(\widetilde{X}_{t,\kappa})_{t\geq 0}$ for $\kappa \in\,]0, \infty[$. Then there are coefficients $a_{n,l} \in \mathbb{R}$ for $0 \leq l \leq n \leq N$ such that*

$$\mathbb{E}\big(e_n(\cosh(\widetilde{X}_{t,\kappa}))\big) = \sum_{l=0}^{n} a_{n,l}\, e^{r_l t}$$

with $r_0 = 0$ where the coefficients $a_{n,l}$ and the exponents r_l depend on p, q and x only and not on κ. The same holds for $\kappa = \infty$ and starting points x in the interior of C_N^B.

EXAMPLE 5.3. For $x = 0 \in C_N^B$, we have $H_{\lambda(n)}(0) = 1$ and thus

$$\mathbb{E}\big(H_{\lambda(n)}(\widetilde{X}_{t,\kappa})\big) = e^{r_n t} \qquad (n = 0, \ldots, N,\, t \geq 0,\, \kappa \in\,]0, \infty[).$$

We finally turn to an application concerning a determinantal formula. Again we fix some starting point $x \in C_N^B$ as well as p, q and consider the associated diffusions $(\widetilde{X}_{t,\kappa})_{t\geq 0}$ for $\kappa \in\,]0, \infty[$. Then by Corollary 5.2, for all $t \geq 0$ and $y \in \mathbb{C}$,

$$(5.12) \qquad \mathbb{E}\Big(\prod_{j=1}^{N}\big(y - \cosh(\widetilde{X}_{t,\kappa,j})\big)\Big) = \sum_{n=0}^{N}(-1)^n\, \mathbb{E}\big(e_n(\cosh(\widetilde{X}_{t,\kappa}))\big) \cdot y^{N-n}$$

$$= \sum_{n=0}^{N}(-1)^n\, e_n\big(\cosh(\widetilde{X}_{t,\infty})\big) \cdot y^{N-n}$$

$$= \prod_{j=1}^{N}\big(y - \cosh(\widetilde{X}_{t,\infty,j})\big) =: D_{t,x}(y).$$

It is an interesting task to find particularly nice starting points $x \in C_N^B$ for which $D_{t,x}(y)$ can be determined explicitly.

For instance, in the setting of multivariate Bessel processes of types A_{N-1} or B_N and start in the origin, $D_{t,0}(y)$ is a classical one-dimensional Hermite or Laguerre polynomial of degree N in y, which is scaled by a factor $\sqrt{t}$; for the details see [**KVW**]. Moreover, in the setting of Heckman-Opdam Jacobi processes of type BC on the compact alcove $\mathbb{A}_N$, and with the same paramatrization of the multiplicity k by p, q, κ as here, there is a (unique) stationary solution x_0 of the ODE in the interior of $\mathbb{A}_N$, whose coordinates are the ordered zeroes of some classical one-dimensional Jacobi polynomial of degree N in y. The indices of this Jacobi polynomial are determined by p, q. This means that for this particular

starting point x_0, the function $D_{t,x_0}(y)$ is just this specific Jacobi polynomial and is independent of $t \geq 0$ (due to stationarity). We refer to [**V**] for further details.

We expect that in our non-compact BC setting, $D_{t,x}(y)$ should be of particular interest when the associated ODE (5.6) starts in $x = 0 \in \partial C_N^B$, where we expect that the corresponding initial value problem is uniquely solvable, as in the Dunkl setting in [**VW**]. It seems that the explicit solution of this initial value problem is more involved than in the cases considered in [**KVW, V**].

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT PADERBORN, WARBURGER STR. 100, D-33102
PADERBORN, GERMANY
    Email address: `roesler@math.upb.de`

FAKULTÄT MATHEMATIK, TECHNISCHE UNIVERSITÄT DORTMUND, VOGELPOTHSWEG 87, D-
44221 DORTMUND, GERMANY
    Email address: `michael.voit@math.tu-dortmund.de`