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On the Dual of a Commutative Signed Hypergroup

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Signed hypergroups are convolution structures similar to hypergroups, though being not necessarily positivity-preserving. We prove a generalized Plancherel theorem for positive definite measures on a commutative signed hypergroup, with an analogue of the classical Plancherel theorem as a special case. Moreover, signed hypergroups with subexponential growth are studied. As an application, the dual of the Laguerre convolution structure on $\mathbb{R}_+$ is determined.

1. Introduction

In [16] we introduced an axiomatic frame for convolution algebras on locally compact Hausdorff spaces which generalizes the hypergroup axiomatics in several points, mainly in abandoning positivity of the convolution. We called these structures "signed hypergroups", thus emphasizing their structurally close connection to hypergroups in the sense of Jewett [12], Spector [18] and Dunkl [4]. (When saying "hypergroup" in this article, we shall always refer to the axiomatics of Jewett [12], which has meanwhile been widely adopted.) An important example of a signed hypergroup which is indeed not positivity-preserving is provided by the Laguerre convolution on $\mathbb{R}_+$. It has been studied by several authors including McCully [14], Görlich and Markett [9], [10], Stempak [19] and Thangavelu [21]; see also the discussion of this example in Rösler [16]. For the compact case, our signed hypergroups are closeley related to the compact hypercomplex systems of Vainerman [23]; in the finite case, they essentially coincide with Wildberger's finite signed hypergroups, which were formerly called "ensembles", see [27], [28]. Somewhat stronger axioms than ours for compact and discrete signed hypergroups have recently been proposed by Ross [17]. As to further concepts of not necessarily positive convolution systems, in particular the hypercomplex systems of Berezansky and Krein, we refer to Berezansky, Kaluzhny [1] and the surveys of Litvinov [13] and Vainerman [22].

While in [16] some basic harmonic analysis for signed hypergroups was developed, the present paper shall continue the discussion of the dual of a commutative signed hypergroup. In particular, we prove a generalized Plancherel theorem for positive definite measures on a commutative signed hypergroup. It includes a characterization of the support of the Fourier transform of a positive definite measure which seems not yet to be written down even for hypergroups. As a special case, a straight analogue of the classical Plancherel theorem for commutative hypergroups (see Jewett [12]) is obtained. As far as groups and hypergroups are concerned, Plancherel theorems for positive definite measures can be found in Berg, Forst [2] and in Bloom, Heyer [3] respectively; see also Voit [26]. Our proof for signed hypergroups, however, is derived from the abstract formulation of Plancherel's theorem for positive functionals on a commutative $*$-algebra in Fell, Doran [6]. This general version is due to Godement [7]; see also Godement [8] for the special case of Gelfand pairs.

Just as for hypergroups (cf. Voit [25]), the dual of a commutative signed hypergroup coincides with the support of its Plancherel measure in case of so-called subexponential growth. To give an example of application, we use this criterion in order to determine the dual of the signed Laguerre hypergroup explicitely.

2. Preliminaries

2.1. Notation. For a locally compact Hausdorff space X let $C_b(X), C_0(X)$ and $C_c(X)$ denote the spaces of continuous functions on X which are bounded, those which vanish at infinity and those having compact support respectively. By $M(X)$ and $M^+(X)$ we abbreviate the spaces of Radon measures and positive Radon measures on X. The spaces $M_b(X)$, $M_b^+(X)$ and $M_b^{\mathbf{R}}(X)$ consist of all bounded regular Borel measures on X, those which are positive, and the real ones respectively. For $\mu \in M_b(X)$, $|\mu| \in M_b^+(X)$ is the total variation of μ and $\delta_x \in M_b^+(X)$ denotes the point measure at $x \in X$. The weak-$*$-topology $\sigma(M_b(X), C_0(X))$ shall be referred to as the τ_*-topology on $M_b(X)$. Finally, for a normed space N, we abbreviate by $B(N)$ the algebra of bounded linear operators on N.

2.2. Signed hypergroups. Let X be a locally compact Hausdorff space and $\omega : X \times X \to M_b(X)$, $(x, y) \mapsto \delta_x * \delta_y$, be a τ_*-continuous convolution of point measures on X such that

(i) $\|\delta_x * \delta_y\| \le C$ for all $x, y \in X$ with some constant $C > 0$,

(ii) for $f \in C_c(X)$ and $x \in X$, the mappings $T^x f : y \mapsto \delta_x * \delta_y(f)$ and $T_x f : y \mapsto \delta_y * \delta_x(f)$ again belong to $C_c(X)$.

Then in view of Theorem 1 in Pym [15],

$$\mu * \nu(f) := \int_{X \times X} \delta_x * \delta_y(f) \, d(\mu \otimes \nu)(x, y), \quad f \in C_0(X), \ \mu, \nu \in M_b(X),$$

establishes a bilinear and separately τ_*-continuous multiplication on $M_b(X)$ which we call the canonical continuation of ω. As usual, we shall write $f(x * y)$ for $\delta_x * \delta_y(f)$, $f \in C_b(X)$.

A *signed hypergroup* is a triple (X, m, ω) consisting of a locally compact, σ-compact Hausdorff space X, a distinguished positive Radon measure $m \in M^+(X)$ with $supp\, m = X$ and a τ_*-continuous mapping $\omega : X \times X \to M_b^{\mathbf{R}}(X)$, $(x, y) \mapsto \delta_x * \delta_y$, satisfying the following axioms:

(A1) For each $x \in X$ and $f \in C_b(X)$, the translates $T^x f : y \mapsto \delta_x * \delta_y(f)$ and $T_x f : y \mapsto \delta_y * \delta_x(f)$ again belong to $C_b(X)$. Furthermore, for $f \in C_c(X)$ and any compact subset $K \subset X$, the set $\bigcup_{x \in K} (supp(T^x f) \cup supp(T_x f))$ is relatively compact in X.

(A2) $\|\delta_x * \delta_y\| \le C$ for all $x, y \in X$ with some constant $C > 0$.

(A3) The canonical continuation of ω is associative.

(A4) There exists a neutral element $e \in X$, which means that

$$\delta_e * \mu = \mu * \delta_e = \mu \quad \text{for all } \mu \in M_b(X).$$

(A5) There exists an involutive homeomorphism $^-$ on X such that for all $f, g \in C_c(X)$ and $x \in X$ the following adjoint relation holds:

$$\int_X (T^x f) g dm = \int_X f(T^{\overline{x}} g) dm.$$

We point out that axiom (A5) implies the following important property of $^-$:

$$(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}} \quad \text{for all } x, y \in X,$$

where $\mu^-(A) = \mu(A^-)$ for Borel measures μ on X and Borel sets $A \subseteq X$. In fact, by the adjoint relation we obtain for $x, y \in X$ and any $f, g \in C_c(X)$:

$$\int_X f(z) \left((\delta_x * \delta_y)^- * \delta_z \right)(g) \, dm(z) = \int_X (\delta_x * \delta_y * \delta_z)(f) \, g(z) \, dm(z) =$$

$$\int_X T^y (T^x f) \, g \, dm = \int_X T^x f \, T^{\overline{y}} g \, dm = \int_X f(z) \, (\delta_{\overline{y}} * \delta_{\overline{x}} * \delta_z)(g) \, dm(z).$$

As $supp\, m = X$, this proves that $(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}}$.

We shall from now on always assume that (X, m, ω) is commutative, that is, $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in X$. The algebra $(M_b(X), *)$ then becomes a commutative Banach-*-algebra with unit δ_e, the involution $\mu \mapsto \mu^*$, $\mu^*(A) := \overline{\mu(A^-)}$, and the norm $\|\mu\|' := \|L_\mu\|$, where $L_\mu \in B(M_b(X))$ is given by $L_\mu(\nu) := \mu * \nu$. $L^1(X, m)$ with the same multipication and norm is a closed $*$-ideal in $(M_b(X), *, \|.\|')$, see Theorem 3.7 of Rösler [16]; for abbreviation, we shall refer to it as $L^1(X)$. In particular, with $f^*(x) := \overline{f(\overline{x})}$, the identity $f^*m = (fm)^*$ holds for all $f \in L^1(X)$.

2.3. The left-regular representation. According to Lemma 3.8 in [16], the mapping $\lambda : L^1(X) \to B(L^2(X, m))$, $\lambda(f)g := f * g$ is well-defined and norm-continuous. We call it the left-regular representation of $L^1(X)$. λ is a faithful $*$-representation of $L^1(X)$: The second property follows from the fact that for all $f, g, h \in C_c(X)$,

$$\int_X (f^* * g)\,\overline{h}\,dm = (f^* * g) * h^*(e) = g * (f * h)^*(e) = \int_X g\,(\overline{f * h})\,dm.$$

In order to check that λ is faithful, suppose that $f \in L^1(X), f \not\equiv 0$. It follows that there exists some $g \in C_c(X)$ with $\lambda(f)g\,(e) = \int_X f^- g\,dm \neq 0$. But as $f * g$ belongs to $C_0(X)$, $\lambda(f)g$ must be different from 0 in $L^2(X, m)$.

2.4. Multiplicative functions. As for commutative hypergroups, the spectrum $\Delta(L^1(X))$ and its symmetric part $\Delta_S(L^1(X))$ can be identified with spaces of multiplicative functions on X in a canonical way. More explicitely, let $\mathcal{X}_b(X)$ and $\widehat{X}$ denote the character space and the symmetric character space of X respectively, that is

$$\mathcal{X}_b(X) := \{\varphi \in C_b(X) : \varphi \not\equiv 0,\ \varphi(x * y) = \varphi(x)\,\varphi(y) \text{ for all } x, y \in X\},$$
$$\widehat{X} := \{\varphi \in \mathcal{X}_b(X) : \varphi(\overline{x}) = \overline{\varphi(x)} \text{ for all } x \in X\}.$$

Then equipped with the topology of uniform convergence on compact subsets of X, $\mathcal{X}_b(X)$ and $\widehat{X}$ are homeomorphic to $\Delta(L^1(X))$ and $\Delta_S(L^1(X))$ respectively via the mapping $\varphi \mapsto F_\varphi$, $F_\varphi(\mu) = \int_X \varphi\,d\mu$ for $\mu \in M_b(X)$. For details, we refer to Rösler [16].

In contrast to the hypergroup situation, the constant function 1 on X need not belong to $\widehat{X}$; so we have to make sure that $\widehat{X}$ is not empty. This can be seen as follows: Let A denote the subalgebra $A = \{\lambda(f) : f \in L^1(X)\}$ of $B(L^2(X, m))$. The left-regular representation λ being faithful, we have $A \neq \{0\}$. The norm-closure $\overline{A}$ of A is a commutative C^*-algebra, hence its spectrum $\Delta(\overline{A})$ is not empty and symmetric. Now if $\varphi \in \Delta(\overline{A})$, then $\varphi\,|_A \not\equiv 0$ by continuity of φ, and the mapping $\widetilde{\varphi} : f \mapsto \varphi(\lambda(f))$ obviously defines an element of $\Delta_S(L^1(X))$.

Finally, Fourier transforms of measures and functions as well as inverse Fourier transforms are introduced in the usual way, see Rösler [16].

2.5. Approximate identities. A sequence $(g_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ is called an *approximate identity*, if $g_n \geq 0$, $\int_X g_n\, dm = 1$ for all n and $supp\, g_n \to \{e\}$ in the sense that for any open neighbourhood U of e, there exists an index $n(U) \in \mathbb{N}$ such that $supp\ g_n \subseteq U$ for all $n \geq n(U)$.

The following result will be of use later on:

2.6. Lemma. *Let $(g_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ be an approximate identity. Then for any $f \in C_c(X)$, the set $\bigcup_{n\in\mathbb{N}} supp\,(f * g_n)$ is relatively compact, and $lim_{n\to\infty}\|f * g_n - f\|_\infty = 0$.*

Proof. Take $f \in C_c(X)$. Then

$$f * g_n(x) = \int_{supp\, g_n} f(\overline{y} * x) g_n(y)\, dm(y),$$

and as $\bigcup_{n\in\mathbb{N}} supp\, g_n$ is relatively compact in X, axiom (A1) for signed hypergroups yields that $\bigcup_{n\in\mathbb{N}} supp\,(f * g_n)$ is relatively compact as well. Now let $K \subseteq X$ be compact with $supp\, f \subseteq K$ and $\bigcup_{n\in\mathbb{N}} supp(f * g_n) \subseteq K$, and choose any $\epsilon > 0$. The mapping $(x, y) \mapsto f(\overline{y} * x)$ being continuous on $X \times X$, there exists a neighbourhood U of e such that $|f(\overline{y}*x) - f(x)| < \epsilon$ for all $y \in U$ and $x \in K$. As $supp\, g_n$ is contained in U for $n \geq n(U)$, it follows that $\|f * g_n - f\|_\infty = \|f * g_n - f\|_K < \epsilon$ for all $n \geq n(U)$.

For hypergroups, the property $e \in supp\,(\delta_x * \delta_y) \iff y = \overline{x}$ is axiomatically required. In our more general setting, we can at least deduce the following:

2.7. Corollary. *If $x \in X$ satisfies $e \notin supp\,(\delta_x * \delta_y)$ for all $y \neq \overline{x}$, then e is contained in $supp\,(\delta_x * \delta_{\overline{x}})$.*

Proof. Suppose $e \notin supp\,(\delta_x * \delta_{\overline{x}})$. Then our assumption on x entails that $e \notin supp\,(\delta_x * \mu)$ for all $\mu \in M_b(X)$. Now let $(g_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ be an approximate identity. Then for $h \in C_c(X)$ with $h(x) \neq 0$ there exists an index $N \in \mathbb{N}$ such that $supp\, g_n \cap supp\,(\delta_x * h^- m) = \emptyset$ for $n \geq N$. Thus

$$h * g_n(x) = \int_X h(\overline{y} * x) g_n(y)\, dm(y) = \int_X g_n(x * y) h^-(y)\, dm(y) = 0.$$

On the other hand, the above lemma says that $\lim_{n\to\infty} h*g_n(x) = h(x) \neq 0$; a contradiction.

3. A generalized Plancherel theorem for positive definite measures

Let (X, m, ω) be a commutative signed hypergroup. The following definition takes over the notion of positive definiteness in the case of groups (Berg, Forst [2]) and hypergroups (Bloom, Heyer [3]):

3.1. Definition. *A Radon measure $\mu \in M(X)$ is called positive definite on (X, m, ω), if*

$$\int_X (f * f^*)\, d\mu \geq 0 \text{ for all } f \in C_c(X).$$

3.2. Examples. 1. For $\mu \in M_b(X)$ the measure $\mu * \mu^*$ is positive definite; indeed, for $f \in C_c(X)$ a short calculation leads to

$$\int_X (f * f^*)\, d(\mu * \mu^*) = \int_X |\mu^- * f|^2\, dm \geq 0.$$

2. Let P be the set of all (not necessarily bounded) positive definite functions, that is

$$P = \{f \in C(X) : \int_X |f|\, d|\delta_x * \delta_y| < \infty \text{ and } \int_X f\, d(\mu * \mu^*) \geq 0$$
$$\text{for all } \mu \in M_b(X) \text{ with compact support}\}.$$

(The first condition meets the fact that $supp\,(\delta_x * \delta_y)$ need not be compact.) Then for $f \in C(X)$, fm is positive definite if and only if $f \in P$.

For the proof, we first note that for any $g \in C_c(X)$ and $f \in C(X)$ satisfying $\int_X |f|\, d|\delta_x * \delta_y| < \infty$ for all $x, y \in X$, the adjoint relation

$$\int_X (T^x f)\, g\, dm = \int_X f(T^{\overline{x}} g) dm$$

holds for all $x \in X$. This is shown exactly in the same way as in Lemma 3.2 of Rösler [16] for $f \in C_b(X)$ and $g \in C_c(X)$.

Now take $f \in P$. Then for any $g \in C_c(X)$, the above adjoint relation yields

$$\int_X (g * g^*)(x) f(x)\, dm(x) = \int_X g^*(y) \Big(\int_X g(\overline{y} * x) f(x) dm(x) \Big) dm(y) =$$
$$\int_X g^*(y) \Big(\int_X g(x) f(y * x) dm(x) \Big) dm(y) = \int_X f\, d(g^* m * gm) \geq 0.$$

So fm is positive definite. The reverse direction is clear.

3. Let X^* be the set of semicharacters on X, that is

$$X^* = \{\varphi \in C(X) : \int_X |\varphi|\, d|\delta_x * \delta_y| < \infty,\ \varphi(\overline{x}) = \overline{\varphi(x)} \text{ and}$$
$$\varphi(x * y) = \varphi(x)\varphi(y) \text{ for all } x, y \in X\}.$$

As $X^* \subseteq P$, φm is positive definite for every $\varphi \in X^*$.

4. Let X be a commutative hypergroup and H a compact subhypergroup of X with normalized Haar measure m_H. For $\rho \in \widehat{H}$, the identity

$$\Big(\int_X |\rho|^2 dm_H\Big) \cdot \rho m_H = \rho m_H * \rho m_H$$

holds, where $*$ denotes the convolution in X. Thus ρm_H, as an element of $M_b(X)$, is positive definite.

Besides positive definiteness, the decisive property of a measure $\mu \in M(X)$ in order to allow a generalized Plancherel theorem will be the condition

$$\text{(B)} \qquad \int_X f * g * g^* * f^* \, d\mu \leq M \cdot \|f\|_{1,m}^2 \cdot \int_X g * g^* \, d\mu$$

for all $f, g \in C_c(X)$, with some constant $M > 0$.

This condition is not very comfortable to handle with; in fact, the proof of the generalized Plancherel theorem on commutative hypergroups in Bloom, Heyer [3] does not involve it, but instead requires *shift-boundedness* of the positive definite measure under consideration, see Voit [26]. Indeed, shift-boundedness ensures that (B) is satisfied, as we shall now see in the slightly more general setting of signed hypergroups. As for hypergroups, we call a measure $\mu \in M(X)$ shift-bounded, if for every $f \in C_c(X)$ the function

$$\mu * f(x) := \int_X f(\overline{y} * x) d\mu(y) = \int_X T^{\overline{x}} f^- d\mu$$

belongs to $C_b(X)$. (Note that $T^{\overline{x}} f^-$ has compact support according to Axiom (A1) for signed hypergroups, and that continuity of $\mu * f$ follows from Lemma 3.3 in Rösler [16].)

3.3. Lemma. *If a positive definite measure μ on (X, m, ω) is shift-bounded, then it satisfies condition (B).*

Proof. For any $f, g \in C_c(X)$, we can write

$$\int_X f * g * g^* * f^* \, d\mu = h_1 * h_2(e) = \int h_2(\overline{x}) h_1(x) dm(x),$$

where $h_1 := f * f^* \in C_c(X)$, and $h_2 := \mu^- * g * g^* \in C_b(X)$ is positive definite. In particular, $|h_2(x)| \leq C \cdot h_2(e)$ for all $x \in X$, which leads to

$$\int_X f * g * g^* * f^* \, d\mu \leq C \cdot h_2(e) \cdot \|h_1\|_{1,m} \leq C^2 \cdot \|f\|_{1,m}^2 \cdot \int_X g * g^* \, d\mu.$$

3.4. Examples. 1. Every measure $\mu \in M_b(X)$ is clearly shift-bounded.

2. For $\varphi \in X^*$, φm is shift-bounded if and only if φ is bounded. This follows immediately from the identity $\varphi m * f(x) = \widehat{f}(\varphi) \cdot \varphi(x)$ for $f \in C_c(X)$.

Let $\mathcal{C}(X)$ denote the dense subalgebra $(C_c(X), *)$ of $L^1(X)$. If $\mu \in M(X)$ is positive definite, then it induces a positive functional p_μ on $\mathcal{C}(X)$ by $p_\mu(f) := \int_X f \, d\mu$. Now assume that μ also satisfies condition (B). Then $\mathcal{N}_\mu(X) := \{f \in C_c(X) : p_\mu(f*f^*) = 0\}$ is an ideal in $\mathcal{C}(X)$, and p_μ gives rise to an inner product $\langle \cdot \, , \cdot \rangle_\mu$ on the quotient space $\mathcal{C}_\mu(X) := \mathcal{C}(X)/\mathcal{N}_\mu(X)$ according to

$$\langle \rho(f), \rho(g) \rangle_\mu := p_\mu(f * g^*) \quad \text{for } f, g \in \mathcal{C}(X),$$

where ρ is the quotient map $\mathcal{C}(X) \to \mathcal{C}_\mu(X)$. We set $\|h\|_\mu := \sqrt{\langle h, h \rangle_\mu}$ for $h \in \mathcal{C}_\mu(X)$, and denote by $H_\mu(X)$ the Hilbert space which is obtained from $\mathcal{C}_\mu(X)$ by completion with respect to $\|.\|_\mu$. The natural representation $\widetilde{T}$ of $\mathcal{C}(X)$ on $\mathcal{C}_\mu(X)$, given by $\widetilde{T}(f)(\rho(g)) := \rho(f * g)$, is easily seen to preserve the *-operation. Moreover, condition (B) ensures that $\widetilde{T}$ is norm-continuous. Hence it can be uniquely extended to a norm-continuous *-representation $T_\mu : L^1(X) \to B(H_\mu(X))$. We shall refer to the operator norm of $T_\mu(f)$, $f \in L^1(X)$, as $\|T_\mu(f)\|$.

We are now going to apply the generalized Plancherel theorem for positive functionals on a commutative *-algebra, see Fell, Doran [6], 21.4 and 21.6, to functionals of the form p_μ on $\mathcal{C}(X)$. Before doing so, we have to ensure the following density criterion on $\mathcal{C}(X)$:

3.5. Lemma. *For any $f \in \mathcal{C}(X)$ and $\epsilon > 0$, there exist $g, h \in \mathcal{C}(X)$ such that $p_\mu((f - g * h) * (f - g * h)^*) < \epsilon$.*

Proof. Choose an approximate identity $(g_n)_{n \in \mathbb{N}} \subset C_c(X)$ and set $h_n := f - f * g_n \in C_c(X)$. Then $\bigcup_{n \in \mathbb{N}} supp\, h_n$ is relatively compact according to Lemma 2.6; hence there exist compact subsets $K, L \subseteq X$ with $supp\, h_n \subseteq K$ and $supp\,(h_n * h_n^*) \subseteq L$ for all $n \in \mathbb{N}$. Furthermore, $\|h_n\|_\infty < \epsilon$ if n is large enough. So for such n, we obtain

$$p_\mu(h_n * h_n^*) \leq \int_L \|h_n * h_n^*\|_\infty \, d\mu \leq C \, \epsilon^2 \cdot m(K) \cdot \mu(L).$$

3.6. Theorem. *Let $\mu \in M(X)$ be positive definite and satisfy condition (B). Then there exists a unique measure $\sigma_\mu \in M^+(\widehat{X})$, called the Fourier transform of μ, such that*

(1) *$\widehat{f} \in L^2(\widehat{X}, \sigma_\mu)$ for all $f \in C_c(X)$ and*

$$\int_X f * g^* \, d\mu = \int_{\widehat{X}} \widehat{f}\, \overline{\widehat{g}} \, d\sigma_\mu \quad \textit{for all } f, g \in C_c(X).$$

Furthermore, the following hold:

(2) *The Fourier transform* $\widehat{\ }$ *on* $C_c(X)$ *gives rise to a unique isometric isomorphism* $\Phi: H_\mu \to L^2(\widehat{X}, \sigma_\mu)$ *with* $\Phi(\rho(f)) = \widehat{f}$ *for* $f \in C_c(X)$.

(3) *The representantion* T_μ *is unitarily equivalent under* Φ *with the* **-representation* W *of* $L^1(X)$ *on* $L^2(\widehat{X}, \sigma_\mu)$ *given by*

$$W(f)h = \widehat{f}h.$$

(4) $supp\, \sigma_\mu = \{\varphi \in \widehat{X} : |\widehat{f}(\varphi)| \le \|T_\mu(f)\|$ *for all* $f \in C_c(X)\}$.

Proof. Items (1) to (3) result directly from the generalized Plancherel theorem in Fell, Doran [6], 21.4 and 21.6, when applied to the dense *-subalgebra $\mathcal{C}(X)$ of $L^1(X)$ and the positive functional p_μ. (In [6], 21.4, the additional assumption is made that the underlying Banach-*-algebra should be symmetric; however, a careful reading shows that this condition is not necessary.)
For the proof of (4), set

$$S_\mu := \{\varphi \in \widehat{X} : |\widehat{f}(\varphi)| \le \|T_\mu(f)\| \text{ for all } f \in C_c(X)\}.$$

Let M_g, $g \in C_b(\widehat{X})$, denote the multiplication operator on $L^2(\widehat{X}, \sigma_\mu)$ defined by $M_g(h) = gh$. Its operator norm is given by

$$\|M_g\|_{\sigma_\mu} = \|g\|_{supp\sigma_\mu} := \sup\{|g(\varphi)| : \varphi \in supp\, \sigma_\mu\}.$$

Thus according to (3), we obtain that

$$\|T_\mu(f)\| = \|M_{\widehat{f}}\|_{\sigma_\mu} = \|\widehat{f}\|_{supp\sigma_\mu}$$

for every $f \in C_c(X)$. This yields the inclusion $supp\, \sigma_\mu \subseteq S_\mu$. For the reverse inclusion, it is enough to show that $|g(\varphi)| \le \|g\|_{supp\sigma_\mu}$ for all $\varphi \in S_\mu$ and $g \in C_0(\widehat{X})$, which means that

$$S_\mu = \{\varphi \in \widehat{X} : |g(\varphi)| \le \|M_g\|_{\sigma_\mu} \text{ for all } g \in C_0(\widehat{X})\}.$$

But this is an immediate consequence of the fact that $\{\widehat{f} : f \in C_c(X)\}$ is dense in $C_0(\widehat{X})$ with respect to $\|.\|_\infty$: indeed, take any $g \in C_0(\widehat{X})$ and $\epsilon > 0$, and choose $f \in C_c(X)$ with $\|\widehat{f} - g\|_\infty < \epsilon$ and $\||\widehat{f}|^2 - |g|^2\|_\infty < \epsilon$. Then if $\varphi \in S_\mu$, we have $|g(\varphi)| \le |\widehat{f}(\varphi)| + \epsilon \le \|M_{\widehat{f}}\|_{\sigma_\mu} + \epsilon$. On the other hand,

$$\|M_{\widehat{f}}h\|_{2,\sigma_\mu}^2 \le \int_{\widehat{X}} |g|^2|h|^2 d\sigma_\mu + \epsilon \int_{\widehat{X}} |h|^2 d\sigma_\mu$$

for all $h \in C_c(\widehat{X})$, and hence $\|M_{\widehat{f}}\|_{\sigma_\mu}^2 \le \|M_g\|_{\sigma_\mu}^2 + \epsilon$. As ϵ was arbitrary, it follows that $|g(\varphi)| \le \|M_g\|_{\sigma_\mu}$ for all $\varphi \in S_\mu$ and $g \in C_0(\widehat{X})$.

We now focus to the special case $\mu = \delta_e \in M_b(X)$. Clearly, δ_e is positive definite on (X, m, ω). It also satisfies condition (B), which in this case reads

$$\|f * g\|_{2,m} \le M \cdot \|f\|_{1,m} \cdot \|g\|_{2,m}$$

for all $f, g \in C_c(X)$, with some constant $M > 0$; this follows from the norm-continuity of the left-regular representation $\lambda : L^1(X) \to B(L^2(X,m))$, see Section 2.3. Of course, $T_{\delta_e} = \lambda$, and thus the above theorem yields an analogue to the well-known classical Plancherel theorem for commutative hypergroups:

3.7. Theorem. *There exists a unique measure $\pi \in M^+(\widehat{X})$, called the Plancherel measure of (X, m, ω), such that*

$$\int_X f\,\overline{g}\,dm = \int_{\widehat{X}} \widehat{f}\,\overline{\widehat{g}}\,d\pi \qquad \text{for all } f, g \in C_c(X).$$

Furthermore, the Fourier transform on $C_c(X)$ uniquely extends to an isometric isomorphism $\mathcal{F} : L^2(X,m) \to L^2(\widehat{X}, \pi)$, called the Plancherel transform on (X, m, ω). Finally,

$$supp\,\pi = \{\varphi \in \widehat{X} : |\widehat{f}(\varphi)| \le \|\lambda(f)\| \ \text{ for all } f \in C_c(X)\}.$$

3.8. Corollary. *The Banach-algebra $L^1(X)$ is semi-simple.*

Proof. From the proof of Theorem 3.6 it is seen that $\|\widehat{f}\|_{supp\pi} = \|\lambda(f)\|$ for all $f \in C_c(X)$. As $C_c(X)$ is dense in $L^1(X)$ with respect to $\|.\|_{1,m}$, the same identity holds for all $f \in L^1(X)$. Now if $\widehat{f} = 0$, then $\lambda(f) = 0$, and thus finally $f = 0$ by injectivity of λ.

3.9. Examples. 1. If $\mu \in M_b(X)$ is positive definite, then $\sigma_\mu = \widehat{\mu^-}\pi$. As the corresponding result for hypergroups in Bloom, Heyer [3], this is obtained from the Plancherel Theorem 3.7 by writing

$$\int_X f * f^*\,d\mu = \mu^- * f * f^*(e) = \int_X \mu^- * f(x)\,\overline{f}(x)\,dm(x) = \int_{\widehat{X}} |\widehat{f}\,|^2 \widehat{\mu^-}\,d\pi.$$

2. Let $\mu = hm$ with $h \in C_b(X)$ being positive definite. Then according to Bochner's theorem (Theorem 4.11 in Rösler [16]), there exists a unique measure $a \in M_b^+(\widehat{X})$ such that $h = \check{a}$. A short calculation now yields

$$\int_X (f * g^*)\,h\,dm = \int_{\widehat{X}} (\widehat{f}\,\overline{\widehat{g}})(\overline{\varphi})\,da(\varphi)$$

for all $f, g \in C_c(X)$. Thus defining $a^- \in M_b^+(\widehat{X})$ by $da^-(\varphi) := da(\varphi^-)$, we have $\sigma_\mu = a^-$. In particular, if $\mu = \varphi m$ with $\varphi \in \widehat{X}$, then $\sigma_\mu = \delta_{\overline{\varphi}}$.

The following proposition is a generalization of the classical inversion theorem:

3.10. Proposition. *Let $\mu \in M_b(X)$ and $\rho \in M_b(\widehat{X})$. Then $\mu = \check{\rho} m$ if and only if $\rho = \widehat{\mu}\pi$.*

Proof. For $f \in C_c(X)$, $\mu * f$ belongs to $L^1(X) \cap L^2(X,m)$, and by Theorem 3.7 we obtain that for all $g \in C_c(X)$,

$$\int_X (f*g)\, d\mu^- = \mu*f*g(e) = \int_X (\mu*f)\overline{g^*}\, dm = \int_{\widehat{X}} \widehat{f}\,\widehat{g}\,\widehat{\mu}\, d\pi = \int_{\widehat{X}} \widehat{f*g}\,\widehat{\mu}\, d\pi.$$

But the set $\{f*g : f,g \in C_c(X)\}$ is dense in $C_0(X)$ according to Lemma 2.6, and the set $\{\widehat{f*g},\ f,g \in C_c(X)\}$ is obviously dense in $C_0(\widehat{X})$. Hence $\int_X h\, d\mu^- = \int_{\widehat{X}} \widehat{h}\,\widehat{\mu}\, d\pi$ for all $h \in C_c(X)$. On the other hand, a short calculation yields that $\int_X h\,\check{\rho}^-\, dm = \int_{\widehat{X}} \widehat{h}\, d\rho$. Together, the assertion follows.

3.11. Corollary. *If $f \in C(X) \cap L^1(X)$ satisfies $\widehat{f} \in L^1(\widehat{X},\pi)$, then*

$$f(x) = \int_{\widehat{X}} \widehat{f}(\varphi)\varphi(x)\, d\pi(\varphi) \quad \textit{for all } x \in X.$$

This is obtained from the above theorem with $\mu = fm$ and $\rho = \widehat{f}\pi$.

At the end of this section we want to point out the exact relationship between condition (B) and shift-boundedness for positive definite measures; in fact, both conditions are equivalent, and moreover, they are equivalent to the existence of a Fourier transform. This is now quite easily seen as a consequence of Theorem 3.6:

3.12. Proposition. *For a Radon measure $\mu \in M(X)$ the following are equivalent:*

(1) *μ is positive definite and shift-bounded.*

(2) *μ is positive definite and satisfies condition (B).*

(3) *There exists a measure $\sigma_\mu \in M^+(\widehat{X})$ such that*

$$\int_X f*g^*\, d\mu = \int_{\widehat{X}} \widehat{f}\,\overline{\widehat{g}}\, d\sigma_\mu \quad \textit{for all } f,\, g \in C_c(X).$$

Proof. It remains to verify $(3) \Rightarrow (1)$. Thus suppose (3) is satisfied. Then positive definiteness of μ is obvious. Furthermore, for $f,\, g \in C_c(X)$ we obtain

$$(*) \qquad \mu^- * f * g\,(x) = \int_X f*(\delta_{\bar{x}}*g)\, d\mu = \int_{\widehat{X}} \widehat{f}(\varphi)\varphi(x)\,\widehat{g}(\varphi)\, d\sigma_\mu(\varphi);$$

in particular, $\mu^- * f * g$ is bounded. Now let $f \in C_c(X)$ be arbitrary and choose an approximate identity $(g_n)_{n\in\mathbb{N}} \subseteq C_c(X)$. Lemma 2.6 ensures that for each $x \in X$, there is a compact subset $K_x \subseteq X$ with $supp\,(f * \delta_{\overline{x}}) \subseteq K_x$ and $\bigcup_{n\in\mathbb{N}} supp\,(f * \delta_{\overline{x}} * g_n) \subseteq K_x$. Moreover, there exists an index $N_x \in \mathbb{N}$ such that

$$\|f * \delta_{\overline{x}} - f * \delta_{\overline{x}} * g_n\|_\infty < \frac{1}{|\mu|(K_x)} \quad \text{for all } n \geq N_x .$$

Hence for $n \geq N_x$ the estimation

$$|\mu^- * f(x) - \mu^- * f * g_n(x)| = \left| \int_{K_x} (f * \delta_{\overline{x}})\, d\mu - \int_{K_x} (f * \delta_{\overline{x}} * g_n)\, d\mu \right| < 1$$

holds. On the other hand, the sequence $(\mu^- * f * g_n)_{n\in\mathbb{N}}$ is uniformly bounded on X, which follows from $(*)$ and the uniform boundedness of the sequence $(\widehat{g}_n)_{n\in\mathbb{N}}$ on $\widehat{X}$. Thus $\mu^- * f$ is bounded, and the proof is finished.

4. Subexponential growth

It has been proved by Vogel [24], that the L^1-algebra of a commutative hypergroup with so-called polynomial or subexponential growth is symmetric. This leads to a useful characterization of the dual of a hypergroup with subexponential growth in terms of the Plancherel measure, see Voit [25]. We shall now carry over this criterion to commutative signed hypergroups; in the following section it will then be applied to the Laguerre convolution structure.

Let (X, m, ω) denote a commutative signed hypergroup. As for hypergroups, we define the convolution of subsets A, $B \subseteq X$ by

$$A * B := \bigcup_{x\in A, y\in B} supp\,(\delta_x * \delta_y).$$

We shall need the following additional condition:

(C) For compact subsets A, $B \subseteq X$, $A * B$ is compact.

This is for example the case if $\delta_x * \delta_y$ has compact support for all $x, y \in X$ and the mapping $(x, y) \mapsto supp\,(\delta_x * \delta_y)$ on $X \times X$ is continuous in the Michael topology; the proof is the same as for hypergroup convolutions (see Jewett [12], 3.2). It is important to note that in contrary to the hypergroup case, the convolution of sets will in general not be associative; this is because for measures $\mu, \nu \in M_b^+(X)$ with compact support the identity $supp\,(\mu * \nu) = supp\,\mu * supp\,\nu$ need not hold. However, the inclusion "$\subseteq$" is easily seen to remain true. Thus setting

$$A^{*0} := \{e\}, \quad A^{*n} := A^{*(n-1)} * A \quad (A \subseteq X,\, n \geq 1),$$

we particularly have the inclusion $supp\,(f^{*n}) \subseteq (supp\,f)^{*n}$ for any $f \in C_c(X)$ and $n \in \mathbb{N}$.

4.1. Definition. *(X, m, ω) has subexponential growth, if for each compact subset $K \subseteq X$ and each constant $a > 1$,*

$$m(K^{*n}) = O(a^n) \quad \text{as } n \to \infty.$$

4.2. Theorem. *If (X, m, ω) satisfies condition (C) and has subexponential growth, then the following hold:*

(1) *$L^1(X)$ is symmetric, that is, $\Delta(L^1(X)) = \Delta_S(L^1(X))$. Consequently, $\mathcal{X}_b(X) = \widehat{X}$.*

(2) *The Plancherel measure π of (X, m, ω) satisfies $supp\, \pi = \widehat{X}$.*

Proof. (1) We follow the proof of Vogel [24] in a slightly simplified way. According to Hulanicki [11] it is enough to show that for $f \in L^1(X)$ with $f = f^*$, the identity $\|\lambda(f)\| = \rho(f)$ holds, where $\rho(f)$ is the spectral radius of f. By the norm-continuity of λ, a standard argument shows that $\|\lambda(f)\| \leq \rho(f)$. For the converse, first suppose that $f = f^*$ belongs to $C_c(X)$. Then the assumptions of the theorem assure that

$$m\big(supp\,(f^{*n})\big) \leq m\big((supp\, f)^{*n}\big) = O(a^n)$$

for $a > 1$. Hence there exists a constant $M(a) > 0$ such that for $n \geq 1$

$$\|f^{*n}\|' \leq C{\cdot}\|f^{*n}\|_1 \leq C{\cdot}m\big(supp\,(f^{*n})\big)^{\frac{1}{2}}\|f^{*n}\|_2 \leq M(a)a^{\frac{n}{2}}{\cdot}\|\lambda(f)\|^{n-1}\,\|f\|_2\,.$$

It follows that

$$\rho(f) = \lim_{n\to\infty} (\|f^{*n}\|')^{\frac{1}{n}} \leq \sqrt{a}\cdot\|\lambda(f)\|$$

for all $a > 1$, and hence $\rho(f) \leq \|\lambda(f)\|$.

Now suppose $f = f^* \in L^1(X)$ is arbitrary. For $\epsilon > 0$, choose $g \in C_c(X)$ with $g = g^*$ and $\|f - g\|_1 < \epsilon$. Then

$$\begin{aligned}\rho(f) &\leq \rho(f-g) + \rho(g) \leq C\cdot\|f-g\|_1 + \|\lambda(g)\| \leq \\ &\leq C\epsilon + \|\lambda(f-g)\| + \|\lambda(f)\| \leq 2C\epsilon + \|\lambda(f)\|.\end{aligned}$$

Hence $\rho(f) \leq \|\lambda(f)\|$, and the proof is finished.

The proof of (2) is exactly the same as in the hypergroup case (Theorem 2.17 in Voit [25].)

5. The dual of the signed Laguerre hypergroup

Let L_n^α, $n \in \mathbb{N}_0$ denote the Laguerre polynomials with parameter $\alpha > -1$ and $\mathcal{L}_n^\alpha$ the Laguerre functions on $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$,

$$\mathcal{L}_n^\alpha(x) := e^{-x/2}\,\frac{L_n^\alpha(x)}{L_n^\alpha(0)}\,.$$

As shown by Görlich and Markett [9], the Laguerre functions with parameter $\alpha \geq 0$ satisfy a product formula of the kind

$$\mathcal{L}_n^\alpha(x)\mathcal{L}_n^\alpha(y) = \int_0^\infty \mathcal{L}_n^\alpha(z)d\mu_{x,y}^\alpha(z) \quad \text{for all } x, y \geq 0;$$

hereby the measures $\mu_{x,y}^\alpha$ belong to $M_b^{\mathbb{R}}(\mathbb{R}_+)$, satisfying $\|\mu_{x,y}^\alpha\| \leq 1$ and $supp\, \mu_{x,y}^\alpha = [(\sqrt{x} - \sqrt{y})^2, (\sqrt{x} + \sqrt{y})^2]$ for all $x, y \geq 0$.
The above product formula gives rise to a commutative signed hypergroup on $\mathbb{R}_+$, namely $X_\alpha = (\mathbb{R}_+, m_\alpha, \omega_\alpha)$ with $dm_\alpha(x) = x^\alpha dx$ and $\omega_\alpha(x, y) = \mu_{x,y}^\alpha$. Its unit element is 0, and the involution is given by the identity mapping on $\mathbb{R}_+$. For further details, see Rösler [16]. The Laguerre functions are uniformly bounded according to

$$|\mathcal{L}_n^\alpha(x)| \leq 1 \quad \text{for all } x \geq 0, \alpha \geq 0$$

(cf. Erdélyi et al. [5], 10.18(14)). So by definition of the convolution, the $\mathcal{L}_n^\alpha$ belong to the dual $\widehat{X_\alpha}$ of X_α. We are going to show that these are in fact the only characters of X_α. This is also stated in Theorem 2.4 of Stempak [19], but a rigorous proof is missing there.

5.1. Theorem. *Let X_α be the signed Laguerre hypergroup with parameter $\alpha \geq 0$. Then*

$$\widehat{X_\alpha} = \mathcal{X}_b(X_\alpha) = \{\mathcal{L}_n^\alpha, n \in \mathbb{N}_0\},$$

and $\widehat{X_\alpha}$ is homeomorphic to $\mathbb{N}_0$ with the discrete topology.

Proof. In order to apply Theorem 4.2, we have to assure condition (C) and the subexponential growth of X_α. (C) is clear by the continuity of the mapping $(x, y) \mapsto supp\, \mu_{x,y}^\alpha$ in the Michael topology. Furthermore, if $K := [0, x]$ with $x > 0$, then $K^{*n} = [0, n^2 x]$ for all $n \in \mathbb{N}$, and thus

$$m_\alpha(K^{*n}) = \int_0^{n^2 x} t^\alpha \, dt = \frac{1}{\alpha + 1} x^{\alpha+1} n^{2\alpha+2},$$

from which the subexponential growth of X_α follows. Now set

$$c_{n,\alpha} := \frac{1}{\Gamma(\alpha + 1)} \sqrt{\frac{\Gamma(n + \alpha + 1)}{n!}} \quad (n \in \mathbb{N}_0).$$

Then the functions $c_{n,\alpha} \mathcal{L}_n^\alpha$, $n \in \mathbb{N}_0$, provide an orthonormal basis of $L^2(\mathbb{R}_+, m_\alpha)$; see Szegö [20], Theorem 5.7.1. In particular, for $f \in C_c(\mathbb{R}_+)$ Parseval's identity

$$\int_0^\infty |f|^2(x)\, dm_\alpha(x) = \sum_{n=0}^\infty c_{n,\alpha}^2 |\widehat{f}(n)|^2 = \frac{1}{\Gamma(\alpha + 1)} \sum_{n=0}^\infty \binom{n+\alpha}{n} |\widehat{f}(n)|^2$$

holds with $\widehat{f}(n) := \int_0^\infty \mathcal{L}_n^\alpha(x) f(x)\, dm_\alpha(x) = \widehat{f}(\mathcal{L}_n^\alpha)$. Therefore by uniqueness of the Plancherel measure π_α of X_α, it follows that

$$\pi_\alpha = \frac{1}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \binom{n+\alpha}{n} \delta_{\mathcal{L}_n^\alpha}.$$

Set $A := \{\mathcal{L}_n^\alpha, n \in \mathbb{N}_0\}$. Then Theorem 4.2 yields that

$$\widehat{X_\alpha} = \mathcal{X}_b(X_\alpha) = supp(\pi_\alpha) = \overline{A},$$

where the closure is taken with respect to the topology τ_c of uniform convergence on compact subsets of $\mathbb{R}_+$. We claim that $\overline{A} = A$; otherwise, there would exist some $\varphi \in \widehat{X_\alpha}$ and a sequence $(f_k) \subseteq A$ converging to φ in τ_c. In particular, $f_k \to \varphi \in C_b(\mathbb{R}_+)$ uniformly on $[0,1]$. But this is impossible, because $f_k(0) = 1$ for all $k \in \mathbb{N}$, whereas at the same time, 0 is a limit point of zeros of the f_k, $k \in \mathbb{N}$.

Finally, the discreteness of $\widehat{X_\alpha}$ is obtained as usual from the orthogonality of the $\mathcal{L}_n^\alpha$ with respect to m_α: As $\mathcal{L}_n^\alpha \in L^1(X_\alpha, m_\alpha)$, the Fourier transform $\widehat{\mathcal{L}_n^\alpha}$ is continuous on $\widehat{X_\alpha}$ with

$$\widehat{\mathcal{L}_n^\alpha}(\mathcal{L}_m^\alpha) = \int_0^\infty \mathcal{L}_n^\alpha \mathcal{L}_m^\alpha \, dm_\alpha(x) = \frac{1}{c_{n,\alpha}^2} \delta_{n,m}.$$

Hence $\mathcal{L}_n^\alpha$ must be an isolated point of $\widehat{X_\alpha}$.

5.2. Remarks. 1. The signed Laguerrre hypergroup provides an example of the somewhat striking fact that the dual of a non-compact signed hypergroup may be discrete. This situation cannot occur for hypergroups, which is clear by a result of Voit [25]: if X is a commutative hypergroup, then the (unique) positive character contained in the support of the Plancherel measure of X is isolated in $\widehat{X}$ if and only if X is compact.

2. It is well-known that for a compact hypergroup X with normalized Haar measure m, $L^2(X,m)$ becomes a Hilbert-$*$-algebra in the natural way (see Jewett [12], 7.2). The same is true for the signed Laguerre hypergroup after a suitable renormalization of m_α: Consider the renormalized signed hypergroup $(X_\alpha, \widetilde{m}_\alpha, \omega_\alpha)$ with $\widetilde{m}_\alpha = \Gamma(\alpha+1)\, m_\alpha$. The associated Plancherel measure $\widetilde{\pi}_\alpha = \Gamma(\alpha+1)\pi_\alpha$ satisfies $\widetilde{\pi}_\alpha(\varphi) \geq 1$ for all $\varphi \in \widehat{X_\alpha}$. Hence for $g \in C_c(X)$, the inequality $\|\widehat{g}\|_\infty \leq \|g\|_2$ holds, where $\|.\|_2$ is taken with respect to $\widetilde{m}_\alpha$; this is because $\sum_{\varphi \in \widehat{X_\alpha}} |\widehat{g}(\varphi)|^2 \, d\widetilde{\pi}_\alpha(\varphi) = \|g\|_2^2$ according to the Plancherel theorem. Thus for any $f, g \in C_c(X)$, we obtain

$$\|f * g\|_2 = \|\widehat{f * g}\|_2 \leq \|\widehat{g}\|_\infty \cdot \|\widehat{f}\|_2 \leq \|f\|_2 \cdot \|g\|_2.$$

The unique bilinear and $\|.\|_2$-continuous continuation of $*$ (and as well of the involution *) to $L^2(X, \widetilde{m}_\alpha)$ now clearly makes $L^2(X, \widetilde{m}_\alpha)$ into a H^*-algebra.

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