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# The Heat Semigroup in the Compact Heckman–Opdam Setting and the Segal–Bargmann Transform

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In the first part of this paper, we study the heat equation and the heat kernel associated with the Heckman–Opdam Laplacian in the compact, Weyl-group invariant setting. In particular, this Laplacian gives rise to a Feller–Markov semigroup on a fundamental alcove of the affine Weyl group. The second part of the paper is devoted to the Segal–Bargmann transform in our context. A Hilbert space of holomorphic functions is defined such that the $L^2$-heat transform becomes a unitary isomorphism.

## 1 Introduction

Heckman–Opdam theory provides a powerful generalization of the theory of non-compact and compact Riemannian symmetric spaces and their spherical functions. In this theory (see e.g. [5, 10, 11]), the system of invariant differential operators on a Riemannian symmetric space is replaced by a commuting algebra of differential reflection operators, called Dunkl operators, which depend on some root system and on multiplicity parameters on the roots. The joint spectral problem for these operators is solved by multivariable hypergeometric functions and hypergeometric polynomials which include the spherical functions of Riemannian symmetric spaces for certain discrete values of the multiplicities.

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In the context of Dunkl operators, the heat equation has already been studied in various settings. The rational case was treated by Rösler in [12], while Schapira [13] studied the heat equation in the noncompact Heckman–Opdam theory. In the present paper, we investigate the compact symmetric case, where we assume invariance under the Weyl group $W$. We are concerned with the heat equation for the Heckman–Opdam Laplacian $L_m$ on a closed fundamental alcove $A_0$ for the affine Weyl group. This Laplacian generalizes the Laplace–Beltrami operator on a Riemannian symmetric space of compact type. We prove that $L_m$ has a closure that generates a Feller–Markov semigroup on the alcove, the Heckman–Opdam heat semigroup. We study smoothness properties of the heat kernel and also develop an $L^p$-theory for the heat equation on $A_0$.

The second main topic of this paper is the Segal–Bargmann transform. Several generalizations of the classical Segal-Bargmann transform to different settings are known. The Segal-Bargmann transform for compact Lie groups was introduced by Hall [4], where also the case of compact symmetric spaces was considered. Different approaches in the case of compact symmetric spaces were given by Stenzel [16] and Faraut [3]. In the framework of Dunkl theory, the rational case has been studied by several authors, see [1, 14, 15]. Apart from the rank 1 case, an explicit description of the Segal–Bargmann space as an $L^2$-space of holomorphic functions has so far not been found in this setting. Ben Saïd and Ørsted [1] instead gave a description as a Fock space generated by a certain reproducing kernel (which is given by the rational Dunkl kernel). The noncompact, symmetric Heckman–Opdam case was investigated in 2007 by Ólafsson and Schlichtkrull [9].

In this paper, we study the Segal–Bargmann transform in the compact symmetric Heckman–Opdam setting. We extend the heat transform to a unitary isomorphism from the weighted $L^2$-space on the alcove $A_0$ to a Segal–Bargmann space $\mathcal{H}_t$, which is a Hilbert space of holomorphic functions. Its inner product is described as an $L^2$-product, involving the heat kernel from the noncompact theory as a weight.

The organization of this paper is as follows: in Section 2, we recall some basics of trigonometric Dunkl theory. In Section 3, the heat equation and the heat semigroup on the fundamental alcove $A_0$ are studied. Finally, the Segal–Bargmann transform is developed in Section 4.

## 2 Fundamentals of Trigonometric Dunkl Theory

We start with a short review of the fundamentals of trigonometric Dunkl theory which will be needed in this article. For details, we refer to the work of Heckman and Opdam [5, 10, 11], and the references cited there.

Let $\mathfrak{a}$ be a finite-dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$, which is extended to a complex bilinear form on the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$. We identify $\mathfrak{a}$ with its dual space $\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ via the given inner product. Let $\Sigma \subset \mathfrak{a}$ be a (not necessarily reduced) root system. For $\alpha \in \Sigma$ we write $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ for the coroot of $\alpha$ and denote by $s_\alpha(x) = x - \langle\alpha^\vee, x\rangle\alpha$ the reflection in the hyperplane $H_\alpha$ perpendicular to $\alpha$.

The reflections $\{s_\alpha : \alpha \in \Sigma\}$ generate the Weyl group $W = W(\Sigma)$. We define the root lattice $Q := \mathbb{Z}.\Sigma$ and the coroot lattice $Q^\vee = \mathbb{Z}.\Sigma^\vee$. Further, we fix some positive subsystem $\Sigma^+$ of $\Sigma$. An element $\lambda \in \mathfrak{a}$ is called (strictly) dominant, if $\langle\lambda,\alpha\rangle \geq 0$ (respectively, $>0$) for all $\alpha \in \Sigma^+$. We write

$$\mathfrak{a}^+ := \{\lambda \in \mathfrak{a} : \langle\lambda, \alpha^\vee\rangle > 0 \ \forall \alpha \in \Sigma^+\}$$

for the Weyl chamber of strictly dominant elements.

For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}$, let

$$\lambda_\alpha := \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}.$$

The *weight lattice* is given by

$$\Lambda := \{\lambda \in \mathfrak{a} : \lambda_\alpha \in \mathbb{Z} \ (\forall \alpha \in \Sigma)\},$$

and the set

$$\Lambda^+ := \{\lambda \in \mathfrak{a} : \lambda_\alpha \in \mathbb{Z}^+ \ (\forall \alpha \in \Sigma^+)\}$$

is called the lattice of dominant weights. Here we use the notation $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$. The positive root lattice $Q^+ = \mathbb{Z}^+.\Sigma^+$ defines a partial ordering $\preceq$ on $\mathfrak{a}$:

$$\mu \preceq \lambda \iff \lambda - \mu \in Q^+.$$

This ordering is called the dominance ordering. Two simple properties are given in the following lemma.

**Lemma 2.1.**

(i) Let $\gamma \in \overline{\mathfrak{a}^+}$ be dominant. Then $w\gamma \preceq \gamma$ for all $w \in W$.

(ii) Let $\lambda, \mu \in \Lambda^+$ be dominant weights with $\mu \preceq \lambda$. Then $|\mu| \leq |\lambda|$. □

**Proof.** Part (i) is Lemma 10.3B in [7]. For the proof of (ii), notice that $\lambda+\mu$ is also dominant and $\lambda-\mu$ is a sum of positive roots. Therefore,

$$0 \le \langle \lambda+\mu, \lambda-\mu\rangle = |\lambda|^2 - |\mu|^2. \qquad \blacksquare$$

A *multiplicity function* is a $W$-invariant map $m : \Sigma \to \mathbb{C}$, $\alpha \mapsto m_\alpha$. We denote the set of multiplicity functions by $\mathcal{M}$. In this article, we consider only nonnegative multiplicities, that is, $m_\alpha \ge 0$ for all $\alpha \in \Sigma$. Define

$$\rho = \rho(m) := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

**Definition 2.2.** Let $\xi \in \mathfrak{a}_{\mathbb{C}}$ and $m \in \mathcal{M}$. The *Dunkl–Cherednik operator* associated with $\Sigma$ and $m$ is given by

$$T_\xi = T(\xi, m) := \partial_\xi + \sum_{\alpha \in \Sigma^+} m_\alpha \langle \alpha, \xi\rangle \frac{1}{1-\mathrm{e}^{-2\alpha}} (1 - s_\alpha) - \langle \rho, \xi\rangle,$$

where $\partial_\xi$ is the usual directional derivative and $\mathrm{e}^{\lambda}(\xi) := \mathrm{e}^{\langle \lambda, \xi\rangle}$ for $\lambda, \xi \in \mathfrak{a}_{\mathbb{C}}$. $\square$

**Remark 2.3.** Heckman and Opdam use a slightly different notation. They consider a root system $R$ with multiplicity $k$, which is connected to our notation via

$$R = 2\Sigma, \quad k_{2\alpha} = \frac{1}{2} m_\alpha.$$

Our notation comes from the theory of symmetric spaces. $\square$

For fixed multiplicity $m$, the operators $T_\xi$, $\xi \in \mathfrak{a}_{\mathbb{C}}$ commute. Therefore, the assignment $\xi \mapsto T(\xi, m)$ uniquely extends to a homomorphism on the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ over $\mathfrak{a}_{\mathbb{C}}$, which may be identified with the algebra of complex polynomials on $\mathfrak{a}_{\mathbb{C}}$. Let $T(p, m)$ be the operator which in this way corresponds to $p \in S(\mathfrak{a}_{\mathbb{C}})$. If $p \in S(\mathfrak{a}_{\mathbb{C}})^W$, the subspace of $W$-invariant polynomials on $\mathfrak{a}_{\mathbb{C}}$, then $T(p, m)$ acts as a differential operator on the space of $W$-invariant analytic functions on $\mathfrak{a}$.

The solution of the joint spectral problem for these differential operators is due to Heckman and Opdam, see [5, 10]:

**Theorem 2.4.** For each fixed spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}$, the so-called hypergeometric system

$$T(p, m)\phi = p(\lambda)\phi \quad \text{for all} \;\; p \in S(\mathfrak{a}_{\mathbb{C}})^W$$

has a unique $W$-invariant solution $\phi = F_\lambda(m; \cdot) = F(\lambda, m; \cdot)$ which is analytic on $\mathfrak{a}$ and satisfies $F_\lambda(m; 0) = 1$. Moreover, there is a $W$-invariant tubular neighborhood $U$ of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbb{C}}$ such that $F$ extends to a (single-valued) holomorphic function $F : \mathfrak{a}_{\mathbb{C}} \times \mathcal{M}^{\text{reg}} \times U \to \mathbb{C}$. □

The function $F(\lambda, m; x)$ is $W$-invariant in both $\lambda$ and $x$. It is called the *hypergeometric function* associated with $\Sigma$. For certain spectral parameters $\lambda$, the functions $F_\lambda$ are actually trigonometric polynomials, the so-called Heckman–Opdam polynomials. In order to make this precise, we need some more notation.

Let $\mathcal{T} := \text{lin}\{e^{i\lambda} : \lambda \in \Lambda\}$ be the space of trigonometric polynomials associated with $\Lambda$. Trigonometric polynomials are $\pi Q^\vee$-periodic, and $T_\xi \mathcal{T} \subset \mathcal{T}$. Consider the torus $T = \mathfrak{a}/\pi Q^\vee$ with the $W$-invariant weight function

$$w_m := \prod_{\alpha \in \Sigma^+} |e^{i\alpha} - e^{-i\alpha}|^{m_\alpha}.$$

Let

$$M_\lambda := \sum_{\mu \in W.\lambda} e^{i\mu}, \qquad \lambda \in \Lambda^+$$

denote the $W$-invariant orbit sums. They form a basis of the space of $W$-invariant trigonometric polynomials $\mathcal{T}^W$. For $\lambda \in \Lambda^+$, the *Heckman–Opdam polynomials* associated with $\Sigma$ are defined by

$$P_\lambda = P_\lambda(m; \cdot) := \sum_{\mu \in \Lambda^+, \mu \preceq \lambda} c_{\lambda\mu}(m) M_\mu$$

where the coefficients $c_{\lambda\mu}(m)$ are uniquely determined by the conditions

(i) $c_{\lambda\lambda}(m) = 1$

(ii) $P_\lambda$ is orthogonal to $M_\mu$ in $L^2(T; w_m)$ for all $\mu \in \Lambda^+$ with $\mu \prec \lambda$.

The Jacobi polynomials $P_\lambda$ form an orthogonal basis of $L^2(T, w_m)^W$, the subspace of $W$-invariant elements from $L^2(T, w_m)$.

**Remark 2.5.** Notice that our notation slightly differs from that of Heckman and Opdam [5, 11], namely by a factor $i$ in the spectral variable. This choice of notation will be more convenient for our purposes. □

The connection between the Jacobi polynomials and the hypergeometric function is as follows:

**Lemma 2.6** (see [5]). For all $z \in \mathfrak{a}_{\mathbb{C}}$ and $\lambda \in \Lambda^+$,

$$P_\lambda(m; z) = c(\lambda + \rho, m)^{-1} F_{\lambda+\rho}(m; iz),$$

where the $c$-function $c(\lambda + \rho, m) = P_\lambda(m; 0)^{-1}$ is given by

$$c(\lambda + \rho, m) = \prod_{\alpha \in \Sigma^+} \frac{\Gamma(\lambda_\alpha + \rho_\alpha + \frac{1}{4}m_{\alpha/2})\Gamma(\rho_\alpha + \frac{1}{4}m_{\alpha/2} + \frac{1}{2}m_\alpha)}{\Gamma(\lambda_\alpha + \rho_\alpha + \frac{1}{4}m_{\alpha/2} + \frac{1}{2}m_\alpha)\Gamma(\rho_\alpha + \frac{1}{4}m_{\alpha/2})}.$$ □

We shall work with the renormalized Jacobi polynomials, defined by

$$R_\lambda(z) := R_\lambda(m; z) := c(\lambda + \rho, m) P_\lambda(m; z) = F_{\lambda+\rho}(m; iz) =: F_{\lambda+\rho}(iz).$$

They satisfy

$$R_\lambda(0) = 1.$$

Dividing the torus $T = \mathfrak{a}/\pi Q^\vee$ by the action of the Weyl group $W$ gives the closed fundamental alcove

$$A_0 = \{x \in \mathfrak{a} : 0 \le \langle \alpha, x \rangle \le \pi \quad (\forall \alpha \in \Sigma^+)\}.$$

We may consider $W$-invariant trigonometric polynomials as functions on $A_0$. Another way of considering a $W$-invariant and $\pi Q^\vee$-periodic function $f$ on $\mathfrak{a}$ is to say that $f$ is $W_{\mathrm{aff}}$-invariant, where

$$W_{\mathrm{aff}} = \pi Q^\vee \rtimes W$$

is the affine Weyl group. The closed alcove $A_0$ is a fundamental domain for the action of $W_{\mathrm{aff}}$ on $\mathfrak{a}$.

The Jacobi polynomials $R_\lambda$ are orthogonal with respect to the inner product

$$\langle f, g\rangle_m = \int_{A_0} f(x)\overline{g(x)} w_m(x)\,\mathrm{d}x,$$

but they are not orthonormal. We put

$$r_\lambda := \frac{1}{\|R_\lambda\|_m^2}.$$

Then the set $\{\sqrt{r_\lambda}R_\lambda : \lambda \in \Lambda^+\}$ is an orthonormal basis of $L^2(A_0, w_m)$.

**Remark 2.7.** We shall need the following facts about the Jacobi polynomials $P_\lambda$ and $R_\lambda$:

(a) The $L^2(A_0)$-norm of $P_\lambda$ is given by

$$\|P_\lambda\|_m^2 = \prod_{\alpha\in\Sigma^+} \frac{\Gamma(\lambda_\alpha + \rho_\alpha - \frac{1}{4}m_{\alpha/2} - \frac{1}{2}m_\alpha + 1)}{\Gamma(\lambda_\alpha + \rho_\alpha - \frac{1}{4}m_{\alpha/2} + 1)} \cdot \frac{\Gamma(\lambda_\alpha + \rho_\alpha + \frac{1}{4}m_{\alpha/2} + \frac{1}{2}m_\alpha)}{\Gamma(\lambda_\alpha + \rho_\alpha + \frac{1}{4}m_{\alpha/2})}$$

see Theorem 3.5.5 in [5]. (Notice that $P_\lambda$ is $W$-invariant.)

(b) The coefficients $c_{\lambda\mu}(m)$ of the $P_\lambda$ are rational functions in $m_\alpha$, $\alpha \in \Sigma^+$. Moreover, their numerator and denominator polynomials have nonnegative integral coefficients. This was observed in [8], Par. 11. As a consequence, the renormalized polynomial $R_\lambda$ is, for nonnegative $m$, a convex combination of exponentials $\mathrm{e}^{\mathrm{i}\gamma}$:

$$R_\lambda = \sum_{\substack{\gamma\in W.\mu \\ \mu\in\Lambda^+, \mu\preceq\lambda}} d_{\lambda,\gamma}\, \mathrm{e}^{\mathrm{i}\gamma}$$

with coefficients $d_{\lambda,\gamma} \geq 0$ and $\sum_\gamma d_{\lambda,\gamma} = 1$.

(c) Because of $\overline{\mathrm{e}^{\mathrm{i}\langle\mu,x\rangle}} = \mathrm{e}^{-\mathrm{i}\langle\mu,x\rangle}$, we have

$$\overline{R_\lambda(x)} = R_\lambda(-x), \quad x \in \mathfrak{a},$$

and more general for $z \in \mathfrak{a}_\mathbb{C}$:

$$\overline{R_\lambda(-\bar{z})} = R_\lambda(z), \quad \overline{R_\lambda(\bar{z})} = R_\lambda(-z). \qquad \square$$

## 3 The Heat Equation on the Alcove

In this section, we consider the $W$-invariant part of the Heckman–Opdam Laplacian on the alcove, which coincides with the radial part of the Laplace Beltrami operator of a compact symmetric space $U/K$ in geometric cases. We study the associated heat semigroup—the *Heckman–Opdam heat semigroup*—and its integral kernel $\Gamma_m$. In particular, we show that this heat kernel can be holomorphically extended to $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$, which will be important for the following section, where we study the Segal–Bargmann transform.

The *Heckman–Opdam Laplacian* is defined by

$$\Delta_m := \sum_{i=1}^{q} T(\xi_i, m)^2 - |\rho|^2$$

where $T(\xi_i, m)$ is the Dunkl–Cherednik operator of Definition 2.2 and $\{\xi_1, \dots, \xi_q\}$ is an orthonormal basis of $\mathfrak{a}$. The operator $\Delta_m$ does not depend on the choice of the basis and has the explicit form

$$\Delta_m f(x) = \Delta f(x) + \sum_{\alpha\in\Sigma^+} m_\alpha \coth\langle\alpha, x\rangle \partial_\alpha f(x) - \sum_{\alpha\in\Sigma^+} \frac{m_\alpha |\alpha|^2}{2\sinh^2\langle\alpha, x\rangle}(f(x) - f(s_\alpha x))$$

where $\Delta$ denotes the Euclidean Laplacian on $\mathfrak{a}$ (see [13] and recall $R = 2\Sigma$ and $k_{2\alpha} = \frac{1}{2}m_\alpha$).

We now restrict our attention to $W$-invariant functions. Keeping in mind that our notation differs by a factor $i$ from that of Heckman and Opdam (see Remark 2.5), we consider the operator

$$L_m := \Delta + \sum_{\alpha\in\Sigma^+} m_\alpha \cot\langle\alpha, x\rangle \partial_\alpha$$

on $C^2(\mathfrak{a})^W$. Then for $f(x) = g(ix)$, we just have

$$L_m f(x) = -(\Delta_m g)(ix). \tag{3.1}$$

**Remark 3.1.** Consider a compact symmetric space $U/K$ on which $K$ acts from the left with restricted root system $\Sigma$ and geometric multiplicity $m$. Then $L_m$ is just the radial part of the Laplace–Beltrami operator on $U/K$. See Proposition 3.11, Chapter II in [6]. □

The Jacobi polynomials $R_\lambda$ are eigenfunctions of $L_m$:

$$L_m R_\lambda = -\langle \lambda, \lambda + 2\rho \rangle R_\lambda, \quad \lambda \in \Lambda^+. \tag{3.2}$$

This follows from equation (3.1). The eigenvalues are negative,

$$-\langle \lambda, \lambda + 2\rho \rangle = -|\lambda|^2 - 2\langle \lambda, \rho \rangle \leq 0$$

since $\lambda$ and $\rho$ are both contained in the Weyl chamber $\overline{\mathfrak{a}^+}$ and therefore $\langle \lambda, \rho \rangle \geq 0$. We shall use the abbreviation

$$\theta_\lambda := \langle \lambda, \lambda + 2\rho \rangle.$$

The Heckman–Opdam heat equation on $A_0$ is given by

$$L_m u = \partial_t u.$$

A formal derivation via Heckman–Opdam transform

$$\hat{f}(\lambda) := \int_{A_0} f(x) R_\lambda(-x) w_m(x)\, \mathrm{d}x$$

motivates the following:

**Definition 3.2.** The *heat kernel* $\Gamma_m$ on $A_0 \times A_0 \times (0, \infty)$ is defined by

$$\Gamma_m(x, y, t) := \sum_{\lambda \in \Lambda^+} r_\lambda\, \mathrm{e}^{-\theta_\lambda t} R_\lambda(x) R_\lambda(-y).$$ □

We shall also consider $\Gamma_m$ as a function on $\mathfrak{a} \times \mathfrak{a} \times (0, \infty)$ which is $W_{\text{aff}}$-invariant in the first and second argument. We still have to show that the series converges. This will be a consequence of the following lemma which states that the growth of the $r_\lambda$ is polynomial in $\lambda_\alpha$, $\alpha \in \Sigma^+$. We start with some simple observations. First, recall from Remark 2.7(b) that the coefficients $d_{\lambda,\gamma}$ in the exponential expansion of the Jacobi polynomials $R_\lambda$ are nonnegative and sum up to 1. Therefore,

$$|R_\lambda(x)| \leq R_\lambda(0) = 1 \quad \forall x \in \mathfrak{a}. \tag{3.3}$$

For the summands of $\Gamma_m$, this implies

$$|r_\lambda \,\mathrm{e}^{-\theta_\lambda t} R_\lambda(x) R_\lambda(-y)| \le r_\lambda \,\mathrm{e}^{-\theta_\lambda t}. \tag{3.4}$$

**Lemma 3.3.** There exists a constant $C > 0$ such that

$$|r_\lambda| \le C \cdot \prod_{\alpha\in\Sigma^+,\lambda_\alpha\neq 0} \lambda_\alpha^{m_\alpha}.$$

□

**Proof.** According to Lemma 2.6 and Remark 2.7(a), we have

$$\begin{aligned} r_\lambda &= \frac{1}{\|R_\lambda\|_m^2} = \frac{1}{\|c(\lambda+\rho)P_\lambda\|_m^2} \\ &= \left(\prod_{\alpha\in\Sigma^+} \frac{\Gamma(\lambda_\alpha+\rho_\alpha+\frac{1}{4}m_{\alpha/2}+\frac{1}{2}m_\alpha)\Gamma(\rho_\alpha+\frac{1}{4}m_{\alpha/2})}{\Gamma(\lambda_\alpha+\rho_\alpha+\frac{1}{4}m_{\alpha/2})\Gamma(\rho_\alpha+\frac{1}{4}m_{\alpha/2}+\frac{1}{2}m_\alpha)}\right)^2 \\ &\quad \cdot \prod_{\alpha\in\Sigma^+} \frac{\Gamma(\lambda_\alpha+\rho_\alpha-\frac{1}{4}m_{\alpha/2}+1)}{\Gamma(\lambda_\alpha+\rho_\alpha-\frac{1}{4}m_{\alpha/2}-\frac{1}{2}m_\alpha+1)} \cdot \frac{\Gamma(\lambda_\alpha+\rho_\alpha+\frac{1}{4}m_{\alpha/2})}{\Gamma(\lambda_\alpha+\rho_\alpha+\frac{1}{4}m_{\alpha/2}+\frac{1}{2}m_\alpha)} \\ &= c\cdot \prod_{\alpha\in\Sigma^+} f_\alpha(\lambda_\alpha) \end{aligned}$$

where $c > 0$ is a constant depending only on $m$ and

$$f_\alpha(\lambda_\alpha) = \frac{\Gamma(\lambda_\alpha+\rho_\alpha+\frac{1}{4}m_{\alpha/2}+\frac{1}{2}m_\alpha)\Gamma(\lambda_\alpha+\rho_\alpha-\frac{1}{4}m_{\alpha/2}+1)}{\Gamma(\lambda_\alpha+\rho_\alpha+\frac{1}{4}m_{\alpha/2})\Gamma(\lambda_\alpha+\rho_\alpha-\frac{1}{4}m_{\alpha/2}-\frac{1}{2}m_\alpha+1)}.$$

We use the well-known asymptotics of the $\Gamma$-function:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}.$$

Then, for all $\alpha\in\Sigma^+$ and $\lambda_\alpha\to\infty$, this implies the asymptotic

$$f_\alpha(\lambda_\alpha) \sim \lambda_\alpha^{\frac{1}{2}m_\alpha} \cdot \lambda_\alpha^{\frac{1}{2}m_\alpha} = \lambda_\alpha^{m_\alpha}.$$

Since $\Sigma^+$ is finite, we find a constant $M > 0$ such that for all positive roots

$$|f_\alpha(\lambda_\alpha)| \le 2\lambda_\alpha^{m_\alpha} \quad \text{for } \lambda_\alpha \ge M. \tag{3.5}$$

Fix such $M$, and denote by $L > 0$ a common upper bound such that

$$|f_\alpha(\lambda_\alpha)| \leq L \quad \text{for } \lambda_\alpha < M \ (\forall \alpha \in \Sigma^+). \tag{3.6}$$

Now let us temporarily fix a $\lambda \in \Lambda^+$. We decompose the set of positive roots in two disjoint sets $\Sigma^+ = \Sigma_1^+ \cup \Sigma_2^+$, where

$$\Sigma_1^+ := \{\alpha \in \Sigma^+ : \lambda_\alpha < M\}, \quad \Sigma_2^+ := \{\alpha \in \Sigma^+ : \lambda_\alpha \geq M\}.$$

Application of estimates (3.5) and (3.6) then yields

$$\left|\prod_{\alpha\in\Sigma^+} f_\alpha(\lambda_\alpha)\right| \leq L^{|\Sigma_1^+|} \cdot \prod_{\alpha\in\Sigma_2^+} 2\lambda_\alpha^{m_\alpha}.$$

Without loss of generality we may assume $L \geq 1$. Note that $\lambda_\alpha \neq 0$ implies $\lambda_\alpha \geq 1$ (since $\lambda_\alpha \in \mathbb{Z}^+$). Therefore, we can extend the above estimate:

$$\left|\prod_{\alpha\in\Sigma^+} f_\alpha(\lambda_\alpha)\right| \leq L^{|\Sigma^+|} \cdot \prod_{\alpha\in\Sigma^+, \lambda_\alpha\neq 0} 2\lambda_\alpha^{m_\alpha}.$$

This holds independently of $\lambda$ and implies the lemma. ■

The consequence of this lemma is that the growth of the summands in the heat kernel $\Gamma_m$ for $\lambda \to \infty$ is dominated by $\mathrm{e}^{-\theta_\lambda t}$, which decays for fixed $t > 0$ as $\mathrm{e}^{-|\lambda|^2 t}$. With (3.4), we conclude

**Proposition 3.4.** The series defining the heat kernel $\Gamma_m$ converges absolutely and uniformly on $\mathfrak{a} \times \mathfrak{a} \times (0, \infty)$. For all $x, y \in \mathfrak{a}$ and $t > 0$, we have

$$|\Gamma_m(x, y, t)| \leq \sum_{\lambda\in\Lambda^+} r_\lambda \,\mathrm{e}^{-\theta_\lambda t} =: C_t < \infty.$$

The long-time behavior of $\Gamma_m$ is given by

$$\lim_{t\to\infty} \Gamma_m(x, y, t) = r_0 = \frac{1}{\int_{A_0} w_m(x)\,\mathrm{d}x}$$

where the convergence is uniform on $\mathfrak{a} \times \mathfrak{a}$. □

One would expect from classical theory of the heat equation that the heat kernel is smooth. This is also true in our setting.

**Proposition 3.5.** For fixed $t_0 > 0$, the heat kernel $\Gamma_m(\cdot, \cdot, t_0)$ extends to a holomorphic function on $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$ which is $W_{\mathrm{aff}}$-invariant in the real part of both arguments. The holomorphic extension is given by

$$\Gamma_m(z, w, t_0) = \sum_{\lambda \in \Lambda^+} r_\lambda \, \mathrm{e}^{-\theta_\lambda t_0} R_\lambda(z) R_\lambda(-w).$$

In particular, $\Gamma_m \in C^\infty(A_0 \times A_0 \times (0, \infty))$. □

**Proof.** It is obvious that $R_\lambda$ is holomorphic on $\mathfrak{a}_{\mathbb{C}}$. Therefore, each summand in $F(z, w) := \sum_{\lambda \in \Lambda^+} r_\lambda \, \mathrm{e}^{-\theta_\lambda t_0} R_\lambda(z) R_\lambda(-w)$ is holomorphic, and normal convergence of the series will imply that $F$ is holomorphic on $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$. To see this, recall that the Jacobi polynomial $R_\lambda$ is a linear combination of exponentials $\mathrm{e}^{\mathrm{i}\gamma}$ with $\gamma \preceq \lambda$ according to Lemma 2.1(i). Part (ii) of Lemma 2.1 then implies $|\gamma| \leq |\lambda|$. Therefore,

$$|\mathrm{e}^{\mathrm{i}\langle \gamma, z\rangle}| \leq \mathrm{e}^{|\gamma||\mathrm{Im}\, z|} \leq \mathrm{e}^{|\lambda||z|}.$$

For $M > 0$, consider the compact ball $K := \{z \in \mathfrak{a}_{\mathbb{C}} : |z| \leq M\}$. Then, for $z \in K$, we obtain the following estimate:

$$|R_\lambda(z)| = \left| \sum_{\substack{\gamma \in W.\mu \\ \mu \in \Lambda^+, \mu \preceq \lambda}} d_{\lambda,\gamma} \, \mathrm{e}^{\mathrm{i}\langle \gamma, z\rangle} \right| \leq \mathrm{e}^{|\lambda||z|} \leq \mathrm{e}^{|\lambda| M}.$$

Here we used $d_{\lambda,\gamma} \geq 0$ and $\sum_\gamma d_{\lambda,\gamma} = 1$ (see Remark 2.7 (b)). With a similar estimate for $R_\lambda(w)$, we see that the growth behavior of each summand of $F$ is dominated by the term $\mathrm{e}^{-\theta_\lambda t_0}$ on $K \times K$. Thus we have normal convergence on compact subsets of $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$, which implies that $F$ is holomorphic on $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$. Finally, note that termwise differentiating with respect to $t$ gives a factor $-\theta_\lambda$ but does not change the convergence. ■

Next we collect some further basic properties of the heat kernel:

**Lemma 3.6.**

(a) For all $w \in \mathfrak{a}_{\mathbb{C}}$, the function $u(x, t) := \Gamma_m(x, w, t)$ is a solution of the heat equation $L_m u = \partial_t u$ on $A_0 \times (0, \infty)$.

(b) $\displaystyle\int_{A_0} \Gamma_m(z, x, t) w_m(x)\,\mathrm{d}x = 1 \quad (\forall z \in \mathfrak{a}_{\mathbb{C}}).$

(c) $\displaystyle\Gamma_m(z, w, t+s) = \int_{A_0} \Gamma_m(z, x, t)\Gamma_m(x, w, s) w_m(x)\,\mathrm{d}x \quad (\forall z, w, \in \mathfrak{a}_{\mathbb{C}}).$

(d) $\displaystyle\int_{A_0} \Gamma_m(z, x, t) R_\lambda(x) w_m(x)\,\mathrm{d}x = \mathrm{e}^{-\theta_\lambda t} R_\lambda(z) \quad (\forall z \in \mathfrak{a}_{\mathbb{C}}).$ □

**Proof.** Part (a) follows from (3.2) and termwise differentiation. The further statements are obtained by direct calculation, using the orthogonality of the polynomials $R_\lambda$ with respect to $\langle \cdot, \cdot \rangle_m$, and the fact that $R_\lambda(-x) = \overline{R_\lambda(x)}$ for all $x \in \overline{A_0}$. ■

**Definition 3.7.** For $f \in L^1(A_0, w_m)$, we define

$$H(t) f(x) := \begin{cases} \int_{A_0} \Gamma_m(x, y, t) f(y) w_m(y)\,\mathrm{d}y & \text{for } t > 0; \\ f(x) & \text{for } t = 0. \end{cases} \tag{3.7}$$

□

With the Heckman–Opdam transform, we can write

$$H(t) f(x) = \sum_{\lambda \in \Lambda^+} r_\lambda\, \mathrm{e}^{-\theta_\lambda t} \hat{f}(\lambda) R_\lambda(x), \qquad t > 0.$$

Because of $|\hat{f}(\lambda)| \le \|f\|_1$ and Proposition 3.4 this sum converges absolutely and uniformly on $A_0$.

**Lemma 3.8.** Let $f \in L^1(A_0, w_m)$. Then, for each $t > 0$, we have $H(t) f \in C(A_0)$ with

$$\|H(t) f\|_\infty \le C_t \|f\|_1.$$

Moreover,

$$H(t+s) f = H(t) H(s) f \quad (\forall s, t \ge 0).$$ □

**Proof.** This follows directly from Proposition 3.4 and Lemma 3.6(c). ■

Since $A_0$ is compact, we have continuous embedding

$$C(A_0) \hookrightarrow L^p(A_0, w_m) \hookrightarrow L^1(A_0, w_m), \qquad 1 < p < \infty. \tag{3.8}$$

In particular, Lemma 3.8 implies that the family $(H(t))_{t\geq 0}$ forms a semigroup of bounded linear operators on the Banach space $(C(A_0), \|.\|_\infty)$. We shall prove that this semigroup is actually a Feller–Markov semigroup on $C(A_0)$ which is generated by the closure of the Heckman–Opdam Laplacian $L_m$. Feller–Markov means that the semigroup is strongly continuous, contractive, and positive, that is, $f \geq 0$ on $A_0$ implies $H(t)f \geq 0$ on $A_0$. As for the rational Dunkl case and the noncompact trigonometric case in [12, 13], the proof will be based on a variant of the Lumer–Phillips theorem characterizing the generator of a Feller–Markov semigroup, c.f. Theorem 2.2., p. 165, in [2].

**Theorem 3.9.** The family $(H(t))_{t\geq 0}$ is a Feller–Markov semigroup on $(C(A_0), \|.\|_\infty)$. Its generator is given by the closure $\bar{L}_m$ of $L_m$. □

We call $(H(t))_{t\geq 0}$ the *Heckman–Opdam heat semigroup on* $A_0$.

**Proof.** In order to see that $L_m$ has a closure which generates a Feller–Markov semigroup on $(C(A_0), \|.\|_\infty)$, we shall apply Theorem 2.2., p. 165, of [2] where we consider $\mathbb{C}$-valued functions on $A_0$. We thus have to check the following three conditions:

(i) If $f \in \mathcal{D}(L_m) = \mathcal{T}^W$, then also $\bar{f} \in \mathcal{D}(L_m)$ and $L_m(\overline{f}) = \overline{L_m(f)}$.
(ii) There exists a $t > 0$ such that the range of $tI - L_m$ is dense in $C(A_0)$.
(iii) If $f \in \mathcal{D}(L_m)$ is a real valued with a nonnegative maximum in $x_0 \in A_0$, that is, $0 \leq f(x_0) = max_{x\in A_0} f(x)$, then $L_m f(x_0) \leq 0$ (positive maximum principle).

Condition (i) is obvious. For condition (ii), notice that $(tI - L_m)R_\lambda = (t + \theta_\lambda)R_\lambda$ with $t + \theta_\lambda > 0$ for $t > 0$. This implies that the image of $\mathcal{T}^W$ under $tI - L_m$ is dense in $C(A_0)$. The positive maximum principle is obvious when $x_0 \notin H_\alpha$ for all $\alpha \in \Sigma$. If $\langle \alpha, x_0\rangle = 0$ for some $\alpha \in \Sigma$, one has to use a similar argument as in the proof of Lemma 4.1 in [12]: Consider $f$ as a $W_{\text{aff}}$-invariant function on $\mathbb{R}^q$ and let $x \notin H_\alpha$ for all $\alpha \in \Sigma$. Then Taylor expansion yields

$$0 = f(s_\alpha x) - f(x) = -\langle \alpha, x\rangle \partial_\alpha f(x) + \frac{1}{2}\langle \alpha, x\rangle^2 \alpha^T D^2 f(\xi)\alpha,$$

where $\xi$ lies on the line segment between $x$ and $s_\alpha x$. Therefore,

$$\lim_{x\to x_0} \cot\langle \alpha, x\rangle \partial_\alpha f(x) = \frac{1}{2}\alpha^T D^2 f(x_0)\alpha \leq 0.$$

Thus $L_m$ is closable, and its closure generates a Feller–Markov semigroup $(T(t))_{t\geq 0}$ on $(C(A_0), \|.\|_\infty)$. It remains to check that $H(t)$ and $T(t)$ coincide. For this, it suffices to prove that they coincide on the dense subspace $\mathcal{T}^W$ of $(C(A_0), \|.\|_\infty)$. Notice that $L_m R_\lambda = -\theta_\lambda R_\lambda$ and

$$H(t)R_\lambda = e^{-\theta_\lambda t} R_\lambda \quad (\lambda \in \Lambda^+). \tag{3.9}$$

Hence the function $t \mapsto H(t)R_\lambda$ is continuously differentiable on $[0, \infty)$ and solves the abstract Cauchy problem

$$\frac{d}{dt} u(t) = \overline{L_m} u(t) \quad \text{for } t > 0,$$

$$u(0) = R_\lambda.$$

But according to the semigroup theory, this Cauchy problem has a unique continuously differentiable solution which is given by $t \mapsto T(t)R_\lambda$. This shows that $T(t)R_\lambda = H(t)R_\lambda$, and the assertion follows. ■

The positivity of the heat semigroup implies that $\Gamma_m$ is nonnegative on the alcove $A_0$. In fact, we have more.

**Proposition 3.10.** The heat kernel $\Gamma_m$ is strictly positive, that is,

$$\Gamma_m(x, y, t) > 0 \quad \text{for all } (x, y, t) \in A_0 \times A_0 \times (0, \infty).$$ □

**Proof.** Assume that $\Gamma_m(x_0, y_0, t_0) = 0$. According to Lemma 3.6(c), we have

$$\Gamma_m(x_0, y_0, t_0) = \int_{A_0} \Gamma_m\left(x_0, a, \frac{t_0}{2}\right) \Gamma_m\left(a, y_0, \frac{t_0}{2}\right) w_m(a)\, da.$$

Let $F_t(z, w) := \Gamma_m(z, w, t) = \sum_{\lambda \in \Lambda^+} r_\lambda e^{-\theta_\lambda t} R_\lambda(z) R_\lambda(-w)$. Now the positivity of the heat semigroup implies $\Gamma_m \geq 0$. Therefore,

$$F_{\frac{t_0}{2}}(x_0, a) F_{\frac{t_0}{2}}(a, y_0) \equiv 0 \quad \text{on } A_0.$$

But we already know from Proposition 3.5 that $F_t$ is holomorphic in both arguments. If the product of two holomorphic functions vanishes on an open, connected, and nonempty set, then one of them has to be identically zero. But this is a contradiction to $F_t(x_0, x_0) = \sum r_\lambda e^{-\theta_\lambda t} |R_\lambda(x_0)|^2 > 0$. ■

Let us conclude this section with some remarks concerning $L^p$-theory and behavior for $t \to \infty$: so far we have considered the heat semigroup on the space of continuous functions $C(A_0)$. But an $L^p(A_0, w_m)$ theory $(1 \le p < \infty)$ is easily developed.

By Lemma 3.8 and the embeddings (3.8), each $H(t)$ defines a bounded linear operator on $L^p(A_0, w_m)$ for $1 \le p < \infty$.

**Proposition 3.11.** The family $(H(t))_{t\ge 0}$ defines a positive, contractive, and strongly continuous semigroup on $L^p(A_0, w_m)$ for $1 \le p < \infty$. □

**Proof.** The positivity of $\Gamma_m$ implies that $H(t)$ is positive on $L^p(A_0, w_m)$. Moreover, Jensen's inequality implies

$$|H(t)f(x)|^p \le \int_{A_0} |f(y)|^p \, \Gamma_m(x, y, t) w_m(y)\, \mathrm{d}y$$

and therefore $\|H(t)f\|_p \le \|f\|_p$. It remains to check strong continuity of the semigroup on the dense subspace $\mathcal{T}^W$, which is immediate from (3.9). ■

**Remark 3.12.** The $L^2$-theory of the heat semigroup is particularly explicit. The Laplacian $L_m$ with domain $\mathcal{T}^W$ is symmetric in $L^2(A_0, w_m)$ [10, Proposition 2.3], and the Jacobi polynomials $R_\lambda$ form a complete set of eigenfunctions with real eigenvalues. Therefore, $L_m$ is essentially self-adjoint. Its closure is given by orthogonal expansion with respect to the Jacobi basis:

$$\overline{L_m} f = \sum_{\lambda \in \Lambda^+} (-\theta_\lambda) r_\lambda \langle f, R_\lambda \rangle_m R_\lambda = \sum_{\lambda \in \Lambda^+} (-\theta_\lambda) r_\lambda \hat{f}(\lambda) R_\lambda$$

with domain

$$\mathcal{D}(\overline{L_m}) = \left\{ f \in L^2(A_0, w_m) : \sum_{\lambda \in \Lambda^+} r_\lambda \theta_\lambda^2 |\hat{f}(\lambda)|^2 < \infty \right\}.$$

This self-adjoint operator generates a strongly continuous semigroup $\mathrm{e}^{t\overline{L_m}}$ on $L^2(A_0, w_m)$. By Borel functional calculus,

$$\mathrm{e}^{t\overline{L_m}} R_\lambda = \mathrm{e}^{-\theta_\lambda t} R_\lambda$$

and for general $f = \sum_\lambda r_\lambda \hat{f}(\lambda) R_\lambda$,

$$\mathrm{e}^{t\overline{L_m}} f = \sum_{\lambda \in \Lambda^+} r_\lambda \, \mathrm{e}^{-\theta_\lambda t} \hat{f}(\lambda) R_\lambda.$$

This coincides with the heat semigroup $(H(t))_{t\ge 0}$ on $L^2(A_0, w_m)$. □

For $t \to \infty$ the heat (given by the initial distribution $f$) spreads uniformly on the alcove:

**Proposition 3.13.** Let $f \in L^p(A_0, w_m)$, $1 \le p < \infty$. Then

$$\lim_{t\to\infty} H(t) f = \frac{1}{\int_{A_0} w_m(x)\,\mathrm{d}x} \int_{A_0} f(x)\, w_m(x)\,\mathrm{d}x$$

with respect to $\|.\|_p$. □

**Proof.** Write

$$H(t) f(x) = r_0 \hat{f}(0) + \sum_{\lambda\in\Lambda^+,\lambda\neq 0} r_\lambda\, \mathrm{e}^{-\theta_\lambda t} \hat{f}(\lambda) R_\lambda(x)$$

and take the limit $t \to \infty$. ■

## 4 The Segal–Bargmann Transform

In this section, we focus on the $L^2$-setting. The smoothness of the heat kernel implies that the heat transform $f \mapsto H(t) f$ smoothens the arbitrary initial data. We shall see that it actually gives rise to a unitary isomorphism between $L^2(A_0, w_m)$ and a certain Hilbert space $\mathcal{H}_t$ of holomorphic functions on $\mathfrak{a}_{\mathbb{C}}$—the so-called *Segal–Bargmann transform*.

Let us start with a short reminder of the classical situation for the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$; for details, see, for example, [3].

**Example 4.1** (The Segal–Bargmann transform for the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$)**.** We consider functions on $\mathbb{T}$ as functions on $\mathbb{R}$ which are invariant under the action of $2\pi\mathbb{Z}$. Let $f \in L^2(\mathbb{T})$. The heat equation

$$\Delta u = \partial_t u, \quad u(x, 0) = f(x)$$

has the solution

$$u(x, t) = H(t) f(x) := \sum_{n\in\mathbb{Z}} \hat{f}(n)\, \mathrm{e}^{-n^2 t}\, \mathrm{e}^{inx}$$

with the usual Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\, \mathrm{e}^{-inx}\,\mathrm{d}x.$$

The function $H(t)f$ extends holomorphically to a function on $\mathbb{C}$ which is $2\pi\mathbb{Z}$-periodic in the real part of its argument:

$$H(t)\,f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\,\mathrm{e}^{-n^2 t}\,\mathrm{e}^{inz}.$$

Consider the heat kernel on $\mathbb{R}$,

$$\gamma_t^1(y) = \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-|y|^2/4t}$$

and put

$$\rho_t(y) := 2\gamma_{2t}^1(2y).$$

Then the image of

$$H(t) : L^2(\mathbb{T}) \longrightarrow \mathcal{O}(\mathbb{C}/2\pi\mathbb{Z})$$

is the Segal–Bargmann space

$$\mathcal{H}_t := \{F \in \mathcal{O}(\mathbb{C}/2\pi\mathbb{Z}) : \|F\|^2_{\mathcal{H}_t} := \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} |F(x+iy)|^2 \rho_t(y)\,\mathrm{d}x\,\mathrm{d}y < \infty\}.$$

Moreover, the Segal–Bargmann transform $H(t) : L^2(\mathbb{T}) \to \mathcal{H}_t$ is a unitary isomorphism. □

We shall now extend this result to the compact (and Weyl-group invariant) Heckman–Opdam case. The classical example as well as the theory in the case of compact symmetric spaces (see [3]) indicates that the Segal–Bargmann space will depend on the noncompact heat kernel.

**Remark 4.2** (The noncompact heat equation)**.** The noncompact Heckman–Opdam heat equation was studied by Schapira [13]. Let us recall some results.

Denote by $D_m$ the $W$-invariant part of $\Delta_m$, that is,

$$D_m = \Delta + \sum_{\alpha\in\Sigma^+} m_\alpha \coth\langle\alpha, x\rangle\partial_\alpha.$$

The hypergeometric function $F_\lambda$ is an eigenfunction of the operator $D_m$:

$$D_m F_{\lambda+\rho} = \langle\lambda, \lambda + 2\rho\rangle F_{\lambda+\rho} = \theta_\lambda F_{\lambda+\rho}. \tag{4.1}$$

The operator $D_m$ has a closure which generates the noncompact heat semigroup $e^{t\overline{D_m}}$ on $C_0(\mathfrak{a})$. The $W$-invariant noncompact heat kernel is given by

$$\gamma_t^1(x, y) = \int_{i\mathfrak{a}} e^{-t(|\lambda|^2+|\rho|^2)} F_\lambda(x) F_\lambda(-y) \frac{d\lambda}{|c(\lambda)|^2}.$$

We will use the notation $\gamma_t^1(x) := \gamma_t^1(x, 0)$ in the following. Heckman–Opdam transform shows that

$$\int_{\mathfrak{a}} F_{\lambda+\rho}(x)\gamma_t^1(x)\delta_m(x)\, dx = e^{t\theta_\lambda} \tag{4.2}$$

where

$$\delta_m := \prod_{\alpha\in\Sigma^+} |e^\alpha - e^{-\alpha}|^{m_\alpha}$$

is Opdam's weight function. □

We now turn to the holomorphic extension of the heat transform on $L^2(A_0, w_m)$. Recall that the heat kernel

$$\Gamma_m(z, w, t) = \sum_{\lambda\in\Lambda^+} r_\lambda\, e^{-\theta_\lambda t} R_\lambda(z) R_\lambda(-w)$$

is holomorphic in $(z, w)$, where the series converges normally on compact subsets of $\mathfrak{a}_\mathbb{C} \times \mathfrak{a}_\mathbb{C}$. Thus for $f \in L^2(A_0, w_m)$ and $t > 0$, the heat transform $H(t)f$ given by (3.7) extends to a holomorphic function on $\mathfrak{a}_\mathbb{C}$,

$$H(t)f(z) = (e^{t\overline{L_m}} f)(z) = \int_{A_0} \Gamma_m(z, y, t) f(y) w_m(y)\, dy.$$

Alternatively, this can be written as

$$H(t)f(z) = \sum_{\lambda\in\Lambda^+} r_\lambda\, e^{-\langle\lambda,\lambda+2\rho\rangle t} \hat{f}(\lambda) R_\lambda(z) \tag{4.3}$$

where the sum converges normally on compact subsets of $\mathfrak{a}_\mathbb{C}$.

**Definition 4.3.** In the following, $H(t)$ shall always stand for the analytic continuation of the heat transform to $\mathfrak{a}_\mathbb{C}$. □

**Definition 4.4.** Define $\mathcal{F}_t := \mathrm{Im}(H(t)) \subset \mathcal{O}(\mathfrak{a}_{\mathbb{C}})$ as the holomorphic image of the heat transform with inner product

$$\langle H(t) f, H(t) g\rangle_{\mathcal{F}_t} := \langle f, g\rangle_{L^2(A_0, w_m)}.$$ □

**Proposition 4.5.** The space $(\mathcal{F}_t, \|.\|_{\mathcal{F}_t})$ is a Hilbert space with reproducing kernel

$$K_t(z, w) = \Gamma_m(w, \bar{z}, 2t).$$

The set of $W$-invariant trigonometric polynomials $\mathcal{T}^W$ is dense in $(\mathcal{F}_t, \|.\|_{\mathcal{F}_t})$. □

**Proof.** The heat transform $L^2(A_0, w_m) \to \mathcal{F}_t$ is by definition a unitary isomorphism. Hence $(\mathcal{F}_t, \|.\|_{\mathcal{F}_t})$ is a Hilbert space. The set $\mathcal{T}^W$ is dense in $L^2(A_0, w_m)$ and therefore also in $\mathcal{F}_t$ (recall that $H(t)$ maps $\mathcal{T}^W$ onto itself). Finally, let $F = H(t) f \in \mathcal{F}_t$. Then

$$\begin{aligned} F(z) = H(t) f(z) &= \int_{A_0} f(y) \Gamma_m(z, y, t) w_m(y)\, \mathrm{d}y \\ &= \langle f, \Gamma_m(\cdot, \bar{z}, t)\rangle_{L^2(A_0, w_m)} = \langle H(t) f, H(t)\Gamma_m(\cdot, \bar{z}, t)\rangle_{\mathcal{F}_t} = \langle F, K_{t,z}\rangle_{\mathcal{F}_t} \end{aligned}$$

with

$$\begin{aligned} K_{t,z}(w) = H(t)\Gamma_m(\cdot, \bar{z}, t)(w) &= \int_{A_0} \Gamma_m(w, y, t)\Gamma_m(y, \bar{z}, t) w_m(y)\, \mathrm{d}y \\ &= \Gamma_m(w, \bar{z}, 2t) \end{aligned}$$

according to Lemma 3.6(c). ■

We are interested in a more explicit description of the image of $H(t)$ as a Hilbert space of holomorphic functions.

**Definition 4.6.** Assume that the Fourier coefficients of $f \in L^2(A_0, w_m)$ satisfy the growth condition

$$\sum_{\lambda \in \Lambda^+} r_\lambda |\hat{f}(\lambda)|^2\, \mathrm{e}^{|\lambda+\rho||x|} < \infty \quad (\forall x \in \mathfrak{a}). \tag{4.4}$$

For such $f$ and $x, y \in \mathfrak{a}$, we define a generalized translation by

$$\tau_{ix} f(y) := f(-ix * y) := \sum_{\lambda \in \Lambda^+} r_\lambda \hat{f}(\lambda) R_\lambda(y) R_\lambda(-ix).$$

□

Recalling that $R_\lambda(-ix) = F_{\lambda+\rho}(x)$, we observe that $\tau_{ix} f \in L^2(A_0, w_m)$ if and only if

$$\|\tau_{ix} f\|_2^2 < \infty \iff \sum_{\lambda \in \Lambda^+} r_\lambda |\hat{f}(\lambda)|^2 |F_{\lambda+\rho}(x)|^2 < \infty. \tag{4.5}$$

Notice also that $x \mapsto \tau_{ix} f(y)$ is $W$-invariant on $\mathfrak{a}$. According to [10, Proposition 6.1], the hypergeometric function satisfies a growth estimate

$$|F_\lambda(x)| \leq C\, \mathrm{e}^{|\lambda||x|} \quad \forall x \in \mathfrak{a}$$

with a constant $C$ independent of $x$ and $\lambda$. This shows that condition (4.5) is implied by our growth condition (4.4), and the above translation is indeed well defined. Moreover,

$$R_\lambda(-ix * y) = R_\lambda(-ix) R_\lambda(y).$$

We shall now describe the target space of the Segal–Bargmann transform in a different way.

**Definition 4.7.** Let

$$\mathcal{H}_t := \{F \in \mathcal{O}(\mathfrak{a}_\mathbb{C}) : F \text{ is } W_{\mathrm{aff}}\text{-invariant in the real part of its argument}; \|F\|_{\mathcal{H}_t} < \infty\},$$

with the inner product

$$\langle F, G \rangle_{\mathcal{H}_t} := \sum_{\lambda \in \Lambda^+} r_\lambda \hat{F}(\lambda) \overline{\hat{G}(\lambda)}\, \mathrm{e}^{2t\theta_\lambda}.$$

Here the (Heckman–Opdam) Fourier transform $\hat{F}(\lambda) := \int_{A_0} F(x) R_\lambda(-x) w_m(x)\, \mathrm{d}x$ is with respect to the real part $x$ of the variable. □

Note that functions in $\mathcal{H}_t$ automatically satisfy growth condition (4.4) (with respect to the real part of the variable). Moreover, the identity theorem implies that each $F \in \mathcal{H}_t$ is invariant under the action of $W$ on $\mathfrak{a}_\mathbb{C}$, that is, it is $W$-invariant also in the imaginary part of its argument.

**Proposition 4.8.** The space $\mathcal{H}_t$ is a Hilbert space of holomorphic functions. □

**Proof.** Let us recall the definition: a *Hilbert space of holomorphic functions* $\mathcal{H}$ on a domain $D$ is a subspace of $\mathcal{O}(D)$ with the structure of a Hilbert space such that the embedding $\mathcal{H} \hookrightarrow \mathcal{O}(D)$ is continuous.

This means, for every compact subset $K \subset \mathfrak{a}_{\mathbb{C}}$, we have to find a constant $C_K$ such that

$$|F(z)| \leq C_K \|F\|_{\mathcal{H}_t} \quad \forall z \in K.$$

We claim that each $F \in \mathcal{H}_t$ can be written as

$$F(z) = \sum_{\lambda \in \Lambda^+} r_\lambda \hat{F}(\lambda) R_\lambda(z) \tag{4.6}$$

where the series converges normally on each compact subset $K$ of $\mathfrak{a}_{\mathbb{C}}$. Indeed, let $z \in K$ with $|\mathrm{Im}\, z| \leq M$. Then

$$\begin{aligned}\sum |r_\lambda \hat{F}(\lambda) R_\lambda(z)| &\leq \sum r_\lambda |\hat{F}(\lambda)|\, \mathrm{e}^{t\theta_\lambda}\, \mathrm{e}^{-t\theta_\lambda} |R_\lambda(z)| \\ &\leq \left(\sum r_\lambda |\hat{F}(\lambda)|^2\, \mathrm{e}^{2t\theta_\lambda}\right)^{1/2} \cdot \left(\sum r_\lambda |R_\lambda(z)|^2\, \mathrm{e}^{-2t\theta_\lambda}\right)^{1/2}.\end{aligned}$$

Now the first factor is just $\|F\|_{\mathcal{H}_t}$ and for the second factor we use the estimate $|R_\lambda(z)|^2 \leq \mathrm{e}^{2|\lambda||\mathrm{Im}\, z|} \leq \mathrm{e}^{2M|\lambda|}$ to obtain a constant $C_K$ such that

$$\sum \|r_\lambda \hat{F}(\lambda) R_\lambda\|_{\infty,K} \leq C_K \|F\|_{\mathcal{H}_t}. \tag{4.7}$$

As a consequence, the sum in (4.6) defines a holomorphic function on $\mathfrak{a}_{\mathbb{C}}$. On the other hand, for given $f \in L^2(A_0)$ the sum $\sum_{\lambda \in \Lambda^+} r_\lambda \hat{f}(\lambda) R_\lambda$ is just the expansion of $f$ with respect to the orthonormal basis of Heckman–Opdam polynomials. Since $F \in \mathcal{H}_t$ is continuous and bounded in the real part (as a $W_{\mathrm{aff}}$-invariant function), we have $F \in L^2(A_0)$ and therefore $F(x) = \sum_{\lambda \in \Lambda^+} r_\lambda \hat{F}(\lambda) R_\lambda(x)$ a.e. on $A_0$. Now the identity theorem implies the claim (4.6).

Relation (4.6) together with (4.7) shows that the embedding $\mathcal{H}_t \hookrightarrow \mathcal{O}(\mathfrak{a}_{\mathbb{C}})$ is continuous. It remains to check that $\mathcal{H}_t$ is complete with respect to the given inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_t}$. For this, consider a Cauchy sequence $(F_n)$ in $\mathcal{H}_t$. Because of the continuous embedding, it converges uniformly on compact subsets of $\mathfrak{a}_{\mathbb{C}}$ to some limit $F \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}})$ which is

$W_{\mathrm{aff}}$-periodic in its real part. Further, $F_n = H(t) f_n$ with $f_n \to f \in L^2(A_0, w_m)$. This shows that $F_n \to H(t) f$ in $\|.\|\mathcal{H}_t$, and therefore also locally uniformly. ■

In the next proposition, we give another representation of the inner product of $\mathcal{H}_t$ and state its reproducing kernel.

**Proposition 4.9.**

(1) The inner product of $\mathcal{H}_t$ can be written as

$$\langle F, G\rangle_{\mathcal{H}_t} = \int_{\mathfrak{a}} \int_{A_0} \tau_{ix} F(y)\, \overline{G(y)} w_m(y) \gamma^1_{2t}(x) \delta_m(x)\, \mathrm{d}y\, \mathrm{d}x. \tag{4.8}$$

(2) The reproducing kernel of $\mathcal{H}_t$ is given by $K_t(z, w) = \Gamma_m(w, \bar{z}, 2t)$. □

**Remarks 4.10.**

1. The above representation of $\mathcal{H}_t$ is close to, but not exactly, a weighted Bergman space on $\mathfrak{a}_{\mathbb{C}}$, because the generalized translate acts only on one of the functions $F, G$.
2. In (4.8), $\int_{\mathfrak{a}}$ can be replaced by $|W| \int_{\mathfrak{a}^+}$ because $x \mapsto \tau_{ix} F(y)$ and the heat kernel $\gamma^1_{2t}$ are $W$-invariant. □

**Proof.** We already remarked that $\tau_{ix} F$ is well defined for functions in $\mathcal{H}_t$. Using dominated convergence, we calculate

$$\begin{aligned}
&\int_{\mathfrak{a}} \int_{A_0} \tau_{ix} F(y)\, \overline{G(y)} w_m(y) \gamma^1_{2t}(x) \delta_m(x)\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{\mathfrak{a}} \int_{A_0} \sum_{\lambda} r_\lambda \hat{F}(\lambda) R_\lambda(-ix) R_\lambda(y) \overline{G(y)} w_m(y) \gamma^1_{2t}(x) \delta_m(x)\, \mathrm{d}y\, \mathrm{d}x \\
&= \sum_{\lambda} r_\lambda \hat{F}(\lambda) \int_{\mathfrak{a}} \left( \int_{A_0} R_\lambda(y) \overline{G(y)} w_m(y)\, \mathrm{d}y \right) R_\lambda(-ix) \gamma^1_{2t}(x) \delta_m(x)\, \mathrm{d}x \\
&= \sum_{\lambda} r_\lambda \hat{F}(\lambda) \overline{\hat{G}(\lambda)} \int_{\mathfrak{a}} F_{\lambda+\rho}(x) \gamma^1_{2t}(x) \delta_m(x)\, \mathrm{d}x = \sum_{\lambda} r_\lambda \hat{F}(\lambda) \overline{\hat{G}(\lambda)}\, \mathrm{e}^{2t\theta_\lambda} = \langle F, G\rangle_{\mathcal{H}_t}.
\end{aligned}$$

Here we used $R_\lambda(-ix) = F_{\lambda+\rho}(x)$ and (4.2). For the reproducing kernel property note that for $K_{t,z}(w) = \Gamma_m(w, \bar{z}, 2t)$, we have

$$\hat{K}_{t,z}(\lambda) = \mathrm{e}^{-2t\theta_\lambda}\overline{R_\lambda(z)}.$$

Thus

$$\langle F, K_{t,z}\rangle_{\mathcal{H}_t} = \sum_{\lambda\in\Lambda^+} r_\lambda \hat{F}(\lambda) R_\lambda(z)$$

which is equal to $F(z)$ according to (4.6). ∎

So, $\mathcal{F}_t$ and $\mathcal{H}_t$ are Hilbert spaces of holomorphic functions with the same reproducing kernel. But then by general Hilbert space theory they have to coincide (see Proposition II.1.5 in [3]). The holomorphic heat transform $H(t): L^2(A_0, w_m) \to \mathcal{F}_t$ is by definition a unitary isomorphism. We conclude

**Theorem 4.11.** The Segal–Bargmann transform

$$L^2(A_0, w_m) \longrightarrow \mathcal{H}_t, \quad f \mapsto H(t)f$$

defined by (4.3) is a unitary isomorphism from the $L^2$-space on the alcove $A_0$ onto the Hilbert space of holomorphic functions $\mathcal{H}_t$. □

**Remark 4.12.** We finish with a remark concerning the geometric cases related to compact symmetric spaces, where we refer to the exposition of [3]. Let $X = U/K$ be a Riemannian symmetric space of compact type and $X_\mathbb{C} = U_\mathbb{C}/K_\mathbb{C}$ its complexification. Let further $G/K$ be the noncompact dual of $X$, of real rank $r$ and with restricted root system $\Sigma$ and geometric multiplicity $m$. Let $\mathfrak{a}^+$ denote the Weyl chamber associated with a fixed positive subsystem of $\Sigma$. The Laplace–Beltrami operator of $X$ generates the heat semigroup $(H(t))_{t\geq 0}$ on $L^2(X)$, and the Segal–Bargmann transform $B_t : f \mapsto H(t)f$ is a unitary isomorphism from $L^2(X)$ onto the weighted Bergman space

$$\mathcal{B}^2(X_\mathbb{C}, p_t) = \left\{F \in \mathcal{O}(X_\mathbb{C}) : \|F\|^2 = \int_{X_\mathbb{C}} |F(z)|^2 p_t(z)\, \mathrm{d}m(z) < \infty\right\}.$$

Here $m$ is a $U_\mathbb{C}$-invariant measure on $X_\mathbb{C}$ and $p_t(z) = 2^r \gamma^1_{2t}(\mathrm{e}^{2H}\cdot o)$ for $z = u\mathrm{e}^H$, with $\gamma^1_t$ denoting the heat kernel of $G/K$. Suppose now $F \in \mathcal{B}^2(X_\mathbb{C}, p_t)$ is invariant under the left action of $K_{\mathbb{C}}$ on $X_{\mathbb{C}}$. Then $F$ is the holomorphic extension of a unique $K$-invariant function on $X$ and can be identified, via its spherical expansion, with a holomorphic function on $\mathfrak{a}_{\mathbb{C}}$ which is $W_{\text{aff}}$-invariant in the real part of its argument. Following the proof of Theorem V.3.2 in [3], one obtains

$$\|F\|^2 = \sum_{\lambda\in\Lambda^+} r_\lambda \, e^{2t\Theta_\lambda} |\hat{F}(\lambda)|^2 = \|F\|^2_{\mathcal{H}_t}$$

where $\hat{F}$ is the Heckman–Opdam transform corresponding to the underlying root data. Thus, the subspace of $K_{\mathbb{C}}$-invariant functions from $\mathcal{B}^2(X_{\mathbb{C}}, p_t)$ can be identified with $\mathcal{H}_t$. □

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