One-parameter Semigroups related to abstract Quantum Models of Calogero Type

Margit Rösler
Zentrum Mathematik, Technische Universität München
Arcisstr. 21, D-80290 München, Germany
e-mail: roesler@mathematik.tu-muenchen.de

Abstract

We study various classes of strongly continuous one-parameter semigroups which are generated by abstract versions of linear Calogero-Moser-Sutherland Hamiltonians for arbitrary root systems. These Hamiltonians contain modifications by exchange terms and can be written in terms of Dunkl operators. The semigroups under consideration include the generalized heat semigroup and the Schrödinger semigroup related with the free abstract Calogero Hamiltonian, as well as the semigroup generated by the Calogero Hamiltonian with harmonic confinement. The latter one is closely related with a Dunkl-type generalization of the classical Ornstein-Uhlenbeck semigroup.

1 Introduction

In recent years, quantum many particle models of Calogero-Moser-Sutherland (CMS) type have gained considerable interest in theoretical physics. These models describe systems of identical particles on a circle or line which interact pairwise through long range potentials of inverse square type. They are exactly solvable and are therefore of great interest for the understanding of quantum many body physics. CMS models have in particular attracted some attention in conformal field theory, and they are being used to test the ideas of fractional statistics ([Ha], [Hal]). While explicit spectral resolutions of such models were already obtained by Calogero and Sutherland ([Ca], [Su]), a new aspect in the understanding of their algebraic structure and quantum integrability was only recently initiated by [Po] and [He]. The Hamiltonian under consideration is hereby modified by certain exchange operators, which allow to write it in a decoupled form. These exchange modifications can be expressed in terms of Dunkl operators of type A_{N-1}. Dunkl operators, as introduced and first studied by C.F. Dunkl ([D1], [D2]), are parametrized differential-reflection operators associated with root systems. They extend the usual partial derivatives by additional reflection terms. Besides their important role in the context of quantum integrable many particle systems, Dunkl operators provide a key tool in the analysis of special functions related with root systems. In the present paper, we study several classes of one-parameter semigroups which are generated by second order Dunkl operators. These operators can be seen as abstract versions of linear

CMS operators which are associated with arbitrary root systems and are modified by exchange terms in the sense of [Po]. After a brief survey on Dunkl operators in Section 2, the connection of these operators with quantum Calogero models is described in Section 3. We then turn to the basic one-parameter semigroup in the Dunkl setting, namely the generalized heat semigroup introduced in [R1]; it is discussed in Section 4 on various function spaces besides $(C_0(\mathbb{R}^N), \|.\|_\infty)$. When considered for imaginary times, the Dunkl-type heat semigroup in a suitably weighted L^2-space leads to the solution of the time-dependent Schrödinger equation for the free quantum Calogero model. This is contained in Section 5. Finally, the last section is devoted to the semigroup generated by the Calogero Hamiltonian with harmonic confinement. It can be interpreted as the Dunkl-type version of the classical oscillator semigroup, and is closely related with the Ornstein-Uhlenbeck semigroup studied in [R-V].

2 Some basic facts from the theory of Dunkl operators

Let R be a (reduced, not necessarily crystallographic) root system in $\mathbb{R}^N$, i.e. a finite subset of $\mathbb{R}^N \setminus \{0\}$ with $R \cap \mathbb{R}\cdot\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. Here σ_α denotes the reflection in the hyperplane orthogonal to α, which is given by $\sigma_\alpha(x) = x - \langle \alpha, x\rangle \cdot \alpha$, with $\langle .,.\rangle$ denoting the standard Euclidean scalar product. We hereby assume that the root system R is normalized, i.e. $|\alpha|^2 = 2$ for all $\alpha \in R$, where $|\,.\,|$ is the Euclidean norm. We further denote by G the finite reflection group generated by $\{\sigma_\alpha,\, \alpha \in R\}$. A function $k : R \to \mathbb{C}$ is called a multiplicity function on the root system R, if it invariant under the natural action of G on R. We fix some multiplicity-function k on R, which is throughout this paper assumed to be non-negative, i.e. $k(\alpha) \geq 0$ for all $\alpha \in R$. The Dunkl operators on $\mathbb{R}^N$ associated with G and k are defined by

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}\,,$$

where R_+ is an (arbitrary) positive subsystem of R, i.e. $\langle \alpha, \beta\rangle > 0$ for some $\beta \in \mathbb{R}^N$ and all $\alpha \in R_+$. The operators T_i can be considered as a perturbation of the usual partial derivatives in the parameter k, and many properties of the usual partial derivatives carry over to them ([D1], [D2], [dJ]); here we mention only the following ones:

(i) The set $\{T_i\,,\, i = 1, \ldots, N\}$ generates a commutative algebra of differential-reflection operators on $\mathbb{R}^N$.

(ii) The operators T_i are homogeneous of degree -1 on the space $\Pi^N := \mathbb{C}[\mathbb{R}^N]$ of polynomial functions in N variables, i.e. if $p \in \Pi^N$ has total degree k, then $T_i\, p$ has total degree $k - 1$.

(iii) If $f \in C^k(\mathbb{R}^N)$ with $k \geq 1$, then $T_i f \in C^{k-1}(\mathbb{R}^N)$; moreover, if f belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$, then also $T_i f \in \mathscr{S}(\mathbb{R}^N)$.

Of particular importance in our context is the generalized Laplacian, which is defined by $\Delta_k := \sum_{i=1}^N T_i^2$. It is given explicitly by

$$\Delta_k \,=\, \Delta + \sum_{\alpha\in R} k(\alpha)\delta_\alpha \tag{2.1}$$

with

$$\delta_\alpha f(x) \,=\, \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x)-\sigma_\alpha f(x)}{\langle \alpha, x\rangle^2};$$

here Δ and ∇ denote the usual Laplacian and gradient respectively.

2.1 Example. (Dunkl operators of type A_{N-1}). These belong to the symmetric group $G = S_N$, which acts in a canonical way on $\mathbb{R}^N$ by permuting the standard basis vectors $e_1, \ldots, e_N$. Each transposition (ij) acts as a reflection σ_{ij}, sending $e_i - e_j$ to its negative. On $C^1(\mathbb{R}^N)$, σ_{ij} acts by transposing the coordinates x_i and x_j with respect to the standard basis. The attached root system, of type A_{N-1}, is given by $R = \{e_i - e_j\,,\ 1 \le i,j \le N,\ i \ne j\}$. Since all transpositions are conjugate in S_N, the vector space of multiplicity functions on R is one-dimensional. The Dunkl operators associated with the multiplicity parameter $k \in \mathbb{C}$ are given by

$$T_i^S \,=\, \partial_i \,+\, k\cdot \sum_{j\ne i} \frac{1-\sigma_{ij}}{x_i - x_j} \quad (i = 1, \ldots, N),$$

and the generalized Laplacian is

$$\Delta_k^S \,=\, \Delta + 2k \sum_{1\le i<j\le N} \frac{1}{x_i - x_j}\Big[(\partial_i - \partial_j) - \frac{1-\sigma_{ij}}{x_i - x_j}\Big].$$

The Dunkl theory provides also a counterpart to the usual exponential function, called the Dunkl kernel $E_k(x,y)$. For each fixed $y \in \mathbb{R}^N$, the function $x \mapsto E_k(x,y)$ can be characterized as the unique solution of the system $T_i f = y_i f$ $(i = 1, \ldots, N)$ with $f(0) = 1$; see [O]. The kernel $E_k(x,y)$ is symmetric in its arguments and has a unique holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$. It satisfies $E_k(z,0) = 1$ and $E_k(\lambda z, w) = E_k(z, \lambda w)$ for all $z, w \in \mathbb{C}^N$ and all $\lambda \in \mathbb{C}$. Moreover, E_k has a Bochner-type representation of the form

$$E_k(x,z) \,=\, \int_{\mathbb{R}^N} e^{\langle \xi, z\rangle} d\mu_x^k(\xi), \quad \text{for all } z \in \mathbb{C}^N,$$

where μ_x^k is a compactly supported probability measure on $\mathbb{R}^N$ with $\operatorname{supp} \mu_x^k$ being contained in the convex hull of the orbit $\{gx, g \in G\}$, see [R2]. It follows that $|E_k(x, iy)| \le 1$ for all $x, y \in \mathbb{R}^N$, and that

$$\min_{g\in G} e^{\langle gx, y\rangle} \,\le\, E_k(x,y) \,\le\, \max_{g\in G} e^{\langle gx, y\rangle}. \tag{2.2}$$

In particular, $E_k(x,y) > 0$ for all $x, y \in \mathbb{R}^N$. We mention that this positivity result was first deduced in [R1] from the positivity of the associated heat semigroup. The Dunkl kernel gives rise to a corresponding integral transform on $\mathbb{R}^N$ with respect to the weight function

$$w_k(x) \,=\, \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}.$$

Notice that w_k is G-invariant and homogeneous of degree 2γ, with the index

$$\gamma := \gamma(k) = \sum_{\alpha\in R_+} k(\alpha).$$

The Dunkl transform on $L^1(\mathbb{R}^N, w_k)$ is defined by

$$\widehat{f}^k(\xi) := c_k^{-1} \int_{\mathbb{R}^N} f(x)E_k(-i\xi, x)\, w_k(x)dx,$$

where c_k is the Mehta-type constant

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x)dx.$$

This integral transform has many properties which are completely analogous to those of the classical Fourier transform. A thorough investigation is given in [dJ]. We recall from there that the Dunkl transform is a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$, satisfying $(T_j f)^{\wedge k}(\xi) = i\xi_j \widehat{f}^k(\xi)$. Moreover, it has a unique Plancherel-type extension to an isometric isomorphism of $L^2(\mathbb{R}^N, w_k)$, which is also denoted by $f \mapsto \widehat{f}^k$. The inverse transform is given by $f^{\vee k}(x) = \widehat{f}^k(-x)$.

3 Quantum Calogero models

We continue with a short explanation of linear Calogero-Moser-Sutherland models and the relevance of Dunkl operators in their algebraic description. The Hamiltonian of the so-called quantum Calogero model with harmonic confinement in $L^2(\mathbb{R}^N)$ is given by

$$\mathcal{H}_C \,=\, -\Delta + \omega^2|x|^2 \,+\, 2k(k-1)\sum_{1\leq i<j\leq N} \frac{1}{(x_i - x_j)^2}; \tag{3.1}$$

here $\omega > 0$ is a frequency parameter and $k \geq 0$ is a coupling constant. In case $\omega = 0$, (3.1) describes the free Calogero model. The study of this Hamiltonian was initiated by Calogero ([Ca]); he computed its spectrum and determined the structure of the eigenfunctions and scattering states in the confined and free case, respectively. Perelomov [Pe] observed that (3.1) is completely quantum integrable, i.e. there exist N commuting, algebraically independent symmetric linear operators in $L^2(\mathbb{R}^N)$ including $\mathcal{H}_C$. We mention that the complete integrability of the classical Hamiltonian systems associated with (3.1) goes back to Moser [Mo]. There exist generalizations of the classical Calogero-Moser-Sutherland models in the context of abstract root systems, see e.g. [O-P1], [O-P2]. In particular, if R is an arbitrary root system on $\mathbb{R}^N$ and k is a nonnegative multiplicity function on it, then the corresponding abstract Calogero Hamiltonian with harmonic confinement is given by

$$\widetilde{\mathcal{H}}_k \,=\, -\widetilde{\mathcal{F}}_k + \omega^2|x|^2$$

with the formal expression

$$\widetilde{\mathcal{F}}_k \,=\, \Delta - 2\sum_{\alpha\in R_+} k(\alpha)(k(\alpha)-1)\frac{1}{\langle\alpha, x\rangle^2}.$$

If R is of type A_{N-1}, then $\widetilde{\mathcal{H}}_k$ just coincides with $\mathcal{H}_C$. For both the classical and the quantum case, partial results on the integrability of this model are due to Olshanetsky and Perelomov [O-P1], [O-P2]. A new aspect in the understanding of the algebraic structure and the quantum integrability of CMS systems was later initiated by Polychronakos [Po] and Heckman [He]. The underlying idea is to construct quantum integrals for CMS models from differential-reflection operators. Polychronakos introduced them in terms of an "exchange-operator formalism" for (3.1). He thus obtained a complete set of commuting observables for (3.1) in an elegant way. In [He] it was observed in general that the complete algebra of quantum integrals for free, abstract Calogero models is intimately connected with the corresponding algebra of Dunkl operators. Since then, there has been an extensive and ongoing study of CMS models and explicit operator solutions for them via differential-reflection operator formalisms; among the broad literature, we refer to [L-V], [K], [BHKV], [BF], and [U-W]. Let us briefly describe the connection of abstract Calogero models with Dunkl operators: Consider the following modification of $\widetilde{\mathcal{F}}_k$, involving reflection terms:

$$\mathcal{F}_k \,=\, \Delta - 2\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}\,(k(\alpha)-\sigma_\alpha)\,. \tag{3.2}$$

In order to avoid singularities in the reflecting hyperplanes, it is suitable to carry out a gauge transform by $\sqrt{w_k}$. One obtains (c.f. Lemma 3.1. of [R3]) that $\mathcal{F}_k$ is essentially self-adjoint when considered as a linear operator in $L^2(\mathbb{R}^N)$ with domain $\mathcal{D}(\mathcal{F}_k) := \{w_k^{1/2}f : f\in\mathscr{S}(\mathbb{R}^N)\}$. Moreover,

$$\mathcal{F}_k \,=\, w_k^{1/2}\Delta_k\, w_k^{-1/2}\,,$$

where Δ_k is the Dunkl Laplacian in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathscr{S}(\mathbb{R}^N)$. Consider now the algebra of G-invariant polynomials on $\mathbb{R}^N$:

$$(\Pi^N)^G \,=\, \{p\in\Pi^N : g\cdot p = p \;\text{ for all }\; g\in G\}.$$

It follows easily from equivariance properties of the Dunkl operators (c.f. [dJ]) that for every $p\in(\Pi^N)^G$, the Dunkl operator $p(T)$ leaves $(\Pi^N)^G$ invariant. For such p we denote the restriction of $p(T)$ to $(\Pi^N)^G$ by $\operatorname{Res}(p(T))$. Then, as observed in [He], the family

$$\left\{\operatorname{Res}(p(T)) :\, p\in(\Pi^N)^G\right\}$$

is a commutative algebra of differential operators, containing the operator

$$\operatorname{Res}(\Delta_k) \,=\, w_k^{-1/2}\widetilde{\mathcal{F}}_k\, w_k^{1/2}.$$

This implies the integrability of the free Calogero Hamiltonian $\widetilde{\mathcal{F}}_k$. Polychronakos [Po] also succeeded to determine a complete set of quantum integrals for the classical, i.e. S_N-type Calogero Hamiltonian with harmonic confinement - at least in the physically relevant bosonic and fermionic subspaces of $L^2(\mathbb{R}^N)$. He constructed the integrals by a Lax formalism involving suitable lowering and raising operators. For the abstract Calogero operator $\widetilde{\mathcal{H}}_k$ with harmonic confinement, the general question of how to obtain an algebra of quantum integrals is, to the author's knowledge, still open. It is, however, easy to achieve a complete spectral analysis of $\widetilde{\mathcal{H}}_k$. We again work with the gauge-transformed version with reflection terms,

$$\mathcal{H}_k := w_k^{-1/2}(-\mathcal{F}_k + \omega^2|x|^2)\, w_k^{1/2} \;=\; -\Delta_k + \omega^2|x|^2.$$

This operator is symmetric and densely defined in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathcal{D}(\mathcal{H}_k) := \mathscr{S}(\mathbb{R}^N)$. Notice that in case $k = 0$, $\mathcal{H}_k$ is just the Hamiltonian of the N-dimensional isotropic harmonic oscillator. We further consider the Hilbert space $L^2(\mathbb{R}^N, m_k^\omega)$, where m_k^ω is the probability measure

$$m_k^\omega(x) \;:=\; c_k^{-1}(2\omega)^{\gamma+N/2}\, e^{-\omega|x|^2} w_k(x)dx \;\in M^1(\mathbb{R}^N) \quad (\omega > 0). \tag{3.3}$$

Moreover, we introduce the operator

$$\mathcal{J}_k := \; -\Delta_k + 2\omega \sum_{j=1}^N x_j \partial_j$$

in $L^2(\mathbb{R}^N, m_k^\omega)$, with the dense domain $\mathcal{D}(\mathcal{J}_k) := \Pi^N$ (the polynomials in N variables). The following connection between $\mathcal{H}_k$ and $\mathcal{J}_k$ is established in the same way as part (2) of Theorem 3.4.(2) in [R1].

3.1 Lemma. *On* $\mathcal{D}(\mathcal{J}_k) = \Pi^N$,

$$\mathcal{J}_k \;=\; e^{\omega|x|^2/2}\left(\mathcal{H}_k - (2\gamma + N)\omega\right) e^{-\omega|x|^2/2}.$$

In particular, $\mathcal{J}_k$ *is symmetric in* $L^2(\mathbb{R}^N, m_k^\omega)$.

We conclude with a complete description of the spectral properties of $\mathcal{H}_k$ and $\mathcal{J}_k$; these results generalize well-known facts for the corresponding classical operators. In the following, $\mathcal{P}_n^N$ denotes the space of polynomials from Π^N which are homogeneous of degree n. Notice also that by the homogeneity of Δ_k, the operator $e^{c\Delta_k}$ is well defined on polynomials and preserves the total degree.

3.2 Theorem. *For* $\omega > 0$ *and* $n \in \mathbb{Z}_+$ *define*

$$V_n^\omega := \{e^{-\Delta_k/4\omega}\, p \,:\, p \in \mathcal{P}_n^N\} \subset \Pi^N \quad \text{and} \quad W_n^\omega := \{e^{-\omega|x|^2/2}\, q(x),\, q \in V_n^\omega\} \subset \mathscr{S}(\mathbb{R}^N).$$

Then the following assertions hold:

(1) *The spaces* $L^2(\mathbb{R}^N, m_k^\omega)$ *and* $L^2(\mathbb{R}^N, w_k)$ *admit the orthogonal Hilbert space decompositions*

$$L^2(\mathbb{R}^N, m_k^\omega) \;=\; \bigoplus_{n\in\mathbb{Z}_+} V_n^\omega \qquad \text{and} \qquad L^2(\mathbb{R}^N, w_k) \;=\; \bigoplus_{n\in\mathbb{Z}_+} W_n^\omega\,;$$

here V_n^ω *is the eigenspace of* $\mathcal{J}_k$ *corresponding to the eigenvalue* $2n\omega$, *and* W_n^ω *is the eigenspace of* $\mathcal{H}_k$ *corresponding to the eigenvalue* $(2n + 2\gamma + N)\omega$.

(2) *The operators* $\mathcal{H}_k$ *and* $\mathcal{J}_k$ *are essentially self-adjoint; the spectra of their closures are discrete and given by* $\sigma(\overline{\mathcal{H}_k}) = \{(2n + 2\gamma + N)\omega,\ n \in \mathbb{Z}_+\}$ *and* $\sigma(\overline{\mathcal{J}_k}) = \{\, 2n\omega,\ n \in \mathbb{Z}_+\}$ *respectively.*

Proof. (1) It was shown in Theorem 3.4.(1) of [R1] that in case $\omega = 1$, each function from V_n^ω is an eigenfunction of $\mathcal{J}_k$ corresponding to the eigenvalue $2n\omega$. For arbitrary ω, the corresponding result is obtained by rescaling. Moreover, $V_n^\omega \perp V_m^\omega$ for $n \neq m$ by the symmetry of $\mathcal{J}_k$. This proves the statements for $\mathcal{J}_k$, because $\Pi^N = \bigoplus V_n^\omega$ is dense in $L^2(\mathbb{R}^N, m_k^\omega)$. The statements for $\mathcal{H}_k$ are then immediate by the previous Lemma.
(2) follows from (1) by a well-known criterion for self-adjointness of symmetric operators on a Hilbert space which have a complete set of orthogonal eigenfunctions within their domain (Lemma 1.2.2 of [Da3]). □

By the G-equivariance of Δ_k, the spectral resolution of the Calogero Hamiltonian $\widetilde{\mathcal{H}}_k$ in the bosonic subspace $L^2(\mathbb{R}^N)^G$ is now an easy consequence of Theorem 3.2.

3.3 Corollary. *For $n \in \mathbb{Z}_+$, put $W_n^{\omega,G} = \{e^{-\omega|x|^2/2} e^{-\Delta_k/4\omega} p : p \in \mathcal{P}_n^N \cap (\Pi^N)^G\}$. Then*

$$L^2(\mathbb{R}^N)^G = \bigoplus_{n \in \mathbb{Z}_+} W_n^{\omega,G},$$

and $W_n^{\omega,G}$ is the eigenspace of $\widetilde{\mathcal{H}}_k$ in $L^2(\mathbb{R}^N)^G$ corresponding to the eigenvalue $(2n+2\gamma+N)\omega$.

4 Heat semigroups associated with finite reflection groups

This section deals with the Dunkl-type analogues of the classical heat semigroup on several Banach spaces. These semigroups are generated by the Dunkl Laplacian, and they are governed by a generalized heat kernel which was introduced in [R1] and replaces the usual Gaussian kernel in the Dunkl setting.

4.1 Definition. The generalized heat kernel Γ_k associated with the reflection group G and the multiplicity function k is defined by

$$\Gamma_k(t,x,y) := \frac{M_k}{t^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x,y \in \mathbb{R}^N,\ t>0$$

with $M_k = (2^{\gamma+N/2}c_k)^{-1}$.

The strict positivity of E_k for real arguments implies that Γ_k is strictly positive as well. In the following, we collect some further important properties of this kernel.

4.2 Lemma. (1) $\displaystyle \frac{M_k}{t^{\gamma+N/2}} \min_{g\in G} e^{-|gx-y|^2/4t} \leq \Gamma_k(t,x,y) \leq \frac{M_k}{t^{\gamma+N/2}} \max_{g\in G} e^{-|gx-y|^2/4t}.$

(2) $\displaystyle \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, w_k(y)dy = 1.$

(3) *For fixed t and x, the function $y \mapsto \Gamma_k(t,x,y)$ belongs to $\mathscr{S}(\mathbb{R}^N)$, with $\Gamma_k(t,x,.)^{\wedge k}(\xi) = c_k^{-1} e^{-t|\xi|^2} E_k(-ix,\xi)$.*

(4) $\displaystyle \Gamma_k(t+s,x,y) = \int_{\mathbb{R}^N} \Gamma_k(t,x,z)\,\Gamma_k(s,y,z)\, w_k(z)dz.$

(5) *For fixed $y \in \mathbb{R}^N$, the function $u(t,x) := \Gamma_k(t,x,y)$ solves the generalized heat equation $\Delta_k u = \partial_t u$ on $(0,\infty) \times \mathbb{R}^N$.*

Proof. The estimates (1) are immediate from the bounds (2.2) on E_k. Properties (2) and (5) have been shown in [R1]. The first part of (3) is easily deduced from (1), while the second statement follows from the reproducing identity for E_k (c.f. [D3]),

$$\int_{\mathbb{R}^N} E_k(x,z)\, E_k(x,w)\, e^{-|x|^2/2} w_k(x)dx \;=\; c_k\, e^{(\langle z,z\rangle+\langle w,w\rangle)/2} E_k(z,w) \quad (z,w\in\mathbb{C}^N). \tag{4.1}$$

For the proof of (4), we use (3) and the Plancherel theorem for the Dunkl transform to obtain

$$\int_{\mathbb{R}^N} \Gamma_k(t,x,z)\,\Gamma_k(s,y,z)\, w_k(z)dz \;=\; c_k^{-1}\int_{\mathbb{R}^N} e^{-t|\xi|^2} E_k(ix,\xi)\Gamma_k(s,y,.)^{\wedge k}(\xi)w_k(\xi)d\xi$$
$$= c_k^{-2}\int_{\mathbb{R}^N} e^{-(s+t)|\xi|^2} E_k(ix,\xi)\, E_k(-iy,\xi)w_k(\xi)d\xi \;=\; \Gamma_k(t+s,x,y).$$

□

We next introduce the generalized heat operators associated with the kernel Γ_k.

4.3 Definition. For $f\in L^p(\mathbb{R}^N,w_k)$ $(1\le p\le\infty)$ and $t\ge 0$ define

$$H_k(t)f(x) := \begin{cases}\displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,y)f(y)\,w_k(y)dy & \text{if } t>0,\\ f(x) & \text{if } t=0\end{cases}$$

Notice that the decay properties of Γ_k assure that the integral defining $H_k(t)f(x)$ converges for all $t>0$, $x\in\mathbb{R}^N$. We recall the following properties of the operators $H_k(t)$ on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ from[R1]:

4.4 Theorem. *Let $f\in\mathscr{S}(\mathbb{R}^N)$. Then $u(t,x):=H_k(t)f(x)$ belongs to $C_b([0,\infty)\times\mathbb{R}^N)\cap C^2((0,\infty)\times\mathbb{R}^N)$ and solves the Cauchy problem*

$$\begin{cases}(\Delta_k-\partial_t)\,u=0 & \text{on } (0,\infty)\times\mathbb{R}^N,\\ u(0,.)=f.\end{cases}$$

Moreover, $H_k(t)f$ has the following properties:

(1) *$H_k(t)f\in\mathscr{S}(\mathbb{R}^N)$ for all $t>0$.*
(2) *$H_k(t+s)\,f \;=\; H_k(t)H_k(s)f$ for all $s,t\ge 0$.*
(3) *$\|H_k(t)f-f\|_\infty\to 0$ with $t\to 0$.*

4.5 Lemma. *For every $t>0$, $H_k(t)$ defines a continuous linear operator on each of the Banach spaces $L^p(\mathbb{R}^N,w_k)$ $(1\le p\le\infty)$, $(C_b(\mathbb{R}^N),\|.\|_\infty)$ and $(C_0(\mathbb{R}^N),\|.\|_\infty)$, with norm $\|H_k(t)\|\le 1$.*

Proof. The estimates for the kernel Γ_k in Lemma 4.2(3) and its normalization ensure that for every $f\in L^\infty(\mathbb{R}^N,w_k)$, we have $H(t)f\in C_b(\mathbb{R}^N)$ with $\|H_k(t)f\|_\infty\le\|f\|_\infty$. Moreover, if $f\in L^p(\mathbb{R}^N,w_k)$, then Jensen's inequality implies that

$$|H_k(t)f(x)|^p \;\le\; \int_{\mathbb{R}^N}\Gamma_k(t,x,y)\,|f(y)|^p\,w_k(y)dy,$$

and therefore $\|H_k(t)f\|_{p,w_k}\le\|f\|_{p,w_k}$. Finally, the invariance of $C_0(\mathbb{R}^N)$ under $H_k(t)$ follows from part (1) of the previous theorem, together with the density of $\mathscr{S}(\mathbb{R}^N)$ in $C_0(\mathbb{R}^N)$. □

In the following, X is one of the Banach spaces $L^p(\mathbb{R}^N, w_k)(1 \leq p < \infty)$ or $(C_0(\mathbb{R}^N), \|.\|_\infty)$. We consider the Dunkl Laplacian Δ_k as a linear operator in X with dense domain $\mathcal{D}(\Delta_k) := \mathscr{S}(\mathbb{R}^N)$.

4.6 Theorem. (1) *$(H_k(t))_{t\geq 0}$ is a strongly continuous, positivity-preserving contraction semigroup on X.*

(2) *Δ_k is closable, and its closure $\overline{\Delta}_k$ is the generator of the semigroup $(H_k(t))_{t\geq 0}$ on X.*

In view of this result, we call $(H_k(t))_{t\geq 0}$ the generalized Gaussian or heat semigroup on X.

Proof. (1) Theorem 4.4(2), together with Lemma 4.5 and the density of $\mathscr{S}(\mathbb{R}^N)$ in X, ensures that $(H_k(t))_{t\geq 0}$ forms a semigroup of continuous linear operators on X. Its positivity is clear by the positivity of Γ_k. Moreover, in case $X = (C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$, its strong continuity follows from part (3) of Theorem 4.4. It remains to check strong continuity in the case $X = L^p(\mathbb{R}^N, w_k)$, $1 \leq p < \infty$. In view of Lemma 4.5, and as $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N, w_k)$, it suffices to show that $\lim_{t\downarrow 0} \|H(t)f - f\|_{p,w_k} = 0$ for all $f \in C_c(\mathbb{R}^N)$; hereby we may further assume that $f \geq 0$. We then obtain

$$\|H_k(t)f\|_{1,w_k} = \int_{\mathbb{R}^N} H_k(t)f(x)\, w_k(x)dx = \int_{\mathbb{R}^N} f(x)w_k(x)dx = \|f\|_{1,w_k} \quad \text{for } t > 0.$$

As $\lim_{t\downarrow 0} \|H_k(t)f - f\|_\infty = 0$, a well-known convergence criterion (see for instance Theorem (13.47) of [H-St]) implies that $\lim_{t\downarrow 0} \|H_k(t)f - f\|_{1,w_k} = 0$. The estimation

$$\|H_k(t)f - f\|_{p,w_k}^p \leq \|H_k(t)f - f\|_{1,w_k} \cdot \|H_k(t)f - f\|_{\infty,w_k}^{p-1}$$

then entails that $\lim_{t\downarrow 0} \|H_k(t)f - f\|_{p,w_k} = 0$ as well.

(2) Let A be the generator of the semigroup $(H_k(t))_{t\geq 0}$ on X. As A is closed, it suffices to prove that $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k$, and that $A = \overline{A|_{\mathscr{S}(\mathbb{R}^N)}}$, i.e. $\mathscr{S}(\mathbb{R}^N)$ is a core of A. The proof of these statements is similar to the classical case. To begin with, let $f \in \mathscr{S}(\mathbb{R}^N)$. Then by Theorem 4.4(1), $H_k(t)f \in \mathscr{S}(\mathbb{R}^N)$ for all $t > 0$, and application of the Dunkl transform yields

$$\Big[\frac{1}{t}\big(H_k(t) - id\big)\, f\Big]^{\wedge k}(\xi) = \frac{1}{t}\big(e^{-t|\xi|^2} - 1\big)\widehat{f}^k(\xi).$$

It is easily checked that with $t \downarrow 0$, this tends to $-|\xi|^2\widehat{f}^k(\xi)$ in the topology of $\mathscr{S}(\mathbb{R}^N)$. The Dunkl transform being a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$, we therefore obtain

$$\lim_{t\downarrow 0} \frac{1}{t}\big(H_k(t) - id\,\big)f = (-|\xi|^2\widehat{f}^k)^{\vee k} = \Delta_k f$$

in the topology of $\mathscr{S}(\mathbb{R}^N)$, and therefore in $\|\,.\,\|_{p,w_k}$ as well. It follows that f belongs to the domain $\mathcal{D}(A)$ of A. Thus $\mathscr{S}(\mathbb{R}^N) \subset \mathcal{D}(A)$, and $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k$. Moreover, $\mathscr{S}(\mathbb{R}^N)$ is dense in X and invariant under $(H_k(t))_{t\geq 0}$. A well-known characterization of cores for the generators of strongly continuous semigroups (see, for instance, Theorem 1.9 of [Da1]) now implies that $\mathscr{S}(\mathbb{R}^N)$ is a core of A. □

The above theorem says in particular that $(H_k(t))_{t\geq 0}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^N, w_k)$ in the following sense:

4.7 Definition. ([Da2]) Let $\mu \in M^+(\mathbb{R}^N)$ be a positive Radon measure on $\mathbb{R}^N$. A strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on $L^2(\mathbb{R}^N, \mu)$ is called a symmetric Markov semigroup, if it satisfies the following conditions:

(1) The generator A of $(T(t))_{t\geq 0}$ is self-adjoint and non-positive, i.e. $\langle Af, f\rangle \leq 0$ for all $f \in \mathcal{D}(A)$;

(2) $(T(t))_{t\geq 0}$ is positivity-preserving for all $t \geq 0$, i.e. $T(t)f \geq 0$ for $f \geq 0$;

(3) If $f \in L^\infty(\mathbb{R}^N, \mu) \cap L^2(\mathbb{R}^N, \mu)$, then $\|T(t)f\|_{\infty,\mu} \leq \|f\|_{\infty,\mu}$ for all $t \geq 0$.

Theorem 1.4.2 of [Da2] implies the following

4.8 Corollary. *For $1 < p < \infty$, the semigroup $(H_k(t))_{t\geq 0}$ on $L^p(\mathbb{R}^N, w_k)$ is a bounded holomorphic semigroup (in the sense of [Da1]) in the sector*

$$\Big\{z \in \mathbb{C} : |arg(z)| < \pi \cdot \min\Big(\frac{1}{p}, \frac{1}{q}\Big)\Big\},$$

where q is the conjugate index defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Remarks. 1. For $X = (C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$, Theorem 4.6 just says that the generalized heat semigroup is a Feller-Markov semigroup, i.e. a (strongly continuous) positive contraction semigroup on $C_0(\mathbb{R}^N)$. This observation was the starting point in [R-V] for the construction of an associated semigroup of Markov kernels on $\mathbb{R}^N$. It leads to a Markov process in $\mathbb{R}^N$ which admits a càdlàg version (i.e., there exists an equivalent process whose paths are right-continuous and have limits from the left), and which obeys a modified notion of translation-invariance. For a detailed study of this Dunkl-type Brownian motion we refer to [R-V].

2. It is a basic fact from semigroup theory that for given initial data $f \in \mathcal{D}(\overline{\Delta}_k) \subset X$, the function $u(t) := H_k(t)f$ provides the unique classical solution of the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\,u(t) \,=\, \overline{\Delta}_k u(t) \quad \text{for } \ t > 0, \\ u(0) \,=\, f; \end{cases}$$

here "classical" means $u \in C^1([0,\infty), X)$ with $u(t) \in \mathcal{D}(\overline{\Delta}_k)$ for all $t \geq 0$. We refer to [R1] for the solution of the classical initial-boundary value problem for the Dunkl-type heat equation, with initial data taken from $C_b(\mathbb{R}^N)$.

5 The free, time-dependent Schrödinger equation

Consider again the self-adjoint Dunkl Laplacian $\overline{\Delta}_k$ in $L^2(\mathbb{R}^N, w_k)$. By Stone's Theorem, the skew-adjoint operator $i\overline{\Delta}_k$ generates a strongly continuous unitary semigroup $\big(e^{it\,\overline{\Delta}_k}\big)_{t\geq 0}$ on $L^2(\mathbb{R}^N, w_k)$. The explicit determination of this semigroup can be achieved by standard arguments, see for instance Chapter IX. 1.8 of [Kat] for the classical case. First, notice that the heat kernel Γ_k extends naturally to complex "time" arguments, by

$$\Gamma_k(z, x, y) \,=\, \frac{M_k}{z^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4z}\, E_k\Big(\frac{x}{2z}\,, y\Big)$$

for $x, y \in \mathbb{R}^N$ and $z \in \mathbb{C}_- := \mathbb{C} \setminus \{w \in \mathbb{R} : w \leq 0\}$; here $z^{\gamma+N/2}$ is the holomorphic branch in $\mathbb{C}_-$ with $1^{\gamma+N/2} = 1$. We next determine the Schrödinger semigroup on a sufficiently large subset of $\mathscr{S}(\mathbb{R}^N)$.

5.1 Lemma. *If $f(x) = e^{-b|x|^2}$ with a parameter $b > 0$, then*

$$e^{it\overline{\Delta}_k} f \;=\; \int_{\mathbb{R}^N} \Gamma_k(it, ., y)\, f(y)\, w_k(y) dy \quad \textit{for all}\;\; t > 0. \tag{5.1}$$

Proof. Consider the function

$$u(t,x) := \frac{1}{(1+4ibt)^{\gamma+N/2}}\, e^{-b|x|^2/(1+4ibt)} \quad (t \geq 0,\;\; x \in \mathbb{R}^N).$$

The same calculation as in Lemma 4.3. of [R1] shows that u satisfies the generalized Schrödinger equation

$$\partial_t u \;=\; i\Delta_k u \quad \text{on} \;\; (0,\infty) \times \mathbb{R}^N,$$

with $u(0,x) \;=\; e^{-b|x|^2}$. It is also easily verified that the function $t \mapsto u(t, .)$ belongs to $C^1([0,\infty), L^2(\mathbb{R}^N, w_k))$. This shows that $e^{it\overline{\Delta}_k} f \;=\; u(t, .)$ for $t \geq 0$. Finally, the reproducing identity (4.1) for E_k implies that for $t \geq 0$,

$$\frac{1}{(1+4bt)^{\gamma+N/2}}\, e^{-b|x|^2/(1+4bt)} \;=\; \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, e^{-b|y|^2} w_k(y) dy.$$

By analytic continuation, this identity remains true if t is replaced by it. This completes the proof. $\square$

In the following, we shall need the notion of a generalized translation on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$, c.f. [R1]. Its definition is natural:

$$L_k^y f(x) := c_k^{-1} \int_{\mathbb{R}^N} \widehat{f}^k(\xi)\, E_k(ix,\xi) E_k(iy,\xi)\, w_k(\xi) d\xi \quad (x,\, y \in \mathbb{R}^N,\;\; f \in \mathscr{S}(\mathbb{R}^N)). \tag{5.2}$$

Notice that that for $k = 0$, we just have $L_0^y f(x) = f(x+y)$. Important properties of the usal group translation on $\mathbb{R}^N$ carry over to the generalized translation for arbitrary k. It is, for example, easily checked that $L_k^y f$ belongs to $\mathscr{S}(\mathbb{R}^N)$ again with $(L_k^y f)^{\wedge k}(\xi) = E_k(iy,\xi)\, \widehat{f}^k(\xi)$. Moreover, $L_k^y f(x) = L_k^x f(y)$ for all $x, y \in \mathbb{R}^N$, and the operators L_k^y commute with the corresponding Dunkl operators T_i on $\mathscr{S}(\mathbb{R}^N)$. The following statement is obtained exactly as its classical analogue in [Kat], by using the Plancherel formula and the injectivity of the Dunkl transform.

5.2 Lemma. *The $\mathbb{C}$-linear hull $\langle M \rangle$ of the set*

$$M := \left\{ x \mapsto L_k^a\, e^{-b|x|^2}, \quad a \in \mathbb{R}^N,\, b > 0 \right\}$$

is dense in $L^2(\mathbb{R}^N, w_k)$.

We thus have shown that on the dense subspace $\langle M \rangle$ of $L^2(\mathbb{R}^N, w_k)$, the linear operators

$$S_k(t) f := \int_{\mathbb{R}^N} \Gamma_k(it, \,.\,, y) f(y)\, w_k(y) dy\,, \quad t > 0,$$

coincide with the unitary operators $e^{it\overline{\Delta}_k}$. They can therefore be extended uniquely to unitary operators on $L^2(\mathbb{R}^N, w_k)$, which are written in the same way, the integral now being understood in the L^2-sense. In this sense, we have for all $f \in L^2(\mathbb{R}^N, w_k)$,

$$e^{it\overline{\Delta}_k} f = \begin{cases} \int_{\mathbb{R}^N} \Gamma_k(it, \,.\,, y) f(y)\, w_k(y) dy & \text{if } t > 0, \\ f & \text{if } t = 0. \end{cases} \tag{5.3}$$

6 The semigroup of the Calogero Hamiltonian with harmonic confinement

For a fixed parameter $\omega > 0$, consider the Hamiltonian

$$\mathcal{J}_k = -\Delta_k + 2\omega \sum_{j=1}^N x_j \partial_j$$

with domain $\mathcal{D}(\mathcal{J}_k) := \Pi^N$ in the weighted Hilbert space $L^2(\mathbb{R}^N, m_k^\omega)$ (c.f. Section 3). Notice that $\mathcal{J}_k$ can be interpreted as the Dunkl-type generalization of the classical oscillator Hamiltonian in $L^2(\mathbb{R}^N)$. In the following, we shall work with generalized Hermite polynomials with respect to the measure m_k^ω. Generalized Hermite polynomials were introduced in [R1] (for $\omega = 1$) by means of homogeneous orthogonal systems with respect to a certain bilinear form on polynomials. We give an equivalent definition, which is more convenient on the basis of Theorem 3.2:

6.1 Definition. A family $\{H_\nu = H_\nu(\omega, \,.\,),\ \nu \in \mathbb{Z}_+^N\} \subset \Pi^N$ of real-valued polynomials is called a system of generalized Hermite polynomials (associated with the reflection group G, the multiplicity parameter k and the frequency parameter ω), if the following are satisfied:

(i) $\{H_\nu\,,\ |\nu| = n\}$ is a $\mathbb{C}$-basis of V_n^ω for every $n \in \mathbb{Z}_+$.

(ii) The $H_\nu\,,\ \nu \in \mathbb{Z}_+^N$ are orthogonal with respect to the probability measure m_k^ω on $\mathbb{R}^N$.

We now consider a fixed system $\{H_\nu\,,\ \nu \in \mathbb{Z}_+^N\}$ of generalized Hermite polynomials associated with G and k. We assume in addition that the H_ν are even orthonormal with respect to m_k^ω. By definition, they form a basis of eigenfunctions of $\mathcal{J}_k$ in $L^2(\mathbb{R}^N, m_k^\omega)$ with

$$\mathcal{J}_k H_\nu = 2|\nu|\omega \cdot H_\nu. \tag{6.1}$$

We shall need the following Mehler formula, which was shown in [R1] for $\omega = 1$ and is obtained for general ω by rescaling:

6.2 Lemma. *(Mehler-formula for the generalized Hermite polynomials.) The polynomials $H_\nu = H_\nu(\omega;\,.\,)$ satisfy*

$$\sum_{\nu \in \mathbb{Z}_+^N} H_\nu(x) H_\nu(y)\, r^{|\nu|} = M_k(r, x, y) \tag{6.2}$$

with the generalized Mehler kernel

$$M_k(r,x,y) \,=\, \frac{1}{(1-r^2)^{\gamma+N/2}}\exp\Big\{-\frac{\omega r^2(|x|^2+|y|^2)}{1-r^2}\Big\}\,E_k\Big(\frac{2\omega r x}{1-r^2},\,y\Big).$$

The sum on the left hand side of (6.2) *converges absolutely for all* $x,y\in\mathbb{R}^N$ *and* $0<r<1$.

According to Theorem 3.2, $\mathcal{J}_k$ is essentially self-adjoint in $L^2(\mathbb{R}^N, m_k^\omega)$. Let $\langle\,.\,,.\,\rangle$ denote the scalar product in $L^2(\mathbb{R}^N, m_k^\omega)$. Then the closure of $\mathcal{J}_k$ is given by

$$\overline{\mathcal{J}_k}(f) \,=\, \sum_{\nu\in\mathbb{Z}_+^N} 2|\nu|\omega\,\langle f, H_\nu\,\rangle\, f\,,$$

with domain

$$\mathcal{D}(\overline{\mathcal{J}_k}) \,=\, \{f\in L^2(\mathbb{R}^N, m_k^\omega):\, \sum_{\nu\in\mathbb{Z}_+^N}|\nu|^2|\langle f, H_\nu\,\rangle|^2 \,<\,\infty\}.$$

The spectral resolution of $\overline{\mathcal{J}_k}$ directly implies that $-\overline{\mathcal{J}_k}$ generates a strongly continuous contraction semigroup on $L^2(\mathbb{R}^N, m_k^\omega)$, namely

$$e^{-t\,\overline{\mathcal{J}_k}}f \,=\, \sum_{\nu\in\mathbb{Z}_+^N} e^{-2|\nu|\omega t}\,\langle f, H_\nu\,\rangle H_\nu \quad \text{for all } \ t\geq 0.$$

According to (6.2), we have

$$\sum_{\nu\in\mathbb{Z}_+^N} e^{-2|\nu|\omega t}H_\nu(x)H_\nu(y) \,=\, M_k(e^{-2t},x,y)$$

for all $t>0$. It is easily seen from the absolute convergence of the sum on the left, together with the orthogonality of the generalized Hermite polynomials, that the function $y\mapsto M_k(e^{-2t},x,y)$ belongs to $L^2(\mathbb{R}^N, m_k^\omega)$ for each fixed $x\in\mathbb{R}^N$. This shows that for $t>0$,

$$e^{-t\,\overline{\mathcal{J}_k}}f(x) \,=\, \int_{\mathbb{R}^N} M_k(e^{-2t},x,y)f(y)\,m_k^\omega(y) \quad a.e.$$

6.3 Proposition. $\big(e^{-t\overline{\mathcal{J}}_k}\big)_{t\geq 0}$ *is a symmetric Markov semigroup on* $L^2(\mathbb{R}^N, m_k^\omega)$ *in the sense of Definition 4.7.*

Proof. $\overline{\mathcal{J}}_k$ is self-adjoint and non-negative, and the semigroup $\big(e^{-t\,\overline{\mathcal{J}_k}}\big)_{t\geq 0}$ is positivity-preserving on $L^2(\mathbb{R}^N, m_k^\omega)$, because the kernel M_k is strictly positive. The $\{H_\nu\,,\,\nu\in\mathbb{Z}_+^N\}$ being orthonormal with $H_0 \,=\, 1$, we further have

$$\int_{\mathbb{R}^N} M_k(e^{-2t},x,y)\,dm_k^\omega(y) \,=\, 1 \quad \text{for all } \ t>0,\ x\in\mathbb{R}^N\,. \tag{6.3}$$

This implies that the operators $e^{-t\overline{\mathcal{J}}_k}$, $t\geq 0$ are also contractive with respect to $\|\,.\,\|_\infty$. □

As a consequence, the generalized oscillator semigroup $\big(e^{-t\,\overline{\mathcal{J}_k}}\big)_{t\geq 0}$ also allows an extension to a strongly continuous contraction semigroup on each of the Banach spaces $L^p(\mathbb{R}^N, m_k^\omega)$. We introduce the following notation:

6.4 Definition. For $f\in L^1(\mathbb{R}^N, m_k^\omega)$ and $t\geq 0$ set

$$O_k(t)f(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} M_k(e^{-2t}, x, y) f(y)\, dm_k^\omega(y) & \text{if } \ t>0,\\ f(x) & \text{if } \ t=0\end{cases} \tag{6.4}$$

6.5 Corollary. *$(O_k(t))_{t\geq 0}$ is a strongly continuous, positivity-preserving contraction semigroup on each of the Banach spaces $L^p(\mathbb{R}^N, m_k^\omega)$, $1\leq p<\infty$. For $p>1$ it is a bounded holomorphic semigroup in the sector*

$$\Big\{z\in\mathbb{C}:\, |arg(z)|\,<\,\pi\cdot\min\Big(\frac{1}{p},\frac{1}{q}\Big)\Big\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. This follows from Proposition 6.3 together with Theorems 1.4.1 and 1.4.2 of [Da2]. □

Direct inspection shows that the Mehler kernel is related to the Gaussian kernel Γ_k via

$$M_k(e^{-2t}, x, y)\, m_k^\omega(y) \,=\, \Gamma_k\Big(\frac{1-e^{-4\omega t}}{4\omega},\, e^{-2\omega t}x, y\Big)\, w_k(y)dy \quad (t>0,\ x\in\mathbb{R}^N). \tag{6.5}$$

The operators $O_k(t)$ can be expressed in terms of the heat operators $H_k(t)$:

$$O_k(t)f(x) \,=\, H_k\Big(\frac{1-e^{-4\omega t}}{4\omega}\Big) f\big(e^{-2\omega t}x\big) \tag{6.6}$$

for all $f\in C_0(\mathbb{R}^N)$ and all $t>0$. This implies that $(O_k(t))_{t\geq 0}$ leaves both $C_0(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$ invariant. It provides in fact a Feller-Markov semigroup on $(C_0(\mathbb{R}^N), \|.\|_\infty)$, which is a generalization of the classical Ornstein-Uhlenbeck semigroup to the Dunkl setting. The essential parts of the following result are contained in Section 10 of [R-V]:

6.6 Proposition. *$(O_k(t))_{t\geq 0}$ defines a strongly continuous, positivity-preserving contraction semigroup on $(C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$. The Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is a core of its generator A, and $A|_{\mathscr{S}(\mathbb{R}^N)} \,=\, \Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j$.*

Proof. The first part of the statement has been shown in [R-V]. The proof given there implies also that $\mathscr{S}(\mathbb{R}^N)$ is contained in the domain of A, and that $A|_{\mathscr{S}(\mathbb{R}^N)} \,=\, \Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j$. Since $\mathscr{S}(\mathbb{R}^N)$ is invariant under $(O_k(t))_{t\geq 0}$, it is in fact a core of A. □

Remark. It is also shown in [R-V] that for each $f\in C_b(\mathbb{R}^N)$, the function $u(t,x) := O_k(t)f(x)$ belongs to $C_b([0,\infty)\times\mathbb{R}^N)\cap C^2((0,\infty)\times\mathbb{R}^N)$ and solves the initial value problem

$$\begin{cases}\partial_t u \,=\, (\Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j)\, u & \text{on } (0,\infty)\times\mathbb{R}^N,\\ u(0,.)=f.\end{cases}$$

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