RIESZ DISTRIBUTIONS AND LAPLACE TRANSFORM IN THE DUNKL SETTING OF TYPE A

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ABSTRACT. We study Riesz distributions in the framework of rational Dunkl theory associated with root systems of type A. As an important tool, we employ a Laplace transform involving the associated Dunkl kernel, which essentially goes back to Macdonald [M3], but was so far only established at a formal level. We give a rigorous treatment of this transform based on suitable estimates of the type A Dunkl kernel. Our main result is a precise analogue in the Dunkl setting of a well-known result by Gindikin, stating that a Riesz distribution on a symmetric cone is a positive measure if and only if its exponent is contained in the Wallach set. For Riesz distributions in the Dunkl setting, we obtain an analogous characterization in terms of a generalized Wallach set which depends on the multiplicity parameter on the root system.

1. INTRODUCTION

Riesz distributions play a prominent role in the harmonic analysis on symmetric cones and the study of the wave equation, but also in multivariate statistics due to their close relation to Wishart distributions, see [FK] for some important aspects. To motivate our results, let us describe a typical example: Consider the set Ω_n of positive definite $n \times n$-matrices over $\mathbb{R}$, which is an open (and actually symmetric) cone in the space $\mathrm{Sym}_n = \{x \in M_n(\mathbb{R}) : x = x^t\}$. The latter is a Euclidean space (actually, a Euclidean Jordan algebra) with scalar product $(x|y) = \mathrm{tr}(xy)$. For indices $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu > \mu_0 := (n-1)/2$, the Riesz distributions R_μ associated with Ω_n are defined as the complex Radon measures on Sym_n which are defined by

$$R_\mu(\varphi) = \frac{1}{\Gamma_{\Omega_n}(\mu)} \int_{\Omega_n} \varphi(x)(\det x)^{\mu-\mu_0-1} dx$$

where Γ_{Ω_n} is the Gindikin gamma function

$$\Gamma_{\Omega_n}(\mu) = \int_{\Omega_n} e^{-\mathrm{tr}\,x}(\det x)^{\mu-\mu_0-1} dx.$$

Considered as tempered distributions on Sym_n, the Riesz measures R_μ satisfy the recursion

$$\det\Big(\frac{\partial}{\partial x}\Big) R_\mu = R_{\mu-1},$$

see [FK]. Thus the mappping $\mu \mapsto R_\mu$ uniquely extends to a holomorphic mapping on $\mathbb{C}$ with values in the space of tempered distributions $\mathcal{S}'(\mathrm{Sym}_n)$. Note that for

$n = 1$, the Riesz distributions are just the homogeneous distributions on $\mathbb{R}_+ =]0, \infty[$ which are obtained by holomorphic extension from the Riemann-Liouville measures

$$R_\mu(\varphi) = \frac{1}{\Gamma(\mu)} \int_0^\infty \varphi(x) x^{\mu-1} dx \quad (\operatorname{Re}\mu > 0).$$

It is a famous result due to Gindikin [G] that a Riesz distribution associated with a symmetric cone is actually a positive measure if and only if its index μ belongs to the so-called Wallach set. The Wallach set plays an important role in the study of Hilbert spaces of holomorphic functions on symmetric domains, see [FK, Chapter XIII]. In the case of the symmetric cone Ω_n, it is given by

$$\left\{ 0, \frac{1}{2}, \ldots, \frac{n-1}{2} \right\} \cup \left] \frac{n-1}{2}, \infty \right[.$$

In the present paper, we study Riesz distributions in the framework of Dunkl operator theory associated with the root system

$$A_{n-1} = \{ \pm(e_i - e_j) : 1 \le i < j \le n \} \subset \mathbb{R}^n. \tag{1.1}$$

(Rational) Dunkl operators are commuting differential-reflection operators associated with a root system on some Euclidean space which were intoduced by C.F. Dunkl in [D]. There is a well-developed harmonic analysis associated with these operators which generalizes both the classical Euclidean Fourier analysis as well as the radial harmonic analysis on Riemannian symmetric spaces of Euclidean type. For a general background see e.g. [DX, dJ1, R3]. Among the more recent results in harmonic analysis associated with Dunkl operators let us mention [ADH, Ve].

For A_{n-1}, the Dunkl operators in the directions of the standard basis $(e_i)_{1 \le i \le n}$ of $\mathbb{R}^n$ are given by

$$T_i(k) = \frac{\partial}{\partial x_i} + k \cdot \sum_{j \ne i} \frac{1}{x_i - x_j} (1 - \sigma_{ij}) \tag{1.2}$$

where σ_{ij} is the reflection in $\mathbb{R}^n$ which acts on functions by exchanging the coordinates x_i and x_j, and $k \in \mathbb{C}$ is a so-called multiplicity parameter. For some values of k, Dunkl theory of type A_{n-1} is closely related to the harmonic analysis on symmetric cones, as will be explained in Sections 2 and 3. For example, analysis on the cone Ω_n for structures which depend only on the eigenvalues boils down to Dunkl analysis of type A_{n-1} for structures on $\mathbb{R}_+^n$ which are invariant under the symmetric group S_n; the multiplicity is hereby $k = 1/2$. In this paper we shall consider the general, non-symmetric Dunkl setting associated with the root system A_{n-1} and arbitrary nonnegative multiplicities. For fixed multiplicity $k \ge 0$, the Riesz measures of type A_{n-1} on $\mathbb{R}^n$ are defined by

$$R_\mu(\varphi) := \frac{1}{d_n(k)\Gamma_n(\mu; k)} \int_{\mathbb{R}_+^n} \varphi(x) D(x)^{\mu-\mu_0-1} \omega_k(x) dx, \quad \operatorname{Re}\mu > \mu_0 = k(n-1)$$

where $d_n(k) > 0$ is a certain normalization constant, $\Gamma_n(\mu; k)$ is a multivariate version of the Gamma function (see Section 5),

$$\omega_k(x) = \prod_{1 \le i < j \le n} |x_i - x_j|^{2k} \tag{1.3}$$

is the Dunkl weight function associated with A_{n-1} and multiplicity k, and

$$D(x) := \prod_{i=1}^{n} x_i \,.$$

It turns out that the Riesz measures R_μ satisfy the distributional recursion

$$D\big(T(k)\big) R_\mu = R_{\mu-1}$$

with the Dunkl operator $D(T(k)) := \prod_{i=1}^{n} T_i(k)$, and hence the mappping $\mu \mapsto R_\mu$ extends uniquely to a holomorphic mapping on $\mathbb{C}$ with values in $\mathcal{S}'(\mathbb{R}^n)$. This was already observed in the recent thesis [L], where Dunkl-type Riesz distributions were introduced to study questions related to Huygens' principle. In this paper, we carry out a more detailed study of these distributions. As in the case of symmetric cones, an important tool will be a suitable version of the Laplace transform, given by

$$\mathcal{L}_k f(z) = \int_{\mathbb{R}_+^n} f(x) E_k^A(-x, z) \omega_k(x) dx \tag{1.4}$$

where E_k^A denotes the Dunkl kernel associated with A_{n-1} and multiplicity k. This is a non-symmetric variant of a Laplace transform which was first introduced on a purely formal level by Macdonald in his manuscript [M3] and was further studied for $n = 2$ by Yan [Y2]. The transform (1.4) was used by Baker and Forrester [BF2] and by Sahi and Zhang [SZ], but due to a lack of knowledge about the decay properties of the Dunkl kernel, convergence issues could not be properly settled.

We shall give in Section 3 a rigorous treatment of the Laplace transform (1.4), based on suitable estimates for the Dunkl kernel which were conjectured in [BF2] and are specific to root systems of type A. In particular, we provide a Cauchy-type inversion theorem, which improves the injectivity statements for the Laplace transform in [Y2] and [BF2]. Let us mention at this point that in connection with the Laplace transform, specific properties of the type A Dunkl kernel are decisive. In Section 4, we extend the type A Laplace transform to distributions. Section 5 is then devoted to the study of the Riesz distributions R_μ in the Dunkl setting. We compute their Laplace transforms and study for which indices they are actually measures. Our main result (Theorem 5.15) is a precise analogue of Gindikin's result for Riesz distributions on symmetric cones: The Dunkl type Riesz distribution R_μ on $\mathbb{R}^n$ is a positive measure exactly if μ belongs to the generalized Wallach set

$$\{0, k, \ldots, k(n-1)\} \cup]k(n-1), \infty[\,. \tag{1.5}$$

The Riesz distributions associated with the discrete Wallach points kr, $0 \le r \le n-1$ can be determined explicitly; they are supported in the strata of the cone $\overline{\mathbb{R}_+^n}$, see Theorem 5.11. The proofs of Theorems 5.11 and 5.15 are based on analysis for multivariable hypergeometric functions which are given in terms of Jack polynomial expansions in the sense of [Kan, M3], combined with methods of Sokal [So] and a variant of the Shanbhag principle from [CL]. We finally mention that the generalized Wallach set (1.5) also plays an interesting role in connection with integral representations of Sonine type between Bessel functions of type B_n and the positivity of intertwining operators in the B_n-case, see [RV2].

2. Dunkl theory for root system A_{n-1}

In this section, we provide a brief introduction to the relevant concepts from rational Dunkl theory; for a background the reader is referred to [DX, dJ1, O, R3].

We shall consider the root system A_{n-1} in the Euclidean space $\mathbb{R}^n$ with the usual scalar product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, which we extend to $\mathbb{C}^n \times \mathbb{C}^n$ in a bilinear way. The corresponding finite reflection group is the symmetric group S_n on n elements. For fixed multiplicity parameter k, the associated Dunkl operators $T_i(k)$ (as defined in (1.2)) commute, c.f. [D]. Therefore the assignment $x_i \mapsto T_i(k)$ extends to a unital algebra homomorphism

$$p \mapsto p(T(k)), \ \mathbb{C}[\mathbb{R}^n] \to \mathrm{End}(\mathbb{C}[\mathbb{R}^n]).$$

In this paper, we shall always assume that $k \geq 0$. Then for each spectral parameter $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ there exists a unique analytic function $f = E_k^A(\,.\,, y)$ satisfying

$$T_i(k)f = y_i f \quad \text{for } i = 1, \ldots n; \quad f(0) = 1,$$

see [DX, O]. The function E_k^A is called the Dunkl kernel of type A_{n-1}. It extends to a holomorphic function on $\mathbb{C}^n \times \mathbb{C}^n$ with

$$E_k^A(x, y) = E_k^A(y, x), \quad E_k^A(\lambda x, y) = E_k^A(x, \lambda y), \quad E_k^A(\sigma x, \sigma y) = E_k^A(x, y)$$

for all $\lambda \in \mathbb{C}$ and $\sigma \in S_n$. According to [R2], E_k^A has a positive integral representation. More precisely, for each $x \in \mathbb{R}^n$ there exists a unique probability measure $\mu_x^k \in M^1(\mathbb{R}^n)$ such that

$$E_k^A(x, z) = \int_{\mathbb{R}^n} e^{\langle \xi, z \rangle} d\mu_x^k(\xi) \quad \text{for all } z \in \mathbb{C}^n. \tag{2.1}$$

The support of μ_x is contained in $C(x)$, the convex hull of the orbit of x under the action of S_n. Notice that

$$E_k^A(x, y) > 0 \quad \text{and} \quad |E_k^A(ix, y)| \leq 1 \quad \text{for all } x, y \in \mathbb{R}^n.$$

Lemma 2.1. *For* $x \in \mathbb{R}^n$ *and* $y, z \in \mathbb{C}^n$,

$$|E_k^A(x, y + z)| \leq E_k^A(x, \mathrm{Re}\, z) \cdot e^{\max_{\sigma \in S_n} \langle \sigma x, \mathrm{Re}\, y \rangle}.$$

In particular,

$$|E_k^A(x, z)| \leq E_k^A(x, \mathrm{Re}\, z).$$

Proof. According to (2.1),

$$|E_k^A(x, y + z)| \leq \int_{C(x)} e^{\langle \xi, \, \mathrm{Re}\, y + \mathrm{Re}\, z \rangle} d\mu_x^k(\xi)$$

$$\leq e^{\max_{\sigma \in S_n} \langle \sigma x, \, \mathrm{Re}\, y \rangle} \cdot \int_{C(x)} e^{\langle \xi, \mathrm{Re}\, z \rangle} d\mu_x^k(\xi),$$

which implies the assertion.   □

Remark 2.2. The statement of Lemma 2.1 holds in the context of arbitrary root systems: Consider some (reduced, not necessarily crystallographic) root system R in a Euclidean space $(\mathfrak{a}, \langle\,.\,,\,.\,\rangle)$ with associated reflection group W and a multiplicity function $k \geq 0$ on R (i.e. $k : R \to \mathbb{C}$ is W-invariant). Denote by E_k the associated Dunkl kernel. Then for each $x \in \mathfrak{a}$ there exists a unique probability measure μ_x^k on $\mathfrak{a}$, supported in the convex hull of the W-orbit of x, such that

$$E_k(x, z) = \int_{\mathfrak{a}} e^{\langle \xi, z \rangle} d\mu_x^k(\xi)$$

for all $z \in \mathfrak{a}_{\mathbb{C}}$ (the complexification of $\mathfrak{a}$). Thus by the same argument as above,

$$|E_k(x, y + z)| \leq E_k(x, \mathrm{Re}\, z) \cdot e^{\max_{w \in W} \langle wx, \mathrm{Re}\, y \rangle} \quad \forall x \in \mathfrak{a}, \ y, z \in \mathfrak{a}_{\mathbb{C}}.$$

We return to the root system A_{n-1} with multiplicity $k \geq 0$. Recall the weight function ω_k introduced in (1.3). The associated Dunkl transform on $L^1(\mathbb{R}^n, \omega_k)$ is defined by

$$\widehat{f}^k(y) = \frac{1}{c_k} \int_{\mathbb{R}^n} f(x) E_k^A(x, -iy) \omega_k(x) dx$$

with the normalization constant

$$c_k = \int_{\mathbb{R}^n} e^{-|x|^2/2} \omega_k(x) dx.$$

This is the classical Mehta integral, whose value is given by

$$c_k = c_{k,n} = (2\pi)^{n/2} \prod_{j=1}^{n} \frac{\Gamma(1+jk)}{\Gamma(1+k)}, \tag{2.2}$$

c.f. [M1]. The Dunkl transform is a homeomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and there is an L^1-inversion theorem, see [dJ1]: if $f \in L^1(\mathbb{R}^n, \omega_k)$ with $\widehat{f}^k \in L^1(\mathbb{R}^n, \omega_k)$, then $f(x) = (\widehat{f}^k)^{\wedge k}(-x)$ for almost all $x \in \mathbb{R}^n$. The Dunkl operators act continuously on $\mathcal{S}(\mathbb{R}^n)$ and therefore also on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$, via

$$\langle T_\xi(k)u, \varphi \rangle := -\langle u, T_\xi(k)\varphi \rangle, \quad u \in \mathcal{S}'(\mathbb{R}^n), \ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.3}$$

Besides the Dunkl kernel E_k^A, we shall also need the Bessel function of type A_{n-1},

$$J_k^A(x, y) := \frac{1}{n!} \sum_{\sigma \in S_n} E_k^A(\sigma x, y)$$

which is symmetric (i.e. S_n-invariant) in both arguments.

Remark 2.3. For certain values of k, the Bessel function J_k^A has an interpretation in the context of symmetric spaces. In fact, consider for one of the (skew) fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the set $H_n(\mathbb{F}) := \{x \in M_n(\mathbb{F}) : x = \overline{x}^t\}$ of Hermitian $n \times n$-matrices over $\mathbb{F}$. The unitary group $U_n(\mathbb{F})$ acts on $H_n(\mathbb{F})$ by conjugation $x \mapsto uxu^{-1}$, and $X_n := U_n(\mathbb{F}) \ltimes H_n(\mathbb{F})/U_n(\mathbb{F})$ is a symmetric space of Euclidean type, which can be identified with the tangent space of the symmetric cone $\Omega_n(\mathbb{F}) = \{x \in H_n(\mathbb{F}) : x \text{ positive definite}\}$ in the point I_n (the identity matrix). It is well known that the spherical functions of X_n, considered as functions of the spectra of matrices from $H_n(\mathbb{F})$, can be identified with the Bessel functions $J_k^A(\,.\,, z), z \in \mathbb{C}^n$ with multiplicity $k = d/2$, where $d = \dim_\mathbb{R} \mathbb{F} \in \{1, 2, 4\}$. For details see [dJ2] and [RV1].

It will be important in this paper that for $k > 0$, the Bessel function J_k^A has a series expansion in terms of Jack polynomials. To describe this as well as some related facts, we have to introduce further notation; references are [Sta], [Kan], [BF1] and [FW].

Let Λ_n^+ denote the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ with at most n parts. The number of parts of λ is also called its length and denoted by $l(\lambda)$. We consider the Jack polynomials $C_\lambda^\alpha := C_\lambda^{(\alpha)}$ in n variables with parameter $\alpha > 0$, which are indexed by partitions $\lambda \in \Lambda_n^+$ and are normalized such that

$$(z_1 + \cdots + z_n)^m = \sum_{|\lambda|=m} C_\lambda^\alpha(z) \qquad \text{for all } m \in \mathbb{N}_0, z \in \mathbb{C}^n, \tag{2.4}$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$ denotes the weight of λ. The polynomials C_λ^α are symmetric and homogeneous of degree $|\lambda|$. If $k > 0$, then according to the relations (3.22) and (3.37) of [BF2], J_k^A is a ${}_0F_0$-hypergeometric function in two variables:

$$J_k^A(z, w) = \sum_{\lambda \in \Lambda_n^+} \frac{1}{|\lambda|!} \cdot \frac{C_\lambda^\alpha(z) C_\lambda^\alpha(w)}{C_\lambda^\alpha(\underline{1})} =: {}_0F_0^\alpha(z, w) \quad \text{with } \alpha = 1/k \qquad (2.5)$$

where we use the notation $\underline{1} := (1, \ldots, 1) \in \mathbb{R}^n$. It is known from [KS] that

$$C_\lambda^\alpha(z) = \sum_{|\mu| = |\lambda|} c_{\lambda,\mu}^\alpha z^\mu \qquad (2.6)$$

with nonnegative coefficients $c_{\lambda,\mu}^\alpha \geq 0$. Denoting $\|z\|_\infty = \sup_{1 \leq i \leq n} |z_i|$, we therefore have

$$|C_\lambda^\alpha(z)| \leq C_\lambda^\alpha(\underline{1}) \cdot \|z\|_\infty^{|\lambda|} \qquad (2.7)$$

and thus by (2.4)

$$\sum_{|\lambda| = m} \frac{|C_\lambda^\alpha(z) C_\lambda^\alpha(w)|}{C_\lambda^\alpha(\underline{1})} \leq (n \|z\|_\infty \|w\|_\infty)^m. \qquad (2.8)$$

This implies that the series (2.5) converges locally uniformly on $\mathbb{C}^n \times \mathbb{C}^n$.

Remark 2.4. An alternative proof of the expansion (2.5) is obtained by symmetrization from an analogous expansion of the Dunkl kernel in terms of non-symmetric Jack polynomials, see [R1], Lemma 3.1 and Example 3.6.

3. THE TYPE A LAPLACE TRANSFORM

We again consider the root system A_{n-1} with some fixed multiplicity $k \geq 0$. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we write $x \leq y$ (and $y \geq x$) if $x_i \leq y_i$ for all $i = 1, \ldots n$. This defines a partial order on $\mathbb{R}^n$. For $s \in \mathbb{C}$, we use the abbreviation

$$\underline{s} := (s, \ldots, s) \in \mathbb{C}^n.$$

We further write $\mathbb{R}_+ :=]0, \infty[$ and put

$$\|z\|_1 := \sum_{i=1}^n |z_i| \quad \text{for } z \in \mathbb{C}^n.$$

The rigorous foundation of the Laplace transform in the Dunkl setting associated with A_{n-1} will be based on the following factorization and exponential decay of the Dunkl kernel E_k^A.

Lemma 3.1.   (1) *For all $x, z \in \mathbb{C}^n$ and $s \in \mathbb{C}$,*

$$E_k^A(x, z + \underline{s}) = e^{\langle x, \underline{s} \rangle} \cdot E_k^A(x, z).$$

     (2) *Let $x \in \overline{\mathbb{R}_+^n}$ and $a \in \mathbb{R}^n$. Then for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq a$,*

$$|E_k^A(-x, z)| \leq E_k^A(-x, a) \leq \exp\left(-\|x\|_1 \cdot \min_{1 \leq i \leq n} a_i\right).$$

In particular, if $\operatorname{Re} z \geq \underline{s}$ for some $s > 0$, then

$$|E_k^A(-x, z)| \leq e^{-s\|x\|_1} \quad \forall x \in \overline{\mathbb{R}_+^n}.$$

Properties (1) and (2) are also valid for the Bessel function J_k^A.

Proof. (1) By analyticity, it suffices to consider $x \in \mathbb{R}^n$. Then the assertion follows easily from Proposition 3.19 of [BF2], where $s = 1$. It can also be deduced from formula (2.1), as follows: We have $\langle \xi, \underline{1} \rangle = \langle x, \underline{1} \rangle$ for all ξ in the S_n-orbit of x, which implies that $\langle \xi, \underline{s} \rangle = \langle x, \underline{s} \rangle$ for all $\xi \in C(x)$ and all $s \in \mathbb{C}$. The statement is then immediate from formula (2.1).

(2) In view of Lemma (2.1) it suffices to consider $z \in \mathbb{R}^n$ with $z \geq a$. Our assumption $x \geq 0$ implies that $\xi \geq 0$ for all $\xi \in C(x)$ and therefore also $\langle \xi, z-a \rangle \geq 0$. Thus by (2.1),

$$|E_k^A(-x, z)| = \int_{C(x)} e^{-\langle \xi, z-a \rangle} e^{-\langle \xi, a \rangle} d\mu_x^k(\xi) \leq \int_{C(x)} e^{-\langle \xi, a \rangle} d\mu_x^k(\xi) = E_k^A(-x, a).$$

For the second inequality, we start with the last identity in the above formula and write $\xi \in C(x)$ as

$$\xi = \sum_{\sigma \in S_n} \lambda_\sigma \sigma x \quad \text{with} \quad \lambda_\sigma \geq 0, \ \sum_{\sigma \in S_n} \lambda_\sigma = 1.$$

Using the estimate

$$\langle \sigma x, a \rangle \geq \|x\|_1 \cdot \min_{1 \leq i \leq n} a_i \quad (\sigma \in S_n)$$

we obtain that $\langle \xi, a \rangle \geq \|x\|_1 \cdot \min a_i$. This implies that $E_k^A(-x, a) \leq e^{-\|x\|_1 \cdot \min a_i}$. The same assertions for J_k^A are immediate. $\qquad\square$

Following [BF2, Section 3.4], we define the type A Laplace transform of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+, \omega_k)$ as

$$\mathcal{L}_k f(z) := \int_{\mathbb{R}^n_+} f(x) E_k^A(-x, z) \omega_k(x) dx \quad (z \in \mathbb{C}^n), \tag{3.1}$$

provided the integral exists.

Remarks 3.2. 1. In [M3, formula (2) on p. 38], Macdonald defines the Laplace transform with the S_n-invariant kernel

$$e(-x, z) = J_k^A(-x, z)$$

instead of $E_k^A(-x, z)$ (see [M3, p.26] for the definition of e). If f is S_n-invariant, then in (3.1) the Dunkl kernel $E_k^A(-x, z)$ may be replaced by $J_k^A(-x, z)$ without affecting the value of the integral. So in the symmetric case, our definition coincides with that of Macdonald up to a constant factor.

2. Macdonald's definition of the Laplace transform in [M3] is closely related to the well-known Laplace transform on symmetric cones. To explain this relation, suppose that V is a simple Euclidean Jordan algebra with Jordan multiplication $(x, y) \mapsto xy$ and scalar product $(x|y) = \mathrm{tr}(xy)$ where tr denotes the Jordan trace on V, i.e. $\mathrm{tr}(x)$ is the sum of eigenvalus of x. Let $\Omega \subset V$ be the associated symmetric cone. It can be written as a Riemannian symmetric space $\Omega = G/K$ where G is the identity component of the automorphism group of Ω and $K = G \cap O(V)$. We refer to [FK] for these facts and a general introduction to the analysis on symmetric cones. The Laplace transform of a function $F \in L^1_{\mathrm{loc}}(\Omega)$ is defined by

$$\mathcal{L}F(y) = \int_\Omega F(x) e^{-(x|y)} dx \quad (y \in V),$$

provided the integral exists. Suppose the rank of V is n. Then the possible ordered spectra of elements from Ω are given by the set

$$C_+ = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n : \xi_1 \geq \cdots \geq \xi_n > 0\}.$$

If F is K-invariant, it can be uniquely written as $F(x) = f(\operatorname{spec}(x))$, where $\operatorname{spec}(x) \in C_+$ denotes the set of eigenvalues of x ordered by size and $f : C_+ \to \mathbb{C}$ is measurable. Fix some Jordan frame $(c_1, \ldots, c_n)$ of V (that is, the c_j form a complete system of orthogonal primitive idempotents in V). For $\xi \in C_+$ let $\underline{\xi} := \sum_{j=1}^n \xi_j c_j \in \Omega$. Then according to Theorem VI.2.3 of [FK],

$$\mathcal{L}F(y) = c_0 \int_{C_+} f(\xi) \left(\int_K e^{-(k\underline{\xi}|y)} dk \right) \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j)^d \, d\xi$$

where $d \in \mathbb{N}$ denotes the Peirce constant of V and $c_0 > 0$ is some normalization constant depending on V. In order to identify the integral over K, we recall the spherical (or zonal) polynomials Z_λ on V which are indexed by partitions $\lambda \in \mathbb{N}_0^n$ and normalized such that for each $m \in \mathbb{N}_0$,

$$(\operatorname{tr} x)^m = \sum_{|\lambda|=m} Z_\lambda(x) \quad (x \in V), \tag{3.2}$$

see Section XI.5 of [FK]. The Z_λ are K-invariant and thus depend only on the eigenvalues of their argument. As such, they are given by Jack polynomials:

$$Z_\lambda(x) = C_\lambda^\alpha\big(\operatorname{spec}(x)\big) \quad \text{with } \alpha = \frac{2}{d}, \tag{3.3}$$

see the notes to Chap. XI in [FK]. Further, the Z_λ satisfy the product formula

$$\frac{Z_\lambda(x) Z_\lambda(y)}{Z_\lambda(e)} = \int_K Z_\lambda \big(P(\sqrt{x})ky \big) \, dk \quad (x \in \Omega, y \in V)$$

where P denotes the quadratic representation of V. This is immediate from [FK, Corollary XI.3.2] and the fact that $P(\sqrt{x})e = x$.

Now consider the type A Bessel function J_k^A with multiplicity $k = d/2$. Let $x \in \Omega$, $y \in V$ and $\xi = \operatorname{spec}(x)$, $\eta = \operatorname{spec}(y)$. Then by relations (3.2) and (3.3),

$$\begin{aligned} J_{d/2}^A(\xi, \eta) &= \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \frac{Z_\lambda(x) Z_\lambda(y)}{Z_\lambda(e)} \\ &= \int_K e^{\operatorname{tr}(P(\sqrt{x})ky)} dk = \int_K e^{(x|ky)} dk. \end{aligned}$$

Therefore $\mathcal{L}F(y)$ depends only on $\eta = \operatorname{spec}(y)$ and is given by

$$\mathcal{L}F(y) = c_0 \int_{C_+} f(\xi) J_{d/2}^A(-\eta, \xi) \, \omega_{d/2}(\xi) d\xi.$$

Extending f to a symmetric function on $\mathbb{R}_+^n$, this becomes

$$\mathcal{L}F(y) = \frac{c_0}{n!} \int_{\mathbb{R}_+^n} f(\xi) J_{d/2}^A(-\xi, \eta) \, \omega_{d/2}(\xi) d\xi,$$

which coincides, up to a constant, with Macdonald's Laplace transform for $k = d/2$.

Let us now continue the study of the type A Laplace transform $\mathcal{L}_k$.

Lemma 3.3. *Let $f \in L^1_{loc}(\mathbb{R}^n_+)$ and suppose that $\mathcal{L}_k f(a)$ exists for some $a \in \mathbb{R}^n$, that is*

$$\int_{\mathbb{R}^n_+} |f(x)| E_k^A(-x, a)\, \omega_k(x) dx < \infty.$$

Then the following hold.

(1) *$\mathcal{L}_k f(z)$ exists for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq a$, and $\mathcal{L}_k f$ is holomorphic on the half space*

$$H_n(a) := \{ z \in \mathbb{C}^n : \operatorname{Re} z > a \}.$$

(2) *If $p \in \mathbb{C}[\mathbb{R}^n]$ is a polynomial, then $\mathcal{L}_k(fp)(z)$ exists for all $z \in H_n(a)$, and*

$$p(-T(k))(\mathcal{L}_k f) = \mathcal{L}_k(fp) \quad on \ H_n(a).$$

Proof. Part (1) is immediate from Lemma 3.1(2) and standard theorems for holomorphic parameter integrals. For part (2), let $z \in H_n(a)$ and choose $\epsilon > 0$ such that $\operatorname{Re} z > a + \underline{\epsilon}$. Then for $x \in \mathbb{R}^n_+$ we have $|E_k^A(-x, z)| \leq e^{-\epsilon \|x\|_1} E_k^A(-x, a)$, due to Lemma 3.1. This implies that $\mathcal{L}_k(fp)(z)$ exists for each polynomial p, and differentiation under the integral gives

$$-T_{e_i}(k)\mathcal{L}_k f(z) = \int_{\mathbb{R}^n_+} x_i f(x) E_k^A(-x, z)\omega_k(x) dx, \quad 1 \leq i \leq n.$$

The statement now follows by induction. $\qquad\square$

Example 3.4. Suppose that f is measurable on $\mathbb{R}^n_+$ and exponentially bounded according to

$$|f(x)| \leq C e^{s\|x\|_1}$$

with constants $C > 0$ and $s \in \mathbb{R}$. Then by Lemma 3.1(2), $\mathcal{L}_k f(z)$ exists for all $z \in H_n(\underline{s})$.

We continue with some further elementary properties of the Laplace transform:

Lemma 3.5. *Suppose that f is measurable on $\mathbb{R}^n_+$ with $|f(x)| \leq C \cdot e^{s\|x\|_1}$ for some $s \in \mathbb{R}$. Then*

(1) *For $z \in H_n(0)$, $\mathcal{L}_k\big(e^{-\langle x, \underline{s}\rangle} f\big)(z) = \mathcal{L}_k(f)(z + \underline{s})$.*

(2) *Let $y \in \mathbb{R}^n$. Then $\mathcal{L}_k f(x + iy) \longrightarrow 0$ as $\min x_i \to \infty$.*

(3) *Let $x > \underline{s}$. Then $\mathcal{L}_k f(x + iy) \longrightarrow 0$ as $\min y_i \to \infty$.*

Proof. (1) This is obvious from Lemma 3.1(1) (c.f. also [BF2]).

(2) Write $x = x' + \underline{\xi}$ with $\xi = \min x_i$ and $x' \geq 0$. Then

$$\mathcal{L}_k f(x + iy) = \int_{\mathbb{R}^n_+} e^{-\langle u, \underline{\xi}\rangle} E_k^A(-u, x' + iy) f(u)\omega_k(u) du$$

where

$$\big| e^{-\langle u, \underline{\xi}\rangle} E_k^A(-u, x' + iy) f(u)\big| \leq C \cdot e^{(s-\xi)\|u\|_1} E_k^A(-u, x') \leq C \cdot e^{(s-\xi)\|u\|_1}.$$

As $\xi \to \infty$, the dominated convergence theorem yields the assertion.

(3) As above, write $y = y' + \underline{\eta}$ with $\eta = \min y_i$ and $y' \geq 0$. Then

$$\mathcal{L}_k f(x + iy) = \int_{\mathbb{R}^n_+} f(u) E_k^A(-u, x + iy') e^{-i\langle u, \underline{\eta}\rangle}\omega_k(u) du.$$

The statement now follows from Lemma 3.1 together with the Riemann-Lebesgue Lemma for the classical Fourier transform.

$\qquad\square$

Our next result is a Cauchy-type inversion theorem for the Laplace transform. We extend the weight function ω_k to $\mathbb{C}^n$ by

$$\omega_k(z) := \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2k}.$$

Theorem 3.6. *Suppose that $\mathcal{L}_k f(\underline{s})$ exists for some $s \in \mathbb{R}$, and that*

$$y \mapsto \mathcal{L}_k f(\underline{s} + iy) \in L^1(\mathbb{R}^n, \omega_k).$$

Then f has a continuous representative f_0, and

$$\frac{(-i)^n}{c_k^2} \int_{\mathrm{Re}\, z = \underline{s}} \mathcal{L}_k f(z) E_k^A(x, z)\, \omega_k(z) dz = \begin{cases} f_0(x) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (3.4)$$

Here dz is understood as an n-fold line integral.

Proof. Lemma 3.3 assures that $\mathcal{L}_k f(\underline{s} + iy)$ indeed exists for all $y \in \mathbb{R}^n$. By Lemma 3.1, the left-hand side of (3.4) can also be written as

$$\frac{1}{c_k^2}\, e^{\langle x, \underline{s} \rangle} \int_{\mathbb{R}^n} \mathcal{L}_k f(\underline{s} + iy) E_k^A(x, iy)\, \omega_k(y) dy.$$

This integral is absolutely convergent by our assumption. Extend f to $\mathbb{R}^n$ by

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and put $F(x) = e^{-\langle x, \underline{s} \rangle} \widetilde{f}(x)$. As $e^{-\langle x, \underline{s} \rangle} = E_k^A(-x, \underline{s})$, we have $F \in L^1(\mathbb{R}^n, \omega_k)$. In view of Lemma 3.1, the Dunkl transform of F is given by

$$\widehat{F}^{\,k}(y) = \frac{1}{c_k} \int_{\mathbb{R}^n_+} e^{-\langle x, \underline{s} \rangle} f(x) E_k^A(-ix, y) \omega_k(x) dx = \frac{1}{c_k} \mathcal{L}_k f(\underline{s} + iy), \ y \in \mathbb{R}^n.$$

From this relation, our assumption and the L^1-inversion theorem for the Dunkl transform it follows that F has a continuous representative with

$$c_k^2 F(x) = c_k \int_{\mathbb{R}^n} \widehat{F}^{\,k}(y) E_k^A(ix, y) \omega_k(y) dy = \int_{\mathbb{R}^n} \mathcal{L}_k f(\underline{s} + iy) E_k^A(iy, x) \omega_k(y) dy.$$

Hence $\widetilde{f}$ has a continuous representative as well, satisfying

$$\widetilde{f}(x) = \frac{1}{c_k^2} \int_{\mathbb{R}^n} \mathcal{L}_k f(\underline{s} + iy) E_k^A(x, \underline{s} + iy) \omega_k(\underline{s} + iy) dy$$

$$= \frac{(-i)^n}{c_k^2} \int_{\mathrm{Re}\, z = \underline{s}} \mathcal{L}_k f(z) E_k^A(x, z) \omega_k(z) dz.$$

This implies the assertion. $\square$

As an immediate consequence of this theorem, we obtain

Corollary 3.7 (Injectivity of the Laplace transform)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n_+, \omega_k)$ and $s \in \mathbb{R}$ such that $\mathcal{L}_k f(\underline{s})$ exists and $\mathcal{L}_k f(\underline{s} + iy) = 0$ for all $y \in \mathbb{R}^n$. Then $f = 0$ a.e.*

Remark 3.8. Following a method of Yan [Y2], Baker and Forrester [BF2] proved a weaker injectivity result for the Laplace transform $\mathcal{L}_k$ on a certain weighted L^2-space by applying suitable Dunkl operators to the Laplace integral.

4. THE TYPE A LAPLACE TRANSFORM OF TEMPERED DISTRIBUTIONS

In this section, we define the Laplace transform of tempered distributions in the Dunkl setting. We follow the classical approach, see e.g. [V]. Denote by $\mathcal{S}(\mathbb{R}^n)$ the classical Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. Let further

$$\mathcal{S}'_+(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\mathbb{R}^n_+}\}$$

denote the set of tempered distributions supported in $\overline{\mathbb{R}^n_+}$. In order to define the Laplace transform of $u \in \mathcal{S}'_+(\mathbb{R}^n)$, choose a cutoff function $\chi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \chi \subseteq]-\epsilon, \infty[^n$ for some $\epsilon > 0$ and $\chi(x) = 1$ in a neighborhood of $\overline{\mathbb{R}^n_+}$.

Lemma 4.1. For each $z \in H_n(0)$ the function $x \mapsto \chi(x)E_k^A(x, -z)$ belongs to $\mathcal{S}(\mathbb{R}^n)$.

Proof. We use the following estimates from [dJ1] for the partial derivatives of the Dunkl kernel: There are constants $C_\nu > 0$, $\nu \in \mathbb{N}_0^n$, such that for all $x \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$,

$$|\partial_x^\nu E_k^A(x, z)| \leq C_\nu \cdot \|z\|_\infty^{|\nu|} e^{\max_{\sigma \in S_n} \langle \sigma x, \operatorname{Re} z \rangle}. \tag{4.1}$$

A short computation shows that for $x \in \mathbb{R}^n$ with $x > -\underline{\epsilon}$, $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq \underline{s} > 0$ and $\sigma \in S_n$,

$$\langle \sigma x, \operatorname{Re} z \rangle \geq s \cdot \sum_{i=1}^n x_i - \epsilon \cdot \|\operatorname{Re} z - \underline{s}\|_1.$$

Therefore

$$|\partial_x^\nu E_k(x, -z)| \leq C_\nu \|z\|_\infty^{|\nu|} e^{\epsilon\|\operatorname{Re} z - \underline{s}\|_1} \cdot e^{-s \sum_{i=1}^n x_i}. \tag{4.2}$$

This easily implies the assertion. $\qquad\square$

Definition 4.2. The Laplace transform of $u \in \mathcal{S}'_+(\mathbb{R}^n)$ is defined by

$$\mathcal{L}_k u : H_n(0) \to \mathbb{C}, \ \mathcal{L}_k u(z) := \langle u_x, \chi(x)E_k^A(x, -z)\rangle,$$

where the notation u_x indicates that u acts on functions of the variable x, and the cutoff function χ is as above. As u is supported in $\overline{\mathbb{R}^n_+}$, this definition is independent of the choice of χ.

Remark 4.3. If $m \in \mathcal{S}'_+(\mathbb{R}^n)$ is of order zero, i.e. a complex tempered Radon measure supported in $\overline{\mathbb{R}^n_+}$, then its Laplace transform is given by

$$\mathcal{L}_k m(z) = \int_{\overline{\mathbb{R}^n_+}} E_k^A(x, -z)dm(x), \quad z \in H_n(0).$$

The exponential decay properties of E_k^A in Lemma 3.1 together with Morera's theorem imply that $\mathcal{L}_k m$ is holomorphic on $H_n(0)$ and may be differentiated under the integral.

Example 4.4. Denote by δ_x the Dirac distribution in the point $x \in \mathbb{R}^n$. Then $\mathcal{L}_k(\delta_0) = 1$.

Theorem 4.5 (Injectivity of the Laplace transform of tempered distributions)**.** *Let $u \in \mathcal{S}'_+(\mathbb{R}^n)$ and suppose that there is some $s \in]0, \infty[$ such that $\mathcal{L}_k u(\underline{s} + iy) = 0$ for all $y \in \mathbb{R}^n$. Then $u = 0$.*

Proof. Fix a cutoff function χ as above, and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. By Lemma 4.1, the function $\xi \mapsto \chi(\xi) E_k(-\underline{s} - iy, \xi)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ for each $y \in \mathbb{R}^n$, and it is easily checked that

$$\psi(\xi, y) := \chi(\xi) E_k^A(-\underline{s} - iy, \xi)\varphi(y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Consider the weight function ω_k as a regular tempered distribution on $\mathbb{R}^n$ in the usual way. Then by Fubini's theorem for tensor products of tempered distributions (see e.g. [V, Section 5.5]),

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n} \mathcal{L}_k u(\underline{s} + iy)\varphi(y)\,\omega_k(y)dy \\
&= \int_{\mathbb{R}^n} \left\langle u_\xi, \chi(\xi) E_k^A(-\underline{s} - iy, \xi)\right\rangle \varphi(y)\,\omega_k(y)dy \\
&= \int_{\mathbb{R}^n} \langle u_\xi, \psi(\xi, y)\rangle\,\omega_k(y)dy = \langle u \otimes \omega_k, \psi\rangle = \left\langle u_\xi, \int_{\mathbb{R}^n} \psi(\xi, y)\,\omega_k(y)dy\right\rangle \\
&= \left\langle u_\xi, \chi(\xi)e^{-\langle \underline{s}, \xi\rangle} \int_{\mathbb{R}^n} E_k^A(-iy, \xi)\varphi(y)\,\omega_k(y)dy\right\rangle \\
&= \left\langle e^{-\langle \underline{s}, \cdot\rangle}u, \widehat{\varphi}^{\,k}\right\rangle.
\end{aligned}
$$

Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and the Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$, it follows that $u = 0$. $\qquad\square$

5. Riesz distributions in the type A Dunkl setting

In this section we assume that $k > 0$. We put $\mu_0 := k(n-1)$ and introduce the normalization constant

$$d_n(k) := \prod_{j=1}^n \frac{\Gamma(1 + jk)}{\Gamma(1 + k)} = (2\pi)^{-n/2}c_{k,n}$$

with the Mehta constant $c_{k,n}$ from (2.2), as well as the multivariable Gamma function and generalized Pochhammer symbol

$$\Gamma_n(\mu; k) := \prod_{j=1}^n \Gamma\big(\mu - k(j-1)\big), \quad [\mu]_\lambda^k := \prod_{j=1}^{l(\lambda)} \big(\mu - k(j-1)\big)_{\lambda_j}$$

for $\mu \in \mathbb{C}$ and partitions $\lambda \in \Lambda_n^+$. Here $(a)_n = a(a+1)\cdots(a+n-1)$. Notice that the pole set of $\Gamma_n(\,.\,; k)$ is given by $\{0, k, \ldots, k(n-1)\} - \mathbb{N}_0$.

Before turning to Riesz distributions, we provide some Laplace transform formulas which will be useful in the sequel. Recall the Jack polynomials C_λ^α in n variables with $\alpha > 0$. It will be convenient to work with the renormalized polynomials

$$\widetilde{C}_\lambda^\alpha(z) := \frac{C_\lambda^\alpha(z)}{C_\lambda^\alpha(\underline{1})}, \quad \underline{1} = (1, \ldots, 1) \in \mathbb{R}^n.$$

Recall our notation $D(x) = \prod_{i=1}^n x_i$ for $x \in \mathbb{R}^n$. We need the following integral formula which is due to Macdonald.

Lemma 5.1 ([M3], formula (6.18)). *For $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \mu_0$ and $\lambda \in \Lambda_n^+$,*

$$\int_{\mathbb{R}_+^n} \widetilde{C}_\lambda^{1/k}(x)e^{-\langle x, \underline{1}\rangle} D(x)^{\mu - \mu_0 - 1}\omega_k(x)dx = d_n(k)\Gamma_n(\mu; k) \cdot [\mu]_\lambda^k. \qquad (5.1)$$

For $\lambda = 0$ this becomes

$$\int_{\mathbb{R}^n_+} e^{-\langle x, \underline{1}\rangle} D(x)^{\mu-\mu_0-1}\omega_k(x)dx = d_n(k)\Gamma_n(\mu; k).$$

Formula (5.1) was deduced in [M3] from the Kadell integral ([K], see also [FW, (2.46)]),

$$\int_{[0,1]^n} \widetilde{C}_\lambda^{1/k}(y)D(y)^{\mu-\mu_0-1}D(\underline{1}-y)^{\nu-\mu_0-1}\omega_k(y)dy = d_n(k)\cdot\frac{\Gamma_n(\mu; k)\Gamma_n(\nu; k)}{\Gamma_n(\mu+\nu; k)}\cdot\frac{[\mu]_\lambda^k}{[\mu+\nu]_\lambda^k}$$

by putting $y_j = \frac{x_j}{\nu-\mu_0-1}$ and taking the limit $\nu \to \infty$. We mention that in a similar way, the classical Mehta integral $c_{k,n}$ had been evaluated by Bombieri, c.f. [M1].

Theorem 5.2. *Let $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu > \mu_0$ and $z \in H_n(0)$. Then*

$$\int_{\mathbb{R}^n_+} E_k^A(-x, z)D(x)^{\mu-\mu_0-1}\omega_k(x)dx = d_n(k)\Gamma_n(\mu; k) \cdot D(z)^{-\mu}, \qquad (5.2)$$

where

$$D(z)^a := \prod_{j=1}^n z_j^a \quad \text{for } a \in \mathbb{C}$$

and $\zeta \mapsto \zeta^a$ denotes the principal branch of the power function on $\mathbb{C}\backslash]-\infty, 0]$, satisfying $1^a = 1$.

Note that the Laplace integral in (5.2) indeed converges and defines a holomorphic function on $H_n(0)$, as a consequence of the decay properties of E_k^A.

Proof of Theorem 5.2. As D and ω_k are S_n-invariant, it suffices to prove (5.2) with E_k^A replaced by J_k^A. Consider first $z \in \mathbb{C}^n$ with $\|z\|_\infty < \frac{\epsilon}{n}$, $0 < \epsilon < 1$, and put $\alpha := 1/k$. By the factorization of J_k^A according to Lemma 3.1(1), its hypergeometric expansion (2.5) as well as Lemma 5.1, we obtain

$$\int_{\mathbb{R}^n_+} J_k^A(-x, z+\underline{1})D(x)^{\mu-\mu_0-1}\omega_k(x)dx$$

$$= \int_{\mathbb{R}^n_+} \Big(\sum_{\lambda\in\Lambda_n^+} \frac{C_\lambda^\alpha(-z)}{|\lambda|!}\widetilde{C}_\lambda^\alpha(x)\Big) e^{-\langle x,\underline{1}\rangle} D(x)^{\mu-\mu_0-1}\omega_k(x)dx$$

$$= \sum_\lambda \frac{C_\lambda^\alpha(-z)}{|\lambda|!}\int_{\mathbb{R}^n_+} \widetilde{C}_\lambda^\alpha(x)\, e^{-\langle x,\underline{1}\rangle} D(x)^{\mu-\mu_0-1}\omega_k(x)dx$$

$$= d_n(k)\Gamma_n(\mu; k)\cdot\sum_\lambda [\mu]_\lambda^k\frac{C_\lambda^\alpha(-z)}{|\lambda|!}.$$

Here the interchange of the summation with the integral is justified by the dominated convergence theorem, since

$$\sum_\lambda \frac{1}{|\lambda|!}|C_\lambda^\alpha(-z)\widetilde{C}_\lambda^\alpha(x)| \leq \sum_{m=0}^\infty \frac{1}{m!}(n\|z\|_\infty\|x\|_\infty)^m = e^{n\|z\|_\infty\|x\|_\infty} \leq e^{\epsilon\langle x,\underline{1}\rangle}$$

where estimate (2.8) has been used. It is known from [Kan] that for each $a \in \mathbb{C}$, the hypergeometric series

$$\sum_{\lambda\in\Lambda_n^+} [a]_\lambda^k\frac{C_\lambda^\alpha(z)}{|\lambda|!} =: {}_1F_0^\alpha(a; z)$$

converges absolutely for $\|z\|_\infty < \rho$, provided $\rho \in]0,1[$ is small enough. Moreover, in this case Yan's [Y2, Prop. 3.1] binomial formula states that

$$_1F_0^\alpha(a;z) = D(\underline{1}-z)^{-a}. \tag{5.3}$$

Thus for $\|z\|_\infty$ small enough, we obtain that

$$\int_{\mathbb{R}_+^n} J_k^A(-x, z+\underline{1})D(x)^{\mu-\mu_0-1}\omega_k(x)dx = d_n(k)\Gamma_n(\mu;k)D(z+\underline{1})^{-\mu},$$

and the general statement follows by analytic continuation with respect to z. $\square$

We now proceed to the definition of Riesz distributions associated with root systems of type A. For a parameter $\mu \in \mathbb{C}$, define $g_\mu \in L^1_{\text{loc}}(\mathbb{R}_+^n, \omega_k)$ by

$$g_\mu(x) := \frac{1}{d_n(k)\Gamma_n(\mu;k)} \cdot D(x)^{\mu-\mu_0-1}, \quad x \in \mathbb{R}_+^n.$$

Theorem 5.2 says that for $\operatorname{Re}\mu > \mu_0$, the Laplace transform $\mathcal{L}_k g_\mu$ exists on $H_n(0)$ and is given by

$$\mathcal{L}_k g_\mu(z) = D(z)^{-\mu}. \tag{5.4}$$

Definition 5.3. For $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \mu_0$ define the (type A) Riesz measure $R_\mu \in M(\mathbb{R}^n)$ by

$$\langle R_\mu, \varphi \rangle := \int_{\mathbb{R}_+^n} \varphi(x)g_\mu(x)\omega_k(x)dx$$

$$= \frac{1}{d_n(k)\Gamma_n(\mu;k)} \int_{\mathbb{R}_+^n} \varphi(x)D(x)^{\mu-\mu_0-1}\omega_k(x)dx, \ \varphi \in C_c(\mathbb{R}^n).$$

We shall regard the complex Radon measure R_μ also as a tempered distribution on $\mathbb{R}^n$ with support $\overline{\mathbb{R}_+^n}$. Notice that the mapping $\mu \to R_\mu$ is holomorphic on $\{\mu \in \mathbb{C} : \operatorname{Re}\mu > \mu_0\}$ with values in $\mathcal{S}'(\mathbb{R}^n)$, i.e. $\mu \mapsto \langle R_\mu, \varphi \rangle$ is holomorphic for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

The Riesz measures R_μ have already been introduced in the (unpublished) thesis [L]. There also the subsequent Bernstein identity as well as Corollary 5.6 concerning the distributional extension of the Riesz measures with respect to μ were proven. For the reader's convenience, we shall nevertheless include proofs of these results, where our proof of the Bernstein identity is slightly different from that in [L]. The Bernstein identity is based on the operator

$$D\big(T(k)\big) := \prod_{i=1}^n T_i(k)$$

with $T_i(k)$ the A_{n-1} Dunkl operators. Notice that $D\big(T(k)\big)$ acts as a linear differential operator of order n on $C^r(\mathbb{R}^n)^{\text{rad}}$, the subspace of S_n-invariant functions from $C^r(\mathbb{R}^n)$ with $r \geq n$.

Lemma 5.4 (Bernstein identity). *For $x \in \mathbb{R}_+^n$ and $a \in \mathbb{C}$,*

$$D\big(T(k)\big)D(x)^a = b_k(a)D(x)^{a-1},$$

with

$$b_k(a) = \prod_{i=1}^n (a + k(i-1)).$$

Proof. We claim that for $a \in \mathbb{C}$ and $i = 1, \ldots, n$,

$$T_i(k)\big(D(x)^{a-1} \cdot x_i \cdots x_n\big) = \big(a + k(i-1)\big)D(x)^{a-1} \cdot x_{i+1} \cdots x_n. \qquad (5.5)$$

For the proof of this identity, note that for $f, g \in C^1(\mathbb{R}^n_+)$ with f or g S_n-invariant, the Dunkl operators satisfy the product rule

$$T_i(k)(fg) = T_i(k)f \cdot g + f \cdot T_i(k)g.$$

Therefore

$$T_i(k)\big(D(x)^{a-1} \cdot x_i \cdots x_n\big) = T_i(k)(x_i \cdots x_n) \cdot D(x)^{a-1} + x_i \cdots x_n \cdot \partial_i\big(D(x)^{a-1}\big).$$

Further,

$$T_i(k)(x_i \cdots x_n) = \partial_i(x_i \cdots x_n) + k \sum_{j \neq i} \frac{(x_i \cdots x_n) - \sigma_{ij}(x_i \cdots x_n)}{x_i - x_j}$$

$$= \big(1 + k(i-1)\big) \cdot x_{i+1} \cdots x_n.$$

This gives formula (5.5), from which the statement of the Lemma follows by recursion. $\qquad \square$

Remark 5.5. An alternative, less direct proof of the Bernstein identity can be obtained from the Laplace transform identity (5.4), similar as in [FK], Propos. VII.1.4 for symmetric cones: For $a \in \mathbb{C}$ with $\operatorname{Re} a < -\mu_0$ we have

$$D(x)^a = \mathcal{L}_k g_{-a}(x) \quad \text{for } x \in \mathbb{R}^n_+$$

and thus by Lemma 3.3,

$$D(T(k))D(x)^a = (-1)^n \mathcal{L}_k(D\, g_{-a})(x) = (-1)^n \frac{\Gamma_n(1-a; k)}{\Gamma_n(-a; k)} \mathcal{L}_k(g_{-a+1})(x)$$

$$= b_k(a)D(x)^{a-1}.$$

For general $a \in \mathbb{C}$, the Bernstein identity then follows by analytic extension.

Corollary 5.6. *The mapping* $\mu \mapsto R_\mu$, $\{\operatorname{Re} \mu > \mu_0\} \to \mathcal{S}'(\mathbb{R}^n)$ *extends uniquely to an* $\mathcal{S}'(\mathbb{R}^n)$*-valued holomorphic mapping on* $\mathbb{C}$ *satisfying the recursion*

$$D\big(T(k)\big)R_\mu = R_{\mu-1} \quad (\mu \in \mathbb{C}).$$

Proof. The proof is analogous to the case of symmetric cones, c.f. Chapter VII of [FK]. First notice that

$$\frac{\Gamma_n(\mu; k)}{\Gamma_n(\mu - 1; k)} = b_k(\mu - \mu_0 - 1).$$

For $\operatorname{Re} \mu > \mu_0$ we extend g_μ to a locally integrable, tempered function on $\mathbb{R}^n$ by putting $g_\mu := 0$ on $\mathbb{R}^n \setminus \mathbb{R}^n_+$. If $\operatorname{Re} \mu > \mu_0 + n + 1$, then $g_\mu \in C^n(\mathbb{R}^n)$, and Lemma 5.4 implies that

$$D\big(T(k)\big) g_\mu = g_{\mu-1} \quad \text{on } \mathbb{R}^n.$$

From (2.3) and the skew-symmetry of the Dunkl operators $T_i(k)$ in $L^2(\mathbb{R}^n, \omega_k)$ it now follows that

$$D\big(T(k)\big)R_\mu = R_{\mu-1}. \qquad (5.6)$$

This formula recursively defines tempered distributions R_μ for all $\mu \in \mathbb{C}$ in such a way that the mapping $\mu \mapsto R_\mu$ is holomorphic on $\mathbb{C}$. The uniqueness is clear. $\qquad \square$

Definition 5.7. For a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\sigma \in S_n$ define $u^\sigma \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle u^\sigma, \varphi \rangle := \langle u, \varphi^{\sigma^{-1}} \rangle,$$

where $\varphi^\sigma := \varphi \circ \sigma^{-1}$ for functions $\varphi : \mathbb{R}^n \to \mathbb{C}$.

Lemma 5.8. *The Riesz distributions R_μ, $\mu \in \mathbb{C}$ have the following properties:*

  (1) R_μ *is S_n-invariant, i.e. $R_\mu^\sigma = R_\mu$ for all $\sigma \in S_n$.*
  (2) *The support of R_μ is contained in $\overline{\mathbb{R}_+^n}$, i.e. $R_\mu \in \mathcal{S}'_+(\mathbb{R}^n)$.*
  (3) $D(x)R_\mu = \displaystyle\prod_{j=1}^{n}(\mu - k(j-1)) \cdot R_{\mu+1}.$

All three properties are obvious for $\operatorname{Re}\mu > \mu_0$ and follow for general μ by analytic continuation.

Theorem 5.9. *For all $\mu \in \mathbb{C}$ and $z \in H_n(0)$,*

$$\mathcal{L}_k R_\mu(z) = D(z)^{-\mu}. \tag{5.7}$$

Proof. Let $z \in H_n(0)$. We have

$$\mathcal{L}_k R_\mu(z) = \langle R_\mu, \chi E_k^A(\,.\,, -z) \rangle,$$

where $\chi E_k^A(\,.\,, -z) \in \mathcal{S}(\mathbb{R}^n)$. Thus by the previous corollary, the mapping $\mu \mapsto \mathcal{L}_k R_\mu(z)$ extends analytically to $\mathbb{C}$. On the other hand, we already know that for $\operatorname{Re}\mu > \mu_0$ and $z \in H_n(0)$, $\mathcal{L}_k R_\mu(z) = \mathcal{L}_k g_\mu(z) = D(z)^{-\mu}$. Analytic continuation implies the assertion. $\square$

Corollary 5.10. $R_0 = \delta_0$.

Proof. Theorem 5.9 shows that $\mathcal{L}_k R_0 = 1$ on $H_n(0)$, and the statement follows by Example 4.4 and the injectivity of the Dunkl-type Laplace transform of tempered distributions (Theorem 4.5). $\square$

In the analysis on symmetric cones, there is a famous result by Gindikin characterizing those Riesz distributions which are actually positive measures. Their indices are exactly those belonging to the so-called Wallach set, which plays for example an important role in the study of Hilbert spaces of holomorphic functions on symmetric domains.

Motivated by these facts, we are now going to investigate for which indices the Dunkl type Riesz distributions R_μ are actually (positive) measures. We expect that in analogy to the case of symmetric cones, the distribution R_μ is a positive measure if and only if μ belongs to the generalized Wallach set

$$W_k = \{0, k, \ldots, k(n-1) = \mu_0\} \cup \,]\mu_0, \infty[\,.$$

We know already that R_μ is a positive measure if $\mu = 0$ or if $\mu \in \mathbb{R}$ with $\mu > \mu_0$. In the following theorem, we consider the Wallach points rk with $r \in \{1, 2, \ldots, n-1\}$. We start with some notation. For an integer r with $0 \le r \le n-1$, we denote by $\partial_r(\mathbb{R}_+^n)$ the rank r part of the (stratified) boundary $\partial(\mathbb{R}_+^n)$, which is given by

$$\partial_r(\mathbb{R}_+^n) = \bigcup_{\sigma \in S_n} \big\{ x \in \overline{\mathbb{R}_+^n} : x_{\sigma(r+1)} = \cdots = x_{\sigma(n)} = 0 \big\}.$$

Now fix $r \in \{1, \ldots, n-1\}$ and consider the factorization $\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}$. We write $x \in \mathbb{R}^n$ as

$$x = (x', x'') \text{ with } x' = (x_1, \ldots, x_r), \ x'' = (x_{r+1}, \ldots, x_n).$$

We further introduce the notations

$$D(x') := x_1 \cdots x_r, \quad \omega_k(x') := \prod_{1 \le i < j \le r} |x_i - x_j|^{2k}.$$

Note that $\omega_k(x') = 1$ if $r = 1$.

We denote by R'_μ the Dunkl-type Riesz distribution on $\mathbb{R}^r$ associated with root system A_{r-1} and the same multiplicity $k > 0$. Notice that here $k_0 = k(r-1)$. If $r = 1$, then the Dunkl setting degenerates, and R'_μ coincides with the classical Riesz distribution (also called Riemann-Liouville distribution) on $\mathbb{R}$, which is defined by

$$\langle R'_\mu, \varphi \rangle = \frac{1}{\Gamma(\mu)} \int_{\mathbb{R}_+} \varphi(x) x^{\mu-1} dx \quad \text{for } \operatorname{Re} \mu > 0.$$

This distribution extends holomorphically to all $\mu \in \mathbb{C}$ such that $\frac{d}{dx}(R'_\mu) = R'_{\mu-1}$ for all $\mu \in \mathbb{C}$, c.f. [FJ, Section 2.3].

Theorem 5.11. *For $r \in \{1, \ldots, n-1\}$, the Riesz distribution R_{kr} is a positive Radon measure, namely*

$$R_{kr} = \frac{1}{n!} \cdot \sum_{\sigma \in S_n} (R'_{kn} \otimes \delta''_0)^\sigma, \tag{5.8}$$

where δ''_0 denotes the point measure in $0 \in \mathbb{R}^{n-r}$. The support of R_{kr} is given by $\partial_r(\mathbb{R}^n_+)$.

Proof. Notice first that the Riesz distribution R'_{kn} on $\mathbb{R}^r$ is a positive measure, as $kn > k(r-1)$. Its support is $\overline{\mathbb{R}^r_+}$, and therefore the distribution on the right side of (5.8) is an S_n-invariant positive tempered Radon measure which we denote by m_{kr}. It is clear that $m_{kr} \in \mathcal{S}'_+(\mathbb{R}^n)$ with $\operatorname{supp}(m_{kr}) = \partial_r(\mathbb{R}^n_+)$. By the injectivity of the Laplace transform $\mathcal{L}_k$ on $\mathcal{S}'_+(\mathbb{R}^n)$ (Theorem 4.5) and in view of Theorem 5.9, it suffices to prove that

$$\mathcal{L}_k(m_{kr})(z) = D(z)^{-kr} \quad \forall z \in H_n(0). \tag{5.9}$$

Consider first an arbitrary S_n-invariant test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\langle m_{kr}, \varphi \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \langle (R'_{kn} \otimes \delta''_0)^\sigma, \varphi \rangle = \langle R'_{kn} \otimes \delta''_0, \varphi \rangle =$$

$$= \frac{1}{d_r(k)\Gamma_r(kn;k)} \int_{\mathbb{R}^r_+} \varphi(x', 0) \cdot D(x')^{k(n-r+1)-1} \omega_k(x') dx'. \tag{5.10}$$

Now let $z \in H_n(0)$. As m_{kr} is S_n-invariant, we have

$$\mathcal{L}_k(m_{kr})(z) = \langle m_{kr}, \chi E^A_k(., -z) \rangle = \langle m_{kr}, \chi J^A_k(., -z) \rangle$$

with some S_n-invariant cutoff function $\chi \in C^\infty(\mathbb{R}^n)$ as in the definition of the type-A Laplace transform of distributions. Identity (5.10) therefore implies that

$$\mathcal{L}_k(m_{kr})(z) = \frac{1}{d_r(k)\Gamma_r(kn;k)} \int_{\mathbb{R}^r_+} J^A_k((x', 0), -z) D(x')^{k(n-r+1)-1} \omega_k(x') dx'.$$

The proof of Eq. (5.9) will now be finished by the following Lemma.   □

Lemma 5.12. *Fix $r \in \{1, \ldots, n-1\}$. Then for all $z \in H_n(0)$,*

$$\frac{1}{d_r(k)\Gamma_r(kn,k)} \int_{\mathbb{R}_+^r} J_k^A\big((x',0),-z\big) D(x')^{k(n-r+1)-1} \omega_k(x')dx' = D(z)^{-kr}.$$

Proof. As the number of variables is relevant in this proof, we shall write 1_n for the element $\underline{1} \in \mathbb{R}^n$. Both sides of the stated identity are holomorphic as functions of $z \in H_n(0)$, and therefore it suffices to verify the identity for all arguments of the form $z + 1_n$, $z \in \mathbb{C}^n$ with $\|z\|_\infty < \epsilon$ for some $\epsilon \in]0,1[$. As in the proof of Theorem 5.2, we shall use the identity $J_k^A(x,-z-1_n) = J_k^A(x,-z)e^{-\langle x,1_n \rangle}$ as well as the hypergeometric expansion (2.5) of J_k^A. The Jack polynomials in n variables have the stability property (cf. [Sta, Poposition 2.5] together with [Kan, formula (16)]):

$$C_\lambda^\alpha(z_1, \ldots, z_r, 0, \ldots, 0) = \begin{cases} C_\lambda^\alpha(z_1, \ldots, z_r) & \text{if } l(\lambda) \leq r \\ 0 & \text{otherwise.} \end{cases}$$

Hence with $\alpha = 1/k$,

$$J_k^A\big((x',0),-z\big) = \sum_{\lambda \in \lambda_n^+} \frac{C_\lambda^\alpha(x',0)\,C_\lambda^\alpha(-z)}{|\lambda|!\,C_\lambda^\alpha(1_n)} = \sum_{\lambda \in \Lambda_r^+} \frac{1}{|\lambda|!} C_\lambda^\alpha(x')\,\frac{C_\lambda^\alpha(-z)}{C_\lambda^\alpha(1_n)}.$$

We therefore obtain

$$\begin{aligned} I(z) &:= \int_{\mathbb{R}_+^r} J_k^A\big((x',0),-(z+1_n)\big) D(x')^{k(n-r+1)-1} \omega_k(x')dx' \\ &= \int_{\mathbb{R}_+^r} \Big(\sum_{\lambda \in \Lambda_r^+} \frac{1}{|\lambda|!} \frac{C_\lambda^\alpha(-z)}{C_\lambda^\alpha(1_n)} C_\lambda^\alpha(x')\Big) e^{-\langle x',1_r \rangle} D(x')^{k(n-r+1)-1} \omega_k(x')dx' \\ &= \sum_{\lambda \in \Lambda_r^+} \frac{C_\lambda^\alpha(-z)}{|\lambda|!} \cdot \frac{C_\lambda^\alpha(1_r)}{C_\lambda^\alpha(1_n)} \int_{\mathbb{R}_+^r} \widetilde{C}_\lambda^\alpha(x') e^{-\langle x',1_r \rangle} D(x')^{k(n-r+1)-1} \omega_k(x')dx'. \end{aligned}$$

$$(5.11)$$

Here the interchange of the summation and integral is justified by the dominated convergence theorem for $\|z\|_\infty < \epsilon$ with $\epsilon < \frac{1}{r}$, because by estimate (2.7),

$$\sum_{\lambda \in \Lambda_r^+} \frac{1}{|\lambda|!} \big| \widetilde{C}_\lambda^\alpha(-z)\,C_\lambda^\alpha(x') \big| \leq \sum_{m=0}^\infty \frac{1}{m!} (r\|z\|_\infty \|x'\|_\infty)^m \leq e^{r\|z\|_\infty \|x'\|_1}.$$

For $\lambda \in \Lambda_n^+$ we further have

$$\frac{C_\lambda^\alpha(1_r)}{C_\lambda^\alpha(1_n)} = \frac{[kr]_\lambda^k}{[kn]_\lambda^k},$$

see e.g. [Kan, formula (17)]. Inserting this in (5.11) and evaluating the integral by means of formula (5.1), we obtain (for $\|z\|_\infty$ small enough) that

$$I(z) = d_r(k)\Gamma_r(kn;k) \sum_{\lambda \in \Lambda_r^+} [kr]_\lambda^k \cdot \frac{C_\lambda^\alpha(-z)}{|\lambda|!}. \qquad (5.12)$$

On the other hand, Yan's binomial formula (5.3) gives, for small $\|z\|_\infty$,

$$D(z+1_n)^{-kr} = {}_1F_0^\alpha(kr;-z) = \sum_{\lambda \in \Lambda_n^+} [kr]_\lambda^k \cdot \frac{C_\lambda^\alpha(-z)}{|\lambda|!}.$$

If $l(\lambda) > r$, then

$$[kr]_\lambda^k = \prod_{j=1}^{l(\lambda)} \big(k(r - j + 1)\big)_{\lambda_j} = 0.$$

Therefore

$$D(z + 1_n)^{-kr} = \sum_{\lambda \in \Lambda_r^+} [kr]_\lambda^k \cdot \frac{C_\lambda^\alpha(-z)}{|\lambda|!}.$$

This shows that for $\|z\|_\infty$ sufficiently small,

$$I(z) = d_r(k)\Gamma_r(kn; k) \cdot D(z + 1_n)^{-kr},$$

which finishes the proof of the lemma. $\qquad\qquad\Box$

We mention that some partial results on the distributions R_{kr} are sketched in [L].

We are now aiming at necessary conditions under which the Riesz distribution R_μ is a complex or even a positive measure. We know already that R_μ is a positive measure if μ belongs to the generalized Wallach set W_k, and that it is a complex (non-positive) measure if $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Re}\mu > \mu_0 = k(n - 1)$. We shall use methods of Sokal [So] for Riesz distributions on symmetric cones, as well as the following variant of a principle due to Shanbhag, Casalis and Letac (see [CL] as well as [So]):

Lemma 5.13 (Shanbhag-Casalis-Letac prinicple in the Dunkl setting). *Suppose that $m \in \mathcal{S}_+'(\mathbb{R}^n)$ is a positive tempered Radon measure and $p \in \mathbb{C}[\mathbb{R}^n]$ is a polynomial which is real-valued and non-negative on $\operatorname{supp} m$, then*

$$p\big(-T(k)\big)\mathcal{L}_k m \geq 0 \ \ on \ \ \mathbb{R}_+^n.$$

Proof. In view of Remark 4.3 we have $\mathcal{L}_k m \in \mathbb{C}^\infty(\mathbb{R}_+^n)$ and may differentiate under the integral. Thus for $x \in \mathbb{R}_+^n$,

$$p\big(-T(k)\big)\mathcal{L}_k m(x) = \int_{\mathbb{R}_+^n} p(y)E_k^A(y, -x)dm(y) \geq 0.$$

$\qquad\qquad\Box$

Recall from Theorem 5.9 that for each $\mu \in \mathbb{C}$,

$$\mathcal{L}_k R_\mu(x) = D(x)^{-\mu} \ \ \text{on} \ \ \mathbb{R}_+^n.$$

The following evalutation formula involving Dunkl operators associated with the (renormalized) Jack polynomials will be crucial; it is an analogue of the formula on top of p. 245 in [FK] for the spherical functions on symmetric cones.

Lemma 5.14. *Let $\alpha = 1/k$. Then for $\mu \in \mathbb{C}$ and each partition λ of length $l(\lambda) \leq n$,*

$$\widetilde{C}_\lambda^\alpha(T(k))D(\underline{1} - x)^{-\mu}\big|_{x=0} = [\mu]_\lambda^k. \qquad\qquad (5.13)$$

Proof. Suppose first that $\operatorname{Re}\mu > \mu_0$. Then according to Theorem 5.2,

$$D(\underline{1} - x)^{-\mu} = \frac{1}{d_n(k)\Gamma_n(\mu; k)} \int_{\mathbb{R}_+^n} E_k^A(-y, \underline{1} - x)D(y)^{\mu - \mu_0 - 1}\omega_k(y)dy$$

$$= \frac{1}{d_n(k)\Gamma_n(\mu; k)} \int_{\mathbb{R}_+^n} E_k^A(y, x)e^{-\langle y, \underline{1}\rangle} D(y)^{\mu - \mu_0 - 1}\omega_k(y)dy.$$

Differentiating under the integral gives

$$\widetilde{C}^\alpha_\lambda(T(k))D(\underline{1}-x)^{-\mu} = \frac{1}{d_n(k)\Gamma_n(\mu;k)} \int_{\mathbb{R}^n_+} \widetilde{C}^\alpha_\lambda(y)E^A_k(y,x)e^{-\langle y,\underline{1}\rangle}D(y)^{\mu-\mu_0-1}\omega_k(y)dy.$$

Thus by Lemma 5.1,

$$\widetilde{C}^\alpha_\lambda(T(k))D(\underline{1}-x)^{-\mu}\big|_{x=0} = \frac{1}{d_n(k)\Gamma_n(\mu;k)} \int_{\mathbb{R}^n_+} \widetilde{C}^\alpha_\lambda(y)\, e^{-\langle y,\underline{1}\rangle}D(y)^{\mu-\mu_0-1}\omega_k(y)dy$$

$$= [\mu]^k_\lambda.$$

Both sides of formula (5.13) are holomorphic in $\mu \in \mathbb{C}$ (for the left side note that $\widetilde{C}^\alpha_\lambda(T(k))$ acts as a differential operator on S_n-invariant functions), and therefore the stated identity extends to all $\mu \in \mathbb{C}$. □

Theorem 5.15. *Consider the Riesz distributions R_μ, $\mu \in \mathbb{C}$.*

(1) *If R_μ is a complex measure, then either $\mathrm{Re}\,\mu > \mu_0 = k(n-1)$, or μ is contained in the finite set*

$$[0,\infty[\,\cap\,\big(\{0,k,\ldots k(n-1)\} - \mathbb{N}_0\,\big).$$

(2) *R_μ is a positive measure if and only if μ is contained in the generalized Wallach set $W_k = \{0,k,\ldots,k(n-1)\}\cup\,]k(n-1),\infty[.$*

Proof. (1) We shall apply Proposition 2.3 of [So]. For this, consider the regular distributions $u_\mu \in \mathcal{D}'(\mathbb{R}^n_+)$, $\mu \in \mathbb{C}$, which are defined by the densities

$$f_\mu(x) = \frac{1}{d_n(k)\Gamma_n(\mu;k)} \cdot D(x)^{\mu-\mu_0-1}\omega_k(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+), \tag{5.14}$$

that is

$$\langle u_\mu, \varphi\rangle = \int_{\mathbb{R}^n_+} \varphi(x)f_\mu(x)dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^n_+).$$

Notice that $\mu \mapsto f_\mu(x)$ is holomorphic on $\mathbb{C}$ for each $x \in \mathbb{R}^n_+$ and that $\mu \mapsto u_\mu$ is holomorphic on $\mathbb{C}$ with values in $\mathcal{D}'(\mathbb{R}^n_+)$. If $\mathrm{Re}\,\mu > \mu_0$, then the Riesz distribution R_μ provides an extension of the distribution u_μ to a distribution on $\mathbb{R}^n$ in the sense that the restriction of R_μ to $\mathbb{R}^n_+$ coincides with u_μ. Moreover, in this case R_μ is also a complex measure, i.e. of order zero. In addition we know that the mapping $\mu \mapsto R_\mu$ is holomorphic on all of $\mathbb{C}$. We are therefore in the situation of [So, Proposition 2.3], which yields the following conclusions: First, R_μ extends u_μ for each $\mu \in \mathbb{C}$, and second, if R_μ is a complex measure on $\mathbb{R}^n$, then the density f_μ (extended by zero to all of $\mathbb{R}^n$) must belong to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. But this implies that either $\mathrm{Re}\,\mu > \mu_0$ or μ is a pole of $\Gamma_n(\,.\,;k)$, i.e. $\mu \in \{0,k,\ldots k(n-1)\} - \mathbb{N}_0$. (Note that in the latter case, f_μ is identical zero.) If $\mu \in\,]-\infty,0[$, then it follows from the Laplace transform formula (5.7) that R_μ cannot be a measure. Indeed, suppose that R_μ is a (tempered) measure. Then for $x \in \mathbb{R}$ with $x > 0$ we have

$$\mathcal{L}_k R_\mu(\underline{x}) = \int_{\mathbb{R}^n_+} E^A_k(y,-\underline{x})dR_\mu(y) = \int_{\mathbb{R}^n_+} e^{-\langle y,\underline{x}\rangle}dR_\mu(y),$$

which is a usual Laplace transform. Thus by Lemma 3.6. of [So], $x \mapsto \mathcal{L}_k R_\mu(\underline{a}+\underline{x})$ is bounded on $[0,\infty[$ for each $a > 0$. But on the other hand, $\mathcal{L}_k R_\mu(\underline{a}+\underline{x}) = (a+x)^{-n\mu}$, which is unbounded as $x \to \infty$.

(2) In view of Corollary 5.10 and Theorem 5.11 it remains to prove the "only if" part. Suppose that R_μ is a positive measure. We have to exclude the possibility

that μ belongs to one of the open intervalls $]k(r-1), kr[$ with $r \in \{1, \ldots, n-1\}$. For this, we apply the Shanbhag-Casalis-Letac principle to R_μ and the Jack polynomials $\widetilde{C}_\lambda^\alpha$, which are non-negative on $\overline{\mathbb{R}_+^n}$ as a consequence of their non-negative monomial expansion (2.6). Thus by Lemma 5.13,

$$\widetilde{C}_\lambda^\alpha(-T(k))(\mathcal{L}_k R_\mu) \geq 0 \quad \text{on } \mathbb{R}_+^n.$$

Employing Lemma 5.14 we therefore obtain that for all $\lambda \in \Lambda_n^+$,

$$[\mu]_\lambda^k = \widetilde{C}_\lambda^\alpha(T(k))D(\underline{1} - x)^{-\mu}|_{x=0} = \left(\widetilde{C}_\lambda^\alpha(-T(k))\mathcal{L}_k R_\mu\right)(\underline{1}) \geq 0. \tag{5.15}$$

Here for the second equality, it was used that for $f \in C^1(\mathbb{R}^n)$ and $g(x) := f(\underline{1} - x)$, the Dunkl operators satisfy $(T_i(k)f)(\underline{1} - x) = -T_i(k)g(x)$. Now suppose that $\mu \in \,]k(r - 1), kr[, \; r \in \{1, \ldots, n - 1\}$. Choose $\lambda := (1, \ldots, 1, 0, \ldots) \in \Lambda_n^+$ with exactly $r + 1$ parts equal to 1. Then

$$[\mu]_\lambda^k = \prod_{j=1}^{r+1}(\mu - k(j - 1)) \, < 0,$$

in contradiction to (5.15). $\qquad \square$

Remark 5.16. We mention that part (2) can be proven directly without referring to part (1) and Proposition 2.3 of [So]. Indeed, if R_μ is a positive measure, then $\mathcal{L}_k R_\mu$ is non-negative on $\mathbb{R}_+^n$ and thus by Theorem 5.9, μ must be real-valued. Then it remains to exclude the intervals $]k(r - 1), kr[$ as above.

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