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A multivariate version of the disk convolution

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ABSTRACT

We present an explicit product formula for the spherical functions of the compact Gelfand pairs $(G, K_1) = (SU(p+q), SU(p) \times SU(q))$ with $p \geq 2q$, which can be considered as the elementary spherical functions of one-dimensional K-type for the Hermitian symmetric spaces G/K with $K = S(U(p) \times U(q))$. Due to results of Heckman, they can be expressed in terms of Heckman–Opdam Jacobi polynomials of type BC_q with specific half-integer multiplicities. By analytic continuation with respect to the multiplicity parameters we obtain positive product formulas for the extensions of these spherical functions as well as associated compact and commutative hypergroup structures parametrized by real $p \in\,]2q-1, \infty[$. We also obtain explicit product formulas for the involved continuous two-parameter family of Heckman–Opdam Jacobi polynomials with regular, but not necessarily positive multiplicities. The results of this paper extend well known results for the disk convolutions for $q = 1$ to higher rank.

1. Introduction

It is well-known that the spherical functions of Riemannian symmetric spaces of compact type can be considered as Heckman–Opdam polynomials, which are the polynomial variants of Heckman–Opdam hypergeometric functions. In particular, the spherical functions of Grassmann manifolds $SU(p+q, \mathbb{F})/S(U(p, \mathbb{F}) \times U(q, \mathbb{F}))$ with $p \geq q$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ can be realized as Heckman–Opdam polynomials of type BC_q with certain multiplicities given by the root data, namely

$$k = (d(p-q)/2, (d-1)/2, d/2) \quad \text{with } d = \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\},$$

corresponding to the short, long and middle roots, respectively. We refer to [5,4,16] for the foundations of Heckman–Opdam theory, and to [17] for the compact Grassmann case. Recall that the spherical functions

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of a Gelfand pair (G, K) can be characterized as the continuous, K-biinvariant functions on G satisfying the product formula

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)dk \quad (g, h \in G) \tag{1.1}$$

where dk is the normalized Haar measure on K. In [17], the product formula for the spherical functions of $SU(p+q, \mathbb{F})/S(U(p, \mathbb{F}) \times U(q, \mathbb{F}))$, considered as functions on the fundamental alcove

$$A_q = \{t \in \mathbb{R}^q : \ \pi/2 \geq t_1 \geq t_2 \geq \ldots \geq t_q \geq 0\}, \tag{1.2}$$

was extended by analytic interpolation to all real parameters $p \in]2q-1, \infty[$. This led to positive product formulas for the associated BC-Heckman–Opdam polynomials as well as commutative hypergroup structures on A_q with the Heckman–Opdam polynomials as characters. For a background on hypergroups see [9], where hypergroups are called convos. For the non-compact Grassmannians over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, similar constructions had previously been carried out in [19]. In the case $\mathbb{F} = \mathbb{C}$, where the non-compact Grassmannian $SU(p, q)/S(U(p) \times U(q))$ is a Hermitian symmetric space, the approach of [19] was extended to the spherical functions of $U(p, q)/U(p) \times SU(q)$ in [25]. Here the analysis was based on the decisive fact that the spherical functions of this (non-symmetric) space are intimately related to the generalized spherical functions of the symmetric space $SU(p, q)/S(U(p) \times U(q))$ of one-dimensional K-type. These in turn can be expressed in terms of Heckman–Opdam hypergeometric functions according to [5, Section 5].

In the present paper, we consider the pair (G, K_1) with $G = SU(p+q)$ and $K_1 = SU(p) \times SU(q)$ (over $\mathbb{F} = \mathbb{C}$), where we assume $p > q$. In this case, (G, K_1) is a Gelfand pair. By a decomposition of Cartan type, we identify the spherical functions of G/K_1 as functions on the compact cone

$$X_q := \{(zr_1, r_2, \ldots, r_q) : \quad 0 \leq r_1 \leq \ldots r_q \leq 1, \ z \in \mathbb{T}\} \subset \mathbb{C} \times \mathbb{R}^{q-1}$$

with the torus $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$. Similarly as in the non-compact case [25], the spherical functions of G/K_1 can be considered as elementary spherical functions of K-type χ_l $(l \in \mathbb{Z})$ for the Hermitian symmetric space G/K with $K = S(U(p) \times U(q))$. According to the results of Heckman [5], they can be expressed in terms of Heckman–Opdam polynomials of type BC_q, depending on the integral parameter $p > q$ and the spectral parameter $l \in \mathbb{Z}$. Under the stronger requirement $p \geq 2q$, we proceed similarly as in [17] and write down the product formula for the spherical functions of G/K_1 as product formulas on X_q.

In Section 4 we then extend this formula to a product formula on X_q for a continuous range of parameters $p \in]2q-1, \infty[$ by means of Carlson's theorem, a principle of analytic continuation. In particular, we obtain a continuous family of associated commutative hypergroup structures on X_q. We determine the dual spaces and Haar measures of these hypergroups. For $q = 1$, the space X_q is the complex unit disk, and the associated hypergroups are the well-known disk hypergroups studied in [1,10,2], where the associated product formulas are based on the work of Koornwinder [12].

For each real $p \in]2q-1, \infty[$, the associated hypergroup structure on X_q contains a compact subgroup isomorphic to the one-dimensional torus $(\mathbb{T}, \cdot)$. Moreover, the quotient $X_q/\mathbb{T}$ can be identified with the alcove A_q and carries associated quotient convolution structures; see [9] and [23] for the general background. In this way we in particular recover the above-mentioned hypergroup structures of [17] on A_q for $\mathbb{F} = \mathbb{C}$. More generally, we obtain in Section 5 explicit continuous product formulas and convolution structures on A_q for all Jacobi polynomials of type BC_q with multiplicities

$$k = (k_1, k_2, k_3) = (p - q - l, 1/2 + l, 1)$$

with $p \in]2q-1, \infty]$ and $l \in \mathbb{R}$. Unfortunately, for general $l \neq 0$, the positivity of these product formulas remains open.

2. Preliminaries

We start our considerations with the compact Grassmann manifolds G/K over $\mathbb{C}$, where $G = SU(p+q)$ and $K = S(U(p) \times U(q))$ with $p \geq q \geq 1$. From the Cartan decomposition of G (see [17] or Theorem VII.8.6 of [6]) it follows that a system of representatives of the double coset space $G//K$ is given by the matrices

$$a_t = \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & \cos \underline{t} & -\sin \underline{t} \\ 0 & \sin \underline{t} & \cos \underline{t} \end{pmatrix}, \quad t \in A_q$$

with A_q as in (1.2); here $\underline{t}$ is the $q \times q$-diagonal matrix with the components of t as entries, and $\cos \underline{t}$, $\sin \underline{t}$ are understood componentwise. We recall from [17] how the double coset representatives can be determined explicitly:

For $X \in M_q(\mathbb{C})$ denote by $\sigma_{sing}(X) = \sqrt{\text{spec}(X^*X)} = (\sigma_1 \ldots, \sigma_q) \in \mathbb{R}^q$ the vector of singular values of X, decreasingly ordered by size. Write $g \in G$ in $(p \times q)$-block notation as

$$g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}, \tag{2.1}$$

and suppose that $g \in K b_t K$. Then

$$t = \arccos(\sigma_{sing}(D(g))). \tag{2.2}$$

By [17], the spherical functions of G/K are given by Heckman–Opdam polynomials, as follows: Consider $\mathbb{R}^q$ with the standard inner product $\langle .\,, . \rangle$ and denote by $F_{BC_q}(\lambda, k; t)$ the Heckman–Opdam hypergeometric function associated with the root system

$$R = 2BC_q = \{\pm 2e_i, \pm 4e_i, 1 \leq i \leq q\} \cup \{\pm 2e_i \pm 2e_j : 1 \leq i < j \leq q\} \subset \mathbb{R}^q$$

and with multiplicity parameter $k = (k_\alpha)_{\alpha \in R}$ in the notion of [19]. We also write $k = (k_1, k_2, k_3) \in \mathbb{C}^3$ where k_1, k_2, k_3 belong to the short, long and middle roots, respectively. Recall that there exists an open regular parameter set $K^{reg} \subset \mathbb{C}^3$ with $[0, \infty[^3 \subset K^{reg}$ such that for all $k \in K^{reg}$ and $\lambda \in \mathbb{C}^q$ the function $F_{BC_q}(\lambda, k; \,.\,)$ exists in a suitable tubular neighbourhood of $\mathbb{R}^q \subset \mathbb{C}^q$. Fix the positive subsystem $R_+ = \{2e_i, 4e_i, 2e_i \pm 2e_j : 1 \leq i < j \leq q\}$. Writing $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$, the associated set of dominant weights is

$$P_+ := \{\lambda \in \mathbb{R}^q : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_+ \ \forall \, \alpha \in R_+\} = \{\lambda \in 2\mathbb{Z}_+^q : \ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q\}. \tag{2.3}$$

Notice that our normalization of root systems and multiplicities is in accordance with [5,16] but differs from the "geometric" notion of [17], where both are rescaled by a factor 2.

We consider the renormalized Heckman–Opdam polynomials associated with R and k in trigonometric notion as in [17], which are defined by

$$R_\lambda(k; t) = F_{BC_q}(\lambda + \rho(k), k; it), \quad \lambda \in P_+\,. \tag{2.4}$$

Here

$$\rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha.$$

They are related to the Heckman–Opdam polynomials $P(\lambda, k; .)$ in the normalization of [5] by

$$R_\lambda(k; t) = c(\lambda + \rho(k), k) \cdot P(\lambda, k; \exp(it))$$

where

$$c(\lambda,k)=\prod_{\alpha\in R_+}\frac{\Gamma\big(\langle\lambda,\alpha^\vee\rangle+\frac{1}{2}k_{\alpha/2}\big)}{\Gamma\big(\langle\lambda,\alpha^\vee\rangle+\frac{1}{2}k_{\alpha/2}+k_\alpha\big)}\cdot\prod_{\alpha\in R_+}\frac{\Gamma\big(\langle\rho(k),\alpha^\vee\rangle+\frac{1}{2}k_{\alpha/2}+k_\alpha\big)}{\Gamma\big(\langle\rho(k),\alpha^\vee\rangle+\frac{1}{2}k_{\alpha/2}\big)}$$

is the generalized c-function, with the convention $k_{\alpha/2}:=0$ if $\alpha/2\notin R$; see formula (4.4.10) of [5]. The polynomials R_λ are holomorphic on $\mathbb{C}^q$. Indeed, $F_{BC_q}(\lambda+\rho(k),k;\,.\,)$ is holomorphic on all of $\mathbb{C}^q$ if and only if $\lambda\in P_+$, see [5].

According to Theorem 4.3 of [17], the spherical functions of $G/K=SU(p+q)/S(U(p)\times U(q))$ can be considered as trigonometric polynomials on the alcove $A_q\subset\mathbb{R}^q$ and are given by

$$\varphi_\lambda(t)=R_\lambda(k;t),\quad \lambda\in P_+$$

with the multiplicity

$$k=(k_1,k_2,k_3)=(p-q,1/2,1)\in K^{reg}.$$

For fixed $q\geq 1$ and $p\geq 2q$, the product formula (1.1) for the φ_λ was written in [17] as a product formula on A_q depending on p in a way which allowed extension to all real parameters $p>2q-1$ by analytic continuation; see Theorem 4.4 of [17].

3. Spherical functions of $(SU(p+q),SU(p)\times SU(q))$ and their product formula

Let us turn to the pair $(G,K_1):=(SU(p+q),SU(p)\times SU(q))$, where we assume $p>q$. In this case, (G,K_1) is a Gelfand pair according to the classification of [14]. In this section, we derive an explicit product formula for the spherical functions of (G,K_1). First, we determine a system of representatives for the double coset space $G//K_1$. For this, consider the compact set

$$X_q:=\{x=(zr_1,r_2,\ldots,r_q): z\in\mathbb{T},\ r_i\in\mathbb{R}\text{ with }0\leq r_1\leq\ldots\leq r_q\leq 1\}\\ \subseteq\mathbb{C}\times\mathbb{R}^{q-1}.$$

If $q=1$, this is just the closed unit disc in $\mathbb{C}$. For $q\geq 2$, the set X_q can be interpreted as a cone of real dimension $q+1$ with apex 0 and basis set $X_{q-1}\times\{1\}$.

For $x=(zr_1,r_2,\ldots,r_q)\in X_q$ we write

$$x=[r,z]\ \text{ with }\ r=(r_1,\ldots,r_q).$$

Note that the phase factor $z\in\mathbb{T}$ is arbitrary if $r_1=0$. We now relate X_q with the alcove A_q. We define an equivalence relation on $A_q\times\mathbb{T}$ via

$$(t,z)\sim(t',z'):\Longleftrightarrow\quad \Big((t,z)=(t',z')\text{ and }t_1=t_1'<\frac{\pi}{2}\Big)\quad\text{or}\\ \Big(t=t'\ \text{ and }\ t_1=t_1'=\frac{\pi}{2}\Big)$$

Then the mapping $A_q\times\mathbb{T}\to X_q$, $(t,z)\mapsto[\cos t,z]$ induces a homeomorphism between the quotient space $(A_q\times\mathbb{T})/\sim$ and the cone X_q.

Lemma 3.1. *For $z \in \mathbb{T}$, let $h(z) = diag(z, 1 \ldots, 1) \in M_q(\mathbb{C})$. Then a set of representatives of $G//K_1$ is given by the matrices $b_x, x \in X_q$ with*

$$b_x = \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & h(z^{-1})\cos \underline{t} & -\sin \underline{t} \\ 0 & \sin \underline{t} & h(z)\cos \underline{t} \end{pmatrix} \text{ for } x = [\cos t, z] \text{ with } t \in A_q. \tag{3.1}$$

In particular, the double coset space $G//K_1$ is naturally homeomorphic with the cone X_q.

Proof. We first check that each double coset has a representative of the stated form. In fact, by the known Cartan decomposition of G with respect to K, each $g \in G$ can be written as

$$g = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix} a_t \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}$$

with $t \in A_q$ and $u_i \in U(p)$, $v_i \in U(q)$ satisfying $\det(u_i) \cdot \det(v_i) = 1$. As $U(q) = SU(q) \rtimes H_q$ with $H_q = \{h(z) : z \in \mathbb{T}\}$, there are $z_1, z_2 \in \mathbb{T}$ such that $\widetilde{v}_1 := v_1 h(z_1)^{-1} \in SU(q)$ and $\widetilde{v}_2 := h(z_2)^{-1} v_2 \in SU(q)$. For $z, w \in \mathbb{T}$, define $H(w, z) = diag(w, 1, \ldots, 1, z, 1, \ldots, 1) \in M_p(\mathbb{C})$ with the entry z in position $p + 1$. Put $\widetilde{u}_1 := u_1 \cdot H(z_1/z_2, z_2)$ and $\widetilde{u}_2 := H(z_2/z_1, z_1) \cdot u_2 \in U(p)$. Then $\widetilde{u}_1, \widetilde{u}_2 \in SU(p)$, and a short calculation in $p \times q$-blocks gives

$$\begin{aligned} g &= \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix} \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & \cos \underline{t} & -\sin \underline{t} \\ 0 & \sin \underline{t} & \cos \underline{t} \end{pmatrix} \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{u}_1 & 0 \\ 0 & \widetilde{v}_1 \end{pmatrix} \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & h(z_1)^{-1}h(z_2)^{-1}\cos \underline{t} & -\sin \underline{t} \\ 0 & \sin \underline{t} & h(z_1)h(z_2)\cos \underline{t} \end{pmatrix} \begin{pmatrix} \widetilde{u}_2 & 0 \\ 0 & \widetilde{v}_2 \end{pmatrix}. \end{aligned}$$

Thus $g \in K_1 b_x K_1$ with $x = [\cos t, z_1 z_2]$.

In order to show that the b_x are contained in different double cosets for different x, we analyze how the parameter x depends on $g \in G$. Let

$$g = \begin{pmatrix} \widetilde{u}_1 & 0 \\ 0 & \widetilde{v}_1 \end{pmatrix} b_x \begin{pmatrix} \widetilde{u}_2 & 0 \\ 0 & \widetilde{v}_2 \end{pmatrix}$$

with $\widetilde{u}_i \in SU(p)$, $\widetilde{v}_i \in SU(q)$. Suppose that $x = [r, z]$ with $r = \cos t$, $t \in A_q$. Using the $(p \times q)$-block notation (2.1), we obtain

$$D(g) = \widetilde{v}_1 h(z) \underline{r}\, \widetilde{v}_2, \quad \Delta(D(g)) = z \cdot \prod_{i=1}^{q} r_i = z \cdot |\Delta(D(g))|.$$

Here and throughout the paper, Δ denotes the usual determinant of a complex square matrix. Thus $r = \cos t = \sigma_{sing}(D(g))$. Further, z is uniquely determined by g exactly if $r_1 \neq 0$, which is equivalent to $\prod_{i=1}^{q} r_i \neq 0$. In this case, $z = \frac{\Delta D(g)}{|\Delta D(g)|}$. Thus $(t, z) \in A_q \times \mathbb{T}$ is uniquely determined by g up to the equivalence $\sim$, which proves our statement. □

The proof of the above lemma reveals the following equivalence for $g \in G$:

$$g \in K_1 b_x K_1 \iff x = [r, z] \text{ with } r = \sigma_{sing}(D(g)),\ z = \arg \Delta(D(g)),$$

with the argument $\arg:\mathbb{C}\to\mathbb{T}$ defined by

$$\arg z := \frac{z}{|z|} \text{ if } z\neq 0, \ \arg 0 := 1.$$

We are now going to write the general product formula (1.1) for spherical functions as a product formula on the cone $X_q \cong (A_q\times\mathbb{T})/\sim$. For $x=[\cos t, z_1]$, $x'=[\cos t', z']\in X_q$ and K_1-biinvariant $f\in C(G)$ we have to evaluate the integral

$$\int_{K_1} f(b_x k b_{x'})\,dk.$$

Write $k=\begin{pmatrix} u & 0\\ 0 & v\end{pmatrix}$ with $u\in SU(p)$, $v\in SU(q)$. Then $(p\times q)$-block calculation gives

$$b_x k b_{x'} = \begin{pmatrix} * & *\\ * & D(b_x k b_{x'})\end{pmatrix}$$

where

$$D(b_x k b_{x'}) = (0,\sin \underline{t}\,)\,u\begin{pmatrix} 0\\ -\sin\underline{t}'\end{pmatrix} + h(z)\cos\underline{t}\,v\,h(z')\cos\underline{t}'\,.$$

With the block matrix

$$\sigma_0 := \begin{pmatrix} 0_{(p-q)\times q}\\ I_q\end{pmatrix}\in M_{p,q}(\mathbb{C})$$

this can be written as

$$D(b_x k b_{x'}) = \,-\sin\underline{t}\,\sigma_0^* u\sigma_0\sin\underline{t}' \,+\, h(zz')\,\cos\underline{t}\,v\,\cos\underline{t}'. \tag{3.2}$$

Regarding K_1-biinvariant functions $f\in C(G)$ as continuous functions on X_q, we have

$$\int_{K_1} f(b_x k b_{x'})dk \,= \int_{SU(p)\times SU(q)} f\left([\sigma_{sing}(D(b_x k b_{x'})), \arg\Delta(D(b_x k b_{x'})\,)]\right)dk$$

with $D(b_x k b_{x'})$ from (3.2). Notice that $\sigma_0^* u\sigma_0\in M_q(\mathbb{C})$ is the lower right $q\times q$-block of u and is contained in the closure of the ball

$$B_q := \{w\in M_q(\mathbb{C}):\ w^*w\le I_q\},$$

where $w^*w\le I_q$ means that $I_q - w^*w$ is positive semidefinite. We now assume that $p\ge 2q$ and reduce the $SU(p)$-integration by means of Lemma 2.1 of [19]. Notice first that for continuous g on $\overline{B}_q$,

$$\int_{SU(p)} g(\sigma_0^* u\sigma_0)du = \int_{U(p)} g(\sigma_0^* u\sigma_0)du,$$

where du denotes the normalized Haar measure in each case. Thus Lemma 2.1 of [19] gives

$$\int_{K_1} f(b_x k b_{x'})dk = \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} f\big([\sigma_{sing}(-\sin \underline{t}\, w \sin \underline{t}' + h(zz')\cos \underline{t}\, v \cos \underline{t}'),$$
$$\arg \Delta(-\sin \underline{t}\, w \sin \underline{t}' + h(zz')\cos \underline{t}\, v \cos \underline{t}')]\big)$$
$$\cdot \Delta(I_q - w^* w)^{p-2q} dv dw, \tag{3.3}$$

where

$$\kappa_p = \int_{B_q} \Delta(I_q - w^* w)^{p-2q}\ dw$$

and dw means integration with respect to Lebesgue measure on B_q. After the substitution $w \mapsto h(zz')w$, we finally arrive at the following

Theorem 3.2. *Suppose that $p \geq 2q$. Then the product formula for the spherical functions of the Gelfand pair (G, K_1), considered as functions on the cone X_q, can be written as*

$$\varphi\ ([\cos t, z])\varphi([\cos t', z'])$$
$$= \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} \varphi\big([\sigma_{sing}(-\sin \underline{t}\, w \sin \underline{t}' +\ \cos \underline{t}\, v \cos \underline{t}'),$$
$$zz' \cdot \arg \Delta(-\sin \underline{t}\, w \sin \underline{t}' + \cos \underline{t}\, v \cos \underline{t}')]\big) \cdot \Delta(I_q - w^* w)^{p-2q}\, dv dw.$$

We remark that for $p = 2q - 1$, a degenerate version of this integral formula may be written down by using the coordinates introduced in Section 3 of [18].

We next turn to the classification of the spherical functions of (G, K_1). Note first that $K = K_1 \rtimes H$ with $H = \{H_z, z \in \mathbb{T}\}$ where H_z is the diagonal matrix with entries z in position $p - q + 1$, $1/z$ in position $p + 1$ and 1 else. Let $\chi : K \to \mathbb{T}$ be the homomorphism with kernel K_1 and $\chi(H_z) := z$. Then the characters of K are given by the functions $\chi_l(k) = \chi(k)^l$, $l \in \mathbb{Z}$, and we have the following characterization:

Lemma 3.3. *For $\varphi \in C(G)$ the following properties are equivalent:*

(1) *φ is K_1-spherical, i.e., K_1-biinvariant with $\varphi(g)\varphi(h) = \int_{K_1} \varphi(gkh)dk$ for all $g, h \in G$.*
(2) *φ is an elementary spherical function for (G, K) of type χ_l for some $l \in \mathbb{Z}$, i.e. φ is not identical zero and satisfies the twisted product formula*

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)\chi_l(k)dk \quad \text{for all } g, h \in G. \tag{3.4}$$

Remark 3.4. We here adopt the notion of elementary spherical functions of type χ_l according to [8]. Each such function automatically satisfies $\varphi(e) = 1$ as well as the χ_l-bi-coinvariance condition

$$\varphi(k_1 g k_2) = \chi_l(k_1 k_2)^{-1} \cdot \varphi(g) \quad \text{for all } g \in G, k_1, k_2 \in K, \tag{3.5}$$

see Lemma 3.2 of [8]. The discussion of [5] is based on a different, but equivalent definition of elementary spherical functions of K-type (for the non-compact dual), which requires (3.5) together with a system of invariant differential operators on sections in an associated homogeneous line bundle, see Definition 5.2.1 of [5]. For the equivalence of definitions we refer to Theorem 3.2 of [21].

Proof of Lemma 3.3. The proof of this result, which we expect to be well-known, can be carried out for instance precisely as in Lemma 2.3 of [25]. □

For an arbitrary irreducible Hermitian symmetric space and its compact dual, the elementary spherical functions of type χ_l can be written as modifications of Heckman–Opdam hypergeometric functions, see Section 5 of [5], in particular Theorem 5.2.2 and Corollary 5.2.3, as well as [8]. In the compact case, they correspond to the χ_l-spherical representations of G which were classified in [20]. To become explicit in the particular case of our compact symmetric spaces $G/K = SU(p+q)/S(U(p)\times U(q))$, recall the set P_+ of dominant weights from (2.3) as well as the renormalized Heckman–Opdam polynomials R_λ of type BC_q from (2.4). According to [5], the elementary spherical functions of (G,K) of type χ_l, considered as functions on A_q, are given by

$$t \mapsto \prod_{j=1}^{q} \cos^{|l|} t_j \cdot R_\lambda(k(p,q,l);t), \quad \lambda \in P_+\,,\ l \in \mathbb{Z} \tag{3.6}$$

with multiplicity parameters

$$k(p,q,l) := (p-q-|l|, 1/2+|l|, 1), \quad l \in \mathbb{Z}.$$

Notice at this point that for F_{BC_q}, the set

$$\{k=(k_1,k_2,k_3) : \operatorname{Re} k_3 \geq 0,\ \operatorname{Re}(k_1+k_2)\geq 0\}$$

is contained in K^{reg} (see Remark 4.4.3 of [5]), and so in particular the multiplicities $k(p,q,l)$ are regular. The associated ρ-function is

$$\rho(k(p,q,l)) = (p-q+|l|+1)\sum_{j=1}^{q} e_j \ + 2\sum_{j=1}^{q}(q-j)e_j.$$

We now turn to the spherical functions of (G,K_1), which we again consider as functions on the cone X_q.

Theorem 3.5. *Let $(G,K) = (SU(p+q), S(U(p)\times U(q)))$ and $K_1 = SU(p)\times SU(q)$. Then the spherical functions of the Gelfand pair (G,K_1), considered as functions on X_q, are precisely given by*

$$\varphi^p_{\lambda,l}([\cos t, z]) = z^l \cdot \prod_{j=1}^{q} \cos^{|l|} t_j \cdot R_\lambda(k(p,q,l);t), \quad \lambda \in P_+\,,\ l \in \mathbb{Z} \tag{3.7}$$

with the multiplicity $k(p,q,l) = (p-q-|l|,\ \frac{1}{2}+|l|,\ 1) \in K^{reg}$.

Proof. First observe that $\varphi^p_{\lambda,l}$ is indeed well-defined as a function on X_q, because the right side is zero if $t_1 = \pi/2$, independently of $z \in \mathbb{T}$. Now suppose that $\varphi : G \to \mathbb{C}$ is spherical for (G,K_1). Then by Lemma 3.3 it is χ_l-spherical for some $l \in \mathbb{Z}$. Consider $x = [\cos t, z] \in X_q$ and write $b_x = H_{1/\sqrt{z}}\, a_t H_{1/\sqrt{z}}$ with an arbitrary square root of z. Then in view of (3.5),

$$\varphi(b_x) = z^l \cdot \varphi(a_t),$$

where $\varphi(a_t)$ is of the form (3.6). This proves the statement. □

Example 3.6 *(The rank-one case $q = 1$).* Here $G/K = SU(p+1)/S(U(p) \times U(1)) \cong U(p+1)/U(p)$ and $G//K$ is homeomorphic to the unit disk $X_1 = \{z \in \mathbb{C} : |z| \le 1\} = D$. We shall identify the spherical functions $\varphi^p_{\lambda,l}$ as the well-known disk polynomials on D introduced in [12], which are known to be the spherical functions of $(U(p+1), U(p))$.

We have $R_+ = \{2, 4\} \subset \mathbb{R}$ and $P_+ = 2\mathbb{Z}_+$. According to the example on p. 89 of [16], $F_{BC_1}(\lambda, k; t)$ may be expressed as a ${}_2F_1$- (Gaussian) hypergeometric function. Consider the renormalized one-dimensional Jacobi polynomials

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1(\alpha+\beta+n+1, -n, \alpha+1; (1-x)/2) \quad (x \in \mathbb{R},\ n \in \mathbb{Z}_+) \tag{3.8}$$

for $\alpha, \beta > -1$. Then the Heckman–Opdam polynomials associated with $R = 2BC_1$ and multiplicity $k = (k_1, k_2)$ can be written as

$$R_{2n}(k; t) = R_n^{(\alpha,\beta)}(\cos 2t) \qquad (n \in \mathbb{Z}_+,\ t \in [0, \pi/2]) \tag{3.9}$$

with

$$\alpha = k_1 + k_2 - 1/2, \quad \beta = k_2 - 1/2;$$

cf. equation (5.4) of [17], where a different scaling of roots and multiplicities is used. Writing $r = \cos t$, we have $\cos 2t = 2r^2 - 1$. We thus obtain from Theorem 3.5 the well-known fact that the spherical functions of (G, K) are given, as functions on D, by the so-called disk polynomials

$$\widetilde{\varphi}_{n,l}(zr) = \varphi^p_{2n,l}([r, z]) = z^l r^{|l|} \cdot R_n^{(p-1,|l|)}(2r^2 - 1) \qquad (z \in \mathbb{T},\ r \in [0, 1]) \tag{3.10}$$

with $l \in \mathbb{Z}$, $n \in \mathbb{Z}_+$. Moreover, the product formula of Theorem 3.2 becomes in this case

$$\widetilde{\varphi}_{n,l}(zr) \cdot \widetilde{\varphi}_{n,l}(z's) = \frac{1}{\kappa_p} \int_D \widetilde{\varphi}_{n,l}(zz'(rs - w\sqrt{1-r^2}\sqrt{1-s^2})) \cdot (1 - |w|^2)^{p-2}\, dw \tag{3.11}$$

with

$$\kappa_p = \int_D (1 - |w|^2)^{p-2}\, dw = \frac{\pi}{p-1}.$$

This formula is well known; see for instance [1,10].

4. Convolution algebras on the cone X_q for a continuous family of multiplicities

In this section, we extend the product formula for the spherical functions of (G, K_1) in Theorem 3.2 from integers $p \ge 2q$ to a continuous range of parameters $p \in]2q-1, \infty[$. We show that for each $p \in]2q-1, \infty[$, the corresponding product formula induces a commutative Banach algebra structure on the space of all bounded Borel measures on X_q and an associated commutative hypergroup structure. These hypergroups generalize the known disk hypergroups for $q = 1$ which were studied for instance in [1,10]; see also the monograph [2].

As usual, the basis for analytic continuation will be Carlson's theorem, which we recapitulate for the reader's convenience from [22]:

Theorem 4.1. *Let f be holomorphic in a neighbourhood of $\{z \in \mathbb{C} : \text{Re } z \ge 0\}$ satisfying $f(z) = O(e^{c|z|})$ for some $c < \pi$. If $f(z) = 0$ for all nonnegative integers z, then f is identically zero on $\{\text{Re } z > 0\}$.*

It is now straightforward, but a bit tedious in detail to prove that the product formula of Theorem 3.2 can be extended analytically with respect to the variable p. For the necessary exponential bounds, one has to use that the coefficients of the Heckman–Opdam polynomials $P(\lambda, k; \exp(it)) = c(\lambda + \rho(k), k)^{-1} R_\lambda(k; t)$ are rational in the multiplicity k, see Par. 11 of [15]. As the arguments are the same as those in the proof of Theorem 4.1 of [19] and Theorem 3.2 of [25], we skip the details. We obtain:

Theorem 4.2. *Let $p \in]2q-1, \infty[$, $\lambda \in P_+$, and $l \in \mathbb{Z}$. Then the functions*

$$\varphi^p_{\lambda,l}([\cos t, z]) = z^l \cdot \prod_{j=1}^{q} \cos^{|l|} t_j \cdot R_\lambda(k; t)$$

on X_q with multiplicity $k = (p - q - |l|,\ \frac{1}{2} + |l|,\ 1) \in K^{reg}$ satisfy the product formula

$$\begin{aligned} &\varphi^p_{\lambda,l}([\cos t, z]) \cdot \varphi^p_{\lambda,l}([\cos t', z']) \\ &\quad = \frac{1}{\kappa_p} \int_{B_q} \int_{SU(q)} \varphi^p_{\lambda,l}\big(\big[\sigma_{sing}(-\sin \underline{t}\, w \sin \underline{t}' + \cos \underline{t}\, v \cos \underline{t}'), \\ &\qquad\qquad zz' \cdot \arg \Delta(-\sin \underline{t}\, w \sin \underline{t}' + \cos \underline{t}\, v \cos \underline{t}')\big]\big) \cdot \Delta(I_q - w^* w)^{p-2q}\, dv dw \end{aligned}$$

for $(t, z), (t', z') \in A_q \times \mathbb{T}$.

The positive product formula in Theorem 4.2 for $p \in]2q-1, \infty[$ leads to a continuous series of probability-preserving commutative convolution algebras on the cone X_q. In fact, similarly to the noncompact case [25], we obtain commutative hypergroup structures on X_q with the $\varphi^p_{\lambda,l}$ ($\lambda \in P_+$, $l \in \mathbb{Z}$) as hypergroup characters. To start with, let us briefly recapitulate some notions from hypergroup theory from [9,2].

Definition 4.3. A hypergroup is a locally compact Hausdorff space X with a weakly continuous, associative and bilinear convolution $*$ on the Banach space $M_b(X)$ of all bounded regular Borel measures on X such that the following properties hold:

(1) For all $x, y \in X$, $\delta_x * \delta_y$ is a compactly supported probability measure on X such that the support $\mathrm{supp}(\delta_x * \delta_y)$ depends continuously on x, y with respect to the so-called Michael topology on the space of compact subsets of X (see [9] for details).
(2) There exists a neutral element $e \in X$ with $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in X$.
(3) There exists a continuous involution $x \mapsto \bar{x}$ on X such that $e \in \mathrm{supp}(\delta_x * \delta_y)$ holds if and only if $y = \bar{x}$, and such that $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$, where for $\mu \in M_b(X)$, the measure μ^- denotes the pushforward of μ under the involution.

Due to weak continuity and bilinearity, the convolution of bounded measures on a hypergroup is uniquely determined by the convolution of point measures. A hypergroup is called commutative if so is the convolution. We recall from [9] that for a Gelfand pair (G, K), the double coset space $G//K$ carries the structure of a commutative hypergroup in a natural way. For a commutative hypergroup X the dual space is defined by

$$\widehat{X} = \{\varphi \in C_b(X) : \varphi \not\equiv 0,\ \varphi(x * \bar{y}) := (\delta_x * \delta_{\bar{y}})(\varphi) = \varphi(x)\overline{\varphi(y)}\ \forall x, y \in X\}.$$

Each commutative hypergroup $(X, *)$ admits a (up to a multiplicative constant unique) Haar measure ω_X, which is characterized by the condition $\omega_X(f) = \omega_X(f_x)$ for all continuous, compactly supported $f \in C_c(X)$ and all $x \in X$, where f_x denotes the translate $f_x(y) = (\delta_y * \delta_x)(f)$.

Now let $p \in]2q-1, \infty[$. Using the positive product formula of Theorem 4.2, we introduce the convolution of point measures on X_q by

$$\begin{aligned}(\delta_{[\cos t,z]} *_p \delta_{[\cos t',z']})(f) \\ := \frac{1}{\kappa_p} \int_{B_q} \int_{SU(q)} f\big([\sigma_{sing}(-\sin \underline{t}\, w \sin \underline{t}' + \cos \underline{t}\, v \cos \underline{t}'), \\ zz' \cdot \arg \Delta(-\sin \underline{t}\, w \sin \underline{t}' + \cos \underline{t}\, v \cos \underline{t}')]\big) \Delta(I_q - w^* w)^{p-2q} dv dw \qquad (4.1)\end{aligned}$$

for $f \in C_b(X_q)$.

Theorem 4.4. *Let $q \geq 1$ be an integer and $p \in]2q-1, \infty[$. Then the convolution $*_p$ defined in (4.1) extends uniquely to a bilinear, weakly continuous, commutative convolution on the Banach space $M_b(X_q)$. This convolution is also associative, and $(X_q, *_p)$ is a commutative hypergroup with $[(1, \ldots, 1), 1]$ as identity and with the involution $\overline{[r,z]} := [r, \bar{z}]$. A Haar measure of the hypergroup $(X_q, *_p)$ is given by*

$$d\omega_p([r,z]) = \prod_{j=1}^{q} r_j (1 - r_j^2)^{p-q} \cdot \prod_{1 \leq i < j \leq q} (r_i^2 - r_j^2)^2 \, dr dz$$

with the Lebesgue measure dr on $\mathbb{R}^q$ and the normalized Haar measure dz on $\mathbb{T}$. Finally, the dual space is given by

$$(X_q, *_p)^\wedge = \{\varphi^p_{\lambda,l} : \lambda \in P_+, \, l \in \mathbb{Z}\}.$$

Note that ω_p is the pushforward measure under the mapping $(t,z) \mapsto [\cos t, z], \quad A_q \times \mathbb{T} \to X_q$ of the measure

$$d\widetilde{\omega}_p(t,z) = \prod_{j=1}^{q} \cos t_j \sin^{2p-2q+1} t_j \cdot \prod_{1 \leq i < j \leq q} (\cos(2t_i) - \cos(2t_j))^2 dt dz$$

on $A_q \times \mathbb{T}$.

For the proof of Theorem 4.4, consider the measure ω_p defined above. We start with the following observation:

Lemma 4.5. *The functions*

$$\varphi^p_{\lambda,l} \quad (\lambda \in P_+, \, l \in \mathbb{Z})$$

form an orthogonal basis of $L^2(X_q, \omega_p)$, and their $\mathbb{C}$-span is dense in the space $C(X_q)$ of continuous functions on X_q with respect to $\|.\|_\infty$.

Proof. For $k = k(p,q,l)$, the Heckman–Opdam polynomials $R_\lambda(k;t)$ are orthogonal on the alcove A_q with respect to the weight

$$\begin{aligned}\delta_k(t) &= \prod_{\alpha \in R_+} \left| e^{i\langle \alpha, t\rangle/2} - e^{-i\langle \alpha, t\rangle/2} \right|^{2k_\alpha} \\ &= const \cdot \prod_{j=1}^{q} \sin^{2p-2q+1} t_j \, \cos^{2|l|+1} t_j \cdot \prod_{1 \leq i < j \leq q} (\cos(2t_i) - \cos(2t_j))^2\end{aligned}$$

This immediately implies that the functions $(t,z) \mapsto \varphi^p_{\lambda,l}([\cos t, z])$ are orthogonal on $A_q \times \mathbb{T}$ with respect to $\widetilde{\omega}_p$. Let V denote the subspace of $C(X_q)$ spanned by $\{\varphi^p_{\lambda,l} : \lambda \in P_+,\, l \in \mathbb{Z}\}$. Clearly, V is an algebra which is stable under complex conjugation. Further, V separates points on X_q, because the $R_\lambda(k;\,.\,)$ span the space of Weyl group invariant trigonometric polynomials. Hence by the Stone–Weierstraß theorem, V is dense in $C(X_q)$ with respect to $\|.\|_\infty$. The rest is obvious. □

Proof of Theorem 4.4. The statements are clear for integer values of p, where $*_p$ is just the convolution of the double coset hypergroup $SU(p+q)//SU(p)\times SU(q)$. From the definition of the convolution for general p we see that $\delta_{[r,z]} *_p \delta_{[r',z']}$ is a probability measure and that the mapping $([r,z],[r',z']) \mapsto \delta_{[r,z]} *_p \delta_{[r',z']}$ is weakly continuous. It is now standard (see [9]) to extend the convolution of point measures uniquely in a bilinear, weakly continuous way to a probability preserving convolution on $M_b(X_q)$. For commutativity and associativity, it suffices to consider point measures. Let $[r_i, z_i] \in X_q$, $1 \le i \le 3$. Then for $f = \varphi^p_{\lambda,l}$ with $\lambda \in P_+$ and $l \in \mathbb{Z}$ we have

$$(\delta_{[r_1,z_1]} *_p \delta_{[r_2,z_2]})(f) = f([r_1,z_1])f([r_2,z_2]) = (\delta_{[r_2,z_2]} *_p \delta_{[r_1,z_1]})(f)$$

and in the same way,

$$((\delta_{[r_1,z_1]} *_p \delta_{[r_2,z_2]}) *_p \delta_{[r_3,z_3]})(f) = (\delta_{[r_1,z_1]} *_p (\delta_{[r_2,z_2]} *_p \delta_{[r_3,z_3]}))(f).$$

By Lemma 4.5, both identities remain valid for arbitrary $f \in C(X_q)$. The remaining hypergroup axioms are immediate from the fact that the supports of convolution products of point measures are independent of p. For the statement on the Haar measure, the argumentation is similar to [17]. Notice first that by definition of hypergroup translates, the identity

$$\int_{X_q} f_x \, d\omega_p = \int_{X_q} f d\omega_p \quad \text{for all } x \in X_q$$

holds for $f = \varphi^p_{\lambda,l}$ with arbitrary $\lambda \in P_+$, $l \in \mathbb{Z}$. In view of Lemma 4.5, it extends to arbitrary $f \in C(X_q)$, hence ω_p is a Haar measure. Finally, it is clear that the $\varphi^p_{\lambda,l}$ are hypergroup characters. There are no further ones, because the characters of a compact hypergroup are orthogonal with respect to its Haar measure. □

The hypergroups $(X_q, *_p)$ have a prominent subgroup. For this we recall that a closed, non-empty subset $H \subset X_q$ is a subgroup if for all $x, y \in H$, $\delta_x *_p \delta_{\bar{y}}$ is a point measure with support in H. It is clear from (4.1) that

$$H := \{[1,z] = (z,1,\ldots,1) : z \in \mathbb{T}\}$$

is a subgroup of $(X_q, *_p)$ which is isomorphic to the torus group $\mathbb{T}$.

The cosets

$$x *_p H := \bigcup_{y\in H} \text{supp}\,(\delta_x *_p \delta_y), \quad x \in X_q$$

form a disjoint decomposition of X_q, and the quotient

$$X_q/H := \{x *_p H : \ x \in X_q\}$$

is again a locally compact Hausdorff space with respect to the quotient topology, cf. Section 10.3 of [9]. Moreover,

$$(\delta_{x*_pH} *_p \delta_{y*_pH})(f) := \int_X f(z *_p H)\, d(\delta_x *_p \delta_y)(z), \quad x, y \in X_q\,,\ f \in C_b(X_q/H) \tag{4.2}$$

establishes a well-defined quotient convolution and an associated commutative quotient hypergroup; see [9,23], and the references given there. We may identify X_q/H topologically with the alcove A_q via $[\cos t, z] *_p H \mapsto t$. It is then immediate from (4.1) that the quotient convolution on A_q derived from $*_p$ is given by

$$\begin{aligned}(\delta_t *_p \delta_{t'})(f) &= \\ &= \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} f\big(\arccos(\sigma_{sing}(-\sin \underline{t}\, w \sin \underline{t}' + \cos \underline{t}\, v \cos \underline{t}'))\big) \\ &\qquad \cdot \Delta(I_q - w^*w)^{p-2q} dv dw \end{aligned} \tag{4.3}$$

for $t, t' \in A_q$ and $f \in C_b(A_q)$. These are precisely the hypergroup convolutions studied in Section 6 of [17]. For integers $p \geq 2q$, this connection just reflects the fact that for $G = SU(p+q), K = S(U(p) \times U(q))$ and $K_1 = SU(p) \times SU(q)$, we have

$$(G//K_1)/(K//K_1) \simeq G//K$$

as hypergroups. This is a fact which holds for general commutative double coset hypergroups, see Theorem 14.3 of [9].

Remarks 4.6. 1. We mention at this point that the Haar measure of the hypergroup $(X_q, *_p)$, which was obtained in Theorem 4.4 by an orthogonality argument, can alternatively be calculated by using Weil's integration formula for Haar measures on hypergroups and their quotients (see [7,23]). In the same way as in the non-compact case treated in [25], the Haar measure can thus be obtained from the known Haar measure of the quotient $(X_q, *_p)/H$. For integers $p \geq 2q$, the hypergroup $(X_q, *_p)$ can be identified with the double coset hypergroup $SU(p+q)//SU(p) \times SU(q)$, and its Haar measure therefore coincides by construction (see [9]) with the pushforward measure of the Haar measure on $SU(p+q)$ under the canonical projection

$$SU(p+q) \to SU(p+q)//SU(p) \times SU(q) \simeq X_q\,.$$

2. Concerning their structure, the hypergroups $(X_q, *_p)$ are also closely related to continuous family of commutative hypergroups $(C_q \times \mathbb{R}, *_p)$ with $p \geq 2q - 1$ and the BC_q-Weyl chamber $C_q = \{(x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 \geq \ldots \geq x_q \geq 0\}$ which were studied in [24]. For integral p, $(C_q \times \mathbb{R}, *_p)$ is an orbit hypergroup under the action of $U(p) \times U(q)$ on the Heisenberg group $M_{p,q}(\mathbb{C}) \times \mathbb{R}$. The characters are given in terms of multivariable Bessel and Laguerre functions.

5. Continuous product formulas for Heckman–Opdam Jacobi polynomials

Fix the rank $q \geq 1$ and a parameter $p \in]2q - 1, \infty[$. Recall that for $l \in \mathbb{Z}$ the functions

$$\varphi^p_{\lambda,l}([\cos t, z]) = z^l \cdot \prod_{j=1}^{q} \cos^{|l|} t_j \cdot R_\lambda(k(p, q, l); t)$$

satisfy the product formula of Theorem 4.2. We shall now extend this formula to exponents $l \in \mathbb{R}$ via Carlson's theorem, and write it down as a product formula for the Jacobi polynomials $R_\lambda(k;t)$ with $k = k(p,q,l)$. This will work out smoothly only for non-degenerate arguments $t, t' \in A_q$ with $t_1, t_1' \neq \pi/2$. Notice first that for a product formula for the Jacobi polynomials, we may restrict our attention to $l \in [0, \infty[$ as $k(p,q,l)$ depends on $|l|$ only. In the following, we shall use the abbreviation

$$d(t,t';v,w) := -\sin \underline{t}\, w \sin \underline{t}' \,+\, \cos \underline{t}\, v \cos \underline{t}' \,.$$

The main result of this section is

Theorem 5.1. *Let $q \geq 1$ be an integer, $p \in]2q-1, \infty[$, $l \in [0, \infty[$, and $k = k(p,q,l)$. Then for all $\lambda \in P_+$ and $t, t' \in A_q$ with $t_1, t_1' \neq \pi/2$,*

$$R_\lambda(k;t)R_\lambda(k;t') \,=\, \frac{1}{\kappa_p} \int_{B_q} \int_{SU(q)} R_\lambda\big(k; \arccos(\sigma_{sing}(d(t,t';v,w)))\big) \cdot$$
$$\cdot \operatorname{Re}\left[\left(\frac{\Delta(d(t,t';v,w))}{\Delta(\cos \underline{t}) \cdot \Delta(\cos \underline{t}')}\right)^l\right] \cdot \Delta(I_q - w^*w)^{p-2q} dv dw. \tag{5.1}$$

Proof. For $l \in \mathbb{Z}_+$, $z = z' = 1$ and $t, t' \in A_q$, the product formula in Theorem 4.2 implies

$$\Big(\prod_{j=1}^{q} \cos t_j \cos t_j'\Big)^l \cdot R_\lambda(k;t)R_\lambda(k;t') =$$
$$= \frac{1}{\kappa_p} \int_{B_q} \int_{SU(q)} \Delta(d(t,t';v,w))^l \cdot R_\lambda\big(k; \arccos(\sigma_{sing}(d(t,t';v,w)))\big)$$
$$\cdot \Delta(I_q - w^*w)^{p-2q} dv dw. \tag{5.2}$$

Our condition on t, t' assures that both sides of (5.2) are analytic in l for $\operatorname{Re}(l) > 0$. Moreover, by Section 11 of [15], the coefficients of the Jacobi polynomials $R_\lambda(k;t)$ with respect to the exponentials $e^{i\langle \mu, t\rangle}$ ($\mu \in P_+$) are rational in k. Therefore both sides of (5.2) satisfy the growth condition of Carlson's theorem. This implies that (5.2) remains correct for all $l \in [0, \infty[$. As $R_\lambda(k;t)$ is real for all $\lambda \in P_+$ and $t \in A_q$, the claimed product formula now follows from (5.2) by taking real parts. □

Contrary to the non-compact case in Section 5 of [25], it seems to be difficult in the present setting to derive positivity of the product formula (5.1) except for the known case $l = 0$. This problem already appears in rank one, i.e. for $q = 1$. We discuss this case for illustration.

Example 5.2. Let $q = 1$, $p \geq 2q - 1 = 1$, $l \in [0, \infty[$, and $k = k(p,q,l) = (p - 1 - l, \frac{1}{2} + l)$. Resuming the notions from Example 3.6, we have

$$\alpha = k_1 + k_2 - 1/2 = p - 1 > 0, \quad \beta = k_2 - 1/2 = l.$$

Consider the classical (normalized) one-dimensional Jacobi polynomials $R_n^{(\alpha,\beta)}$ with

$$R_n^{(\alpha,\beta)}(\cos 2\theta) = R_{2n}(k;\theta),$$

cf. (3.9). With the identity $\cos 2\theta = 2\cos^2\theta - 1$, product formula (5.1) becomes

$$R_n^{(\alpha,\beta)}(\cos 2\theta)R_n^{(\alpha,\beta)}(\cos 2\theta') = \\ = \frac{\alpha}{\pi}\int_0^1\int_{-\pi}^{\pi} R_n^{(\alpha,\beta)}(2|b+are^{i\varphi}|^2-1)\cdot\frac{(b+are^{i\varphi})^\beta}{b^\beta}\cdot r(1-r^2)^{\alpha-1}d\varphi\, dr \tag{5.3}$$

for $\theta, \theta' \in [0, \pi/2[$, with $a := \sin\theta\sin\theta'$ and $b := \cos\theta\cos\theta' > 0$. Following Section 5 of [13], we use the change of variables $(r, \varphi) \mapsto (t, \psi)$ with

$$b + are^{i\varphi} = te^{i\psi} \text{ and } t \geq 0.$$

Then $a^2 r\ dr\ d\varphi = t\ dt\ d\psi$ and identity (5.3) becomes, for $0 < \theta, \theta' < \pi/2$:

$$R_n^{(\alpha,\beta)}(\cos 2\theta)R_n^{(\alpha,\beta)}(\cos 2\theta') = \\ = \frac{\alpha}{\pi}\cdot\frac{1}{b^\beta a^{2\alpha}}\int_0^1\int_{-\pi}^{\pi} R_n^{(\alpha,\beta)}(2t^2-1)(te^{i\psi})^\beta(a^2-b^2-t^2+2bt\cos\psi)_+^{\alpha-1}t\, d\psi\, dt \\ = \frac{\alpha}{\pi}\cdot\frac{2}{b^\beta a^{2\alpha}}\int_0^1 R_n^{(\alpha,\beta)}(2t^2-1)\Big(\int_0^\pi e^{i\beta\psi}(a^2-b^2-t^2+2bt\cos\psi)_+^{\alpha-1}d\psi\Big)t^{\beta+1}dt. \tag{5.4}$$

Here the notation

$$(x)_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

is used. Notice for the last equality that $t = |b + are^{i\varphi}| \leq a + b \leq 1$. Notice also that for $\theta = 0$ or $\theta' = 0$, the product formula degenerates in a trivial way due to $R_n^{(\alpha,\beta)}(1) = 1$.

On the other hand, for $\alpha > \beta$, the Jacobi polynomials satisfy the well-known positive product formula [11]

$$R_n^{(\alpha,\beta)}(\cos 2\theta)R_n^{(\alpha,\beta)}(\cos 2\theta') = \\ = c_{\alpha,\beta}\int_0^1\int_0^\pi R_n^{(\alpha,\beta)}(2|b+are^{i\varphi}|^2-1)\cdot(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}\sin^{2\beta}\varphi\, d\varphi dr \tag{5.5}$$

with some constant $c_{\alpha,\beta} > 0$. By the same substitution as above this can be brought into kernel form,

$$R_n^{(\alpha,\beta)}(\cos 2\theta)R_n^{(\alpha,\beta)}(\cos 2\theta') = \\ = \frac{c_{\alpha,\beta}}{a^{2\alpha}}\cdot\int_0^1 R_n^{(\alpha,\beta)}(2t^2-1)\Big(\int_0^\pi(a^2-b^2-t^2+2bt\cos\psi)_+^{\alpha-\beta-1}\sin^{2\beta}\psi d\psi\Big)t^{2\beta+1}dt. \tag{5.6}$$

As the integrals in (5.4) and (5.6) are identical for all n, we conclude that, for all $t \in [0, 1]$,

$$c_{\alpha,\beta}(tb)^\beta\int_0^\pi(a^2-b^2-t^2+2bt\cos\psi)_+^{\alpha-\beta-1}\sin^{2\beta}\psi d\psi = \\ = \frac{2\alpha}{\pi}\int_0^\pi e^{i\beta\psi}(a^2-b^2-t^2+2bt\cos\psi)_+^{\alpha-1}d\psi. \tag{5.7}$$

This identity seems not obvious, and it would be desirable to have an elementary proof for it which possibly might be extended to the higher rank case.

So far, a positive product formula such as formula (5.5) of Koornwinder seems to be a difficult task in rank $q \geq 2$. However, at least the first step above in the case $q = 1$, that is from (5.3) to (5.4), can be partially extended to $q \geq 2$. Indeed, consider the product formula (5.1) for $t, t' \in A_q$. We define the matrices

$$a_1 := \sin \underline{t}\,,\ a_2 := \sin \underline{t}'\,,\ b = b(v) := \cos \underline{t}\, v \cos \underline{t}' \ \in M_q(\mathbb{C})$$

and consider the polar decomposition $b - a_1 w a_2 =: ru$ with positive semidefinite $r \in M_q(\mathbb{C})$ and $u \in U(q)$. We now carry out the change of variables in two steps. First, for $\widetilde{w} := a_1 w a_2 \in M_q(\mathbb{C})$, we have $d\widetilde{w} = const \cdot \Delta(a_1 a_2)^{2q}\, dw$. Moreover, the polar decomposition $b - \widetilde{w} = \sqrt{r}u$ (for non-singular r) leads to $d\widetilde{w} = const \cdot drdu$, where dr means integration with respect to the Lebesgue measure on the cone Ω_q of positive definite matrices as an open subset of the vector space of all Hermitian matrices, and du is the normalized Haar measure of $U(q)$; see Proposition XVI.2.1 of [3]. Formula (5.1) now reads

$$\begin{aligned} &R_\lambda(k;t)R_\lambda(k;t') = \\ &= const \cdot \int_{\Omega_q} R_\lambda\big(k; \arccos(\sigma_{sing}(\sqrt{r}))\big) \cdot \frac{\Delta(r)^{l/2}}{\Delta(\cos \underline{t})^l \Delta(\cos \underline{t}')^l \Delta(\sin \underline{t})^{2q} \Delta(\sin \underline{t}')^{2q}} \cdot \\ &\quad \cdot \Big(\int_{SU(q)} \int_{U(q)} \Delta\big(H(t,t',r,u,v)_+\big) \Delta(u)^l \, dvdu \Big) dr \end{aligned} \tag{5.8}$$

where

$$H(t,t',r,u,v) = I_q \ - \ a_2^{-1}\big(b(v)^* - u^*\sqrt{r}\big)a_1^{-2}\big(b(v) - \sqrt{r}u\big)a_2^{-1}$$

and the subscript $+$ means that this term is put zero for matrices which are not positive definite. The analysis of the origin of these formulas shows that in the outer integral, r in fact runs through the set $\{r \in \Omega_q :\ I_q - r > 0\}$. One may speculate that (5.8) can be brought into a kernel form by replacing the integration over $r \in \Omega_q$ by an integration over the spectrum of r as a subset of $\{\rho \in \mathbb{R}^q :\ 1 \geq \rho_1 \geq \cdots \geq \rho_q \geq 0\}$.

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