LIMITS OF BESSEL FUNCTIONS FOR ROOT SYSTEMS AS THE RANK TENDS TO INFINITY

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In memory of Gerrit van Dijk

Abstract. We study the asymptotic behaviour of Bessel functions associated of root systems of type A_{n-1} and type B_n with positive multiplicities as the rank n tends to infinity. In both cases, we characterize the possible limit functions and the Vershik-Kerov type sequences of spectral parameters for which such limits exist. In the type A case, this generalizes known results about the approximation of the (positive-definite) Olshanski spherical functions of the space of infinite-dimensional Hermitian matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (with the action of the associated infinite unitary group) by spherical functions of finite-dimensional spaces of Hermitian matrices. In the type B case, our results include asymptotic results for the spherical functions associated with the Cartan motion groups of non-compact Grassmannians as the rank goes to infinity, and a classification of the Olshanski spherical functions of the associated inductive limits.

1. Introduction

The asymptotic analysis of multivariate special functions has a long tradition in infinite dimensional harmonic analysis, tracing back to the work of Olshanski, Vershik, and Kerov, see [Ol90, OV96, VK82]. Of particular interest in this context are the behaviour of spherical representations and the limits of spherical functions of increasing families of Gelfand pairs as specific dimensions tend to infinity.

Bessel functions associated with root systems generalize the spherical functions of Riemannian symmetric spaces of Euclidean type, which occur for special values of the multiplicity parameters. They appear naturally in rational Dunkl theory, with an intimate connection to the Dunkl kernel and the associated harmonic analysis. We refer to [Op93] for a general treatment of such Bessel functions and to [Ro03, dJ06, RV08, DX14] for an overview of rational Dunkl theory including the connection with symmetric spaces. There are two classes of particular interest, including applications to β-ensembles in random matrix theory, namely those of type A_{n-1} and type B_n. We refer to [Fo10] for a general background and to [BCG22] for some recent developments. In the cases of type A and B, the Bessel functions can be expressed as hypergeometric series involving Jack polynomials, c.f. Section 2. Bessel functions of type A_{n-1} have a continuous multiplicity parameter $k \geq 0$ and include as special cases the spherical functions of the motion groups $U_n(\mathbb{F}) \ltimes H_n(\mathbb{F})$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where the unitary group $U_n(\mathbb{F})$ acts by conjugation on the space $H_n(\mathbb{F})$ of Hermitian matrices over $\mathbb{F}$. These cases correspond to $k = \frac{d}{2}$ with $d = \dim_{\mathbb{R}}\mathbb{F} \in \{1, 2, 4\}$. Bessel functions of type B_q have non-negative multiplicity

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parameters of the form $\kappa = (k', k)$, with k the multiplicity on the roots $\pm(e_i \pm e_j)$ and k' that on the roots $\pm e_i$. They generalize the spherical functions of the motion groups $(U_p(\mathbb{F}) \times U_q(\mathbb{F})) \ltimes M_{p,q}(\mathbb{F})$, with $p \geq q$. Here the multiplicities are $k = \frac{d}{2}$, $k' = \frac{d}{2}(p - q + 1) - \frac{1}{2}$. In [RV13], the limits of the spherical functions of these motion groups as $p \to \infty$ and the associated Olshanski spherical pairs were studied, where the rank q remained fixed.

In the present paper, we shall study Bessel functions of type A_{n-1} and type B_n with arbitrary positive multiplicities as the rank tends to infinity, in the spirit of the work of Okounkov and Olshanski [OO98, OO06] about Jack polynomials (type A) and multivariate Jacobi polynomials (type BC). See also [Cu18] for a more recent extension of their results.

We obtain explicit asymptotic results for Bessel functions of type A and type B with arbitrary positive multiplicities as the rank goes to infinity. In the type A case, our results generalize those of [OV96] and [Bo07] for the limits of the spherical functions of the Gelfand pairs $(U_n(\mathbb{F}) \ltimes H_n(\mathbb{F}), U_n(\mathbb{F}))$ as $n \to \infty$. In contrast to [Bo07], whose results in the geometric cases are also weaker than ours (c.f. Remark 3.6), we follow the direct approach of [OV96] for $\mathbb{F} = \mathbb{C}$ via spherical expansions of the involved Bessel functions, which are replaced by hypergeometric expansions in terms of Jack polynomials in our general setting. To become more precise, we consider the Bessel functions $J_{A_{n-1}}(i\lambda(n), (x, 0, \ldots, 0))$ with fixed multiplicity $k > 0$ and $x \in \mathbb{R}^r$ for sequences of spectral parameters $\lambda(n) \in \mathbb{R}^n$ as $n \to \infty$. Following [OV96, OO98], we characterize those sequences $(\lambda(n))_{n\in\mathbb{N}}$ for which the associated sequence of Bessel functions converges (locally uniformly), in terms of specific real parameters $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, β, γ with $\gamma \geq 0$. These parameters describe the growth of the so-called Vershik-Kerov sequence $(\lambda(n))$ as $n \to \infty$. In Theorem 3.5, the main result of Section 3, we obtain that for $x \in \mathbb{R}^{(\infty)} = \cup_{n=1}^{\infty} \mathbb{R}^n$,

$$\lim_{n\to\infty} J_{A_{n-1}}(i\lambda(n), x) \;=\; \prod_{j=1}^{\infty} e^{i\beta x_j - \frac{\gamma}{2k} x_j^2} \prod_{l=1}^{\infty} \frac{e^{-i\alpha_l x_j}}{\left(1 - \frac{i\alpha_l x_j}{k}\right)^k},$$

where the convergence is locally uniform on $\mathbb{R}^r$ for each $r \in \mathbb{N}$. In the group cases $k = \frac{d}{2}$, the limiting functions are products of Polya functions (in the sense of [Fa08]). They coincide with the positive definite Olshanski spherical functions of the spherical pairs $(U_\infty(\mathbb{F}) \ltimes \text{Herm}_\infty(\mathbb{F}), U_\infty(\mathbb{F}))$ which were already determined by Pickrell [Pi91]; see also [OV96], where they occur in the characterization of the ergodic measures of $\text{Herm}_\infty(\mathbb{F})$ with respect to the action of $U_\infty(\mathbb{F})$.

In the type B case, we consider Bessel functions $J_{B_n}(\kappa_n, i\lambda(n), (x, 0, \ldots, 0))$ for $n \to \infty$, where the multiplicity is of the form $\kappa_n = (k'_n, k)$, i.e. the first multiplicity parameter may also vary with n. This is motivated by the geometric cases. Again we characterize the sequences $(\lambda(n))$ for which the associated Bessel functions converge locally uniformly on $\mathbb{R}^r$ as $n \to \infty$, and we determine the possible limits, which are now given by the functions

$$\varphi_{(\alpha,\beta)}(x) = \prod_{j=1}^{\infty} e^{-\frac{\beta x_j^2}{4}} \prod_{l=1}^{\infty} \frac{e^{\frac{\alpha_l x_j^2}{4}}}{\left(1 + \frac{\alpha_l x_j^2}{4k}\right)^k}, \quad x \in \mathbb{R}^{(\infty)},$$

with real parameters $\beta \geq 0$ and $\alpha_l \geq 0$ with $\sum_{l=1}^{n} \alpha_l \leq \beta$. It turns out that for $k = \frac{d}{2}$ with $d = 1, 2, 4$, these can be identified with the positive definite Olshanski spherical functions of spherical pairs (G_∞, K_∞), which are obtained as inductive limits of the motion groups $(U_p(\mathbb{F})\times U_q(\mathbb{F}))\ltimes M_{p,q}(\mathbb{F})$ as both dimension parameters p and q tend to infinity.

The organization of this paper is as follows: After a brief overview of Bessel functions associated with root systems in Section 2, the type A case is treated in Section 3. While in this case our results generalize known results in the geometric cases, our results for type B, which are developed in Section 4, seem to be new even in the geometric cases.

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2. Bessel functions of type A and type B

In this section, we recall some basic facts about Bessel functions in Dunkl theory. We shall not go into details, but refer the reader to [DX14, Ro03] for some general background on Dunkl theory, and to [Op93, dJ06, RV08] for Dunkl-type Bessel functions and their relevance in the symmetric space context. For a reduced (but not necessarily essential) root system $R\subset\mathbb{R}^n$ we fix a multiplicity function $k: R\mapsto[0,\infty)$, i.e. k is invariant under the associated Weyl group W. Let $E=E_R(k;\,.\,,\,.\,):\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ denote the associated Dunkl kernel, where $E(\lambda,\,.\,)$ is characterized as the unique analytic solution of the joint eigenvalue problem for the associated rational Dunkl operators with spectral variable $\lambda\in\mathbb{C}^n$, normalized according to $E(\lambda,0)=1$. The kernel E is holomorphic on $\mathbb{C}^n\times\mathbb{C}^n$ and symmetric in its arguments. For each $\lambda\in\mathbb{R}^n$, there exists a compactly supported probability measure μ_λ on $\mathbb{R}^n$ such that

$$E(i\lambda,x)=\int_{\mathbb{R}^n}e^{i\langle\xi,x\rangle}d\mu_\lambda(\xi)\qquad\text{for all } x\in\mathbb{R}^n. \tag{2.1}$$

In particular, the kernel $E(i\lambda,\,.\,)$ is positive-definite on the additive group $\mathbb{R}^n$. If $k=0$, then $E(i\lambda,x)=e^{i\langle\lambda,x\rangle}$. The Bessel function associated with R and k is defined by

$$J(\lambda,z)=\frac{1}{|W|}\sum_{w\in W}E(\lambda,wz).$$

It is W-invariant in both arguments and for $\lambda\in\mathbb{R}^n$, the function $J(i\lambda,\,.\,)$ is also positive-definite on $\mathbb{R}^n$. We shall be concerned with the root systems $A_{n-1}=\{\pm(e_i-e_j):1\le i<j\le n\}\subset\mathbb{R}^n$ and $B_n=\{\pm e_i,\pm(e_i\pm e_j):1\le i<j\le n\}\subset\mathbb{R}^n$, where (e_i) denotes the standard basis of $\mathbb{R}^n$. In both cases, the Bessel functions can be written as hypergeometric series in terms of Jack polynomials. For A_{n-1}, the multiplicity function is given by a single parameter $k\in[0,\infty)$. We write Λ_n^+ for the set of partitions $\kappa=(\kappa_1,\kappa_2,\dots)$ of length $l(\kappa)\le n$ and denote by $C_\kappa^{(n)}$, $\kappa\in\Lambda_n^+$ the (symmetric) Jack polynomials in n variables of index $\alpha=\frac{1}{k}\in[0,\infty]$, normalized such that

$$\sum_{|\kappa|=m}C_\kappa^{(n)}(z)=(z_1+\ldots+z_n)^m,\quad m\in\mathbb{N}_0.$$

The Jack polynomials are stable with respect to the number of variables, i.e. for $\kappa \in \Lambda_r^+$ with $r < n$ we have

$$C_\kappa^{(r)}(z_1, \dots, z_r) = \begin{cases} C_{\kappa'}^{(n)}(z_1, \dots, z_r, \underline{0}_{n-r}) & \text{if } \kappa' = (\kappa, 0, ...) \\ 0 & \text{otherwise;} \end{cases} \tag{2.2}$$

with the notation $\underline{a}_j := (a, \dots, a) \in \mathbb{C}^j$ for $a \in \mathbb{C}$. See [St89, Prop. 2.5] together with [Ka93, formula(16)]. Therefore the Jack polynomials $C_\kappa^{(n)}$ uniquely extend to continuous functions C_κ on $\mathbb{C}^{(\infty)} = \bigcup_{n=1}^\infty \mathbb{C}^n$, equipped with the inductive limit topology. We shall often consider elements from $\mathbb{C}^{(\infty)}$ as sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $x_n \neq 0$ for at most finitely many n; for $\mathbb{R}^{(\infty)}$ accordingly.

By [BF98], the Bessel function of type A_{n-1} with multiplicity k is given by the Jack hypergeometric series

$$J_{A_{n-1}}(\lambda, z) = {}_0F_0(\lambda, z) = \sum_{\kappa \in \Lambda_n^+} \frac{C_\kappa(\lambda) C_\kappa(z)}{|\kappa|!\, C_\kappa(\underline{1}_n)} \tag{2.3}$$

with the Jack polynomials of index $\alpha = 1/k$ as above.

Bessel functions of type A_{n-1} occur as the (zonal) spherical functions of the Gelfand pair $(G_n, K_n) := (U_n(\mathbb{F}) \ltimes \text{Herm}_n(\mathbb{F}), U_n(\mathbb{F}))$, where the unitary group $U_n(\mathbb{F})$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ acts by conjugation on the space $\text{Herm}_n(\mathbb{F})$ of Hermitian matrices over $\mathbb{F}$. Recall that the spherical functions of a Gelfand pair (G, K) can be characterized as the continuous, K-biinvariant, non-zero functions φ on G satisfying the product formula

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)dk \quad (g, h \in G).$$

The following characterization is possibly folklore, but not well documented in the literature. For the reader's convenience, we therefore provide a proof.

Lemma 2.1. *The spherical functions of (G_n, K_n), considered as $U_n(\mathbb{F})$-invariant functions on $Herm_n(\mathbb{F})$, are given by the Bessel functions*

$$\varphi_\lambda(X) = J_{A_{n-1}}\big(\tfrac{d}{2}; i\lambda, \sigma(X)\big), \quad \lambda \in \mathbb{C}^n$$

where $d = dim_\mathbb{R}\mathbb{F}$ and $\sigma(X) \in \mathbb{R}^n$ denotes the eigenvalues of $X \in Herm_n(\mathbb{F})$, decreasingly ordered by size and counted according to their multiplicity. Moreover, $\varphi_\lambda = \varphi_\mu$ iff there exists some $w \in S_n$ with $\mu = w.\lambda$, and φ_λ is positive definite iff $\lambda \in \mathbb{R}^q$.

Proof. Consider the Gelfand pair $(\widetilde{G}_n, \widetilde{K}_n) = (SU_n(\mathbb{F}) \ltimes \text{SHerm}_n(\mathbb{F}), SU_n(\mathbb{F}))$. Note that $\widetilde{G}_n$ is the Cartan motion group of $SL_n(\mathbb{F})$, which is connected and semisimple. Thus by [dJ06, Sect.6] (c.f. also [RV08, Sect.3]), the spherical functions of $(\widetilde{G}_n, \widetilde{K}_n)$ are given, as functions on $\text{SHerm}_n(\mathbb{F})$, by

$$\widetilde{\varphi}_\lambda(X) = J_{A_{n-1}}\big(\tfrac{d}{2}; i\lambda, \sigma(X)\big), \quad \lambda \in \mathbb{C}_0^n := \{z \in \mathbb{C}^n : z_1 + \ldots + z_n = 0\}.$$

Consider the mapping $\pi : \mathbb{C}^n \to \mathbb{C}_0^n$, $z \mapsto z - \frac{1}{n}\langle z, \underline{1}_n\rangle \underline{1}_n$, where the inner product $\langle\, .\, ,\, .\, \rangle$ is extended to $\mathbb{C}^n \times \mathbb{C}^n$ in a bilinear way. The restriction of π to $\mathbb{R}^n$ is the orthogonal projection onto $\mathbb{R}_0^n = \mathbb{R}^n \cap \mathbb{C}_0^n$. Then for $z, w \in \mathbb{C}^n$ and arbitrary multiplicity $k \geq 0$,

$$J_{A_{n-1}}(k; z, w) = e^{\langle z, \underline{1}_n\rangle\langle w, \underline{1}_n\rangle/n} \cdot J_{A_{n-1}}(k; \pi(z), \pi(w)). \tag{2.4}$$

This follows e.g. from [BF98, Propos. 3.19]. Now suppose ψ is a spherical function of (G_n, K_n), considered as an S_n-invariant function on $\mathbb{R}^n$. Then

$$\psi(x)\psi(y) = \int_{K_n} \psi(x + kyk^{-1})d_nk,$$

where $x, y \in \mathbb{R}^n$ are identified with the corresponding $n \times n$-diagonal matrices. It follows that

$$\psi(x) = \psi(x - \pi(x) + \pi(x)) = \psi(x - \pi(x)) \cdot \psi(\pi(x)) = e^{\alpha\langle x, \underline{1}_n\rangle} \cdot \psi(\pi(x)),$$

where $\alpha \in \mathbb{C}$ is a constant and the restriction of ψ to $\mathbb{R}^n_0$ is spherical for $(\widetilde{G}_n, \widetilde{K}_n)$. Conversely, it is easily checked that each spherical function of $(\widetilde{G}_n, \widetilde{K}_n)$ extends to a spherical function of (G_n, K_n) in this way. The assertion now follows from (2.4). The assertion concerning the positive-definite spherical functions follows from [Wo06, Theorem 5.4]. □

For the root system $R = B_n$, we denote the multiplicity (by slight abuse of notation) as (k', k), where k is the value on the roots $\pm(e_i \pm e_j)$ and k' is the value on $\pm e_i$. Put $\nu_n := k' + k(n-1) + \frac{1}{2}$. Then the Bessel function of type B_n with multiplicity (k', k) can be written as

$$J_{B_n}(\lambda, z) = {}_0F_1\big(\nu_n; \tfrac{\lambda^2}{2}, \tfrac{z^2}{2}\big) = \sum_{\kappa\in\Lambda_n^+} \frac{1}{4^{|\kappa|}[\nu_n]_\kappa} \frac{C_\kappa(\lambda^2)C_\kappa(z^2)}{|\kappa|!\, C_\kappa(\underline{1}_n)} \tag{2.5}$$

with the hypergeometric series

$${}_0F_1\big(\nu; \lambda, z\big) = \sum_{\kappa\in\Lambda_n^+} \frac{1}{[\nu]_\kappa} \frac{C_\kappa(\lambda)C_\kappa(z)}{|\kappa|!\, C_\kappa(\underline{1}_n)}\,.$$

Here the squares in the arguments are understood componentwise and again, the Jack polynomials are those of index $\alpha = 1/k$. It is easily seen that both ${}_0F_0$ and ${}_0F_1$ converge locally uniformly on $\mathbb{C}^n \times \mathbb{C}^n$; c.f. [BR23] for precise convergence properties of Jack hypergeometric series.

Bessel functions of type B occur as the spherical functions of the Gelfand pairs (G, K) with

$$G = (U_p(\mathbb{F}) \times U_q(\mathbb{F})) \ltimes M_{p,q}(\mathbb{F}),\ K = U_p(\mathbb{F}) \times U_q(\mathbb{F}), \quad p \geq q,$$

where $M_{p,q}(\mathbb{F})$ is the space of $p \times q$ matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and K acts on $M_{p,q}(\mathbb{F})$ via $(U, V).X = UXV^{-1}$. The group G is the Cartan motion group of the non-compact Grassmann manifold $U(p, q; \mathbb{F})/U_p(\mathbb{F}) \times U_q(\mathbb{F})$ which is of rank q. The spherical functions of (G, K) may be considered as K-invariant functions on $M_{p,q}(\mathbb{F})$ and thus depend only on the singular values of their argument. Again as a consequence of [dJ06], they are given by the Bessel functions

$$\varphi_\lambda(X) = J_{B_q}(\kappa, i\lambda, \sigma_{sing}(X)),\ \lambda \in \mathbb{C}^q,$$

where $\kappa = (k', k) = \big(\frac{d}{2}(p - q + 1) - \frac{1}{2}, \frac{d}{2}\big)$ and $\sigma_{sing}(X) = \sigma(\sqrt{X^*X}) \in \mathbb{R}^q$ denotes the set of singular values of $X \in M_{p,q}(\mathbb{F})$, ordered by size. We may therefore also consider the φ_λ as functions on the closed Weyl chamber

$$\overline{C}_q = \{x = (x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 \geq \ldots \geq x_q \geq 0\}, \tag{2.6}$$

i.e. $\varphi_\lambda(x) = J_{B_q}(\kappa; i\lambda, x)$, $x \in \overline{C}_q$. Moreover, $\varphi_\lambda = \varphi_\mu$ iff there exists some $w \in W = S_n \ltimes \mathbb{Z}_2^n$ with $\mu = w.\lambda$. The positive-definite spherical functions are the φ_λ with $\lambda \in \overline{C}_n$, which again follows from [Wo06, Theorem 5.4].

3. The type A case

We start with some motivation from asymptotic spherical harmonic analysis, see [Ol90, Fa08] for a general background. Suppose that (G_n, K_n), $n \in \mathbb{N}$ is an increasing sequence of Gelfand pairs, where $G_n \subseteq G_{n+1}$, $K_n \subseteq K_{n+1}$ are closed subgroups satisfying $K_n = G_n \cap K_{n+1}$. Then the pair (G_∞, K_∞) with the inductive limit groups $G_\infty := \lim_{n\to} G_n$, $K_\infty := \lim_{n\to} K_n$ is called an Olshanski spherical pair. The spherical functions of (G_∞, K_∞) are defined as the continuous, non-zero and K_∞-biinvariant functions $\varphi : G_\infty \to \mathbb{C}$ satisfying

$$\varphi(g)\varphi(h) = \lim_{n\to\infty} \int_{K_n} \varphi(gkh) d_n k \quad (g, h \in G_\infty),$$

where $d_n k$ is the normalized Haar measure on K_n. We remark that this definition is according to [Fa08], whereas in [Ol90] spherical functions are in addition required to be positive definite. Consider now the sequence of Gelfand pairs $(G_n, K_n) = (U_n(\mathbb{F}) \ltimes \mathrm{Herm}_n(\mathbb{F}), U_n(\mathbb{F}))$ as above. We regard G_n and K_n as closed subgroups of G_{n+1} and K_{n+1} in the usual way. Then (G_∞, K_∞) with the inductive limits

$$K_\infty := \lim_{\rightarrow} K_n = U_\infty(\mathbb{F}), \quad G_\infty := \lim_{\rightarrow} G_n = U_\infty(\mathbb{F}) \ltimes \mathrm{Herm}_\infty(\mathbb{F}) \tag{3.1}$$

is an Olshanski spherical pair. The positive definite spherical functions of (G_∞, K_∞) were completely determined by Pickrell [Pi91, Sect.5], see also [OV96] for $\mathbb{F} = \mathbb{C}$, and [Fa08, Section 3]. As functions on $\mathrm{Herm}_\infty(\mathbb{F})$, they are given by

$$\varphi(X) = \prod_{j=1}^{\infty} e^{i\beta x_j - \frac{\gamma}{d} x_j^2} \prod_{l=1}^{\infty} \frac{e^{-i\alpha_l x_j}}{(1 - i\frac{2}{d}\alpha_l x_j)^{d/2}},$$

where $\beta, \gamma \in \mathbb{R}, \gamma \geq 0$, $\alpha_l \in \mathbb{R}$ with $\sum_{l=1}^\infty \alpha_l^2 < \infty$, and $(x_1, x_2, \dots) \in \mathbb{R}^{(\infty)}$ are the eigenvalues of X, ordered by size and counted according to their multiplicity. The product is invariant under rearrangements of the α_l. For $\mathbb{F} = \mathbb{C}$ it is also noted in [OV96] that the set of positive definite spherical functions is bijectively parametrized by the set

$$\{(\alpha, \beta, \gamma) : \beta \in \mathbb{R}, \gamma \geq 0, \alpha = \{\alpha_1, \alpha_2, \dots\} \text{ a multiset with } \alpha_l \in \mathbb{R} \text{ and } \sum_l \alpha_l^2 < \infty\}.$$

In [OV96], explicit approximations of the positive definite spherical functions by positive definite spherical functions of the pairs (G_n, K_n) with $n \to \infty$ by use of spherical expansions were obtained in the case $\mathbb{F} = \mathbb{C}$. In [Bo07], this was generalized by completely different methods to $\mathbb{F} = \mathbb{R}, \mathbb{H}$.

In the present paper, we shall obtain the result of Pickrell and explicit approximations of Olshanski spherical functions as particular cases of a more general asymptotic result for Bessel functions of type A_{n-1} with an arbitrary multiplicity parameter $k > 0$.

Let us first turn to the spectral parameters to be considered for $n \to \infty$. Instead of working with multisets, it will be convenient for us to work with sequences (or finite tuples) with a prescribed order of their components. We introduce the following order on $\mathbb{R}$:

$$x \ll y \text{ iff either } |x| < |y| \text{ or } |x| = |y| \text{ and } x \leq y.$$

For instance, the sequence $(3, -3, 2, 1, -1, -1, 0, 0, \dots)$ is decreasing w.r.t. $\ll$.

Definition 3.1. Consider $\lambda(n) \in \mathbb{R}^n$ such that its entries are decreasing with respect to $\ll$. We regard $(\lambda(n))_{n\in\mathbb{N}}$ as a sequence in $\mathbb{R}^{(\infty)} = \bigcup_{n=1}^\infty \mathbb{R}^n$ and call it a Vershik-Kerov sequence (VK sequence for short), if the following limits exist:

$$\begin{aligned}\alpha_i &:= \lim_{n\to\infty} \frac{\lambda(n)_i}{n} \quad (i \in \mathbb{N}),\\ \beta &:= \lim_{n\to\infty} \frac{p_1(\lambda(n))}{n},\\ \delta &:= \lim_{n\to\infty} \frac{p_2(\lambda(n))}{n^2},\end{aligned}$$

where

$$p_m(x) = \sum_{i=1}^\infty x_i^m \quad \text{for } m \in \mathbb{N},\ p_0 \equiv 1$$

are the power sum symmetric functions on $\mathbb{R}^{(\infty)}$. They span the space of symmetric functions on $\mathbb{R}^{(\infty)}$, i.e. the polynomial functions in arbitrary many variables.

Lemma 3.2. (i) *If $(\lambda(n))_{n\in\mathbb{N}}$ is a Vershik-Kerov sequence with associated parameters $(\alpha_i), \beta, \delta$ as above, then*

$$\gamma := \delta - \sum_{i=1}^\infty \alpha_i^2 \geq 0.$$

In particular, the sequence $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ is square-summable.

(ii) *If in addition $\lambda(n)_i \geq 0$ for all $i, n \in \mathbb{N}$, then $\gamma = 0$.*

Definition 3.3. Suppose that $(\lambda(n))_{n\in\mathbb{N}}$ is a VK sequence. Then the triple $\omega = (\alpha, \beta, \gamma)$ with $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ are called the VK parameters of the sequence $(\lambda(n))_{n\in\mathbb{N}}$. Note that the entries of α are also ordered w.r.t. $\ll$.

Proof of Lemma 3.2. For fixed $N \in \mathbb{N}$ and all $n \geq N$ one has

$$\sum_{i=1}^N \alpha_i^2 \leq \sum_{i=1}^N \left(\alpha_i^2 - \frac{\lambda(n)_i^2}{n^2}\right) + \sum_{i=1}^n \frac{\lambda(n)_i^2}{n^2}.$$

By definition of (α_i) and δ, the right-hand side tends to δ as $n \to \infty$. This proves part (i).

(ii) By the ordering of the entries of $\lambda(n)$, we obtain for $N \in \mathbb{N}$ and $n \geq N$ that

$$\begin{aligned}\frac{p_2(\lambda(n))}{n^2} &= \sum_{i=1}^{N-1} \left(\frac{\lambda(n)_i}{n}\right)^2 + \sum_{i=N}^n \left(\frac{\lambda(n)_i}{n}\right)^2\\ &\leq \sum_{i=1}^{N-1} \left(\frac{\lambda(n)_i}{n}\right)^2 + \frac{\lambda(n)_N}{n} \sum_{i=1}^n \frac{\lambda(n)_i}{n}.\end{aligned}$$

Taking the limit $n \to \infty$ on both sides, we obtain that

$$\delta \leq \sum_{i=1}^{N-1} \alpha_i^2 + \alpha_N \beta.$$

As $\lim_{N\to\infty} \alpha_N = 0$, this implies that $\delta \leq \sum_{i=1}^\infty \alpha_i^2$ and therefore $\gamma = 0$. $\square$

We shall throughout fix a strictly positive multiplicity $k > 0$ on A_{n-1} and suppress it in our notation.

For sequences $(\lambda(n))_{n\in\mathbb{N}}$ of spectral parameters $\lambda(n) \in \mathbb{R}^n$ with growing dimension n, we are interested in the convergence behaviour of the Bessel functions $J_{A_{n-1}}(i\lambda(n), \,.\,)$ as $n \to \infty$. For this, we consider $J_{A_{n-1}}(\lambda, \,.\,)$ as a function on $\mathbb{C}^r$ for all $r \leq n$ by

$$J_{A_{N-1}}(\lambda, z) := J_{A_{n-1}}\big(\lambda, (z, \underline{0}_{n-r})\big), \quad z \in \mathbb{C}^r\,. \tag{3.2}$$

For later use, we record the following representation.

Proposition 3.4. *For $\lambda \in \mathbb{C}^n$ and $z \in \mathbb{C}^r$ with $r \leq n$,*

$$J_{A_{n-1}}(\lambda, z) = \sum_{\kappa\in\Lambda_r^+} \frac{C_\kappa(\lambda)[kr]_\kappa}{[kn]_\kappa |\kappa|!}\, \mathcal{P}_\kappa(z),$$

with the renormalized Jack polynomials

$$\mathcal{P}_\kappa(z) = \frac{C_\kappa(z)}{C_\kappa(\underline{1}_r)}$$

and the generalized Pochhammer symbol

$$[\mu]_\kappa = \prod_{j=1}^{l(\kappa)} (\mu - k(j-1))_{\kappa_j} \quad (\mu \in \mathbb{C}).$$

Proof. Consider formula (2.3). From [Ka93, formula (17)] it is known that for all $\kappa \in \Lambda_r^+$,

$$\frac{C_\kappa(\underline{1}_r)}{C_\kappa(\underline{1}_n)} = \frac{[kr]_\kappa}{[kn]_\kappa}\,.$$

Together with the stability property (2.2), the assertion follows. □

We shall prove the following theorem:

Theorem 3.5. *Let $(\lambda(n))_{n\in\mathbb{N}}$ be a sequence of spectral parameters $\lambda(n) \in \mathbb{R}^n$ such that each $\lambda(n)$ is decreasing with respect to $\ll$. Then for fixed multiplicity $k > 0$, the following statements are equivalent.*

1. *$(\lambda(n))_{n\in\mathbb{N}}$ is a Vershik-Kerov sequence.*
2. *The sequence of Bessel functions $\big(J_{A_{n-1}}(i\lambda(n), \,.\,)\big)_{n\in\mathbb{N}}$ converges locally uniformly on compact subsets of $\mathbb{R}^{(\infty)}$, i.e. the convergence is locally uniform on each of the spaces $\mathbb{R}^r$, $r \in \mathbb{N}$.*
3. *For each fixed multi-index of length r, the corresponding coefficients in the Taylor of expansion of $J_{A_{n-1}}(i\lambda(n), \,.\,)$ around $0 \in \mathbb{R}^r$ converge as $n \to \infty$.*
4. *For all symmetric functions $f : \mathbb{R}^{(\infty)} \to \mathbb{C}$, the limit*
$$\lim_{n\to\infty} \frac{f(\lambda(n))}{n^{degf}}$$
exists.

Moreover, in this case one has

$$\lim_{n\to\infty} J_{A_{n-1}}(i\lambda(n), x) \;=\; \prod_{j=1}^{\infty} e^{i\beta x_j - \frac{\gamma}{2k}x_j^2} \prod_{l=1}^{\infty} \frac{e^{-i\alpha_l x_j}}{\big(1 - \frac{i\alpha_l x_j}{k}\big)^k}, \tag{3.3}$$

where $(\alpha = (\alpha_i), \beta, \gamma)$ are the VK parameters of the VK sequence $(\lambda(n))_{n\in\mathbb{N}}$, and the product on the right side extends analytically to $\mathbb{R}^r$ for each $r \in \mathbb{N}$.

Remark 3.6. In the geometric case $k = 1$, i.e. for Hermitian matrices over $\mathbb{C}$, this result essentially goes back to [OV96], while in [Bo07], where also $\mathbb{F} = \mathbb{R}$ and $\mathbb{H}$ are considered, only the limit (3.3) is established, by completely different methods and under the additional condition $\gamma = 0$.

Our proof of Theorem 3.5 is inspired by the methods of [OV96], [OO98] and [Fa08, Chapter 3]. We start with the following observation.

Theorem 3.7. *Assume that $(\lambda(n))_{n\in\mathbb{N}}$ is a VK sequence with parameters $\omega = (\alpha, \beta, \gamma)$. Then*

$$\lim_{n\to\infty} \frac{p_m(\lambda(n))}{n^m} = \widetilde{p}_m(\omega) := \begin{cases} 1, & m = 0, \\ \beta, & m = 1, \\ \delta = \gamma + \sum_{i=1}^{\infty} \alpha_i^2\,, & m = 2, \\ \sum_{i=1}^{\infty} \alpha_i^m\,, & m \geq 3, \end{cases}$$

where the series in the last case is absolutely convergent. In particular, for each symmetric function f on $\mathbb{R}^{(\infty)}$, the limit

$$\widetilde{f}(\omega) := \lim_{n\to\infty} \frac{f(\lambda(n))}{n^{degf}}$$

exists.

Proof. We only have to consider the case $m \geq 3$. In view of the ordering of $\lambda(n)$ we have for arbitrary $N \in \mathbb{N}$ that

$$\sum_{i=N}^{\infty} \Big|\frac{\lambda(n)_i}{n}\Big|^m \leq \Big|\frac{\lambda(n)_N}{n}\Big|^{m-2} \cdot \frac{p_2(\lambda(n))}{n^2}. \tag{3.4}$$

The expression on the right side converges to $\alpha_N^{m-2}\delta$ as $n \to \infty$. As α is square-summable by Lemma 3.2, this implies that for each $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\sum_{i=N}^{\infty} |\alpha_i|^m + \sum_{i=N}^{\infty} \Big|\frac{\lambda(n)_i}{n}\Big|^m < \epsilon. \tag{3.5}$$

Estimate (3.5) further leads to

$$\begin{aligned} \Big|\frac{p_m(\lambda(n))}{n^m} - p_m(\alpha)\Big| &\leq \sum_{i=N}^{\infty} |\alpha_i|^m + \sum_{i=N}^{\infty} \Big|\frac{\lambda(n)_i}{n}\Big|^m + \sum_{i=1}^{N-1} \Big|\frac{\lambda(n)_i^m}{n^m} - \alpha_i^m\Big| \\ &\leq \epsilon + \sum_{i=1}^{N-1} \Big|\frac{\lambda(n)_i^m}{n^m} - \alpha_i^m\Big|. \end{aligned}$$

By the definition of a VK sequence, the last sum tends to zero as $n \to \infty$. As $\epsilon > 0$ was arbitrary, this finishes the proof. □

We next consider for $\lambda \in \mathbb{C}^{(\infty)}$ the complex function

$$\Phi(\lambda; z) := \prod_{j=1}^{\infty} \frac{1}{(1 - \lambda_j z)^k}\,,$$

where $\zeta \mapsto \zeta^k$ denotes the principal holomorphic branch of the power function on $\mathbb{C}\setminus]-\infty,0]$. For fixed λ, the product is finite and $\Phi(\lambda;\,.\,)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}$. According to formula (2.9) of [OO98],

$$\Phi(\lambda;z)=\sum_{j=0}^{\infty} g_j(\lambda)z^j$$

with

$$g_j(\lambda)=\sum_{i_1\leq\ldots\leq i_j}\frac{(k)_{m_1}(k)_{m_2}\cdots}{m_1!\,m_2!\cdots}\cdot\lambda_{i_1}\cdots\lambda_{i_j}, \tag{3.6}$$

where $m_l := \#\{r\in\mathbb{N}: i_r=l\}$ denotes the multiplicity of the number l in the tuple $(i_1,\ldots,i_j)$ and $(k)_m=k(k+1)\cdots(k+m-1)$ is the Pochhammer symbol.

Moreover, from [OO98, formula (2.8)] and the connection between the C- and P-normalizations of the Jack polynomials according to formula (12.135) of [Fo10], one calculates that

$$g_j(\lambda)=\frac{(k)_j}{j!}\cdot C_{(j)}(\lambda). \tag{3.7}$$

(For partitions $\kappa=(j)$ with just one part, the Jack polynomials $C_{(j)}$ and $P_{(j)}$ coincide).

Lemma 3.8. *Suppose $\omega=(\alpha,\beta,\gamma)$ are the VK parameters of a Vershik-Kerov sequence. Then the following hold.*

(1) *The infinite product*

$$\Psi(\omega;z):=e^{k\beta z+\frac{k\gamma}{2}z^2}\prod_{l=1}^{\infty}\frac{e^{-k\alpha_l z}}{(1-\alpha_l z)^k}$$

is holomorphic in $S:=\mathbb{C}\setminus\big(\big]-\infty,-\frac{1}{|\alpha_1|}\big]\cup\big[\frac{1}{|\alpha_1|},\infty\big[\big)$. If $\alpha_l\geq 0$ for all $l\in\mathbb{N}$, then $\Psi(\omega;\,.\,)$ is holomorphic in $\widetilde{S}:=\mathbb{C}\setminus\big[\frac{1}{\alpha_1},\infty\big[$.

(2) *ω is uniquely determined by $\Psi(\omega;\,.\,)$.*

Proof. (1) Power series expansion around $z=0$ shows that for $|\alpha_l z|\leq\delta<1$,

$$\Big|1-\frac{e^{-k\alpha_l z}}{(1-\alpha_l z)^k}\Big|\leq C_\delta|\alpha_l z|^2$$

with some constant $C_\delta>0$. Recall that α is decreasing w.r.t. $\ll$ and square-summable. Hence for fixed $n\in\mathbb{N}$, the product

$$\prod_{l=n}^{\infty}\frac{e^{-k\alpha_l z}}{(1-\alpha_l z)^k}$$

defines a holomorphic function in the open disc $\big\{z\in\mathbb{C}:|z|<1/|\alpha_n|\in[0,\infty]\big\}$. Moreover,

$$\prod_{l=1}^{n-1}\frac{e^{-k\alpha_l z}}{(1-\alpha_l z)^k}$$

is holomorphic in S and even in $\widetilde{S}$ if $\alpha_l\geq 0$ for all l. As $\lim_{l\to\infty}\alpha_l=0$, it follows that $\psi(\omega;\,.\,)$ is holomorphic in S or even in $\widetilde{S}$. Unless α is identical zero (which is equivalent to $\alpha_1=0$), $\Psi(\omega;\,.\,)$ has a singularity in $z=\frac{1}{\alpha_1}$.

(2) If $\psi(\omega;\,.)$ is entire, then $\alpha_1=0$. Otherwise $\lim_{z\to 1/\alpha_1}|\Psi(\omega;z)|=\infty$. Thus α_1 is uniquely determined by $\Psi(\omega;\,.\,)$. Multiplying successively by $(1-\alpha_1 z)^k,\ldots,$

we further obtain that $\alpha_2, \alpha_3, \ldots$ are uniquely determined by $\Psi(\omega;\,.\,)$ as well. It is then obvious that also β and γ are uniquely determined by $\Psi(\omega;\,.\,)$. □

Proposition 3.9. *(1) For $\lambda \in \mathbb{C}^{(\infty)}$ and $|z|$ small enough,*

$$\Phi(\lambda; z) = \exp\Big(k \sum_{m=0}^{\infty} p_m(\lambda) \frac{z^m}{m}\Big). \tag{3.8}$$

(2) Moreover, if $(\lambda(n))_{n\in\mathbb{N}}$ is a VK sequence with parameters $\omega = (\alpha, \beta, \gamma)$, then

$$\lim_{n\to\infty} \Phi\Big(\frac{\lambda(n)}{n}; z\Big) = \Psi(\omega; z),$$

where the convergence is uniform in z in some compact neighborhood of 0.

Proof. (1) Since both sides of (3.8) have value 1 in $z = 0$, it suffices to verify that they have the same logarithmic derivative. Let log be the principle holomorphic branch of the logarithm in $\mathbb{C}\setminus]-\infty, 0]$. Then for $|z|$ small enough,

$$\frac{d}{dz}\log \Phi(\lambda; z) = \sum_{j=0}^{\infty} \frac{k\lambda_j}{1-\lambda_j z} \;=\; k\sum_{m=0}^{\infty} p_{m+1}(\lambda) z^m\,.$$

This is exactly the logarithmic derivative of the right-hand side in (3.8).

(2) For the second assertion, note that for $m \geq 2$ we may estimate

$$\Big|\frac{p_m(\lambda(n))}{n^m}\Big| \leq C_m \cdot \frac{p_2(\lambda(n))}{n^2}$$

with a constant $C_m > 0$ independent of n, which follows from (3.4). Since the right-hand side converges for $n \to \infty$, the sequence on the left-hand side is uniformly bounded in n. Hence there exists some $\epsilon > 0$ such that for each n, the series

$$h_n(z) = \sum_{m=0}^{\infty} p_m\Big(\frac{\lambda(n)}{n}\Big)\frac{z^m}{m}$$

converges for $|z| < \epsilon$, and the dominated convergence theorem shows that (h_n) converges for $n \to \infty$ to $\sum_{m=0}^{\infty} \widetilde{p}_m(\omega)\frac{z^m}{m}$ locally uniformly in $\{|z| < \epsilon\}$. Thus

$$\lim_{n\to\infty} \Phi\Big(\frac{\lambda(n)}{n}; z\Big) \;=\; \exp\Big(k\sum_{m=0}^{\infty} \widetilde{p}_m(\omega)\frac{z^m}{m}\Big) \tag{3.9}$$

locally uniformly in $\{z \in \mathbb{C} : |z| < \epsilon\}$.

Now consider $\Psi(\omega;\,.\,)$, which is holomorphic in a neighborhood of 0. Taking the logarithmic derivative as in the proof of [Fa08, Prop. 3.12] and recalling Theorem 3.7, we obtain

$$\frac{d}{dz}\log \Psi(\omega; z) = k\Big[\beta + \gamma z - \sum_{l=1}^{\infty}\Big(\alpha_l - \frac{\alpha_l}{1-\alpha_l z}\Big)\Big] \;=\; k\sum_{m=0}^{\infty} \widetilde{p}_{m+1}(\omega) z^m.$$

The right-hand side in equation (3.9) has the same logarithmic derivative. Since $\Phi\big(\frac{\lambda(n)}{n}; 0\big) = 1 = \Psi(\omega; 0)$, this proves the stated limit. □

We now consider the asymptotic behaviour of the Bessel functions $J_{A_{n-1}}$ as $n \to \infty$. We shall employ the following useful convergence property, see [Fa08, Propos. 3.11]:

Lemma 3.10. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of smooth, positive-definite functions on $\mathbb{R}^r$ with $\varphi_n(0) = 1$ and φ an analytic function in some neighborhood of $0 \in \mathbb{R}^r$. Assume that $\lim_{n\to\infty} \partial^\alpha \varphi_n(0) = \partial^\alpha \varphi(0)$ for all $\alpha \in \mathbb{N}_0^r$, i.e. the Taylor coefficients of φ_n around 0 converge to those of φ. Then φ has an analytic extension to $\mathbb{R}^r$, and the sequence (φ_n) converges to φ locally uniformly on $\mathbb{R}^r$.*

For $z \in \mathbb{C}^{(\infty)}$, we put

$$\widehat{\Psi}(\omega; z) := \prod_{j=1}^{\infty} \Psi(\omega; z_j),$$

which is actually a finite product.

Theorem 3.11. *Assume that $(\lambda(n))_{n\in\mathbb{N}}$ is a VK sequence with parameters $\omega = (\alpha, \beta, \gamma)$. Then for $x \in \mathbb{R}^{(\infty)}$, the Bessel functions of type A_{n-1} with multiplicity $k > 0$ satisfy*

$$\lim_{n\to\infty} J_{A_{n-1}}(i\lambda(n), x) = \widehat{\Psi}\Big(\omega; \frac{ix}{k}\Big) = \prod_{j=1}^{\infty} e^{i\beta x_j - \frac{\gamma}{2k} x_j^2} \prod_{l=1}^{\infty} \frac{e^{-i\alpha_l x_j}}{\big(1 - \frac{i\alpha_l}{k} x_j\big)^k}$$

locally uniformly in $x \in \mathbb{R}^r$ for all $r \in \mathbb{N}$.

Proof. The Cauchy identity for Jack polynomials, see for instance [St89, Prop. 2.1], states for $\lambda \in \mathbb{C}^{(\infty)}$ and $z \in \mathbb{C}^r$ with $|z_j|$ small enough that

$$\sum_{\kappa\in\Lambda_r^+} \frac{[kr]_\kappa}{|\kappa|!} C_\kappa(\lambda) \mathcal{P}_\kappa(z) = \prod_{j,l} \frac{1}{(1 - \lambda_l z_j)^k} = \prod_{j=1}^{r} \Phi(\lambda; z_j).$$

Thus by Proposition 3.9,

$$\sum_{\kappa\in\Lambda_r^+} \frac{[kr]_\kappa C_\kappa\big(\frac{\lambda(n)}{n}\big)}{|\kappa|!\, k^{|\kappa|}} \mathcal{P}_\kappa(iz) \longrightarrow \widehat{\Psi}\big(\omega; \frac{iz}{k}\big) =: \varphi(z) \quad \text{for } n \to \infty,$$

where the convergence is locally uniform in z in some open neighborhood of $0 \in \mathbb{C}^r$. Therefore, the coefficients in the power series expansion of the left side around $z = 0$ must converge (as $n \to \infty$) to the corresponding coefficients of φ. Moreover, by Lemma 3.8 we know that φ extends analytically to $\mathbb{R}^r$, and by Theorem 3.7 we have

$$\lim_{n\to\infty} C_\kappa\Big(\frac{\lambda(n)}{n}\Big) = \widetilde{C}_\kappa(\omega). \tag{3.10}$$

It therefore follows for all z in some neighborhood of $0 \in \mathbb{C}^r$ that

$$\varphi(z) = \sum_{\kappa\in\Lambda_r^+} \frac{[kr]_\kappa \widetilde{C}_\kappa(\omega)}{|\kappa|!\, k^{|\kappa|}} \mathcal{P}_\kappa(iz).$$

Now consider the functions

$$\varphi_n(x) := J_{A_{n-1}}(i\lambda(n), x),$$

which are positive definite on $\mathbb{R}^r$ for $r \le n$. In view of Proposition 3.4,

$$\varphi_n(x) = \sum_{\kappa\in\Lambda_r^+} \frac{[kr]_\kappa C_\kappa((\lambda(n))}{|\kappa|!\, [kn]_\kappa} \mathcal{P}_\kappa(ix). \tag{3.11}$$

As $[kn]_\kappa \sim (kn)^{|\kappa|}$ for $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{C_\kappa(\lambda(n))}{[kn]_\kappa} = \frac{\widetilde{C}_\kappa(\omega)}{k^{|\kappa|}}.$$

So we are able to apply Lemma 3.10 to obtain that $\varphi_n \to \varphi$ locally uniformly on $\mathbb{R}^r$. □

Remark 3.12. The proof shows that for $z \in \mathbb{C}^r$ with $|z|$ small enough,

$$\widehat{\Psi}(\omega; z) = \sum_{\kappa\in\Lambda_r^+} \frac{[kr]_\kappa}{|\kappa|!} \widetilde{C}_\kappa(\omega)\mathcal{P}_\kappa(z).$$

Lemma 3.13. *Consider a sequence $(\lambda(n))_{n\in\mathbb{N}}$ such that each $\lambda(n) \in \mathbb{R}^n$ is decreasing with respect to $\ll$. Suppose that the sequence of Bessel functions $J_{A_{n-1}}(i\lambda(n), \,.\,)$ converges pointwise on $\mathbb{R}$ to a function which is continuous at 0. Then $(\lambda(n))$ is a VK sequence.*

Proof. Put $\varphi_n(x) := J_{A_{n-1}}(i\lambda(n), x)$, $x \in \mathbb{R}$ and $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$. In view of representation (2.1), there exist compactly supported probability measures μ_n on $\mathbb{R}$ such that

$$\varphi_n(x) = \int_{\mathbb{R}} e^{ix\xi} d\mu_n(\xi) \quad \text{for all } x \in \mathbb{R}.$$

By Lévy's continuity theorem, there exists a probability measure μ on $\mathbb{R}$ such that $\mu_n \to \mu$ weakly and

$$\varphi(x) = \int_{\mathbb{R}} e^{ix\xi} d\mu(\xi) \quad \text{for all } x \in \mathbb{R}.$$

In particular, the family of measures $\{\mu_n : n \in \mathbb{N}\}$ is tight. Recall the functions $g_j(\lambda)$ from (3.6). By Proposition 3.4 and formula (3.7) we have

$$\varphi_n(x) = \sum_{j=0}^{\infty} \frac{C_{(j)}(\lambda(n)) \cdot (k)_j}{(kn)_j \cdot j!} (ix)^j = \sum_{j=0}^{\infty} \frac{g_j(\lambda(n))}{(kn)_j} (ix)^j .$$

Hence the moments of the measures μ_n are given by

$$\int_{\mathbb{R}} \xi^j d\mu_n(\xi) = j!\, \frac{g_j(\lambda(n))}{(kn)_j}.$$

We now employ Lemma 5.2 of [OO98]. From the definition of the functions g_j one can find a constant $C > 0$ such that $g_4(\lambda) \le C g_2(\lambda)^2$ for all $\lambda \in \mathbb{R}^{(\infty)}$, which shows that the quotient

$$\frac{\int_{\mathbb{R}} \xi^4 d\mu_n(\xi)}{\left(\int_{\mathbb{R}} \xi^2 d\mu_n(\xi)\right)^2}$$

is bounded as a function of $n \in \mathbb{N}$. Hence we conclude from Lemma 5.1. of [OO98] that the sequence $\left(\int_{\mathbb{R}} \xi^2 d\mu_n(\xi)\right)_{n\in\mathbb{N}}$ is bounded, which in turn implies that the sequence $\left(\frac{g_2(\lambda(n))}{n^2}\right)_{n\in\mathbb{N}}$ is bounded. As $2g_2 = k^2 p_1^2 + kp_2$, the sequences

$$\left(\frac{|p_1(\lambda(n))|}{n}\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\frac{p_2(\lambda(n))}{n^2}\right)_{n\in\mathbb{N}} \tag{3.12}$$

are bounded as well. Standard compactness arguments and a diagonalization argument imply that $(\lambda(n))_{n\in\mathbb{N}}$ has a subsequence which is Vershik-Kerov. Finally, consider two such subsequences $(\lambda_l(n))_{n\in\mathbb{N}}$ with VK parameters ω_l, $l=1,2$. Then by Theorem 3.11 and our assumptions,

$$\varphi(x) = \lim_{n\to\infty} J_{A_{n-1}}(i\lambda_l(n), x) = \Psi\Big(\omega_l; \frac{ix}{k}\Big) \quad \text{for all } x\in\mathbb{R}.$$

Hence $\Psi(\omega_1;\,.\,) = \Psi(\omega_2;\,.\,)$, and Proposition 3.8 implies that $\omega_1=\omega_2$. It follows that the full sequence $(\lambda(n))_{n\in\mathbb{N}}$ is Vershik-Kerov. □

Putting things together, we are now able to finalize the proof of Theorem 3.5.

Proof of Theorem 3.5. The implication (1) ⇒ (2) is contained in Theorem 3.11. The converse implication (2) ⇒ (1) is just Lemma 3.13. Further, Theorem 3.7 proves the implication (1) ⇒ (4). The equivalence of statements (3) and (4) is obvious from expansion (3.11), because the Jack polynomials span the algebra of symmetric functions. It thus remains to prove the implication (4) ⇒ (1). For this, suppose that $(\lambda(n))_{n\in\mathbb{N}}$ is a sequence with each $\lambda(n)\in\mathbb{R}^n$ decreasing w.r.t $\ll$, and such that $\lim_{n\to\infty}\frac{f(\lambda(n))}{n^{\deg f}}$ exists for all symmetric functions f. Then in particular, the sequences $\big(\frac{p_1(\lambda(n))}{n}\big)$ and $\big(\frac{p_2(\lambda(n))}{n^2}\big)$ are bounded. Again by a compactness argument, $(\lambda(n))$ has a subsequence which is Vershik-Kerov. Suppose $(\lambda_l(n))$, $l=1,2$ are two such subsequences with VK parameters ω_l. Then by Theorem 3.11, the sequences $\big(J_{A_{n-1}}(i\lambda_1(n),\,.\,)\big)$ and $\big(J_{A_{n-1}}(i\lambda_2(n),\,.\,)\big)$ converge locally uniformly on $\mathbb{R}^r$ to the same limit, because for each $\kappa\in\Lambda_r^+$, the limit

$$\lim_{n\to\infty}\frac{C_\kappa(\lambda_l(n))}{n^{|\kappa|}} = \lim_{n\to\infty}\frac{C_\kappa(\lambda(n))}{n^{|\kappa|}}$$

is independent of l. Arguing further as in the proof of Lemma 3.13, we obtain that $\omega_1=\omega_2$ and that $(\lambda(n))$ is a VK sequence.This finishes the proof of the theorem. □

We shall now parametrize the possible limit functions in Theorem 3.11. We put

$$\Omega := \Big\{(\alpha,\beta,\gamma) : \beta\in\mathbb{R}, \gamma\geq 0, \alpha=(\alpha_i)_{i\in\mathbb{N}} \text{ with } \alpha_i\in\mathbb{R}, \alpha_{i+1}\ll\alpha_i\,, \sum_{i=1}^\infty \alpha_i^2<\infty\Big\}.$$

Note that for $(\alpha,\beta,\gamma)\in\Omega$, either all entries of α are non-zero, or all entries up to finitely many are zero.

Proposition 3.14. *For any element $\omega=(\alpha,\beta,\gamma)\in\Omega$ there exists a VK sequence $(\lambda(n))$ with VK parameters ω.*

Proof. We divide the proof into several steps.

(i) Assume that $\alpha=0$. Then for arbitrary $\epsilon>0$, there exists a sequence $x=(x_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$|x_i|\leq\epsilon \ \text{ for all } \ i\in\mathbb{N}, \ \sum_{i=1}^\infty x_i=\beta \quad\text{and}\quad \sum_{i=1}^\infty x_i^2=\gamma. \tag{3.13}$$

To see this, choose $N\in\mathbb{N}$ such that $\big(\frac{6\gamma}{\pi^2 N}\big)^{1/2}\leq\epsilon$ and start with the alternating sequence

$$x_i' := \Big(\frac{6\gamma}{\pi^2 N}\Big)^{1/2}\cdot\frac{(-1)^i}{k+1} \quad \text{if } k<\frac{i}{N}\leq k+1,\, k\in\mathbb{N}_0.$$

It satisfies the first and the third condition of (3.13), and by the Riemann rearrangement theorem, there exists a rearrangement $(x_i)_{i\in\mathbb{N}}$ of $(x_i')_{i\in\mathbb{N}}$ satisfying the second condition as well. For each $m\in\mathbb{N}$ we can therefore find a real sequence

$x^{(m)} = (x_i^{(m)})$ and and index $n_m \in \mathbb{N}$ with $n_m \to \infty$ for $m \to \infty$, such that for all $n \geq n_m$,

$$\big|x_i^{(m)}\big| \leq \frac{1}{m} \quad \text{for all } i \in \mathbb{N}, \quad \Big|\sum_{i=1}^n x_i^{(m)} - \beta\Big| \leq \frac{1}{m}, \quad \Big|\sum_{i=1}^n \big(x_i^{(m)}\big)^2 - \gamma\Big| \leq \frac{1}{m}.$$

We may also assume that $n_{m+1} > n_m$ for all m. Rearranging the entries of each tuple $(x_1^{(m)}, \dots, x_{n_m}^{(m)})$ according to $\ll$, we thus obtain a sequence $(\lambda(n_m)')_{m\in\mathbb{N}}$ where each $\lambda(n_m)' \in \mathbb{R}^{n_m}$ is decreasing w.r.t. $\ll$ and satisfies

$$\begin{aligned} &\lim_{m\to\infty} \lambda(n_m)_i' = 0 \quad \text{ for all } i \in \mathbb{N}, \\ &\lim_{m\to\infty} \sum_{i=1}^{n_m} \lambda(n_m)_i' = \beta, \\ &\lim_{m\to\infty} \sum_{i=1}^{n_m} \big(\lambda(n_m)_i'\big)^2 = \gamma. \end{aligned}$$

Finally, put $\lambda(n_m) := n_m\lambda(n_m)'$ and $\lambda(n) := (n\lambda(n_m)', 0, \dots, 0) \in \mathbb{R}^n$ for $n_m < n < n_{m+1}$. Then $(\lambda(n))_{n\geq n_1}$ is a VK sequence with parameters $(\alpha = 0, \beta, \gamma)$.

(ii) Assume that α has finitely many non-zero entries and let $m \in \mathbb{N}$ be maximal such that $\alpha_m \neq 0$. Let $(\lambda(n)')_{n\in\mathbb{N}}$ be a VK sequence with parameters $(0, \beta', \gamma)$, where $\beta' = \beta - \sum_{i=1}^m \alpha_i$. For $n > m$, put

$$\lambda(n) := (n\alpha_1, \dots, n\alpha_m, \lambda(n)_1', \dots, \lambda(n)_{n-m}').$$

For n large enough, say $n \geq n_0$, the entries of $\lambda(n)$ are decreasing with respect to $\ll$, because $\lim_{n\to\infty} \frac{\lambda(n)_i'}{n} = 0$. Then $(\lambda(n))_{n\geq n_0}$ is Vershik-Kerov with parameters (α, β, γ).

(iii) Assume that all entries of α are non-zero. For $m \in \mathbb{N}$, put $\omega^{(m)} := (\alpha^{(m)}, \beta, \gamma)$, where $\alpha^{(m)} = (\alpha_1, \dots, \alpha_m, 0, \dots)$. According to part (ii), there exists a VK sequence $(\lambda^{(m)}(n))_{n\in\mathbb{N}}$ with VK parameters $\omega^{(m)}$. By a diagonalization argument we obtain a sequence $\lambda(n_m) := \lambda^{(m)}(n_m)$ with $n_{m+1} > n_m$ satisfying

$$\begin{aligned} &\lim_{m\to\infty} \lambda(n_m)_i = \alpha_i \quad \text{ for all } i \in \mathbb{N}, \\ &\lim_{m\to\infty} \sum_{i=1}^{n_m} \lambda(n_m)_i = \beta, \\ &\lim_{m\to\infty} \sum_{i=1}^{n_m} \big(\lambda(n_m)_i\big)^2 = \delta = \gamma + \sum_{i=1}^\infty \alpha_i^2. \end{aligned}$$

Finally, for $n \in \mathbb{N}$ with $n_m \leq n < n_{m+1}$ put $\lambda(n) := \big(\frac{n}{n_m}\lambda(n_m), 0, \dots, 0\big) \in \mathbb{R}^n$. Then $(\lambda(n))_{n\geq n_1}$ is Vershik-Kerov with VK parameters (α, β, γ). □

Together with Lemma 3.8, this result shows that the possible limits (for $n \to \infty$) of the Bessel functions $J_{A_{n-1}}(i\lambda(n), x)$ with $x \in \mathbb{R}^r$ and $\lambda(n) \in \mathbb{R}^n$ are exactly all the infinite products $\widehat{\Psi}\big(\omega; \frac{ix}{k}\big)$, of Theorem 3.11, which are in bijective correspondence with the parameters $\omega \in \Omega$.

Let us finally come back to the Olshanski spherical pair (G_∞, K_∞) as in (3.14). From our results, we obtain the following result of Pickrell [Pi91] mentioned at the beginning of this section:

Corollary 3.15. *The set of positive definite spherical functions of the Olshanski spherical pair $(G_\infty, K_\infty) = (U_\infty(\mathbb{F}) \ltimes Herm_\infty(\mathbb{F}), U_\infty(\mathbb{F}))$, considered as $U_\infty(\mathbb{F})$-invariant functions on $Herm_\infty(\mathbb{F})$, is parametrized by the set Ω via*

$$\varphi_\omega(X) = \prod_{j=1}^\infty e^{i\beta x_j - \frac{\gamma}{d}x_j^2} \prod_{l=1}^\infty \frac{e^{-i\alpha_l x_j}}{(1-i\frac{2}{d}\alpha_l x_j)^{d/2}}\,, \quad \omega = (\alpha,\beta,\gamma)\in\Omega.$$

Proof. For a topological group H consider the set

$$P_1(H) = \{\varphi \in C(H) : \varphi \text{ positive definite}, \varphi(e) = 1\}$$

and denote by $\mathrm{ex}(P_1(H))$ the set of its extremal points. In [Ol90, Theorem 22.10] it is proven that each $\varphi \in \mathrm{ex}(P_1(G_\infty))$ can be approximated locally uniformly by a sequence of functions $\varphi_n \in \mathrm{ex}(P_1(G_n))$. An inspection of the proof shows that this statement remains true for biinvariant functions, i.e. each K_∞-biinvariant $\varphi \in \mathrm{ex}(P_1(G_\infty))$ can be approximated locally uniformly by a sequence of K_n-biinvariant functions $\varphi_n \in \mathrm{ex}(P_1(G_n))$. According to [Ol90, Theorems 23.3 and 23.6], the positive definite spherical functions of a spherical pair (G,K) (an Olshanski spherical pair or a Gelfand pair) are exactly those elements of $\mathrm{ex}(P_1(G))$ which are K-biinvariant. Thus, for a positive definite spherical function φ of (G_∞, K_∞), there exists a sequence (φ_n) of positive definite spherical functions of (G_n, K_n) which converges locally uniformly to φ. (This is also noted in [OV96, Theorem 3.5]). By Lemma 2.1, φ_n is given by a positive definite Bessel function $J_{A_{n-1}}$ with multiplicity $k = \frac{d}{2}$, $d = \dim_\mathbb{R}\mathbb{F}$, i.e.

$$\varphi_n(X) = J_{A_{n-1}}\big(\frac{d}{2}; i\lambda(n), \sigma(X)\big) \tag{3.14}$$

with some real spectral parameter $\lambda(n) \in \mathbb{R}^n$. Without loss of generality we may assume that $\lambda(n)$ is decreasing w.r.t $\ll$. From Theorem 3.5 it now follows that $(\lambda(n))$ has to be a VK sequence and that $\varphi = \varphi_\omega$, where ω are the VK parameters of $(\lambda(n))$.

Conversely, starting with $\omega \in \Omega$ we may choose an associated VK sequence $(\lambda(n))$ by Proposition 3.14. Then (3.14) defines a sequence (φ_n) of positive definite spherical functions of (G_n, K_n) which converge to φ_ω locally uniformly according to Theorem 3.5. It is then clear from the definitions that φ_ω is a positive definite Olshanski-spherical function of (G_∞, K_∞). □

4. The type B case

Recall from Section 2 the Bessel functions J_{B_n} of type B_n. As $n \to \infty$, we shall consider them with the multiplicities $\kappa_n := (k_n', k)$ with value $k > 0$ on the roots $\pm(e_i \pm e_j)$ and $k_n' \geq 0$ on the roots $\pm e_i$. It will become clear at the end of this section why the multiplicity parameter k_n is allowed to vary with n. With $\nu_n := k_n' + k(n-1) + \frac{1}{2}$ we have

$$J_{B_n}(\kappa_n; \lambda, z) \;=\; \sum_{\kappa\in\Lambda_n^+} \frac{1}{4^{|\kappa|}[\nu_n]_\kappa} \frac{C_\kappa(\lambda^2)C_\kappa(z^2)}{|\kappa|!\, C_\kappa(\underline{1}_n)},$$

where the Jack polynomials are of index $1/k$. Recall the stability property (2.2) of the Jack polynomials. Adopting the notation from (3.2), we therefore have for

$\lambda \in \mathbb{C}^n$ and $z \in \mathbb{C}^r$ with $r \le n$ the representation

$$J_{B_n}(\kappa_n; \lambda, z) := J_{B_n}(\kappa_n; \lambda, (z, \underline{0}_{n-r}) = \sum_{\kappa \in \Lambda_r^+} \frac{C_\kappa(\lambda^2)[kr]_\kappa}{4^{|\kappa|}[kn]_\kappa [\nu_n]_\kappa |\kappa|!} \mathcal{P}_\kappa(z^2). \qquad (4.1)$$

Note that for $\lambda \in \mathbb{R}^n$, $J_{B_n}(\kappa_n; i\lambda, \,.\,)$ is positive definite on $\mathbb{R}^r$ for each $r \le n$. Recall our notion (2.6) for the positive Weyl chamber of type B.

The following counterpart of Theorem 3.5 will be the main result of this section.

Theorem 4.1. *Let $(\lambda(n))_{n\in\mathbb{N}}$ be a sequence of spectral parameters $\lambda(n) \in \overline{C}_n$. Then the following statements are equivalent.*

(1) *$\big(\frac{\lambda(n)^2}{\nu_n}\big)_{n\in\mathbb{N}}$ is a Vershik-Kerov sequence.*
(2) *The sequence of Bessel functions $\big(J_{B_n}(\kappa_n; i\lambda(n), \,.\,)\big)_{n\in\mathbb{N}}$ converges locally uniformly on each of the spaces $\mathbb{R}^r$, $r \in \mathbb{N}$.*
(3) *For each fixed multi-index of length r, the corresponding coefficients in the Taylor of expansion of $J_{B_n}(\kappa_n; i\lambda(n), \,.\,)$ around $0 \in \mathbb{R}^r$ converge as $n \to \infty$.*
(4) *For all symmetric functions $f : \mathbb{R}^{(\infty)} \to \mathbb{C}$, the limit*
$$\lim_{n\to\infty} \frac{f(\lambda(n)^2)}{(n\nu_n)^{degf}}$$
exists.

In this case, let $\omega = (\alpha, \beta, \gamma)$ be the VK parameters of the sequence $\big(\frac{\lambda(n)^2}{\nu_n}\big)$. Then $\gamma = 0$, $\alpha_l \ge 0$ for all l, and

$$\lim_{n\to\infty} J_{B_n}(\kappa_n; i\lambda(n), x) \;=\; \widehat{\Psi}\Big(\omega; -\frac{x^2}{4k}\Big) \;=\; \prod_{j=1}^{\infty} e^{-\frac{\beta x_j^2}{4}} \prod_{l=1}^{\infty} \frac{e^{\frac{\alpha_l x_j^2}{4}}}{\big(1 + \frac{\alpha_l x_j^2}{4k}\big)^k}.$$

Remarks 4.2. 1. It is a consequence of Lemma 3.2 that the VK parameter γ is 0 in the present situation.

2. Assume that $\lim_{n\to\infty} \frac{k_n'}{n} = 0$. Then $\nu_n \sim kn$ for $n \to \infty$. In particular, $\big(\frac{\lambda(n)^2}{\nu_n}\big)$ is Vershik-Kerov with VK parameters $(\alpha, \beta, 0)$ if and only if $\big(\frac{\lambda(n)^2}{n}\big)$ is Vershik-Kerov with VK parameters $\big(k\alpha, k\beta, 0\big)$.

For the proof of Theorem 4.1, we start with the following

Lemma 4.3. *Let $(\lambda(n))_{n\in\mathbb{N}}$ with $\lambda(n) \in \overline{C}_n$ be such that $\big(\frac{\lambda(n)^2}{\nu_n}\big)$ is a Vershik-Kerov sequence with VK parameters $\omega = (\alpha, \beta, 0)$. Then for $x \in \mathbb{R}^{(\infty)}$,*

$$\lim_{n\to\infty} J_{B_n}(\kappa_n; i\lambda(n), x) = \widehat{\Psi}\Big(\omega; -\frac{x^2}{4k}\Big).$$

The convergence is uniform on compact subsets of $\mathbb{R}^{(\infty)}$, i.e. locally uniform in $x \in \mathbb{R}^r$ for all $r \in \mathbb{N}$.

Proof. For $x \in \mathbb{R}^r$, consider

$$\varphi_n(x) := J_{B_n}(\kappa_n; i\lambda(n), x) \;=\; \sum_{\kappa \in \Lambda_r^+} \frac{[kr]_\kappa \,(n\nu_n)^{|\kappa|}}{4^{|\kappa|}|\kappa|!\,[kn]_\kappa [\nu_n]_\kappa} C_\kappa\Big(\frac{\lambda(n)^2}{n\nu_n}\Big)\mathcal{P}_\kappa(-x^2). \quad (4.2)$$

As $\lim_{n\to\infty} \nu_n = \infty$, we have for $n \to \infty$ the asymptotics

$$[\nu_n]_\kappa \sim \nu_n^{|\kappa|}, \; [kn]_\kappa \sim (kn)^{|\kappa|}.$$

Moreover, by relation (3.10),

$$\lim_{n\to\infty} C_\kappa\Big(\frac{\lambda(n)^2}{n\nu_n}\Big) \,=\, \widetilde{C}_\kappa(\omega).$$

As in the proof of Theorem 3.11, we conclude that the Taylor coefficients of φ_n around 0 converge to those of

$$\varphi(x) = \sum_{\kappa\in\Lambda_r^+} \frac{[kr]_\kappa}{(4k)^{|\kappa|}|\kappa|!}\,\widetilde{C}_\kappa(\omega)\mathcal{P}_\kappa(-x^2) \,=\, \widehat{\Psi}\Big(\omega, -\frac{x^2}{4k}\Big).$$

This function is actually real-analytic on $\mathbb{R}^r$ by Lemma 3.8, since the entries of α are non-negative. As the functions φ_n are positive definite on $\mathbb{R}^n$, Lemma 3.10 yields the assertion. □

Lemma 4.4. *Consider a sequence $(\lambda(n))_{n\in\mathbb{N}}$ with $\lambda(n)\in\overline{C}_n$. Assume that the sequence of Bessel functions $J_{B_n}(i\lambda(n),\,.\,)$ converges pointwise on $\mathbb{R}$ to a function which is continuous at 0. Then the sequence $\big(\frac{\lambda(n)^2}{\nu_n}\big)$ is Vershik-Kerov.*

Proof. The proof is similar to that of Lemma 3.13. For $x\in\mathbb{R}$, put

$$\varphi_n(x) := J_{B_n}(\kappa_n; i\lambda(n), x) \,=\, \int_{\mathbb{R}} e^{ix\xi}d\mu_n(\xi)$$

with certain compactly supported probability measures μ_n on $\mathbb{R}$. By the symmetry properties of J_{B_n}, the measure μ_n is even, hence its odd moments vanish. Let further $\varphi(x) := \lim_{n\to\infty}\varphi_n(x)$. Again by Lévy's continuity theorem, there exists a probability measure μ on $\mathbb{R}$ such that $\mu_n\to\mu$ weakly and

$$\varphi(x) = \int_{\mathbb{R}} e^{ix\xi}d\mu(\xi) \quad \text{for all } x\in\mathbb{R}.$$

Further, the family $\{\mu_n : n\in\mathbb{N}\}$ is tight. From (4.1) and formula (3.7) we deduce that

$$\varphi_n(x) = \sum_{j=0}^{\infty} \frac{g_j\big(\lambda(n)^2\big)}{4^j(\nu_n)_j(kn)_j}\,(-x)^{2j}.$$

This shows that the even moments of μ_n are given by

$$\int_{\mathbb{R}} \xi^{2j}d\mu_n(\xi) = (2j)!\,\frac{g_j(\lambda(n)^2)}{4^j(kn)_j(\nu_n)_j}.$$

As in the proof of Lemma 3.13, we deduce that the quotient

$$\frac{\int_{\mathbb{R}}\xi^8 d\mu_n(\xi)}{\big(\int_{\mathbb{R}}\xi^4 d\mu_n(\xi)\big)^2}$$

is bounded in $n\in\mathbb{N}$. Now we conclude from [OO98, Lemma 5.2] (employing the Lemma for the image measure of μ_n under $\xi\mapsto\xi^2$) that the sequence $\big(\int_{\mathbb{R}}\xi^4 d\mu_n(\xi)\big)$ is bounded. As $(\nu_n)_2\sim\nu_n^2$ and $(kn)_2\sim(kn)^2$ for $n\to\infty$, it follows that the sequence

$$\Big(g_2\Big(\frac{\lambda(n)^2}{n\nu_n}\Big)\Big)_{n\in\mathbb{N}}$$

is bounded as well. Continuing as in the proof of Lemma 3.13 we obtain that $\big(\frac{\lambda(n)^2}{\nu_n}\big)$ is a Vershik-Kerov sequence. □

Proof of Theorem 4.1. From Theorem 3.5 it is clear that the statements (1) and (4) are equivalent. The equivalence of (3) and (4) follows from expansion (4.2) and the fact that the Jack polynomials span the algebra of symmetric functions. By Lemma 4.4, statement (2) implies (1). Finally, Lemma 4.3 shows that statement (1) implies statement (2). □

We finally want to determine the set of all parameters $\omega = (\alpha, \beta, 0)$ which occur as VK parameters of a non-negative Vershik-Kerov sequence as in Theorem 4.1. Recall that in the non-negative case, the parameter γ is automatically zero by Lemma 3.2.

Proposition 4.5. *The set Ω_+ of all pairs (α, β) for which there exists a non-negative VK sequence with parameters $(\alpha, \beta, 0)$ is given by*

$$\Omega_+ = \{(\alpha,\beta) : \beta \geq 0,\, \alpha = (\alpha_i)_{i\in\mathbb{N}} \text{ with } \alpha_i \in \mathbb{R},\, \alpha_1 \geq \alpha_2 \geq \ldots \geq 0,\, \sum_{i=1}^{\infty} \alpha_i \leq \beta\,\}.$$

Proof. 1. If $(\alpha, \beta, 0)$ are the VK parameters of a VK sequence $(\lambda(n))$ with $\lambda(n)_i \geq 0$ for all i, then obviously $\beta \geq 0$ and $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Moreover, for fixed $N \in \mathbb{N}$ and $n \geq N$ we have

$$\sum_{i=1}^{N} \alpha_i \;\leq\; \sum_{i=1}^{N} \Big(\alpha_i - \frac{\lambda(n)_i}{n}\Big) \;+\; \sum_{i=1}^{n} \frac{\lambda(n)_i}{n}.$$

As $n \to \infty$, the first sum tends to 0 and the second sum tends to β. This proves that $\sum_{i=1}^{\infty} \alpha_i \leq \beta$.

2. Conversely, let $(\alpha, \beta) \in \Omega_+$. In order to construct an associated non-negative VK sequence, we proceed in two steps.

(i) Assume that α has at most finitely many non-zero entries. If $\alpha \neq 0$, let $m \in \mathbb{N}$ be maximal such that $\alpha_i \neq 0$ for $i \leq m$. If $\alpha = 0$, let $m := 0$. Put

$$\beta' := \beta - \sum_{i=1}^{m} \alpha_i \geq 0.$$

For $n > m$, define $\lambda(n) \in \mathbb{R}^n$ by

$$\lambda(n)_i := \begin{cases} n\alpha_i & \text{if } i \leq m \\ \frac{n\beta'}{n-m} & \text{if } m < i \leq n. \end{cases}$$

Note that the entries of $\lambda(n)$ are non-negative and decreasing for n large enough, say $n \geq n_0$. It is now straightforward to verify that $(\lambda(n))_{n \geq n_0}$ is a VK sequence with parameters $(\alpha, \beta, 0)$.

(ii) Assume that all entries of α are strictly positive. Then a diagonalization argument as in the proof of Proposition 3.14 shows that there exists a VK sequence with parameters $(\alpha, \beta, 0)$. □

Let us finally turn to consequences in the geometric cases, related to the Cartan motion groups of non-compact Grassmann manifolds.

For strictly increasing sequences of dimensions $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ with $p_n \geq q_n$ consider the sequence of Gelfand pairs (G_n, K_n) with

$$G_n = (U_{p_n}(\mathbb{F}) \times U_{q_n}(\mathbb{F})) \ltimes M_{p_n,q_n}(\mathbb{F}),\;\; K_n = U_{p_n}(\mathbb{F}) \times U_{q_n}(\mathbb{F}) \qquad (4.3)$$

over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. It is easily checked that the associated Olshanski spherical pair (G_∞, K_∞) is independent of the specific choice of the sequences $(p_n), (q_n)$, and so the same holds for its spherical functions. Recall from Section 2 that the positive definite spherical functions of (G_n, K_n), considered as functions on the chamber $\overline{C}_{q_n} \subset \mathbb{R}^{q_n}$, are given by the Bessel functions

$$\varphi_{\lambda_n}(x) = J_{B_{q_n}}(\kappa_n, i\lambda_n, x),\ \lambda_n \in \mathbb{R}^{q_n}$$

with $\kappa_n = (k_n', k) = \big(\frac{d}{2}(p_n - q_n + 1) - \frac{1}{2}, \frac{d}{2}\big)$.

Corollary 4.6. (1) *The set of positive definite spherical functions of the Olshanski spherical pair (G_∞, K_∞), considered as functions on*

$$\overline{C}_\infty := \{x \in \mathbb{R}^{(\infty)} : x_1 \geq x_2 \geq \ldots \geq 0\},$$

is given by the functions

$$\varphi_{(\alpha,\beta)}(x) = \prod_{j=1}^{\infty} e^{-\frac{\beta x_j^2}{4}} \prod_{l=1}^{\infty} \frac{e^{\frac{\alpha_l x_j^2}{4}}}{\big(1 + \frac{\alpha_l x_j^2}{2d}\big)^{d/2}}, \quad (\alpha, \beta) \in \Omega_+\,.$$

(2) *Let (p_n, q_n) be dimensions with $q_n = n$ and $\frac{p_n}{n} \to 1$ for $n \to \infty$. Consider a sequence $(\lambda(n))_{n\in\mathbb{N}}$ with $\lambda(n) \in \overline{C}_n$ such that $\big(\frac{\lambda(n)^2}{n}\big)$ is Vershik-Kerov with VK parameters $(k\alpha, k\beta, 0)$. Then the spherical functions of the Gelfand pairs (G_n, K_n) as in (4.3) satisfy*

$$\lim_{n\to\infty} \varphi(\kappa_n; i\lambda(n), x) \,=\, \varphi_{(\alpha,\beta)}(x),$$

where the convergence is locally uniform on each chamber $\overline{C}_r$.

Proof. For part (1), choose (G_n, K_n) with $q_n = n$. The proof is then the same as that of Corollary 3.15 in the type A case. Part (2) is then immediate from Theorem 3.5 and Remark 4.2. □

Remark 4.7. We mention that for $\mathbb{F} = \mathbb{C}$, part (1) of this corollary is in accordance with results of [Bo19], where for the semigroup $\mathrm{Herm}_\infty^+(\mathbb{C})$ of infinite dimensional positive definite matrices over $\mathbb{C}$, the positive definite Olshanski spherical functions of $(U_\infty(\mathbb{C}) \ltimes \mathrm{Herm}_\infty^+(\mathbb{C}), U_\infty(\mathbb{C}))$ were determined by semigroup methods and a reduction to the type A case.

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