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Convolution Algebras which are not Necessarily Positivity-Preserving

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ABSTRACT. In this paper an axiomatic frame for convolution structures is introduced, which generalizes the hypergroup axiomatics mainly in abandoning positivity of the convolution. Essential facts of harmonic analysis, in particular for the commutative case, are developed. Finally, several examples are discussed.

1. Introduction

The theory of generalized convolution algebras began in the late 1930s with the study of the solutions of Sturm-Liouville eigenvalue problems. At that time, Delsarte and Levitan introduced their generalized translation operators (see [5] and [15], [16]), and a quite far-reaching harmonic analysis for them was developed. Related structures are the hypercomplex systems of Berezansky and Krein, see Berezansky, Kaluzhny [2], [3] and the references cited there. For an overview, we refer to the survey of Litvinov [17]. In all these concepts, convolution occurs as a convolution of L^1-functions on the underlying space, with respect to some distinguished measure. It is not always required to be positivity-preserving — in contrast to the hypergroup axiomatics, which became widely adopted later on with the work of Jewett [12], Spector [23] and Dunkl [6].

In this paper, we shall introduce a further axiomatic frame for generalized convolution algebras on locally compact Hausdorff spaces, which we call signed hypergroups. It is designed to generalize the hypergroup axiomatics in several important points, though sticking close to it in its basic ideas. If X denotes the basis space, then the convolution algebra under consideration is the space $M_b(X)$

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of regular bounded Borel measures on X, just as in the hypergroup case. Convolution need not be positivity-preserving, but should at least provide the structure of a Banach- $*$ -algebra with unit. Hereby, the involution in $M_b(X)$ is assumed to be induced by an involutive homeomorphism on X. Axioms on the support of the convolution products of point measures are omitted. As for generalized translation operators and non-positive hypercomplex systems, it cannot be expected to derive the existence of an invariant measure from the other axioms (see Berezansky, Kaluzhny [2], where a submultiplicative measure, called m-measure, is axiomatically required). So we also presuppose a substitute for Haar measure.

In order to give some motivation, we start with a short discussion of the Laguerre convolution on $\mathbb{R}_+$ as an example. This convolution has been studied by several authors including Askey [1], McCully [18], Görlich and Markett [9], [10]; however, they did not embed their work into a more general frame of harmonic analysis. The properties of the Laguerre convolution are then revealed in the definition of a signed hypergroup. Especially, we do not generally require that $\delta_x * \delta_y(X) = 1$ for all $x, y \in X$; this additional property is quite useful, but it is not satisfied in the Laguerre case. In contrast, in their axiomatic treatment of signed hypergroups, Wildberger [27] and Ross [21] require this property.

After some general analysis on signed hypergroups, the commutative case is studied in more detail. Here duality theory can be developed in close analogy to the hypergroup case; in particular, the spaces of (symmetric) multiplicative functions on X may be canonically identified with the (symmetric) spectrum of $(M_b(X), *)$. Furthermore, a Bochner theorem is valid.

The paper ends with a discussion of some related structures and further examples: the compact hypercomplex systems in the sense of Vainerman (see [24] and [25]), the finite signed hypergroups of Wildberger ([27], [28], [29]), and, finally, an example concerning duals of polynomial hypergroups.

1.1. Notation. Let X be a locally compact Hausdorff space. We denote by $C_b(X)$, $C_0(X)$ and $C_c(X)$ the spaces of continuous functions on X which are bounded, those which vanish at infinity and those having compact support respectively. They shall be endowed with the uniform norm $\|.\|_\infty$. By $M(X)$ and $M^+(X)$ we denote the spaces of Radon measures and positive Radon measures on X. The spaces $M_b(X)$, $M_b^+(X)$, $M_b^{\mathbb{R}}(X)$ and $M_c(X)$ consist of all bounded regular Borel measures on X, those which are positive, the real ones, and those having compact support respectively. The total variation norm on these spaces is abbreviated by $\|.\|$, and the weak- $*$ -topology $\sigma(M_b(X), C_0(X))$ shall be referred to as the τ_* -topology on $M_b(X)$. Finally, $\delta_x \in M_b^+(X)$ denotes the point measure at $x \in X$.

2. The Laguerre convolution

We consider the normalized Laguerre-functions $\mathcal{L}_n^\alpha, n \in \mathbb{N}_0$, with parameter $\alpha > -1$. They are defined by

$$\mathcal{L}_n^\alpha(x) := e^{-x/2}\frac{L_n^\alpha(x)}{L_n^\alpha(0)}, \quad x \geq 0,$$

where L_n^α denotes the Laguerre polynomial

$$L_n^\alpha(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu}\frac{(-x)^\nu}{\nu!}.$$

From the orthogonality relation of the Laguerre-polynomials (see e.g. Chihara [4]), it follows that

$$\int_0^\infty \mathcal{L}_n^\alpha(x)\mathcal{L}_m^\alpha(x)x^\alpha dx = \frac{\Gamma(\alpha+1)}{\binom{n+\alpha}{n}}\delta_{n,m}.$$

Watson [26] proved the following product formula for Laguerre-polynomials in the case $\alpha > -\frac{1}{2}$:

$$L_n^\alpha(x)L_n^\alpha(y) = \\ = \frac{\Gamma(n+\alpha+1)}{n!\sqrt{\pi}}\int_0^\pi L_n^\alpha(x+y+2\sqrt{xy}\cos t)e^{-\sqrt{xy}\cos t}\frac{J_{\alpha-1/2}(\sqrt{xy}\sin t)}{(\frac{1}{2}\sqrt{xy}\sin t)^{\alpha-1/2}}\sin^{2\alpha} t dt.$$

This formula gives rise to a Laguerre convolution operator, which has been introduced by McCully [18] and Askey [1] and further studied by Görlich and Markett (see [9], [10] and the references cited there). We rewrite the above product formula in terms of Laguerre functions:

$$\mathcal{L}_n^\alpha(x)\mathcal{L}_n^\alpha(y) = \int_0^\infty \mathcal{L}_n^\alpha(z)d\mu_{x,y}^\alpha(z), \; x, y \geq 0,$$

with

$$d\mu_{x,y}^\alpha(z) = \begin{cases} K^\alpha(x,y,z)z^\alpha dz, & \text{if } x, y \neq 0 \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0, \end{cases}$$

where the kernel K^α is given by

$$K^\alpha(x,y,z) = \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\sqrt{2\pi}(xyz)^\alpha}J_{\alpha-\frac{1}{2}}(\varrho(x,y,z))\varrho(x,y,z)^{\alpha-\frac{1}{2}}\cdot 1_{[(\sqrt{x}-\sqrt{y})^2,(\sqrt{x}+\sqrt{y})^2]}(z);$$

$$\varrho(x,y,z) = \frac{1}{2}\big(z-(\sqrt{x}-\sqrt{y})^2\big)^{1/2}\big((\sqrt{x}+\sqrt{y})^2-z\big)^{1/2}.$$

Obviously, K^α is not positive, but in case $\alpha \geq 0$, it is at least quasi-positive:

$$\int_0^\infty \left|K^\alpha(x,y,z)\right|z^\alpha dz \leq 1 \qquad \text{for all} \quad x, y > 0, \; \alpha \geq 0,$$

as shown in [9]. This fact gives rise to a convolution of measures as follows:

2.1. Proposition. *For $\alpha \geq 0$ and $x, y \in \mathbb{R}_+$ define $\delta_x * \delta_y \in M_b^{\mathbb{R}}(\mathbb{R}_+)$ by*

$$\delta_x * \delta_y(f) := \int_0^\infty f(z) d\mu_{x,y}^\alpha(z), \quad f \in C_0(\mathbb{R}_+).$$

Then $*$ can be extended to a separately τ_*-continuous, bilinear and commutative convolution on $M_b(\mathbb{R}_+)$ by setting

$$\mu * \nu(f) := \int_{\mathbb{R}_+ \times \mathbb{R}_+} \delta_x * \delta_y(f) d(\mu \otimes \nu)(x, y), \qquad f \in C_0(\mathbb{R}_+).$$

It has the following further properties:

(1) $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$ *for all* $\mu, \nu \in M_b(\mathbb{R}_+)$.
(2) $*$ *is associative.*
(3) $\delta_0 * \mu = \mu$ *for all* $\mu \in M_b(\mathbb{R}_+)$.
(4) *The mapping* $(x, y) \mapsto \delta_x * \delta_y$ *is* τ_**-continuous.*
(5) *For fixed* $x \in \mathbb{R}_+$ *and* $f \in C_b(\mathbb{R}_+)$, *the function* $y \mapsto \delta_x * \delta_y(f)$ *again belongs to* $C_b(\mathbb{R}_+)$.
(6) *For* $f \in C_c(X)$ *with* $\text{supp} f \subseteq [0, k]$, *the support of* $y \mapsto \delta_x * \delta_y(f)$ *is contained in* $[0, (\sqrt{x} + \sqrt{k})^2]$.
(7) *For all* $f, g \in C_c(\mathbb{R}_+)$ *and* $x \in \mathbb{R}_+$,

$$\int_0^\infty \delta_x * \delta_y(f) g(y) y^\alpha dy = \int_0^\infty f(y) \delta_x * \delta_y(g) y^\alpha dy.$$

Proof. As $\|\mu_{x,y}^\alpha\| \leq 1$ for all $x, y \in \mathbb{R}_+$, the product $\mu * \nu$ is well-defined in $M_b(\mathbb{R}_+)$ with $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$. The bilinearity and commutativity of $*$ are clear. Now, fix $f \in C_0(\mathbb{R}_+)$. From the representation

$$\delta_x * \delta_y(f) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}} \int_0^\pi f(x+y+2\sqrt{xy}\cos t) \cdot \frac{J_{\alpha-1/2}(\sqrt{xy}\sin t)}{(\frac{1}{2}\sqrt{xy}\sin t)^{\alpha-1/2}} \sin^{2\alpha} t \, dt \tag{2.1}$$

it is easily seen that the function Tf defined by $Tf(x, y) := \delta_x * \delta_y(f)$ belongs to $C_0(\mathbb{R}_+ \times \mathbb{R}_+)$. Hence for $\mu \in M_b(\mathbb{R}_+)$, the function $y \mapsto \int_0^\infty Tf(x, y) d\mu(x)$ is continuous (use a dominated convergence argument) and vanishes at infinity. For any net (ν_α) in $M_b(\mathbb{R}_+)$ with $\tau_* - \lim \nu_\alpha = \nu \in M_b(\mathbb{R}_+)$ we thus obtain

$$\lim_\alpha (\mu * \nu_\alpha)(f) = \lim_\alpha \int_0^\infty \left(\int_0^\infty Tf(x, y) d\mu(x) \right) d\nu_\alpha(y) =$$
$$= \int_0^\infty \left(\int_0^\infty Tf(x, y) d\mu(x) \right) d\nu(y) = \mu * \nu(f).$$

This is the separate τ_*-continuity of $*$. To prove associativity, it suffices to consider point measures. But for $x, y, z \in \mathbb{R}_+$ and all $n \in \mathbb{N}_0$, the identity

$$\big(\delta_x * (\delta_y * \delta_z)\big)(\mathcal{L}_n^\alpha) = \mathcal{L}_n^\alpha(x) \mathcal{L}_n^\alpha(y) \mathcal{L}_n^\alpha(z) = \big((\delta_x * \delta_y) * \delta_z\big)(\mathcal{L}_n^\alpha)$$

holds. As the convolution of two point measures has compact support, the same is true for $\delta_x * (\delta_y * \delta_z)$ and $(\delta_x * \delta_y) * \delta_z$. So on the union of these supports, continuous functions can be uniformly approximated by polynomials, and the assertion follows. The remaining statements are quite easy to check: (3) is clear from the fact that $\mathcal{L}_n^\alpha(0) = 1$ for all $n \in \mathbb{N}_0$, (4), (5) and (6) are immediate consequences of (2.1), and (7) follows from the symmetry of the kernel K^α in its arguments.

2.2. Remarks.

1. As $\delta_x * \delta_y$ is not positive, but $\|\delta_x * \delta_y\| \leq 1$, it follows that $\delta_x * \delta_y(\mathbb{R}_+) < 1$ for all $x, y \neq 0$.
2. $dm(y) := y^\alpha dy$ is strictly subinvariant with respect to $*$, in the sense that

$$\int_0^\infty Tf(x,y)y^\alpha dy < \int_0^\infty f(y)y^\alpha dy$$

for all $x \in \mathbb{R}_+$ and $f \in C_c(\mathbb{R}_+), f \not\equiv 0$, with $f \geq 0$. Indeed, part (7) of Proposition 2.1 clearly remains true for $g \in C_b(\mathbb{R}_+)$, and with $g = 1$ we get

$$\int_0^\infty Tf(x,y)y^\alpha dy = \int_0^\infty f(y)\delta_x * \delta_y(\mathbb{R}_+)y^\alpha dy < \int_0^\infty f(y)y^\alpha dy.$$

Moreover, from Corollary 3.4.(2) it will follow that there exists no nontrivial invariant measure for the Laguerre convolution on $\mathbb{R}_+$.

3. Signed hypergroups

Let X be a locally compact Hausdorff space and T a linear mapping from $C_0(X)$ to the space $B(X \times X)$ of all bounded functions on $X \times X$. Following Pym [20], we shall call T a generalized translation operator on X. Suppose that
(i) T is continuous with respect to $\|.\|_\infty$ on both $C_0(X)$ and $B(X \times X)$;
(ii) for $f \in C_0(X)$ and $x \in X$, the translates $y \mapsto Tf(x,y)$ and $y \mapsto Tf(y,x)$ again are in $C_0(X)$.
Then for $\mu \in M_b(X)$, the function $y \mapsto \int Tf(x,y)d\mu(x)$ is continuous, and by

$$\mu * \nu(f) := \int_X \int_X Tf(x,y)\, d\mu(x) d\nu(y), \quad f \in C_0(X),$$

a bilinear, separately τ_*-continuous multiplication on $M_b(X)$ is defined; see Pym [20]. We call it the canonical continuation of the mapping $\omega : X \times X \to M_b(X)$, $\omega(x,y)(f) := Tf(x,y)$ for $f \in C_0(X)$.

Now suppose $\omega : X \times X \to M_b(X)$, $(x,y) \mapsto \delta_x * \delta_y$, defines a convolution of point measures such that $\|\delta_x * \delta_y\| \leq C$ for all $x, y \in X$ with some constant $C > 0$, and that for $f \in C_c(X)$ and $x \in X$ the mappings $T^x f : y \mapsto \delta_x * \delta_y(f)$ and $T_x f : y \mapsto \delta_y * \delta_x(f)$ again belong to $C_c(X)$. Then the translation operator T on X defined by $Tf(x,y) := \delta_x * \delta_y(f)$ satisfies the conditions (i) and (ii) above, from which it follows that ω has a canonical continuation to a separately τ_*-continuous multiplication $*$ on $M_b(X)$. In case ω is additionally required to be τ_*-continuous, this multiplication can be written as

$$\mu * \nu(f) = \int_{X\times X} \delta_x * \delta_y(f) d(\mu\otimes\nu)(x,y), \quad f \in C_0(X).$$

We note at this point that τ_*-continuity of ω does not in general involve τ_*-continuity of its canonical continuation.

We are now able to formulate the definition of a signed hypergroup:

3.1. Definition. *Let X be a locally compact, σ-compact Hausdorff space and m a positive Radon measure on it with $supp\, m = X$. Further, let $\omega : X\times X \to M_b^{\mathbb{R}}(X)$, $(x,y)\mapsto \delta_x * \delta_y$, be a τ_*-continuous mapping. Then the triple (X,m,ω) is called a signed hypergroup, if the following axioms are satisfied:*

(A1) *For each $x\in X$ and $f\in C_b(X)$, the translates $T^x f : y\mapsto \delta_x * \delta_y(f)$ and $T_x f : y \mapsto \delta_y * \delta_x(f)$ again belong to $C_b(X)$. Furthermore, for $f\in C_c(X)$ and any compact subset $K\subset X$, the set $\bigcup_{x\in K}\big(supp(T^x f)\cup supp(T_x f)\big)$ is relatively compact in X.*

(A2) *$\|\delta_x * \delta_y\| \le C$ for all $x,y\in X$ with some constant $C>0$.*

(A3) *The canonical continuation of ω is associative.*

(A4) *There exists a neutral element $e\in X$, such that*

$$\delta_e * \mu = \mu * \delta_e = \mu \quad \textit{for all } \mu\in M_b(X).$$

(A5) *There exists an involutive homeomorphism $^-$ on X such that*

$$(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}} \quad \textit{for all } x,y\in X,$$

where for a Borel measure μ on X, the measure μ^- on X is defined by $\mu^-(A) := \mu(A^-)$, $A\subset X$ any Borel set.

(A6) *For all $f,g\in C_c(X)$ and $x\in X$ the following adjoint relation holds:*

$$\int_X (T^x f) g\, dm = \int_X f(T^{\overline{x}} g)\, dm.$$

According to Proposition 2.1, the Laguerre convolution leads to a signed hypergroup, namely $(\mathbb{R}_+, m, \omega)$, where $dm(x) = x^\alpha dx$, $\omega(x,y) = \mu^\alpha_{x,y}$, and the involution on $\mathbb{R}_+$ is the identity mapping. Clearly, any σ-compact hypergroup (in the sense of Jewett [12]) with a Haar measure is in particular a signed hypergroup. Further examples will be discussed in Chapter 5.

Let (X,m,ω) be a signed hypergroup. Before working out some harmonic analysis, we want to discuss the role of the measure m, which obviously serves as a substitute for a left Haar measure on X. We shall see that m does not have the invariance property of a Haar measure in general, but only under some extra requirement on the convolution. For this, we need two lemmata which will also be used later on; the first one is essentially based on the σ-compactness of X.

3.2. Lemma. *For $f \in C_b(X)$, the mapping $(x,y) \mapsto \delta_x * \delta_y(f)$ is measurable, and the following identities hold:*

(1) $$\mu * \nu(f) = \int_{X\times X} \delta_x * \delta_y(f) d(\mu \otimes \nu)(x,y) \qquad \text{for all } \mu, \nu \in M_b(X).$$

(2) $$\int_X (T^x f) g \, dm = \int_X f(T^{\overline{x}} g) dm \qquad \text{for all } x \in X \text{ and } g \in C_c(X).$$

Proof. Choose compact subsets K_n, $n \in \mathbb{N}$ of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$. According to Urysohn's lemma, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $C_c(X)$ with $0 \le g_n \le 1$, $supp\, g_n \subset K_{n+1}^\circ$ and $g_n = 1$ on K_n. Define $h_n \in C_c(X)$ by $h_n := g_n f$. By the dominated convergence theorem, we obtain that $(x,y) \mapsto \delta_x * \delta_y(f) = \lim_{n\to\infty} \delta_x * \delta_y(h_n)$ is measurable, and

$$\mu * \nu(f) = \lim_{n\to\infty} \mu * \nu(h_n) = \lim_{n\to\infty} \int_{X\times X} \delta_x * \delta_y(h_n) d(\mu \otimes \nu)(x,y) =$$
$$= \int_{X\times X} \delta_x * \delta_y(f) d(\mu \otimes \nu)(x,y)$$

for any $\mu, \nu \in M_b(X)$. So (1) is proved. By the same way, (2) results from (A6).

3.3. Lemma. *Let $n \in M^+(X)$ be arbitrary. Then for $f \in C_c(X)$ the mappings Φ_f and $\Psi_f : X \to C_c(X)$ defined by $\Phi_f(x) := T^x f$, $\Psi_f(x) := T_x f$ are continuous with respect to $\|.\|_{1,n}$.*

Proof. We restrict to Φ_f; for Ψ_f the proof is analogous. Fix $x_0 \in X$ and choose a relatively compact neighbourhood U of x_0 in X. According to Axiom (A1), there exists a compact set $M \subset X$, such that $\bigcup_{x\in\overline{U}}\{supp(T^x f)\}$ is contained in M. Now the mapping $(x,y) \mapsto \delta_x * \delta_y(f)$ is uniformly continuous on $\overline{U} \times M$. Thus for $\epsilon > 0$ there exists a neighbourhood $U_\epsilon \subset U$ of x_0 with $|\delta_x * \delta_y(f) - \delta_{x_0} * \delta_y(f)| < \epsilon$ for all $x \in U_\epsilon$ and all $y \in M$. Hence for $x \in U_\epsilon$,

$$\|\Phi_f(x) - \Phi_f(x_0)\|_{1,n} = \int_M |\delta_x * \delta_y(f) - \delta_{x_0} * \delta_y(f)| dn(y) < \epsilon\, n(M).$$

3.4. Corollary.

(1) *The following statements are equivalent:*

(i) *m is left-translation invariant with respect to $*$, that is*

$$\int_X T^x f dm = \int_X f dm \qquad \text{for all } x \in X \text{ and } f \in C_c(X).$$

(ii) *$\delta_x * \delta_y(X) = 1$ for all $x, y \in X$.*

(2) *If* $n \in M(X)$ *is left-translation invariant with respect to* $*$, *then* $n = cm$ *with a constant* $c \in \mathbb{C}$. *In particular, if condition* (1)(ii) *is not satisfied, then there exists no nontrivial left-translation invariant measure on* (X, m, ω).

Proof. (1) (ii) $\Rightarrow$ (i) is clear by setting $f = 1$ in identity (2) of Lemma 3.2. (i) $\Rightarrow$ (ii): Suppose $\delta_x * \delta_y(X) \neq 1$ for some $x, y \in X$, say $\delta_x * \delta_y(X) < 1$. As the mapping $y \mapsto \delta_x * \delta_y(1)$ is continuous by Axiom (A1), there exists some neighbourhood U of y such that $\delta_x * \delta_z(X) < 1$ for all $z \in U$. Take $f \in C_c(X)$, $f \geq 0$, $f \not\equiv 0$, with $supp\, f \subset U$. In view of Lemma 3.2, we obtain

$$\int_X (T^{\overline{x}} f) dm = \int_X f(z)\delta_x * \delta_z(X) dm(z) < \int_U f(z) dm(z) = \int_X f dm.$$

The case $\delta_x * \delta_y(X) > 1$ is treated analogously.

(2) By Lemma 3.3, this is proved in the same way as the uniqueness of Haar measure on a hypergroup; see Jewett [12], 5.2.

We are now going to define the convolution of functions and measures on a signed hypergroup (X, m, ω):

3.5. Lemma.

(1) *For* $f \in C_0(X)$ *and* $\mu \in M_b(X)$ *there are defined functions* $\mu * f$ *and* $f * \mu$ *in* $C_0(X)$ *by*

$$\mu * f(x) := \int_X T^{\overline{y}} f(x) d\mu(y), \; f * \mu(x) := \int_X T_{\overline{y}} f(x) d\mu(y).$$

(2) *If* $f \in C_c(X)$ *and* $\mu \in M_c(X)$, *then* $\mu * f$ *and* $f * \mu$ *belong to* $C_c(X)$ *again.*

(3) *For* $f \in C_c(X)$ *and* $\mu \in M_b(X)$, *the identity* $(\mu * f)m = \mu * fm$ *holds, and*

$$\|\mu * f\|_{1,m} \leq C \cdot \|f\|_{1,m} \cdot \|\mu\|.$$

Proof. (1) follows from Theorem 1.1 in Pym [20] (Note that $\mu * f(x) = \mu^- * \delta_x(f)$ and $f * \mu(x) = \delta_x * \mu^-(f)$ for $f \in C_0(X)$); (2) is an immediate consequence of Axiom (A1).

(3) By (A6), we obtain for each $h \in C_c(X)$:

$$\int_X h(x)(\mu * f)(x) dm(x) = \int_X \int_X h(x) T^{\overline{y}} f(x) d\mu(y) dm(x) =$$
$$= \int_X \int_X T^y h(x) d\mu(y) d(fm)(x),$$

and thus $(\mu * f)m = \mu * fm$. The rest is clear.

3.6. Remark. For $f, g \in C_c(X)$ a function $f * g \in C_c(X)$ is defined by $f * g := fm * g$. This leads to a convolution on $C_c(X)$ which is commutative if and only if the basic convolution on $M_b(X)$ is commutative. In this case, the identity $f * g(e) = g * f(e)$ holds for any $f, g \in C_c(X)$, and from this it is immediately seen that $m^- = m$.

We introduce a norm $\|.\|'$ on $M_b(X)$ by $\|\mu\|' := \max(\|L_\mu\|, \|R_\mu\|)$, where L_μ and R_μ denote the continuous multiplication operators on $M_b(X)$ given by $L_\mu(\nu) := \mu * \nu$ and $R_\mu(\nu) := \nu * \mu$, $\nu \in M_b(X)$. The subset of those measures in $M_b(X)$ which are absolutely continuous with respect to m shall be identified with $L^1(X, m)$ in the sense of the Radon-Nikodym theorem. Furthermore, for $\mu \in M_b(X)$ we define the measure $\mu^* \in M_b(X)$ by $\mu^* := \overline{\mu^-}$, where the outer bar is conjugation.

3.7. Theorem.

(1) $(M_b(X), *, \|.\|')$ *is a Banach-*-algebra with unit element* δ_e *and the involution* $\mu \mapsto \mu^*$.

(2) $(L^1(X,m), *, \|.\|')$ *is a closed left-ideal in* $(M_b(X), *, \|.\|')$.

Proof. (1) From (A2) it follows that $\|\mu * \nu\| \leq C \cdot \|\mu\| \cdot \|\nu\|$ for all $\mu, \nu \in M_b(X)$. Thus taking $\|\mu\|_1 := \|L_\mu\|$ as the norm on $M_b(X)$, $(M_b(X), *)$ becomes a Banach algebra with unit δ_e (see Th.10.2 in Rudin [22]). The same is true with $\|\mu\|_2 := \|R_\mu\|$ as a norm on $M_b(X)$, and therefore, $\|.\|' := \max(\|.\|_1, \|.\|_2)$ is equivalent to $\|.\|$ and makes $(M_b(X), *)$ into a Banach algebra with unit δ_e as well. The operation $\mu \mapsto \mu^*$ on $M_b(X)$ is clearly conjugate-linear with $(\mu^*)^* = \mu$. Further, by (A5) and the fact that the convolution product of point measures belongs to $M_b^{\mathbb{R}}(X)$, we obtain

$$(\mu * \nu)^* = \overline{\nu^- * \mu^-} = \nu^* * \mu^*$$

for all $\mu, \nu \in M_b(X)$; hence $*$ is an involution on $(M_b(X), *)$. It remains to show that $*$ is isometric with respect to $\|.\|'$. But as $\|\mu^*\| = \|\mu\|$ for all $\mu \in M_b(X)$, we have

$$\|L_{\mu^*}\| = \sup_{\|\nu\| \leq 1} \|\mu^* * \nu\| = \sup_{\|\nu\| \leq 1} \|\nu^* * \mu\| = \|R_\mu\|.$$

Analogously, $\|R_{\mu^*}\| = \|L_\mu\|$. It follows that $\|\mu^*\|' = \|\mu\|'$ for all $\mu \in M_b(X)$.

(2) It is clear that $L^1(X,m)$ is $\|.\|'$-closed in $M_b(X)$; thus it remains to show that for any $f \in L^1(X,m)$ and $\mu \in M_b(X)$ the product $\mu * fm$ is absolutely continuous with respect to m. Choose a sequence $(f_n)_{n \in \mathbb{N}} \subset C_c(X)$ with $f = \lim_{n \to \infty} f_n$ in $L^1(X,m)$. According to Lemma 3.5.(3), $(\mu * f_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $L^1(X,m)$; denote its limit by $\mu * f$. By the $\|.\|$-continuity

of $*$ we now obtain

$$\mu * fm = \lim_{n\to\infty} \mu * f_n m = \lim_{n\to\infty} (\mu * f_n)m = (\mu * f)m,$$

from which the assertion follows.

For $\mu \in M_b(X)$ consider the convolution operator $L_\mu : C_c(X) \to C_0(X)$ defined by $L_\mu(f) := \mu * f$. According to the above theorem, it can be uniquely extended to a continuous linear operator on $L^1(X,m)$. We shall now prove that the same is true for any space $L^p(X,m), 1 \le p \le \infty$, thus establishing the convolution of measures with functions of $L^p(X,m)$:

3.8. Lemma. *For every $\mu \in M_b(X)$, the operator L_μ can be extended to a continuous linear operator on $L^p(X,m), 1 \le p \le \infty$, satisfying $\|L_\mu\| \le C \cdot \|\mu\|$.*

Proof. The case $p = 1$ is already clear. By the Fubini theorem, for any $f, g \in C_c(X)$ the relation

$$\int_X (\mu * f)(x) g(x) dm(x) = \int_X f(x)(\mu^- * g)(x) dm(x) \tag{3.1}$$

holds. Now for $f \in L^\infty(X,m)$ define $\mu * f$ as the unique element of $L^\infty(X,m)$, which satisfies (3.1) for all $g \in L^1(X,m)$. This yields a continuation of L_μ to a continuous linear operator on $L^\infty(X,m)$ with $\|L_\mu\| \le C \cdot \|\mu\|$. Now the Riesz-Thorin interpolation theorem can be applied; it shows that for any $p \in]1,\infty[$, L_μ extends uniquely to a continuous linear operator on $L^p(X,m)$ with $\|L_\mu f\|_p \le C \cdot \|\mu\| \cdot \|f\|_p$ for all $f \in L^p(X,m)$.

If $m^- = m$, then all statements concerning left-translations and left-convolutions by measures can be analogously carried over to right-translations and right-convolutions. In particular, for $f \in C_c(X)$ and $\mu \in M_b(X)$ the identity $(f * \mu)m = fm * \mu$ holds, which assures that $(L^1(X,m), *, \|.\|')$ is a two-sided ideal in $(M_b(X), *, \|.\|')$. It is also self-adjoint, because $(fm)^* = f^* m$ with $f^*(x) := \overline{f(\overline{x})}$ for every $f \in L^1(X,m)$.

As already mentioned, the condition $m^- = m$ is satisfied in the abelian case. A further important class of examples is provided by the following lemma:

3.9. Lemma. *Let (X, m, ω) be a compact signed hypergroup, with the additional property that $\delta_x * \delta_y(X) = 1$ for all $x, y \in X$. Then $m^- = m$.*

Proof. According to Corollary 3.4, m is left-invariant with respect to $*$. Now it is immediately checked that for any $x \in X$ the measure $m * \delta_{\overline{x}} \in M_b(X)$ is left-invariant as well. By Corollary 3.4(2), there exists a constant $c(x) \in \mathbb{C}$ with $m * \delta_{\overline{x}} = c(x)m$. Furthermore, $m * \delta_{\overline{x}}(X) = m(X)$, and therefore $c(x) = 1$, which means that m is right-invariant. Hence m^- is left-invariant, and so Corollary 3.4(2) yields $m^- = cm$ with some $c \in \mathbb{C}$. As $m^-(X) = m(X) \neq 0$, it follows that $c = 1$.

4. Harmonic analysis for commutative signed hypergroups

In this chapter, (X, m, ω) shall always be a commutative signed hypergroup.

4.1. DEFINITION. *A character of* (X, m, ω) *is a function* $\varphi \in C_b(X)$, $\varphi \not\equiv 0$, *which satisfies*

$$\delta_x * \delta_y(\varphi) = \varphi(x)\varphi(y) \quad \textit{for all } x, y \in X.$$

We denote the set of characters of (X, m, ω) *by* $\mathcal{X}_b(X)$. *Furthermore,*

$$\widehat{X} := \{\varphi \in \mathcal{X}_b(X) : \varphi(\overline{x}) = \overline{\varphi(x)} \quad \textit{for all } x \in X\}$$

shall denote the subset of those characters in $\mathcal{X}_b(X)$ *which are symmetric.* $\widehat{X}$ *is also called the dual of* X.

4.2. REMARKS.

(1) If $\varphi \in \mathcal{X}_b(X)$, then $\varphi(e) = 1$ and $\|\varphi\|_\infty \leq C$. This follows from the facts

$$\varphi(x) = \delta_x * \delta_e(\varphi) = \varphi(x)\varphi(e) \quad \text{and} \quad |\varphi(x)|^2 = |\delta_x * \delta_{\overline{x}}(\varphi)| \leq C \cdot \|\varphi\|_\infty.$$

(2) If $\varphi \in \widehat{X}$, then $\sigma^*(\varphi) = \overline{\sigma(\varphi)}$ for all $\sigma \in M_b(X)$. This is clear, because $\overline{\varphi^-} = \varphi$.

In the theory of commutative hypergroups, the identification of the space of bounded multiplicative functions on a hypergroup with the maximal ideal space of the associated L^1-convolution algebra (Jewett [12], Chap.6.3) is of decisive importance. This correspondence can be established quite analogously for general commutative signed hypergroups.

$\mathcal{X}_b(X)$ and $\widehat{X}$ shall be equipped with the topology of uniform convergence on compact sets. Further, we denote by Δ the maximal ideal space of $(M_b(X), *, \|.\|')$, equipped with the Gelfand topology τ_G, and by Δ_S the closed subspace of Δ consisting of those functionals which are symmetric with respect to $*$. We introduce the spaces

$$\Delta^0 := \{F \in \Delta : F \not\equiv 0 \text{ on } L^1(X, m)\},$$
$$\Delta^0_S := \Delta^0 \cap \Delta_S.$$

These are open subsets of Δ and Δ_S respectively, and hence in particular locally compact Hausdorff spaces. Just as in the hypergroup case, they can be identified in a natural way with the maximal ideal spaces $\Delta(L^1(X,m))$ and $\Delta_S(L^1(X,m))$ respectively.

4.3. Proposition. *Every functional $F \in \Delta^0$ can be represented as*

$$F(\mu) = \int_X \varphi_F d\mu, \quad \mu \in M_b(X),$$

for a unique character $\varphi_F \in \mathcal{X}_b(X)$. Conversely, each $\varphi \in \mathcal{X}_b(X)$ is of the form $\varphi = \varphi_F$ for some $F \in \Delta^0$. In the same way, the symmetric characters on (X,m,ω) correspond to the functionals from Δ_S^0.

Proof. 1. Take $\varphi \in \mathcal{X}_b(X)$. As $supp\, m = X$, φ represents an element in $L^\infty(X,m)$ different from 0. Hence $F(\mu) := \mu(\varphi)$ defines a functional $F \in M_b(X)^*$ which does not vanish on $L^1(X,m)$. In view of Lemma 3.2, we have

$$F(\mu * \nu) = \int_X \varphi d(\mu * \nu) = \int_{X\times X} \delta_x * \delta_y(\varphi) d(\mu \otimes \nu)(x,y) = \mu(\varphi)\nu(\varphi) = F(\mu)F(\nu)$$

for all $\mu, \nu \in M_b(X)$. It follows that $F \in \Delta^0$ and $\varphi = \varphi_F$.

2. For a given $F \in \Delta^0$, the restriction $\widetilde{F}$ of F to $L^1(X,m)$ is a nonzero element of $L^1(X,m)^*$, represented by a unique $\psi \in L^\infty(X,m), \psi \neq 0$. Now choose some $h \in C_c(X)$ with $\int_X \psi h\, dm = 1$. Then for any $f \in C_c(X)$,

$$\int_X \psi f dm = \widetilde{F}(f)\widetilde{F}(h) = \widetilde{F}(f * h) = \int_X \int_X \psi(y)\ \delta_{\bar{x}} * \delta_y(h) f(x) dm(x) dm(y).$$

Therefore the function $\varphi : X \to \mathbb{C}$ defined by

$$\varphi(x) := \int_X \psi(T^{\bar{x}} h) dm$$

is a representative of ψ, that is, $F(f) = \int_X \varphi f dm$ for all $f \in L^1(X,m)$; furthermore, φ is continuous by Lemma 3.3. Thus for any $f, g \in C_c(X)$,

$$\Big(\int_X \varphi f dm\Big)\Big(\int_X \varphi g dm\Big) = F(f)F(g) = F(f * g) =$$

$$= \int_X \Big(\int_X \delta_y * \delta_x(\varphi) f(y) g(x) dm(y)\Big) dm(x),$$

where Lemma 3.2 was used to obtain the last identity. It follows that

$$\varphi(x) \int_X \varphi f dm = \int_X (T_x \varphi) f\ dm = \int_X (T^x \varphi) f\ dm \tag{4.1}$$

for m-almost all x. Furthermore, the function

$$x \mapsto \int_X (T^x \varphi) f dm = \int_X \varphi(T^{\bar{x}} f) dm$$

is continuous on X according to Lemma 3.3, whence (4.1) must hold for all $x \in X$. But this proves the multiplicativity of φ, because the integrands in both sides of (4.1) are continuous (Axiom (A1)).

Now choose $f \in L^1(X,m)$ with $F(f) = 1$. As $L^1(X,m)$ is an ideal in $M_b(X)$, we can write

$$F(\mu) = F(\mu * f) = \int_X \varphi d(\mu * f) = \int_{X\times X} \delta_x * \delta_y(\varphi) d(\mu \otimes fm)(x,y) =$$
$$= \Big(\int_X \varphi d\mu\Big) \cdot \Big(\int_X \varphi f dm\Big) = \int_X \varphi d\mu$$

for any $\mu \in M_b(X)$. Thus φ is a representing character for F as claimed; its uniqueness is clear.

Finally, in case $F \in \Delta_S^0$ we have $\overline{\varphi(x)} = F(\delta_x^*) = \varphi(\overline{x})$ for all $x \in X$; hence $\varphi \in \widehat{X}$.

4.4. Lemma. *The mapping* $X \times \Delta^0 \to \mathbb{C}$, $(x, F) \mapsto \varphi_F(x)$ *is continuous.*

Proof. Let $\epsilon > 0, x \in X$ and $F \in \Delta^0$ be given; furthermore choose $h \in C_c(X)$ with $F(h) = 1$. According to Lemma 3.3, there exists an open neighbourhood $U(x)$ of x such that $\|(T^{\overline{y}}h - T^{\overline{x}}h)m\|' < \epsilon$ for all $y \in U(x)$.

For any $y \in U(x)$ and $G \in \Delta$ satisfying

$$|G(h) - F(h)| < \epsilon, \ |G(T^{\overline{x}}h) - F(T^{\overline{x}}h)| < \epsilon,$$

we first obtain

$$|\varphi_G(y) - \varphi_F(x)| \le |\varphi_G(y)(F(h) - G(h))| + |(\varphi_G(y) - \varphi_G(x))G(h)| +$$
$$+ |\varphi_G(x)G(h) - \varphi_F(x)F(h)|.$$

By Lemma 3.2,

$$\varphi_G(x)G(h) = \int_X (T^x\varphi_G) h dm = \int_X \varphi_G(T^{\overline{x}}h) dm = G(T^{\overline{x}}h).$$

This leads to the estimation

$$|\varphi_G(y) - \varphi_F(x)| \le \epsilon \cdot \|\varphi_G\|_\infty + |G(T^{\overline{y}}h - T^{\overline{x}}h)| + |G(T^{\overline{x}}h) - F(T^{\overline{x}}h)| \le$$
$$\le C\epsilon + \|(T^{\overline{y}}h - T^{\overline{x}}h)m\|' + \epsilon < \epsilon(C+2),$$

which shows the continuity of $(x, F) \mapsto \varphi_F(x)$.

We have now done all the preliminary work for the following main theorem; its proof is essentially the same as in the group case (see e.g. Hewitt, Ross [11], Th. 23.15) and will therefore be omitted here.

4.5. Theorem. Δ^0 *and* Δ_S^0 *are homeomorphic to* $\mathcal{X}_b(X)$ *and* $\widehat{X}$ *respectively, via the bijection* $\Phi : F \mapsto \varphi_F$. *In particular,* $\mathcal{X}_b(X)$ *and* $\widehat{X}$ *are locally compact Hausdorff spaces.*

4.6. Proposition.
(i) *If X is discrete, then $\widehat{X}$ is compact.*
(ii) *If X is compact, then $\widehat{X}$ is discrete, and all multiplicative functions on X are symmetric and orthogonal in $L^2(X,m)$.*

Proof. (i) is clear by the above theorem; (ii) is shown exactly in the same way as for hypergroups; see Dunkl [6], 3.5.

Next we introduce Fourier transforms on X and $\widehat{X}$ and collect some of their elementary properties. After this, we shall define positive definite functions and prove an analogue to Bochner's theorem for commutative signed hypergroups.

4.7. Definition.
(1) *Given $\mu \in M_b(X)$, the function*

$$\widehat{\mu} : \widehat{X} \to \mathbb{C}, \ \widehat{\mu}(\varphi) := \int_X \overline{\varphi} d\mu$$

is called the Fourier-Stieltjes transform of μ. In case $\mu = fm$, $f \in L^1(X,m)$, $\widehat{\mu}$ is also called the Fourier transform of f.
(2) *For $\alpha \in M_b(\widehat{X})$, the inverse Fourier-Stieltjes transform $\check{\alpha} : X \to \mathbb{C}$ is defined by*

$$\check{\alpha}(x) := \int_{\widehat{X}} \varphi(x) d\alpha(\varphi).$$

4.8. Lemma.
(1) *The Fourier-Stieltjes transform $\widehat{\ }$ is an algebra homomorphism from $(M_b(X), *)$ to $(C_b(\widehat{X}), \cdot)$, satisfying $\widehat{\mu^*} = \overline{\widehat{\mu}}$.*
(2) *$\widehat{f} \in C_0(\widehat{X})$ for all $f \in L^1(X,m)$ (Riemann-Lebesgue-Lemma).*
(3) *The set $A := \{\widehat{f} : f \in C_c(X)\}$ is a dense, self-adjoint subalgebra of $(C_0(\widehat{X}), \|.\|_\infty)$.*
(4) *The inverse transform $\check{\ }$ is an injective mapping from $M_b(\widehat{X})$ to $C_b(X)$.*

Proof. By identification of $\widehat{X}$ with Δ_S^0 according to Theorem 4.5, for each $\mu \in M_b(X)$ the transform $\widehat{\mu}$ is just the restriction to Δ_S^0 of the Gelfand transform of μ^- as an element of $(M_b(X), *, \|.\|')$. This yields (1). In the same way, (2) follows from the fact that for $f \in L^1(X,m)$, $\widehat{f}$ is the Gelfand transform of f^- as an element of $(L^1(X,m), *, \|.\|')$, restricted to $\Delta_S(L^1(X,m))$.

(3) For $f, g \in C_c(X)$ the functions $f * g$ and f^* belong to $C_c(X)$ again, satisfying $(f*g)m = fm*gm$ and $f^*m = (fm)^*$. By use of (1) and (2) it is now seen that A is a self-adjoint subalgebra of $(C_0(\widehat{X}), \|.\|_\infty)$. Finally, A separates points on $\widehat{X}$, and for each $\varphi \in \widehat{X}$ there exists some $f \in C_c(X)$ with $\widehat{f}(\varphi) \neq 0$;

these properties result from the separation properties of the Gelfand transform. Now the Stone-Weierstraß theorem yields the assertion.

(4) The continuity of $\breve{\alpha}$ for $\alpha \in M_b(X)$ follows, just as for hypergroups (12.1.D in [12]), from the continuity of the mapping $X \times \widehat{X} \to \mathbb{C}$, $(x, \varphi) \mapsto \varphi(x)$. Now suppose $\breve{\alpha} = 0$. If $\alpha \neq 0$, then by (3) there would exist some $f \in C_c(X)$ such that

$$0 \neq \int_{\widehat{X}} \widehat{f} d\alpha = \int_X \breve{\alpha} f^- dm,$$

a contradiction.

4.9. Definition. *A function* $f \in C_b(X)$ *is called positive-definite on* (X, m, ω), *if*

$$\int_X f d(\mu * \mu^*) \geq 0 \qquad \textit{for all } \mu \in M_b(X).$$

4.10. Lemma. *If* $f \in C_b(X)$ *is positive-definite on* (X, m, ω), *then*

$$\|f\|_\infty \leq C \cdot f(e) \quad \textit{and} \quad \mu^*(f) = \overline{\mu(f)} \quad \textit{for all } \mu \in M_b(X).$$

Proof. With $\nu := \delta_e + z\delta_x, z \in \mathbb{C}$, the condition $\nu * \nu^*(f) \geq 0$ yields the relations $f(x) = f^*(x)$ and $\|f(x)\|^2 \leq f(e) \cdot \delta_x * \delta_x^*(f) \leq C \cdot f(e)\|f\|_\infty$, from which the assertions follow.

4.11. Theorem. *For* $f \in C_b(X)$ *the following are equivalent:*
(i) f *is positive-definite on* (X, m, ω).
(ii) $f = \breve{\alpha}$ *with a unique measure* $\alpha \in M_b^+(\widehat{X})$.

Proof. (ii) $\Rightarrow$ (i): This is straightforward by the above definition, Remark 4.2.(2) and Lemma 3.2.

(i) $\Rightarrow$ (ii): By $F_f(\mu) := \mu(f)$ there is defined a positive functional on the commutative Banach-*-algebra $(M_b(X), *, \|.\|')$. By a well-known representation theorem (Th.3 in Naimark [19], §20 Chap. IV), F_f is of the form

$$F_f(\mu) = \int_{\Delta_S} \widetilde{\mu}(F) d\beta(F), \quad \mu \in M_b(X),$$

for a measure $\beta \in M_b^+(\Delta_S)$. Here $\widetilde{\mu}$ denotes the Gelfand transform of μ. For $h \in L^1(X, m)$, $\widetilde{hm}$ vanishes outside of Δ_S^0; hence

$$F_f(h) = \int_{\Delta_S^0} \widetilde{hm}(F) d\beta(F) \qquad \text{for all } h \in L^1(X, m).$$

If we define $\alpha \in M_b^+(\widehat{X})$ as the image measure of $\beta|_{\Delta_S^0}$ under the bijection Φ

from Theorem 4.5, this becomes

$$F_f(h) = \int_{\widehat{X}} \widehat{h^-}(\varphi) d\alpha(\varphi) \qquad \text{for all } h \in L^1(X,m).$$

It follows that

$$\int_X fhdm = \int_{\widehat{X}} \int_X \varphi(x)h(x)dm(x)d\alpha(\varphi) = \int_X \alpha^\vee hdm.$$

As f and $\alpha^\vee$ are continuous, the identity $f = \alpha^\vee$ results. The uniqueness of α is clear from the injectivity of $^\vee$.

5. Related structures and further examples

5.1. Hypercomplex systems with a compact basis. In the compact case, the hypercomplex systems introduced by Vainerman ([24], [25]) provide an essentially more general convolution concept than ours; they do not involve a Banach algebra structure on $M_b(X)$, but they allow duality theory in close analogy to the Pontryagin duality principle for compact and discrete groups.

Let X be a second countable, compact Hausdorff space. Then according to [25], $M_b(X)$ is called a real hypercomplex system with a compact basis, if it has the structure of a $*$-algebra with unit I, satisfying the following axioms:

(1) $\overline{\mu * \nu} = \overline{\mu} * \overline{\nu}$ for all $\mu, \nu \in M_b(X)$.

(2) There exists an involutive homeomorphism $x \mapsto \overline{x}$ on X, such that $\delta_x^* = \delta_{\overline{x}}$ for all $x \in X$.

(3) Multiplication in $M_b(X)$ is separately continuous with respect to the weak topology $\sigma(M_b(X), T)$, where T is a $\|.\|_\infty$-dense subspace of $C(X)$ which is invariant with respect to the mappings $f \mapsto \overline{f}$ and $f(x) \mapsto f(\overline{x})$.

For $\mu \in M_b(X)$ and $f \in T$ a function $\mu * f \in T$ is then defined by $\nu(\mu * f) = \overline{\mu}^* * \nu(f),\ \nu \in M_b(X)$.

(4) The mapping $(x, y) \mapsto \delta_x * \delta_y$, is continuous with respect to $\sigma(M_b(X), T)$.

(5) There exists a unique (up to a positive factor) measure $m \in M^+(X)$ with $supp\, m = X$, which satisfies the identities
$\overline{m}^* = m$ and $\mu * fm = (\mu * f)m$ for all $\mu \in M_b(X),\ f \in T$.
(m is called a Plancherel measure on X.)

(6) $\|\mu * f\|_{2,m} \leq C \cdot \|\mu\| \cdot \|f\|_{2,m}$ for all $\mu \in M_b(X)$ and $f \in T$, with some constant $C > 0$.

Every compact, second countable signed hypergroup (X, m, ω) satisfies these axioms with $T = C(X)$ in a canonical way — provided that m is unique up to a multiplicative constant. This follows immediately from Definition 3.1, together with Lemmata 3.9, 3.5 and 3.8.

5.2. Wildberger's finite signed hypergroups. These are introduced in Wildberger [27], [29] as classes of finite objects which satisfy all the requirements of a finite hypergroup except non-negativity of the convolution; see also [28]. He originally called these objects "ensembles". Abelian signed hypergroups occur in a canonical way as the duals of finite abelian hypergroups; see also Example 9.1.C in Jewett [12].

According to [29], a finite signed hypergroup is a pair $(\mathcal{C}, A)$, where $\mathcal{C} = \{c_0, \cdots, c_n\}$ and A is a $*$-algebra over $\mathbb{C}$ with unit c_0, having the following properties:

(E1) $\mathcal{C}$ is a linear basis of A.
(E2) $\mathcal{C}^* \subset \mathcal{C}$, that is $c_i^* = c_{\sigma(i)}$ with a certain permutation σ of $\{0, 1, \cdots, n\}$.
(E3) The structure constants $n_{i,j}^k \in \mathbb{R}$ defined by $c_i c_j = \sum_k n_{i,j}^k c_k$ satisfy the conditions

$$n_{i,\sigma(i)}^0 > 0, \quad n_{i,j}^0 = 0 \text{ for } j \neq \sigma(i).$$

(E4) $\sum_k n_{i,j}^k = 1$ for all i, j.

A mass functional m on $\mathcal{C}$ is defined by

$$m(c_i) := \frac{1}{n_{\sigma(i),i}^0} > 0.$$

Note that as a vector space, A may be canonically identified with $M_b(\mathcal{C})$.

5.2.1. Proposition.

(1) *Let $(\mathcal{C}, A)$ be a finite signed hypergroup in the sense of Wildberger, and let $\mathcal{C}$ be endowed with the discrete topology. Then with $\omega(c_i, c_j) := c_i c_j$, $(\mathcal{C}, m, \omega)$ is a signed hypergroup in the sense of Definition 3.1.*
(2) *Let (X, m, ω) be a finite signed hypergroup in the sense of Definition 3.1, with the additional property that $\delta_x * \delta_y(X) = 1$ for all $x, y \in X$. Then, with the same multiplication and $*$-operation, $(X, M_b(X))$ is a signed hypergroup in the sense of Wildberger. Up to a multiplicative constant, its mass functional coincides with m.*

Proof. (1) Besides Axiom (A6), all properties of a signed hypergroup are clearly satisfied, with $c_i \to c_i^*$ as the involution on $\mathcal{C}$ and c_0 as the unit element. Property (A6) is equivalent to the condition

$$n_{i,j}^k m(c_j) = n_{\sigma(i),k}^j m(c_k) \quad \text{for all } i, j, k. \tag{5.1}$$

To prove this equality, we calculate the coefficient h_0 of the product $c_{\sigma(k)} c_i c_j$ in two ways, according to the associativity of multiplication in A. This leads to

$$h_0 = \sum_l n_{i,j}^l \; n_{\sigma(k),l}^0 = \sum_l n_{\sigma(k),i}^l \; n_{l,j}^0 .$$

Hence, in view of (E3),

$$n_{i,j}^{k}\ m(c_j) = n_{\sigma(k),i}^{\sigma(j)}\ m(c_k).$$

Now as * is an involution on A and the structure constants are real, the identity $n_{\sigma(i),\sigma(j)}^{\sigma(k)} = n_{j,i}^{k}$ holds for all i,j,k. Thus (5.1) is satisfied.

(2) When setting $k=0$ in identity (5.1) (which is equivalent to Axiom (A6) for signed hypergroups), we obtain

$$n_{i,j}^{0} m(c_j) = \delta_{\sigma(i),j} m(c_0).$$

(Here δ denotes the Kronecker symbol.) This proves (E3) and the final statement concerning the mass functional. The rest is clear.

Finally, we note that Ross [21] gives an account of some of Wildberger's results using the notation of harmonic analysis which is similar to our presentation.

5.3. Dual structures of polynomial hypergroups. Let $(P_n)_{n\in\mathbb{N}_0}$ be a real orthogonal polynomial system with respect to $\pi \in M^+(\mathbb{R})$, satisfying the following properties:

(1) There exists some $x_0 \in supp\,\pi$ with $P_n(x_0) = 1$ for all $n \in \mathbb{N}_0$.
(2) The linearization coefficients in

$$P_n P_m = \sum_{k=|n-m|}^{n+m} g(n,m,k) P_k\,,\ n,m \in \mathbb{N}_0$$

are nonnegative.

Then $(P_n)_{n\in\mathbb{N}_0}$ induces the structure of a polynomial hypergroup on $\mathbb{N}_0$, with the convolution derived from the above linearization in the obvious way; for details see Lasser [14]. In particular, $supp\,\pi$ is contained in the compact set

$$D_s := \{x \in \mathbb{R} : (P_n(x))_{n\in\mathbb{N}_0} \text{ is a bounded sequence}\},$$

which is homeomorphic to the dual of $\mathbb{N}_0$.

Suppose further that

(3) For $x,y \in X := supp\pi$, there exists a measure $\mu_{x,y} \in M_b^{\mathbb{R}}(X)$ such that

(i) $$P_n(x)P_n(y) = \int_X P_n(z)\ d\mu_{x,y}(z) \quad \text{for all } x,y \in X$$

(ii) $$\|\mu_{x,y}\| \le C \quad \text{for all } x,y \in X, \text{ with some constant } C>0.$$

Now define $\omega : X \times X \to M_b(X)$ by $\omega(x,y)(f) := \int_X f(z) d\mu_{x,y}(z)$.

5.3.1. Proposition. (X,π,ω) is a commutative signed hypergroup with unit x_0 and the identity mapping as involution; the measure m on X is the orthogonalization measure π.

Proof. τ_*-continuity of ω is proved exactly in the same way as the corresponding statement for weak duals of commutative hypegroups; see Lasser [14], Prop.1. Furthermore, as Axioms (A1) and (A2) are clearly satisfied, ω has a canonical continuation $*$ on $M_b(X)$, which is, by a standard argument, commutative and associative. $x_0 \in X$ serves as a unit, and by the orthogonality of the P_n,

$$\int_X (T^x P_k)(y) P_l(y) d\pi(y) = \int_X P_k(y)(T^x P_l)(y) d\pi(y)$$

holds for all $x \in X$ and $k, l \in \mathbb{N}_0$. As finite linear combinations of the P_n's are uniformly dense in $C(X)$, Axioms (A5) and (A6) are satisfied with the identity mapping as the involution on X.

We add two classes of examples, provided by the Jacobi- and generalized Chebyshev polynomials:

(i) Denote by $(R_n^{\alpha,\beta})_{n\in\mathbb{N}_0}$ the Jacobi polynomials of order (α,β), $\alpha,\beta > -1$, normalized such that $R_n^{(\alpha,\beta)}(1) = 1$ for all $n \in \mathbb{N}_0$. They are orthogonal on $[-1,1]$ with respect to $d\pi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta dx$ (Chihara [4]). As shown by Gasper in [7] and [8], conditions (1) to (3) are satisfied for $(R_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ with $x_0 = 1$, if and only if $\alpha \geq \beta > -1$ and $\alpha + \beta \geq -1$. As $D_s \doteq supp\pi^{(\alpha,\beta)} = [-1,1]$, the dual of the polynomial hypergroup induced by the $R_n^{(\alpha,\beta)}$ is then a commutative signed hypergroup on $[-1,1]$. If the measures $\mu_{x,y}^{(\alpha,\beta)}$ are positive — which is the case exactly if $\beta \geq -1/2$ or $\alpha + \beta \geq 0$ — then the underlying polynomial hypergroup is known to be a Pontryagin hypergroup; see Lasser [14].

(ii) The generalized Chebyshev polynomials $(T_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ are the polynomials orthogonal on $[-1,1]$ with respect to $d\tilde{\pi}^{(\alpha,\beta)}(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}, \alpha,\beta > -1$ (Chihara [4], Chap.V). We assume them to be normalized such that $T_n^{(\alpha,\beta)}(1) = 1$ for all $n \in \mathbb{N}_0$. If $\alpha \geq \beta + 1$, they give rise to a polynomial hypergroup on $\mathbb{N}_0$ (Lasser [14]). The corresponding dual structure has been studied by Laine in [13]; see also [14]. In paticular, condition (3) is satisfied for the above range of parameters. Again $x_0 = 1$ and $D_s = supp\tilde{\pi}^{(\alpha,\beta)} = [-1,1]$. Thus the dual of the polynomial hypergroup induced by the $T_n^{(\alpha,\beta)}$ is a signed hypergroup on $[-1,1]$. However, it is not a hypergroup for any choice of parameters, because hypergroup-specific conditions on the support of the convolution product of point measures are not satisfied.

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