MULTIRESOLUTION ANALYSIS ON SPECTRA OF HERMITIAN MATRICES

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Dedicated to Tom Koornwinder on the occasion of his 80th birthday

Abstract. We establish a multiresolution analysis on the space $\mathrm{Herm}(n)$ of $n \times n$ complex Hermitian matrices which is adapted to invariance under conjugation by the unitary group $U(n)$. The orbits under this action are parametrized by the possible ordered spectra of Hermitian matrices, which constitute a closed Weyl chamber of type A_{n-1} in $\mathbb{R}^n$. The space $L^2(\mathrm{Herm}(n))^{U(n)}$ of radial, i.e. $U(n)$-invariant L^2-functions on $\mathrm{Herm}(n)$ is naturally identified with a certain weighted L^2-space on this chamber.

The scale spaces of our multiresolution analysis are obtained by usual dyadic dilations as well as generalized translations of a scaling function, where the generalized translation is a hypergroup translation which respects the radial geometry. We provide a concise criterion to characterize orthonormal wavelet bases and show that such bases always exist. They provide natural orthonormal bases of the space $L^2(\mathrm{Herm}(n))^{U(n)}$. Furthermore, we show how to obtain radial scaling functions from classical scaling functions on $\mathbb{R}^n$. Finally, generalizations related to the Cartan decompositions for general compact Lie groups are indicated.

1. Introduction

Suppose we are given a discrete subgroup $\Gamma \subseteq \mathrm{GL}(V) \ltimes V$ of the affine group of a Euclidean vector space V generated by translations coming from a lattice subgroup of V and dilations that arise as integer powers of an expansive automorphism which leaves the lattice invariant.

It is a classical problem in wavelet analysis to determine whether there are functions $\psi^1, \dots, \psi^r \in L^2(V)$, often called a wavelet set, such that

$$\{\gamma.\psi^i : \gamma \in \Gamma,\ 1 \leq i \leq r\} \tag{1.1}$$

constitutes an orthonormal Hilbert basis of $L^2(V)$. Here $L^2(V)$ is with respect to the Lebesgue measure on V and Γ acts on $L^2(V)$ via $\gamma.\psi(x) = \psi(\gamma^{-1}x)$. The standard approach to this problem is to obtain the wavelet basis (1.1) from a multiresolution analysis (see e.g. [BMM99, Mad93, Woj97]), and in general the size r of the wavelet set depends on the determinant of the expansive automorphism. In non-Euclidean settings, such as on manifolds, concepts of multiresolution analysis are often less natural. There is a broad literature on wavelet analysis and multiresolution on spheres, see e.g. [FNS18] and [FFP16] for a more general background about wavelet methods on manifolds. Let us also mention [Pap11] among further concepts of non-Euclidean multiresolution analyses and [OOR06] for a non-Euclidean (continuous) wavelet transform on rectangular matrix spaces. In [RR03] a radial multiresolution for $SO(3)$-invariant

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functions on $\mathbb{R}^3$ was introduced. It was based on the natural hypergroup translation on the orbit space $(\mathbb{R}^3)^{SO(3)} \cong [0,\infty[$, and the intimate connection of the characters of this hypergroup (certain Bessel functions) to the Tchebychef polynomials of second kind played a crucial role. For structural reasons, this concept cannot be generalized to $SO(n)$-invariant functions on $\mathbb{R}^n$ for $n > 3$.

In the present paper, we consider the space $V = \mathrm{Herm}(n)$ of Hermitian $n \times n$-matrices as a Euclidean space with the trace form $\langle X, Y\rangle = \mathrm{tr}(XY)$. It is naturally acted upon by the unitary group $U(n)$ via conjugation. In various analytic contexts, such as random matrix theory, one is interested in the space $L^2(\mathrm{Herm}(n))^{U(n)}$ consisting of functions $f \in L^2(\mathrm{Herm}(n))$ which are invariant under this action, i.e. depend only on the eigenvalues of their argument. It seems natural to exploit the additional geometric invariance also in wavelet analysis, in order to obtain a discrete wavelet decomposition of $L^2(\mathrm{Herm}(n))^{U(n)}$. Identifying $U(n)$-orbits in $\mathrm{Herm}(n)$ with their ordered spectrum via the spectral theorem will also reduce the dimension of the underlying space from n^2 to n. In a closely related way, one could consider functions on the space $\mathrm{SHerm}(n) = \{X \in \mathrm{Herm}(n) : \mathrm{tr}\, X = 0\}$, which are radial in the sense of conjugation invariance under $SU(n)$. As we shall describe in the appendix (Section 6), the setting of [RR03] just corresponds to $SU(2)$-invariant analysis on $\mathrm{SHerm}(2)$.

The goal of the present paper is to generalize the concepts of [RR03] to higher rank, namely to analysis on $\mathrm{Herm}(n)$ which is radial in the sense of $U(n)$-invariance. We shall introduce a radial multiresolution on $\mathrm{Herm}(n)$ and characterize radial orthonormal wavelet bases. A key ingredient will be the definition of a generalized translation operator on $L^2(\mathrm{Herm}(n))^{U(n)}$, as classical translations of $U(n)$-invariant function need not to be $U(n)$-invariant again. We identify the orbit space of the action of $U(n)$ on $\mathrm{Herm}(n)$ with the closed cone

$$\overline{\mathfrak{a}_+} = \{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\}$$

of ordered spectra of Hermitian matrices via $U(n).X \mapsto \sigma(X) \in \overline{\mathfrak{a}_+}$. Here $\sigma(X)$ denotes the set of eigenvalues of X, ordered by size. The cone $\overline{\mathfrak{a}_+}$ is a closed Weyl chamber of type A_{n-1}. It carries a natural hypergroup structure, where the convolution of point measures $\delta_x * \delta_y$ is a compactly supported probability measure on $\overline{\mathfrak{a}_+}$ which describes the possible spectra of sums of Hermitian matrices $X + Y$ with given spectra $\sigma(X) = x, \sigma(Y) = y$. For generic x and y the measure $\delta_x * \delta_y$ is absolutely continuous with respect to the Lebesgue measure in a certain affine plane in $\mathbb{R}^n$ with an explicit formula for the density. This is a consequence of results in [GS02], see also the survey [GS16]. The generalized translation of suitable functions on $\overline{\mathfrak{a}_+}$ is then given by $T_x f(y) := \delta_x * \delta_y(f)$, and harmonic analysis of $L^2(\mathrm{Herm}(n))^{U(n)}$ will play out as harmonic analysis on the L^2-space $L^2(\overline{\mathfrak{a}_+}, \omega)$ of this hypergroup, where $\omega(x) = \prod_{i<j} |x_j - x_j|^2$. This is due to the fact that by the Weyl integration formula, the classical Fourier transform of $U(n)$-invariant functions on $\mathrm{Herm}(n)$ coincides with a Hankel transform with respect to the spherical functions of the Cartan motion group $U(n) \ltimes \mathrm{Herm}(n)$, which are multivariate Bessel functions. The concept of a radial multiresolution in $\mathrm{Herm}(n)$ will thus be that of a multiresolution in $L^2(\overline{\mathfrak{a}_+}, \omega)$. The scale spaces $(V_j)_{j\in\mathbb{Z}}$ are obtained by (classical) dyadic dilations from V_0, which is in turn spanned by generalized translations along lattice points of a so-called radial scaling function $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$. Similarly to [RR03], and in contrast to classical notions of multiresolution analysis, while still being characterized by a two-scale relation, the scaling function ϕ itself is not contained in V_0 and the scale spaces are not translation invariant with respect to the generalized translation. While the authors in [RR03] explicitly construct a set of orthonormal wavelets from a given multiresolution consisting (due to dimensionality reasons) of only one wavelet, this approach is not feasible in our higher-rank situation as the number of wavelets needed will grow with n. Instead, we will give a concise criterion characterizing orthonormal wavelet bases of $L^2(\overline{\mathfrak{a}_+}, \omega)$

by checking whether a certain matrix-valued function is almost everywhere unitary. This result is resemblant of a similar criterion in classical wavelet theory ([Woj97]) and will allow us to show that given a multiresolution analysis, associated orthonormal wavelet bases always exist and that they in fact consist of $2^n - 1$ wavelets. As functions on $\mathrm{Herm}(n)$, these wavelets reflect the underlying radial symmetry and require reduced computational effort as classical multiresolution on the vector space $\mathrm{Herm}(n)$ would need $2^{n^2} - 1$ wavelets. We are furthermore able to relate radial scaling functions to classical permutation-invariant scaling functions on $\mathbb{R}^n$, yielding a simple example of a radial wavelet basis analogous to the classical Shannon wavelets.

The paper is organized as follows: In Section 2 we recall facts about radial analysis on $\mathrm{Herm}(n)$. We will introduce the generalized translation operators on $L^2(\overline{\mathfrak{a}_+}, \omega)$ and study their properties. This will be crucial for Section 3, where we introduce a radial multiresolution analysis on $\mathrm{Herm}(n)$. In Section 4, orthonormal wavelet bases are discussed and Section 5 then describes how to obtain radial scaling functions from classical scaling functions in $\mathbb{R}^n$. Finally, Section 6 is dedicated to the discussion of generalizations of the previous results. In fact, many arguments remain valid upon replacing $U(n)$ with an arbitrary connected compact Lie group, the space $V = \mathrm{Herm}(n)$ by $\mathfrak{p} = i\mathrm{Lie}(K) \subseteq \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(K)$, and by considering the adjoint action of K on $\mathfrak{p}$. In this setting one can see the results of [RR03] as a special case of rank 1, as the situation corresponds to that of $SU(2)$ acting on $\mathrm{SHerm}(2)$. However, in the general higher-rank case, it would not be clear how to obtain radial scaling functions from classical scaling functions.

2. Radial Analysis and Generalized Translation

The unitary group $U(n),\ n \geq 2$ acts naturally on Hermitian matrices $\mathrm{Herm}(n)$ by conjugation. By the spectral theorem, the orbit space $\mathrm{Herm}(n)^{U(n)}$ of this action can be (actually topologically) identified with the closed chamber

$$\overline{\mathfrak{a}_+} = \{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\}$$

via $U(n).X \mapsto \sigma(X)$, where $\sigma(X)$ is the ordered spectrum of X. We note that $U(n) \ltimes \mathrm{Herm}(n)$ is the Cartan motion group of the complex Lie group $G = GL_n(\mathbb{C})$ with maximal compact subgroup $K = U(n)$, coming from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{p} = \mathrm{Herm}(n)$. $GL_n(\mathbb{C})$ belongs to the so-called Harish-Chandra class (cf. e.g. [GV88]), but in contrast to $SL_n(\mathbb{C})$ it is not semisimple. For the maximal abelian subspace $\mathfrak{a} = \{\mathrm{diag}(x_1, \ldots, x_n) : x_i \in \mathbb{R}\} \cong \mathbb{R}^n$ of $\mathfrak{p}$, the set $\mathfrak{a}_+ = \{x \in \mathbb{R}^n : x_1 > \ldots > x_n\}$ is the positive Weyl chamber corresponding to the positive subsystem $\Sigma_+ = \{e_i - e_j : 1 \leq i < j \leq n\}$ of the root system A_{n-1} in $\mathbb{R}^n$. We shall often identify elements from $\mathbb{R}^n$ with diagonal matrices in the above way.

Recall the Weyl integration formula ([Far08], Thm. 10.1.4; more generally [GV88], eqn.(2.4.22)) which states that

$$\int_{\mathrm{Herm}(n)} F(X)\, dX \;=\; c \int_{\overline{\mathfrak{a}_+}} \int_{U(n)} F(uxu^{-1}) du\ \omega(x) dx$$

for $F \in C_c(\mathrm{Herm}(n))$. Here, $c > 0$ is a constant independent of F and

$$\omega(x) = \prod_{i<j} (x_i - x_j)^2.$$

We thus obtain an isometric isomorphism

$$\Phi\colon L^2(\mathrm{Herm}(n))^{U(n)} \to L^2(\overline{\mathfrak{a}_+}, c\omega), \quad F \mapsto F|_{\overline{\mathfrak{a}_+}} = f.$$

As the action of $U(n)$ on $\mathrm{Herm}(n)$ is via orthogonal transformations, the Euclidean Fourier transform of a radial function $F \in C_c(\mathrm{Herm}(n))^{U(n)}$ is again radial and becomes a Hankel transform of f: for $Y \in \mathrm{Herm}(n)$ with $y = \sigma(Y)$,

$$\begin{aligned}\widehat{F}(Y) &:= \frac{1}{(2\pi)^{n^2/2}} \int_{\mathrm{Herm}(n)} F(X) e^{-i\langle X,Y\rangle} dX \\ &= \frac{c}{(2\pi)^{n^2/2}} \int_{\overline{\mathfrak{a}_+}} f(x) J(x,-iy)\,\omega(x)dx \ =: \mathcal{H}f(y)\end{aligned}$$

with the Bessel functions

$$J(y,z) = \int_{U(n)} e^{\mathrm{tr}(uyu^{-1}z)} du \ = \ \frac{\prod_{k=1}^{n-1} k!}{\pi(y)\pi(z)} \sum_{w\in\mathcal{S}_n} \varepsilon(w) e^{\langle wy,z\rangle} \quad (y,z\in\mathbb{C}^n) \tag{2.1}$$

given by the Harish-Chandra-Itzykson-Zuber formula ([HC57], [IZ80]; see also [McS21] for a recent overview). Here $\varepsilon(w)$ denotes the sign of w, the scalar product $\langle\,.\,,\,.\,\rangle$ is extended in a bilinear way to $\mathbb{C}^n$, and

$$\pi(z) = \prod_{i<j}(z_i - z_j)$$

denotes the fundamental alternating polynomial, also known as the Vandermonde determinant. Thus we get a Plancherel theorem extending the Hankel transform $\mathcal{H}$ to a unitary operator on $L^2(\overline{\mathfrak{a}_+},\omega)$. Note that the Bessel function J satisfies $J(\lambda x, z) = J(x,\lambda z)$ for $\lambda\in\mathbb{C}$ and that it is $\mathcal{S}_n$-invariant in both arguments. We may therefore extend $\mathcal{H}f$ to an $\mathcal{S}_n$-invariant (i.e. symmetric) function on $\mathbb{R}^n$ whenever convenient.

Usual translations $F(\cdot - y)$ of radial functions on $\mathrm{Herm}(n)$ need not be radial again. We therefore define generalized translations for $f\in C_c(\overline{\mathfrak{a}_+})$ by averaging with respect to the $U(n)$-action:

$$T_x f(y) := \int_{U(n)} f(\sigma(x + uyu^{-1}))\,du \ =: (\delta_x * \delta_y)(f)$$

Note that $\delta_x * \delta_y$ defines a compactly supported probability measure on $\overline{\mathfrak{a}_+}$ and that $\delta_y * \delta_x = \delta_x * \delta_y$. As an immediate consequence of Fubini's theorem and the left-invariance of the Haar measure we obtain the product formula

$$(\delta_x * \delta_y)J(\cdot,z) = J(x,z)J(y,z).$$

This just says that when considered as functions on $\mathbb{R}^n$, the functions $J(\,.\,,z)$, $z\in\mathbb{C}^n$ are spherical functions of the Gelfand pair $(U(n)\ltimes \mathrm{Herm}(n), U(n))$. Indeed, all spherical functions are of this form, which follows e.g. from [Wol06], Theorem 4.4, and the bounded spherical functions are those with $z\in i\mathbb{R}^n$. Moreover, for radial F,G on $\mathrm{Herm}(n)$,

$$\begin{aligned}\int_{\overline{\mathfrak{a}_+}} T_y f(x) g(x)\omega(x)dx &= \int_{\mathrm{Herm}(n)} F(X+y)G(X)dX \\ &= \int_{\mathrm{Herm}(n)} F(X)G(X-y)dX \ = \ \int_{\overline{\mathfrak{a}_+}} f(x) T_{\overline{y}} g(x)\omega(x)dx\end{aligned} \tag{2.2}$$

with $\overline{y} = -(y_n,\dots,y_1)$, provided the integrals exist. In a similar way, it is checked that $\|T_y f\|_{2,\omega} \le \|f\|_{2,\omega}$. Hence the translation operators T_y extend to norm-decreasing linear operators on $L^2(\overline{\mathfrak{a}_+},\omega)$.

Together with the above product formula, relation (2.2) implies that for $f\in L^2(\overline{\mathfrak{a}_+},\omega)$,

$$\mathcal{H}(T_y f)(x) = J(x,iy)\mathcal{H}f(x). \tag{2.3}$$

Indeed, the above convolution $\delta_x * \delta_y$ of point measures δ_x and δ_y defines a commutative orbit hypergroup structure on $\overline{\mathfrak{a}_+}$ in the sense of [Jew75], Sec. 8 (where hypergroups are called convos). The neutral element is $0\in\overline{\mathfrak{a}_+}$, the involution is given by $x\mapsto \overline{x}$ as defined above and $\omega(x)dx$ is a Haar measure. For details on hypergroups related to motion groups as in the present setting, see also [RV08].

In Section 3 we shall need some more refined information on the measures $\delta_x * \delta_y$. Indeed, a combination and adaption of results from Helgason [Hel00], Graczyk, Sawyer [GS02] and Graczyk, Loeb [GL95] for the semisimple case shows that for $x, y \in \mathfrak{a}_+$ the measure $\delta_x * \delta_y$ is absolutely continuous with respect to the Lebesgue measure on a certain affine plane in $\mathbb{R}^n$, and gives an explicit formula for the density. To make this precise, we orthogonally decompose $\mathbb{R}^n = \mathbb{R}^n_0 \oplus \mathbb{R}\,\underline{1}$ with

$$\mathbb{R}^n_0 = \{x \in \mathbb{R}^n : x_1 + \ldots + x_n = 0\}, \quad \underline{1} = (1, \ldots, 1)$$

and denote by x^0 and x^1 the orthogonal projections of $x \in \mathbb{R}^n$ onto $\mathbb{R}^n_0$ and $\mathbb{R}\underline{1}$, respectively. We further put $\mathbb{C}^n_0 := \{z \in \mathbb{C}^n : z_1 + \ldots + z_n = 0\}$ and

$$\mathfrak{a}_{+0} := \mathbb{R}^n_0 \cap \mathfrak{a}_+.$$

A canonical basis of $\mathbb{R}^n_0$ is given by the simple roots

$$\alpha_i := e_i - e_{i+1}, \qquad 1 \le i \le n-1,$$

and we denote the remaining elements in $\Sigma_+ \setminus \{\alpha_1, \ldots, \alpha_{n-1}\}$ by $\alpha_n, \ldots, \alpha_q$. Let $q := |\Sigma_+| = \frac{1}{2}n(n-1)$. Following [GL95, GS02], we express $\alpha_k = \sum_{j=1}^{n-1} a_{kj}\alpha_j$ for $k = n, \ldots, q$ in the basis above and define a subset $\Delta(y_1, \ldots, y_{n-1}) \subseteq \mathbb{R}^{q-n+1}$ via

$$(y_n, \ldots, y_q) \in \Delta(y_1, \ldots, y_{n-1}) \iff y_n, \ldots, y_q \ge 0 \quad \text{and} \quad \sum_{k=n}^{q} y_k a_{kj} \le y_j, \quad j = 1, \ldots, n-1.$$

For $n \ge 3$, we further define a function $T\colon \mathbb{R}^n_0 \to \mathbb{R}$ by

$$T(y_1\alpha_1 + \ldots + y_{n-1}\alpha_{n-1}) = \int_{\Delta(y_1, \ldots, y_{n-1})} dy_n \ldots dy_q.$$

If $n = 2$, then we define T to jump from 1 inside $\overline{\mathfrak{a}_{+0}}$ to 0 on $\mathbb{R}^2_0 \setminus \overline{\mathfrak{a}_{+0}}$.

In order to describe some further properties of T, we introduce in $\mathbb{R}^n_0$ the dual cone of $\overline{\mathfrak{a}_{+0}}$, which is given by

$$^+\mathfrak{a} = \{x \in \mathbb{R}^n_0 : x = \sum_{j=1}^{n-1} c_j\alpha_j,\ c_j > 0\} \subseteq \mathbb{R}^n_0.$$

We first state some general facts which will be useful later on.

Lemma 2.1. (1) *If $x \in \mathfrak{a}_+$, then $x - wx \in \overline{^+\mathfrak{a}}$ for all $w \in \mathcal{S}_n$.*

(2) *If $x \in \overline{\mathfrak{a}_+}$ and $h \in C(x)$, then $x - h \in \overline{^+\mathfrak{a}}$.*

(3) *$\rho \in \mathfrak{a}_{+0}$.*

Proof. See [Hel00], Ch. IV, Lemma 8.3. and observe that $x - wx \in \mathbb{R}^n_0$ for $x \in \mathbb{R}^n$. □

We shall need the following facts about T, established by Graczyk and Loeb:

Lemma 2.2 ([GL95], Prop. 2)**.** *T is supported in $\overline{^+\mathfrak{a}}$ and continuous and nonnegative on $\overline{^+\mathfrak{a}}$. If $x, y \in \overline{^+\mathfrak{a}}$ with $y - x \in \overline{^+\mathfrak{a}}$, then $T(x) \le T(y)$.*

For regular arguments $x, y \in \mathfrak{a}_+$, the convolution product $\delta_x * \delta_y$ is now described as follows in terms of T:

Proposition 2.3. *For $x, y \in \mathfrak{a}_+$, the measure $\delta_x * \delta_y$ is absolutely continuous with respect to the Lebesgue measure on $x^1 + y^1 + \overline{\mathfrak{a}_{+0}} \subseteq x^1 + y^1 + \mathbb{R}^n_0$. The density is given by*

$$k(x,y,h) = \frac{\pi(\rho)\pi(h)}{\pi(x)\pi(y)} \sum_{v,w \in \mathcal{S}_n} \varepsilon(v)\varepsilon(w)\, T(vy + wx - h).$$

Here $\rho := \frac{1}{2}\sum_{i<j}(e_i - e_j) = \frac{1}{2}(n-1, n-3, \dots, -n+1)$ denotes the Weyl vector.

Proof. Throughout the paper, we denote the pushforward of a measure μ under a measurable mapping φ by $\varphi_*(\mu)$. As $U(n) = H \ltimes SU(n)$ with $H = \{d_z = \operatorname{diag}(z, 1, \dots, 1),\, z \in U(1)\}$, Weyl's integration formula gives

$$(\delta_x * \delta_y)(f) = \int_{U(1)} \int_{SU(n)} f(\sigma(x + d_z k y k^{-1} d_z^{-1}))dk\, dz = \int_{SU(n)} f(x^1 + y^1 + \sigma(x^0 + k y^0 k^{-1}))dk$$

for $x, y \in \overline{\mathfrak{a}_+}$. This shows that

$$\delta_x * \delta_y = [\xi \mapsto \xi + x^1 + y^1]_*(\delta_{x^0} *_0 \delta_{y^0}),$$

where

$$(\delta_{x^0} *_0 \delta_{y^0})(g) := \int_{SU(n)} g(\sigma(x^0 + k y^0 k^{-1}))dk$$

for $g \in C(\overline{\mathfrak{a}_{+0}})$. The measure $\delta_{x^0} *_0 \delta_{y^0}$ is a compactly supported probability measure on $\overline{\mathfrak{a}_{+0}}$ which describes the orbit hypergroup convolution on the space $\mathrm{SHerm}(n)^{SU(n)}$, where $\mathrm{SHerm}(n)$ are the Hermitian matrices of trace 0, acted upon by $SU(n)$ via conjugation. Observe that for $z \in \mathbb{C}^n_0$,

$$J(x^0, z) = \int_{SU(n)} e^{\operatorname{tr}(k x^0 k^{-1} z)} dk$$

and

$$(\delta_{x^0} *_0 \delta_{y^0})(J(\cdot, z)) = J(x^0, z) J(y^0, z).$$

The functions $\psi_z = J(\cdot, iz)|_{\mathbb{R}^n_0},\ z \in \mathbb{C}^n_0$ are just the spherical functions of the Cartan motion group $SU(n) \ltimes \mathrm{SHerm}(n)$. They are related to the spherical functions ϕ_z of the Riemannian symmetric space $SL(n, \mathbb{C})/SU(n)$ of complex type via

$$\psi_z(x) = \frac{\widetilde{\Delta}(x)}{\pi(x)} \phi_z(e^x)$$

with

$$\widetilde{\Delta}(x) = \prod_{i<j} \sinh(x_i - x_j),$$

see [Hel00], Ch. IV, Prop. 4.10. and Ch. II, Thm. 3.15, or the nice presentation in [BSO05], Sect. 9 (where the definition of ρ differs by a factor 2 from ours). Now the claim follows from [GS02], Thm. 2.1 and Prop. 3.1, which contain an explicit expression for the density in the product formula of the ϕ_z and thus for $\delta_{x^0} *_0 \delta_{y^0}$. Note that the authors there work with $\mathcal{S}_n$-invariant measures on $\mathfrak{a} = \mathbb{R}^n_0$, while we work with measures supported in $\overline{\mathfrak{a}_{+0}}$. This results in a factor $1/|\mathcal{S}_n|$ in [GS02], Prop. 3.1. We further note that in [GS02] the authors missed a factor $2^{|\Sigma_+|} = 2^{\frac{1}{2}n(n-1)}$ in equation (3), which leads to a missing factor $2^{-|\Sigma_+|}$ in Prop. 3.1. We fix this in our formulation by denoting ρ as the half-sum of roots not weighted with their multiplicities. $\square$

Remark 2.4. Note that with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^n_0 \oplus \mathbb{R}\underline{1}$ we can write

$$\delta_x * \delta_y = (\delta_{x^0} *_0 \delta_{y^0}) \otimes \delta_{x^1+y^1} .$$

For $x \in \mathbb{R}^n$ we define $C(x) = \mathrm{conv}(\mathcal{S}_n.x)$ as the convex hull of the $\mathcal{S}_n$-orbit of x.

Lemma 2.5. *Suppose $x, y \in \mathfrak{a}_+$ such that $y + C(x) \subseteq \mathfrak{a}_+$. Then*

$$\mathrm{supp}\,(\delta_x * \delta_y) \subseteq\; y + C(x).$$

Proof. This follows from [GS02], Cor. 2.2. combined with the fact that $\delta_x * \delta_y = (\delta_{x^0} *_0 \delta_{y^0}) \otimes \delta_{x^1+y^1}$. Note again that the authors there work with an $\mathcal{S}_n$-invariant formulation. As we assumed $y + C(x) \subseteq \mathfrak{a}_+$, the pushforward measures $w_*(\delta_y * \delta_x)$ have disjoint supports, resulting in the expression above. □

Besides the norm-decreasing translation operators T_y on $L^2(\overline{\mathfrak{a}_+}, \omega)$, we will need dilation operators in order to establish a radial multiresolution. These can be defined in the usual way: Note that ω is homogeneous of degree $2|\Sigma_+|$ and put

$$m := \dim_{\mathbb{R}} \mathrm{Herm}(n) + 2|\Sigma_+| \;= 2n^2 - n.$$

Then one easily calculates that for $a > 0$,

$$D_a f(x) := a^{-\frac{m}{2}} f(\tfrac{1}{a}x) \tag{2.4}$$

defines a unitary operator D_a on $L^2(\overline{\mathfrak{a}_+}, \omega)$ which satisfies

$$\mathcal{H}(D_a f)(x) = D_{1/a}(\mathcal{H}f)(x). \tag{2.5}$$

Finally, we will need an orthonormal basis of $L^2(\overline{\mathfrak{a}_+}, \omega)$ which behaves nicely with respect to the Hankel transform. To this end, we consider the Schur polynomials $(s_\lambda)_{\lambda \in P_+}$ in n variables which are indexed by the partitions of length at most n,

$$P_+ = \{\lambda \in \mathbb{N}_0^n : \lambda_1 \geq \ldots \geq \lambda_n\} = \mathbb{Z}^n \cap \overline{\mathfrak{a}_+}$$

and given by

$$s_\lambda(x) = \frac{1}{\Delta(x)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) e^{i\langle \lambda + \rho, wx\rangle} \;=\; \frac{A_{\lambda+\delta}(e^{ix})}{A_\delta(e^{ix})}, \quad x \in \mathbb{R}^n$$

with

$$\Delta(x) = \prod_{i<j} (e^{i(x_i - x_j)/2} - e^{-i(x_i - x_j)/2}) = \sum_{w \in \mathcal{S}_n} \varepsilon(w) e^{i\langle \rho, wx\rangle},$$

$$A_\lambda(e^{ix}) := \det\big(e^{i\lambda_j x_k}\big)_{1 \leq j,k \leq n} \quad \text{and} \quad \delta = (n-1, n-2, \ldots, 0).$$

Note that

$$\Delta(x) = e^{-i\frac{n-1}{2}\langle x, \underline{1}\rangle} A_\delta(e^{ix}),$$

so that in particular $|\Delta|$ is $\mathcal{S}_n$-invariant and I-periodic for

$$I = 2\pi\mathbb{Z}^n.$$

It is well-known that the s_λ, $\lambda \in P_+$ are the Weyl characters of $U(n)$, considered as functions on its maximal torus $T \cong \mathbb{T}^n$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. By the Schur orthogonality relations and the Peter-Weyl theorem, they form an orthogonal basis of the Hilbert space

$$S := L^2(\mathbb{T}^n, |\Delta|^2)^{\mathcal{S}_n}$$

consisting of those functions $f \in L^2(\mathbb{T}^n, |\Delta|^2)$ which are $\mathcal{S}_n$-invariant. More precisely, the Schur polynomials satisfy

$$\frac{1}{n!}\int_{\mathbb{T}^n} s_\lambda(x)\overline{s_\mu(x)}\,|\Delta(x)|^2 dx = \delta_{\lambda,\mu}.$$

For details, see [Bum13], Thm. 22.2 ff. together with Thm. 36.2, or [Far08], Section 11. We consider the following renormalization of Schur polynomials:

$$S_\lambda(x) := \frac{1}{i^{|\Sigma_+|}\sqrt{n!}}\, s_\lambda(x), \quad \lambda \in P_+\,.$$

The $(S_\lambda)_{\lambda\in P_+}$ form an orthonormal basis of S. Moreover, in view of formula (2.1), they are connected with the Bessel function J via

$$S_\lambda(x) = M_\lambda \frac{\pi(x)}{\Delta(x)} J(x, i(\lambda+\rho)) \quad \text{with} \quad M_\lambda = \frac{\pi(\lambda+\rho)}{\sqrt{n!}\,\prod_{k=1}^{n-1} k!}\,. \tag{2.6}$$

This connection and the periodicity of the S_λ will be of crucial importance for our constructions, together with the observation that $\pi^2 = \omega$ on $\mathbb{R}^n$, where ω occurs as the density of the Haar measure of the hypergroup $(\overline{\mathfrak{a}_+}, *)$ discussed before. Sometimes it will be convenient to work on the fundamental domain

$$D = [0, 2\pi[^n$$

of the torus $\mathbb{T}^n = \mathbb{R}^n/I$ and use the identification

$$S \cong \{\alpha : \mathbb{R}^n \to \mathbb{C} : \alpha|_D \in L^2(D, |\Delta|^2)^{\mathcal{S}_n}, \alpha(x+q) = \alpha(x) \text{ a.e. } (\forall q \in I)\}. \tag{2.7}$$

For abbreviation we further introduce the notation

$$T^{(\lambda)} := T_{\lambda+\rho} \quad (\lambda \in P_+).$$

As a consequence of identity (2.3) for the Hankel transform, we have

$$\mathcal{H}(M_\lambda T^{(\lambda)} f)(\xi) = \frac{\Delta(\xi) S_\lambda(\xi)}{\pi(\xi)} \mathcal{H}f(\xi) \tag{2.8}$$

for $f \in L^2(\overline{\mathfrak{a}_+}, \omega)$.

3. Radial Multiresolution Analysis in Herm(n)

We start with the definition of a radial multiresolution analysis (MRA) on Herm(n). It is a higher-rank anlogue of the concept of a radial MRA in $\mathbb{R}^3$ introduced in [RR03].

Definition 3.1. We call a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\overline{\mathfrak{a}_+}, \omega)$ a (dyadic) radial MRA in Herm(n), if it satisfies the following properties:

(1) $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$,

(2) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$,

(3) $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\overline{\mathfrak{a}_+}, \omega)$,

(4) $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$,

(5) there exists a function $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ such that

$$B_\phi := \left\{ M_\lambda T^{(\lambda)} \phi : \lambda \in P_+ \right\}$$

is a Riesz basis of V_0, i.e. $\operatorname{span} B_\phi$ is dense in V_0 and there exist constants $A, B > 0$ such that

$$A\|\alpha\|_2^2 \leq \left\| \sum_{\lambda \in P_+} \alpha_\lambda M_\lambda T^{(\lambda)} \phi \right\|_2^2 \leq B\|\alpha\|_2^2$$

for all $\alpha = (\alpha_\lambda)_{\lambda \in P_+} \in \ell^2(P_+)$.

The function ϕ is called *scaling function* for the MRA $(V_j)_{j\in\mathbb{Z}}$.

For $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ consider the function

$$P_\phi(\xi) := \frac{1}{n!} \sum_{q \in I} |\mathcal{H}\phi(\xi + q)|^2 \qquad (\xi \in \mathbb{R}^n),$$

which is $\mathcal{S}_n$-invariant and I-periodic.

Proposition 3.2. *Let $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ and $A, B > 0$. Then*

$$A\|\alpha\|_2^2 \leq \left\| \sum_{\lambda \in P_+} \alpha_\lambda M_\lambda T^{(\lambda)} \phi \right\|_2^2 \leq B\|\alpha\|_2^2 \quad \text{for all } \alpha \in \ell^2(P_+)$$

if and only if

$$A \leq P_\phi(\xi) \leq B \quad \text{for almost all } \xi \in \overline{\mathfrak{a}_+}.$$

Proof. Let $\alpha \in \ell^2(P_+)$ be a finite sequence. Define

$$\widetilde{\alpha} := \sum_{\lambda \in P_+} \alpha_\lambda S_\lambda \in S.$$

Using the Plancherel theorem for the Hankel transform and Eq. (2.8) we obtain

$$\begin{aligned}
\left\| \sum_{\lambda \in P_+} \alpha_\lambda M_\lambda T^{(\lambda)} \phi \right\|_2^2 &= \left\| \sum_{\lambda \in P_+} \alpha_\lambda \frac{\Delta S_\lambda}{\pi} \mathcal{H}\phi \right\|_2^2 \\
&= \int_{\mathfrak{a}_+} \left| \sum_{\lambda \in P_+} \alpha_\lambda S_\lambda(\xi) \right|^2 |\mathcal{H}\phi(\xi)|^2 \frac{\omega(\xi)}{|\pi(\xi)|^2} |\Delta(\xi)|^2 d\xi \\
&= \frac{1}{n!} \int_{\mathbb{R}^n} |\widetilde{\alpha}(\xi)|^2 \, |\mathcal{H}\phi(\xi)|^2 |\Delta(\xi)|^2 d\xi \\
&= \frac{1}{n!} \sum_{q \in I} \int_{\mathbb{T}^n} |\widetilde{\alpha}(\xi + q)|^2 |\mathcal{H}\phi(\xi + q)|^2 |\Delta(\xi)|^2 d\xi \\
&= \int_{\mathbb{T}^n} |\widetilde{\alpha}(\xi)|^2 P_\phi(\xi) \; |\Delta(\xi)|^2 d\xi.
\end{aligned}$$

The $\{S_\lambda : \lambda \in P_+\}$ forming an orthonormal basis of S, we have

$$\|\alpha\|_2 = \|\widetilde{\alpha}\|_S \,.$$

Now the assertion is immediate, since finite sequences are dense in $\ell^2(P_+)$. ☐

With $A = B = 1$ we obtain

Corollary 3.3. *For $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ the following are equivalent:*

(1) *The set* $B_\phi := \{M_\lambda T^{(\lambda)}\phi : \lambda \in P_+\}$ *is orthonormal in* $L^2(\overline{\mathfrak{a}_+}, \omega)$.

(2) $P_\phi = 1$ *a.e.*

For $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$, put

$$V_\phi := \overline{\operatorname{span} B_\phi} \subseteq L^2(\overline{\mathfrak{a}_+}, \omega).$$

The set B_ϕ is a Riesz basis of V_ϕ if and only if the two equivalent conditions in the previous proposition are satisfied for some constants $A, B > 0$. In this case we say that ϕ satisfies *condition (RB)*.

Lemma 3.4. *Let* $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ *satisfy (RB). Then for* $f \in L^2(\overline{\mathfrak{a}_+}, \omega)$ *we have the equivalence*

$$f \in V_\phi \quad \Longleftrightarrow \quad \mathcal{H}f(\xi) = \frac{\Delta(\xi)\beta(\xi)}{\pi(\xi)}\,\mathcal{H}\phi(\xi) \quad \text{with} \quad \beta \in S.$$

The function $f \in V_\phi$ *corresponding to* $\beta = \sum_{\lambda \in P_+} \alpha_\lambda S_\lambda \in S$ *with* $\alpha \in \ell^2(P_+)$ *is given by* $f = \sum_{\lambda \in P_+} \alpha_\lambda M_\lambda T^{(\lambda)}\phi$.

Proof. As the $M_\lambda T^{(\lambda)}\phi$ form a Riesz basis for V_ϕ, the functions

$$\frac{\Delta S_\lambda}{\pi}\,\mathcal{H}\phi = \mathcal{H}(M_\lambda T^{(\lambda)}\phi)$$

form a Riesz basis of $\mathcal{H}(V_\phi)$, where we again used Eq. (2.8). This yields the assertion, since for $\alpha \in \ell^2(P_+)$,

$$\mathcal{H}\Big(\sum_{\lambda \in P_+} \alpha_\lambda M_\lambda T^{(\lambda)}\phi\Big) = \frac{\Delta}{\pi}\Big(\sum_{\lambda \in P_+} \alpha_\lambda S_\lambda\Big)\mathcal{H}\phi.$$

□

When ϕ is the scaling function of an MRA $\{V_j\}$, then $V_\phi = V_0$. The following corollary shows that in contrast to the classical notion of an MRA, V_0 is not shift-invariant (and similarly the other scale spaces):

Corollary 3.5. *Let* $(V_j)_{j\in\mathbb{Z}}$ *be a radial MRA. Then* $f \in V_0$ *implies* $T^{(\lambda)}f \notin V_0$ *for all* $\lambda \in P_+$.

Proof. Recall that $\mathcal{H}(T^{(\lambda)}f) = J(\,.\,, i(\lambda+\rho))\mathcal{H}f$. But if $\beta \in S$, then $J(\cdot, i(\lambda+\rho))\,\beta \notin S$ as the functions $J(\cdot, i(\lambda+\rho))$ are not periodic. Now the previous lemma implies the assertion. □

Theorem 3.6 (Orthonormalization)**.** *Suppose that* $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ *satisfies condition (RB), i.e. there are constants* $A, B > 0$ *such that* $A \le P_\phi(\xi) \le B$ *a.e. Define* $\phi^* \in L^2(\overline{\mathfrak{a}_+}, \omega)$ *by its Hankel transform*

$$\mathcal{H}\phi^* := \frac{\mathcal{H}\phi}{\sqrt{P_\phi}}. \tag{3.1}$$

Then $B_{\phi^*} = \{M_\lambda T^{(\lambda)}\phi^* : \lambda \in P_+\}$ *forms an orthonormal basis of* $V_\phi = V_{\phi^*}$.

Proof. Note first that as a consequence of condition (RB), the right side in (3.1) belongs to $L^2(\overline{\mathfrak{a}_+}, \omega)$. By definition $P_{\phi^*} = 1$ a.e., so by Corollary 3.3 it only remains to prove $V_\phi = V_{\phi^*}$. For this, it suffices to verify that $M_\lambda T^{(\lambda)}\phi^* \in V_\phi$ and $M_\lambda T^{(\lambda)}\phi \in V_{\phi^*}$ for all $\lambda \in P_+$. We employ Lemma 3.4 and see

$$M_\lambda T^{(\lambda)}\phi^* \in V_\phi \quad \Longleftrightarrow \quad \mathcal{H}(M_\lambda T^{(\lambda)}\phi^*) = \frac{\Delta\beta}{\pi}\mathcal{H}\phi \quad \text{for some } \beta \in S.$$

But

$$\mathcal{H}(M_\lambda T^{(\lambda)}\phi^*) = \frac{\Delta S_\lambda}{\pi}\frac{\mathcal{H}\phi}{\sqrt{P_\phi}}.$$

Analogously we get

$$M_\lambda T^{(\lambda)}\phi \in V_{\phi^*} \quad \Longleftrightarrow \quad \mathcal{H}(M_\lambda T^{(\lambda)}\phi) = \frac{\Delta\widetilde{\beta}}{\pi}\mathcal{H}\phi^* = \frac{\Delta\widetilde{\beta}}{\pi}\frac{\mathcal{H}\phi}{\sqrt{P_\phi}} \quad \text{for some } \widetilde{\beta} \in S$$

and

$$\mathcal{H}(M_\lambda T^{(\lambda)}\phi) = \frac{\Delta S_\lambda}{\pi}\mathcal{H}\phi.$$

This gives, for all $\lambda \in P_+$, the conditions

$$\frac{S_\lambda}{\sqrt{P_\phi}} \in S \quad \text{and} \quad S_\lambda\sqrt{P_\phi} \in S.$$

But these conditions are guaranteed by our assumption on P_ϕ. □

Given a function $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ satisfying (RB) plus some additional conditions (see Proposition 3.7 and Theorem 3.8 below), we will now construct a radial MRA having ϕ as scaling function. Recall the unitary dilations 2.4 and define scale spaces $\{V_j\}_{j\in\mathbb{Z}}$ by

$$V_0 \coloneqq V_\phi, \quad V_j \coloneqq D_{2^{-j}}V_0.$$

Then property (4) of Definition 3.1 is satisfied by construction. Putting

$$\phi_{j,\lambda}(\xi) \coloneqq D_{2^{-j}}(M_\lambda T^{(\lambda)}\phi)(\xi) = 2^{\frac{jm}{2}}M_\lambda(T^{(\lambda)}\phi)(2^j\xi) \quad (j \in \mathbb{Z},\, \lambda \in P_+), \tag{3.2}$$

we have

$$\langle\phi_{j,\lambda}, \phi_{j,\mu}\rangle = \langle\phi_{0,\lambda}, \phi_{0,\mu}\rangle$$

since dilations are unitary. Thus $\{\phi_{j,\lambda} : \lambda \in P_+\}$ is a Riesz basis of V_j with the same Riesz constants $A, B > 0$ as for ϕ. In particular,

$$V_j = \overline{\operatorname{span}\{\phi_{j,\lambda} : \lambda \in P_+\}}.$$

Moreover, if $B_\phi = \{\phi_{0,\lambda} : \lambda \in P_+\}$ is an orthonormal basis of V_0, then $\{\phi_{j,\lambda} : \lambda \in P_+\}$ constitutes an orthonormal basis of V_j. We shall now analyze the further required properties of Definition 3.1 in this case. We start with the condition that the scale spaces should be nested.

Proposition 3.7. *For $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ satisfying (RB) and the spaces V_j defined above, the following statements are equivalent:*

(1) *$V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$.*

(2) *$V_{-1} \subseteq V_0$.*

(3) *There exists a function $\gamma \in S$ such that ϕ satisfies the two-scale relation*

$$\Delta(2\xi)\mathcal{H}\phi(2\xi) = \gamma(\xi)\Delta(\xi)\mathcal{H}\phi(\xi).$$

In this case, if we expand $\phi_{-1,0}$ in V_0 as $\phi_{-1,0} = \sum_{\lambda\in P_+} \alpha_\lambda \phi_{0,\lambda}$ with $\alpha = (\alpha_\lambda)_{\lambda\in P_+} \in \ell^2(P_+)$, then $\gamma = c\cdot \sum_{\lambda\in P_+} \alpha_\lambda S_\lambda$ with $c = 2^{-\frac{n^2}{2}} i^{|\Sigma_+|}\sqrt{n!}$.

Proof. The equivalence of (1) and (2) is immediate by rescaling. Suppose now $V_{-1}\subseteq V_0$. Then by Lemma 3.4 shows that

$$\mathcal{H}\phi_{-1,0} = \frac{\Delta\beta}{\pi}\mathcal{H}\phi$$

with some $\beta\in S$. On the other hand, using equations (2.8) and (2.5) we calculate

$$\begin{aligned}\mathcal{H}\phi_{-1,0}(\xi) &= \mathcal{H}(D_2M_0T^{(0)}\phi)(\xi) \;=\; D_{1/2}\Big(\frac{\Delta S_0}{\pi}\mathcal{H}\phi\Big)(\xi)\\ &= 2^{m/2}\,\frac{\Delta(2\xi)S_0(2\xi)}{\pi(2\xi)}\mathcal{H}\phi(2\xi) \;=\; \frac{2^{n^2/2}}{i^{|\Sigma_+|}\sqrt{n!}}\,\frac{\Delta(2\xi)}{\pi(\xi)}\mathcal{H}\phi(2\xi).\end{aligned}$$

We conclude that

$$\Delta(2\xi)\mathcal{H}\phi(2\xi) = c\cdot\beta(\xi)\Delta(\xi)\mathcal{H}\phi(\xi) \tag{3.3}$$

with c as astated and obtain the desired result with $\gamma = c\beta\in S$. If $\phi_{-1,0} = \sum_{\lambda\in P_+}\alpha_\lambda\phi_{0,\lambda}$, we employ Lemma 3.4 and see that $\beta = c^{-1}\gamma = \sum_{\lambda\in P_+}\alpha_\lambda S_\lambda$.

Conversely, assume that (3) is satisfied. Performing the same calculation as above for $\phi_{-1,\lambda}$ leads to

$$\mathcal{H}\phi_{-1,\lambda} = \text{const.}\,\frac{S_\lambda(2\xi)}{\pi(\xi)}\Delta(2\xi)\mathcal{H}\phi(2\xi)$$

and an application of Lemma 3.4 shows that $\phi_{-1,\lambda}\in V_0$ if and only if

$$S_\lambda(2\xi)\Delta(2\xi)\mathcal{H}\phi(2\xi) = \gamma_\lambda(\xi)\Delta(\xi)\mathcal{H}\phi(\xi) \tag{3.4}$$

for some $\gamma_\lambda\in S$. Recall that ϕ satisfies the two-scale relation (3). Choosing

$$\gamma_\lambda(\xi) := \text{const.}\,\gamma(\xi)S_\lambda(2\xi)\in S$$

we thus conclude $\phi_{-1,\lambda}\in V_0$ and therefore $V_{-1}\subseteq V_0$. □

Theorem 3.8. *Let $\phi\in L^2(\overline{\mathfrak{a}_+},\omega)$ satisfy condition (RB) and assume that the scale spaces are nested, i.e. $V_j\subseteq V_{j+1}$ for all $j\in\mathbb{Z}$. Suppose further that $|\mathcal{H}\phi|$ is continuous in 0. Then $(V_j)_{j\in\mathbb{Z}}$ is a radial MRA if and only if $\mathcal{H}\phi(0)\neq 0$. Moreover, if ϕ is an orthonormal scaling function then $|\mathcal{H}\phi(0)| = 1$.*

Proof. The idea is the same as in the proof of [RR03], Thm. 4.9, but now the proof is much more involved, as sufficient knowledge of the generalized translation will be needed, which was simple and explicit in [RR03]. To start with, we use Theorem 3.6 and obtain an orthonormal scaling function $\phi^*\in L^2(\overline{\mathfrak{a}_+},\omega)$ with the same scale spaces. In order to prove that $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ we have to show $\lim_{j\to-\infty}\|P_jf\|_2 = 0$ for all $f\in L^2(\overline{\mathfrak{a}_+},\omega)$, where

$$P_j\colon L^2(\overline{\mathfrak{a}_+},\omega)\to V_j,\quad f\mapsto \sum_{\lambda\in P_+}\langle f,\phi^*_{j,\lambda}\rangle\,\phi^*_{j,\lambda}$$

denotes the orthogonal projection and $\phi^*_{j,\lambda}$ is defined as in (3.2). As continuous functions with compact support are dense in $L^2(\overline{\mathfrak{a}_+},\omega)$ we may assume that $f\in C_c(\overline{\mathfrak{a}_+})$. Denoting $K:=\operatorname{supp} f$ and using Parseval's identity we obtain

$$\begin{aligned}
\|P_jf\|_2^2 &= \sum_{\lambda\in P_+}|\langle f,\phi^*_{j,\lambda}\rangle|^2\\
&\le \|f\|_2^2\cdot\int_K\sum_{\lambda\in P_+}|\phi^*_{j,\lambda}(\xi)|^2\,\omega(\xi)d\xi\\
&= \text{const.}\int_{2^jK}\sum_{\lambda\in P_+}M_\lambda^2\,|T^{(\lambda)}\phi^*(\xi)|^2\,\omega(\xi)d\xi\\
&= \text{const.}\int_{2^jK\cap\mathfrak{a}_+}\sum_{\lambda\in P_+}M_\lambda^2\,|(\delta_{\lambda+\rho}*\delta_\xi)(\phi^*)|^2\,\omega(\xi)d\xi,
\end{aligned}$$

where above and in the sequel, const. denotes a varying positive constant depending on f and n only. Observe that $\lambda+\rho\in\mathfrak{a}_+$ for all $\lambda\in P_+$ and therefore Proposition 2.3 applies to $\delta_{\lambda+\rho}*\delta_\xi$ in the integral above. We may further assume that $j\in\mathbb{Z}$ is negative and already so small that $\lambda+\rho+C(\xi)\subseteq\mathfrak{a}_+$ for all $\lambda\in P_+$. Hence, by Lemma 2.5, $\operatorname{supp}(\delta_{\lambda+\rho}*\delta_\xi)\in\lambda+\rho+C(\xi)$ for all λ. In view of Proposition 2.3 we obtain

$$(\delta_{\lambda+\rho}*\delta_\xi)(\phi^*)=\int_{\lambda+\rho+C(\xi)}\phi^*(x)\frac{\pi(\rho)\pi(x)}{\pi(\lambda+\rho)\pi(\xi)}\sum_{v,w\in\mathcal{S}_n}\varepsilon(v)\varepsilon(w)T(v\xi+w(\lambda+\rho)-x)\;dx. \quad (3.5)$$

Here dx denotes the Lebesgue volume in the affine hyperplane $\lambda^1+\xi^1+\mathbb{R}^n_0\subseteq\mathbb{R}^n$ which contains $\lambda+\rho+C(\xi)$, where still $\xi\in 2^jK\cap\mathfrak{a}_+$ with $j<0$. Recall at this point the decomposition $\mathbb{R}^n=\mathbb{R}^n_0\oplus\mathbb{R}\underline{1}$, with orthogonal projections x^0 of $x\in\mathbb{R}^n$ onto $\mathbb{R}^n_0$ and x^1 onto $\mathbb{R}\underline{1}$. Using Lemmata 2.2 and 2.1 we further see that for all $x\in\lambda+\rho+C(\xi)$ and $v,w\in\mathcal{S}_n$,

$$T(v\xi+w(\lambda+\rho)-x)\le T(\lambda+\rho+\xi-x).$$

Moreover, as $x-(\lambda+\rho)\in C(\xi)$, we may estimate the argument of T as $\|x-(\lambda+\rho)-\xi\|_2\le 2\|\xi\|_2\le\sup_{x\in K}\|x\|_2$. By its properties noted in Lemma 2.2, T is therefore uniformly bounded on the domain of integration in (3.5), i.e. with a bound independent of λ and ξ. By the Cauchy-Schwarz inequality and recalling that $\pi^2=\omega$, we may therefore estimate

$$|(\delta_{\lambda+\rho}*\delta_\xi)(\phi^*)|^2\;\le\;\frac{\text{const.}}{\pi(\lambda+\rho)^2\omega(\xi)}\int_{\lambda+\rho+C(\xi)}|\phi^*(x)|^2\,\omega(x)\,dx,$$

where the constant is again independent of λ and ξ. Now fix $R>0$ such that $K\subseteq A+B$ with

$$A=\{x=x^0\in\mathbb{R}^n_0\cap\overline{\mathfrak{a}_+}:\|x\|_2\le R\},\quad B=\{x=x^1\in\mathbb{R}\underline{1}\cap\overline{\mathfrak{a}_+}:\|x\|_2\le R\}$$

and recall that $M_\lambda=\frac{\pi(\lambda+\rho)}{\sqrt{n!}\prod_{k=1}^{n-1}k!}$. Then we may continue our previous estimate as follows:

$$\|P_jf\|_2^2\;\le\;\text{const.}\int_{2^j(A+B)}\sum_{\lambda\in P_+}\Big(\int_{\lambda+\rho+C(\xi)}|\phi^*(x)|^2\omega(x)dx\Big)d\xi.$$

Noting that $C(\xi)=\xi^1+C(\xi^0)$ with $C(\xi^0)\subseteq 2^jA$, $\xi^1\in 2^jB$, we conclude that

$$\begin{aligned}
\|P_jf\|_2^2 &\le \text{const.}\int_{2^jA}\sum_{\lambda\in P_+}\Big(\int_{2^jB}\int_{\lambda+\rho+\xi^1+C(\xi^0)}|\phi^*(x)|^2\omega(x)dx\,d\xi^1\Big)d\xi^0\\
&\le \text{const.}\int_{2^jA}\sum_{\lambda\in P_+}\Big(\int_{\lambda+\rho+2^j(A+B)}|\phi^*(t)|^2\omega(t)dt\Big)d\xi^0.
\end{aligned}$$

Choosing $j \in \mathbb{Z}$ sufficiently small, we can achieve that the sets $\lambda + \rho + 2^j(A+B)$ with $\lambda \in P_+$ are pairwise disjoint. This finally leads to the estimate

$$\|P_j f\|_2^2 \leq \text{const.} \int_{2^j A} \Big(\int_{\overline{\mathfrak{a}_+}} |\phi^*(t)|^2 \omega(t) dt \Big) d\xi^0.$$

The inner integral being finite, it follows that $\|P_j f\|_2 \to 0$ for $j \to -\infty$ as claimed. This finishes the proof of condition (2) in the definition of a radial MRA, which is satisfied without requirements on $\mathcal{H}\phi$.

It remains to analyze condition (3) concerning the density of $\bigcup_{j=-\infty}^{\infty} V_j$ in $L^2(\overline{\mathfrak{a}_+}, \omega)$. Here we closely follow [RR03]. Suppose first that $\mathcal{H}\phi(0) \neq 0$ and let $h \in \big(\bigcup_{j=-\infty}^{\infty} V_j\big)^\perp$, i.e. $P_j h = 0$ for all $j \in \mathbb{Z}$. Let $\varepsilon > 0$. By the Plancherel theorem for the Hankel transform, we can find $f \in L^2(\overline{\mathfrak{a}_+}, \omega)$ such that $\mathcal{H}f$ has compact support and $\|f - h\|_2 \leq \varepsilon$. This implies

$$\|P_j f\|_2 = \|P_j(f-h)\|_2 \leq \varepsilon$$

for all $j \in \mathbb{Z}$. By the Riesz basis assumption on ϕ, we further have

$$A \sum_{\lambda \in P_+} |\langle f, \phi_{j,\lambda}\rangle|^2 \leq \|P_j f\|_2^2 \leq B \sum_{\lambda \in P_+} |\langle f, \phi_{j,\lambda}\rangle|^2.$$

Suppose now that $\operatorname{supp} \mathcal{H}f \subseteq K$ with some compact $K \subseteq \overline{\mathfrak{a}_+}$. Then in view of (2.8),

$$\begin{aligned} \langle f, \phi_{j,\lambda}\rangle &= \langle \mathcal{H}f,\ \mathcal{H}\phi_{j,\lambda}\rangle \\ &= \int_K \mathcal{H}f(\xi)\, \overline{\rho_\lambda^j\, \mathcal{H}\phi(2^{-j}\xi)}\ \omega(\xi)\ d\xi \end{aligned}$$

with the dilates $\rho_\lambda^j := D_{2^j}(\frac{\Delta S_\lambda}{\pi})$, $\lambda \in P_+$. The weight ω being homogeneous, they form an orthonormal basis of the space $X_j := L^2(2^j D, \omega)^{\mathcal{S}_n}$, where $D = [0, 2\pi[^n$. Assume j is sufficiently large, so that $2^j D \supseteq K$. Then

$$\langle f, \phi_{j,\lambda}\rangle = \langle \mathcal{H}f\, \overline{\mathcal{H}\phi(2^{-j}\cdot)}, \rho_\lambda^j \rangle_{X_j}.$$

Thus using Parseval's equation for $X_j \cong L^2(2^j D \cap \overline{\mathfrak{a}_+},\, \omega)$ we obtain

$$\sum_{\lambda \in P_+} |\langle f, \phi_{j,\lambda}\rangle|^2 = \|\mathcal{H}f\, \mathcal{H}\phi(2^{-j}\cdot)\|_{X_j}^2 = \int_K |\mathcal{H}f(\xi)|^2 |\mathcal{H}\phi(2^{-j}\xi)|^2\ \omega(\xi)\, d\xi. \tag{3.6}$$

As we assumed that $|\mathcal{H}\phi|$ is continuous in 0, the functions $|\mathcal{H}\phi(2^{-j}\cdot)|$ converge to the constant $|\mathcal{H}\phi(0)| > 0$ uniformly on K as $j \to +\infty$. Hence

$$\varepsilon \geq \limsup_{j\to\infty} \|P_j f\|_2 \geq \sqrt{A}\, |\mathcal{H}\phi(0)|\, \|\mathcal{H}f\|_2 \ \geq\ \sqrt{A}\, |\mathcal{H}\phi(0)| (\|h\|_2 - \varepsilon).$$

But $\varepsilon > 0$ was arbitrary, so that $h = 0$. This proves that $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\overline{\mathfrak{a}_+}, \omega)$.

Conversely, suppose that $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\overline{\mathfrak{a}_+}, \omega)$. Then

$$\lim_{j\to\infty} P_j f = f$$

for all $f \in L^2(\overline{\mathfrak{a}_+}, \omega)$. If $\mathcal{H}f$ is compactly supported, the same calculation as before gives

$$\lim_{j\to\infty} \|P_j f\|_2 \leq \sqrt{B}\, |\mathcal{H}\phi(0)|\, \|\mathcal{H}f\|_2,$$

which enforces $\mathcal{H}\phi(0) \neq 0$.

Finally, if the $\phi_{j,\lambda}, \lambda \in P_+$ are orthonormal, we may choose $A = B = 1$ and calculate

$$\|f\|_2 = \lim_{j\to\infty} \|P_j f\|_2 = |\mathcal{H}\phi(0)|\, \|\mathcal{H}f\|_2 = |\mathcal{H}\phi(0)|\, \|f\|_2,$$

so that $|\mathcal{H}\phi(0)| = 1$. □

Suppose that $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ is an orthonormal scaling function of a radial MRA. Then according to Proposition 3.7, it satisfies the two-scale relation

$$\Delta(2\xi)\mathcal{H}\phi(2\xi) = \gamma(\xi)\Delta(\xi)\mathcal{H}\phi(\xi)$$

with some $\gamma \in S$. Introducing the *filter function*

$$G(\xi) \coloneqq \frac{\gamma(\xi)\Delta(\xi)}{\Delta(2\xi)}, \tag{3.7}$$

the above two-scale relation becomes

$$\mathcal{H}\phi(2\xi) = G(\xi)\mathcal{H}\phi(\xi).$$

Note that G is $\mathcal{S}_n$-invariant and that $|G|$ is I-periodic. Let us now also consider the finer lattice

$$L \coloneqq \frac{1}{2}I = \pi\mathbb{Z}^n \supseteq I = 2\pi\mathbb{Z}^n.$$

Lemma 3.9. *Suppose that $\phi \in L^2(\overline{\mathfrak{a}_+}, \omega)$ is an orthonormal scaling function of a radial MRA. Then the associated filter function G satisfies*

$$\sum_{p\in L/I} |G(\xi + p)|^2 \;=\; 1$$

almost everywhere. As a consequence, G is essentially bounded and contained in $L^2(D)$.

Proof. Using Corollary 3.3 with 2ξ instead of ξ and the two-scale relation, we get

$$\begin{aligned} 1 &= \frac{1}{n!}\sum_{q\in I} |\mathcal{H}\phi(2\xi + q)|^2 \;=\; \frac{1}{n!}\sum_{p\in L} |\mathcal{H}\phi(\xi+p)|^2 |G(\xi+p)|^2 \\ &= \frac{1}{n!}\sum_{p\in L/I} |G(\xi+p)|^2 \sum_{q\in I} |\mathcal{H}\phi((\xi+p)+q)|^2 \;=\; \sum_{p\in L/I} |G(\xi+p)|^2. \end{aligned}$$

□

4. Orthonormal Wavelets

Suppose we are given a radial MRA $(V_j)_{j\in\mathbb{Z}}$ in $\mathrm{Herm}(n)$ with orthonormal scaling function ϕ. The wavelet space W_j is then defined as the orthogonal complement of V_j in V_{j+1}, i.e.

$$V_{j+1} = V_j \oplus W_j.$$

In this section, we will characterize orthonormal wavelets for the given radial MRA. That is, we will give necessary and sufficient conditions for translations and dilations of functions $\psi^1, \ldots, \psi^r \in L^2(\overline{\mathfrak{a}_+}, \omega)$ in order to constitute an orthonormal basis of the wavelet space W_0. As $W_j = D_{2^{-j}} W_0$, we obtain orthonormal bases of all spaces W_j by dilation, and thus finally an orthonormal wavelet basis of

$$L^2(\overline{\mathfrak{a}_+}, \omega) = \widehat{\bigoplus_{j\in\mathbb{Z}}} W_j.$$

Proposition 4.1. *For functions $\psi^1, \ldots, \psi^{r-1} \in L^2(\overline{\mathfrak{a}_+}, \omega)$ the following are equivalent:*

(1) $\{M_\lambda T^{(\lambda)}\psi^i \;:\; 1 \leq i \leq r-1,\; \lambda \in P_+\}$ *is an orthonormal system.*

(2) *The sum*

$$P_{i,j}(\xi) := \frac{1}{n!}\sum_{q\in I} \mathcal{H}\psi^i(\xi+q)\overline{\mathcal{H}\psi^j(\xi+q)}$$

is absolutely convergent with $P_{i,j}(\xi) = \delta_{i,j}$ *for almost all* ξ.

Proof. For $i = j$ this is Corollary 3.3. Now suppose $i \neq j$. Then the Plancherel theorem for the Hankel transform implies that

$$\begin{aligned}\big\langle M_\lambda T^{(\lambda)}\psi^i,\; M_\mu T^{(\mu)}\psi^j\big\rangle &= \Big\langle \frac{\Delta S_\lambda}{\pi}\mathcal{H}\psi^i,\; \frac{\Delta S_\mu}{\pi}\mathcal{H}\psi^j\Big\rangle \\ &= \int_{\overline{\mathfrak{a}_+}} S_\lambda(\xi)\overline{S_\mu(\xi)}\mathcal{H}\psi^i(\xi)\overline{\mathcal{H}\psi^j(\xi)}\; |\Delta(\xi)|^2 d\xi \\ &= \frac{1}{n!}\int_{\mathbb{R}^n} S_\lambda(\xi)\overline{S_\mu(\xi)}\mathcal{H}\psi^i(\xi)\overline{\mathcal{H}\psi^j(\xi)}\; |\Delta(\xi)|^2 d\xi \\ &= \int_{\mathbb{T}^n} S_\lambda(\xi)\overline{S_\mu(\xi)}P_{i,j}(\xi)\; |\Delta(\xi)|^2 d\xi.\end{aligned}$$

This immediately gives (2) $\Rightarrow$ (1), since $(S_\lambda)_{\lambda\in P_+}$ is an orthonormal basis of $S = L^2(\mathbb{T}^n, |\Delta|^2)^{\mathcal{S}_n}$. For the converse implication, note first that the series defining $P_{i,j}$ is absolutely convergent with $|P_{i,j}| \leq 1$ a.e., as a consequence of Corollary 3.3 and the Cauchy-Schwarz inequality in $\ell^2(I)$. Moreover, $P_{i,j}$ is I-periodic and $\mathcal{S}_n$-invariant (recall that the Hankel transform $\mathcal{H}\psi$ is $\mathcal{S}_n$-invariant). Note that $P_{i,j} = \overline{P_{j,i}}$. By our assumption and the above calculation,

$$\big\langle S_\lambda P_{j,i}\,, S_\mu\big\rangle_S = 0$$

for all $\lambda, \mu \in P_+$. Thus $S_\lambda P_{j,i} = 0$ a.e. for every λ and hence $\langle S_\lambda, P_{i,j}\rangle_S = 0$ for every λ. This implies that $P_{i,j} = 0$ a.e. □

Maintaining the above setting, we shall now characterize the space $W_{-1} \subseteq V_0$.

Proposition 4.2. *For* $f \in L^2(\overline{\mathfrak{a}_+}, \omega)$ *the following statements are equivalent:*

(1) $f \in W_{-1}$.

(2) *There is an element* $\beta \in S$ *such that* $\mathcal{H}f(\xi) = \dfrac{\Delta(\xi)\beta(\xi)}{\pi(\xi)}\mathcal{H}\phi(\xi)$ *and*

$$(\beta(\xi+p)\delta(\xi+p))_{p\in L/I} \perp (\gamma(\xi+p)\delta(\xi+p))_{p\in L/I}$$

almost everywhere as vectors in $\mathbb{C}^r$, *where*

$$r = |L/I| = 2^n, \quad \delta(x) := \frac{\Delta(x)}{\Delta(2x)}$$

and $\gamma \in S$ *is the function from the two-scale relation for* ϕ *in Proposition 3.7.*

Proof. By definition $V_0 = W_{-1} \oplus V_{-1}$ and $V_{-1} = D_2V_0$, thus $h \in V_{-1}$ if and only if $D_{1/2}h \in V_0$. In view of Lemma 3.4 this is equivalent to

$$\mathcal{H}(D_{1/2}h)(\xi) = \frac{\Delta(\xi)\eta(\xi)}{\pi(\xi)}\mathcal{H}\phi(\xi)$$

with some $\eta \in S$. From Eq. (2.5) we obtain

$$2^{-\frac{m}{2}}\mathcal{H}h(\tfrac{\xi}{2}) = (D_2\mathcal{H}h)(\xi) = \mathcal{H}(D_{1/2}h)(\xi)$$

and thus

$$h \in V_{-1} \quad \Longleftrightarrow \quad \mathcal{H}h(\xi) = \frac{2^{\frac{m}{2}}\Delta(2\xi)\eta(2\xi)}{\pi(2\xi)}\mathcal{H}\phi(2\xi) = \frac{\beta(2\xi)}{\pi(\xi)}\gamma(\xi)\Delta(\xi)\mathcal{H}\phi(\xi)$$

for some $\beta \in S$. Furthermore, $f \in V_0$ if and only if

$$\mathcal{H}f(\xi) = \frac{\Delta(\xi)\widetilde{\beta}(\xi)}{\pi(\xi)}\mathcal{H}\phi(\xi)$$

for some $\widetilde{\beta} \in S$. We denote $D' := \frac{1}{2}D = [0,\pi[^n$, which is a fundamental domain of the smaller torus $\mathbb{R}^n/L$. As $W_{-1} \perp V_{-1}$ and $\mathcal{H}$ is unitary, we have $f \in W_{-1}$ if and only if $\langle \mathcal{H}f, \mathcal{H}h\rangle = 0$ for all $h \in V_{-1}$, i.e. if and only if for all $\beta \in S$ we have

$$\begin{aligned}
0 &= \int_{\overline{\mathfrak{a}_+}} \frac{\widetilde{\beta}(\xi)}{\pi(\xi)}\,\overline{\frac{\beta(2\xi)}{\pi(\xi)}}\,\overline{\gamma(\xi)}|\mathcal{H}\phi(\xi)|^2\,|\Delta(\xi)|^2\omega(\xi)d\xi \\
&= \frac{1}{n!}\int_{\mathbb{R}^n} \widetilde{\beta}(\xi)\overline{\beta(2\xi)\gamma(\xi)}|\mathcal{H}\phi(\xi)|^2\,|\Delta(\xi)|^2 d\xi \\
&= \frac{1}{n!}\sum_{p\in L/I}\sum_{q\in I}\int_{D'} \overline{\beta(2\xi)}\,\widetilde{\beta}(\xi+p)\overline{\gamma(\xi+p)}\,|\mathcal{H}\phi((\xi+p)+q)|^2\,|\Delta(\xi+p)|^2 d\xi \\
&= \int_{D'} \overline{\beta(2\xi)}\sum_{p\in L/I}\widetilde{\beta}(\xi+p)\Delta(\xi+p)\overline{\gamma(\xi+p)\Delta(\xi+p)}\,P_\phi(\xi+p)d\xi \\
&= \int_{D'} \overline{\beta(2\xi)}\sum_{p\in L/I}\widetilde{\beta}(\xi+p)\delta(\xi+p)\overline{\gamma(\xi+p)\delta(\xi+p)}\,|\Delta(2\xi)|^2 d\xi.
\end{aligned}$$

Here it was used that $P_\phi = 1$ a.e. and that $\beta(2\,\cdot)$ is L-periodic while $\widetilde{\beta}, \gamma$ and $|\Delta|$ are I-periodic. Note that also $|\delta|^2$ is I-periodic and $\mathcal{S}_n$-invariant, and therefore the finite sum

$$\sum_{p\in L/I}\widetilde{\beta}(\xi+p)\delta(\xi+p)\overline{\gamma(\xi+p)\delta(\xi+p)}$$

is $\mathcal{S}_n$-invariant and L-periodic. Since the $\beta(2\,\cdot),\ \beta \in S$ exhaust the space $L^2(D', |\Delta(2\,\cdot)|^2)^{\mathcal{S}_n}$, we conclude that

$$\sum_{p\in L/I}\widetilde{\beta}(\xi+p)\delta(\xi+p)\overline{\gamma(\xi+p)\delta(\xi+p)} = 0 \quad \text{ a.e.}$$

□

Theorem 4.3. *For elements $\psi^1, \ldots, \psi^{r-1} \in L^2(\overline{\mathfrak{a}_+}, \omega)$ the following are equivalent:*

(1) *The set*

$$\{M_\lambda T^{(\lambda)}\psi^i \;:\; 1 \le i \le r-1,\ \lambda \in P_+\}$$

is an orthonormal basis of W_0.

(2) *The number r is given by $r = |L/I| = 2^n$, and*

$$\mathcal{H}\psi^i(2\xi) = \beta^i(\xi)\delta(\xi)\mathcal{H}\phi(\xi)$$

for certain $\beta^i \in S$ such that the matrix

$$(\beta^i(\xi+p)\delta(\xi+p))_{0\leq i\leq r-1,\, p\in L/I} \in \mathbb{C}^{r\times r}$$

is unitary almost everywhere. Here $\beta^0 := \gamma \in S$ denotes the function from the two-scale relation of ϕ in Proposition 3.7.

Proof. Statement (1) is equivalent to the statement that the $M_\lambda D_2 T^{(\lambda)}\psi^i$ are an orthonormal basis of $D_2W_0 = W_{-1}$. Using Proposition 4.2 we see $M_0D_2T^{(0)}\psi^i \in W_{-1}$ if and only if

$$\mathcal{H}(M_0D_2T^{(0)}\psi^i)(\xi) = \frac{\Delta(\xi)\widetilde{\beta}^i(\xi)}{\pi(\xi)}\mathcal{H}\phi(\xi)$$

with certain $\widetilde{\beta}^i \in S$ such that $(\widetilde{\beta}^i(\xi+p)\delta(\xi+p))_{p\in L/I} \perp (\gamma(\xi+p)\delta(\xi+p))_{p\in L/I}$ almost everywhere. Recalling formula (2.8), we calculate the left hand side of the previous equation as

$$\begin{aligned}\mathcal{H}(M_0D_2T^{(0)}\psi^i)(\xi) &= D_{2^{-1}}\Big(\frac{\Delta S_0}{\pi}\,\mathcal{H}\psi^i\Big)(\xi)\\ &= \text{const.}\,\frac{\Delta(2\xi)}{\pi(\xi)}\,\mathcal{H}\psi^i(2\xi).\end{aligned}$$

We conclude that $M_0D_2T^{(0)}\psi^i \in W_{-1}$ if and only if

$$\mathcal{H}\psi^i(2\xi) = \beta^i(\xi)\delta(\xi)\mathcal{H}\phi(\xi) \tag{4.1}$$

for some $\beta^i \in S$ such that $(\beta^i(\xi+p)\delta(\xi+p))_{p\in L/I} \perp (\gamma(\xi+p)\delta(\xi+p))_{p\in L/I}$ almost everywhere. By Proposition 4.1 we see that $\{M_\lambda T^{(\lambda)}\psi^i : 1\leq i\leq r-1,\ \lambda\in P_+\}$ is an orthonormal system if and only if

$$\begin{aligned}\delta_{ij} &= \frac{1}{n!}\sum_{q\in I}\mathcal{H}\psi^i(2\xi+q)\overline{\mathcal{H}\psi^j(2\xi+q)}\\ &= \frac{1}{n!}\sum_{p\in L}\mathcal{H}\psi^i(2(\xi+p))\overline{\mathcal{H}\psi^j(2(\xi+p))}\\ &= \frac{1}{n!}\sum_{p\in L/I}\sum_{q\in I}\beta^i(\xi+p+q)\overline{\beta^j(\xi+p+q)}|\delta(\xi+q+p)|^2\,|\mathcal{H}\phi(\xi+p+q)|^2\\ &= \sum_{p\in L/I}\beta^i(\xi+p)\overline{\beta^j(\xi+p)}|\delta(\xi+p)|^2\,\frac{1}{n!}\sum_{q\in I}|\mathcal{H}\phi((\xi+p)+q)|^2\\ &= \sum_{p\in L/I}\beta^i(\xi+p)\overline{\beta^j(\xi+p)}|\delta(\xi+p)|^2\end{aligned}$$

almost everywhere. This is equivalent to the condition that the set

$$M := \{(\beta^i(\xi+p)\delta(\xi+p))_{p\in L/I} : 1\leq i\leq r-1\}$$

is an orthonormal system in $\mathbb{C}^{|L/I|}$ a.e. Furthermore, the vector $(\beta^0(\xi+p)\delta(\xi+p))_{p\in L/I}$ is almost everywhere normalized in $\mathbb{C}^{|L/I|}$ according to Lemma 3.9, and is orthogonal to M according to Proposition 4.2. We conclude that $r\leq |L/I|$ is a necessary condition by dimensionality. Moreover, if the ψ^i satisfy (4.1), then a short calculation gives

$$\mathcal{H}(M_\lambda D_2T^{(\lambda)}\psi^i)(\xi) = \text{const.}\cdot\frac{\Delta(\xi)S_\lambda(2\xi)\beta^i(\xi)}{\pi(\xi)}\mathcal{H}\phi(\xi).$$

So indeed, $M_\lambda D_2 T^{(\lambda)}\psi^i \in V_0$ for all $\lambda \in P_+$ by Lemma 3.4. As $S_\lambda(2\,\cdot)$ is L-periodic, we have

$$\big(S_\lambda(2(\xi+p))\beta^i(\xi+p)\delta(\xi+p)\big)_{p\in L/I} \in \mathbb{C}\,\big(\beta^i(\xi+p)\delta(\xi+p)\big)_{p\in L/I}$$

and thus $M_\lambda D_2 T^{(\lambda)}\psi^i \in W_{-1} \subseteq V_0$ by Proposition 4.2. We conclude that $r = |L/I|$ is also a necessary condition. □

Corollary 4.4. *Every radial MRA $(V_j)_{j\in\mathbb{Z}}$ with orthonormal scaling function ϕ admits an orthonormal wavelet basis consisting of $r-1$ wavelets, where $r = |L/I| = 2^n$, i.e. there are functions $\psi^1,\dots,\psi^{r-1} \in L^2(\overline{\mathfrak{a}_+},\omega)$ such that*

$$\{M_\lambda D_{2^{-j}} T^{(\lambda)}\psi^i \;:\; 1 \le i \le r-1,\ \lambda \in P_+\}$$

is an orthonormal basis of the complementary space W_j.

Proof. Recall that $\Delta(x) = \alpha(x)A_\delta(e^{ix})$ with the phase factor $\alpha(\xi) = e^{-i\frac{n-1}{2}\langle\xi,\underline{1}\rangle}$. Hence $\alpha\Delta$ is I-periodic. We now put $\eta^0(\xi) = \alpha(-\xi)G(\xi)$ with the filter function G associated to ϕ according to formula (3.7). We next choose functions $\eta^i \in L^2(D\cap\overline{\mathfrak{a}_+}), 1 \le i \le r-1$, in such a way that the matrix

$$(\eta^i(\xi+p))_{0\le i\le r-1,\, p\in L/I} \in \mathbb{C}^{r\times r}$$

is unitary for almost all $\xi \in D\cap\overline{\mathfrak{a}_+}$; here again $D = [0,2\pi[^n$. This amounts to constructing a unitary matrix with a given first row in a measurable way. We then extend the η^i to all of D via $\eta^i(wx) := \eta^i(x)$ for $w \in \mathcal{S}_n$ and then to $\mathbb{R}^n$ in an I-periodic fashion. We thus obtain functions $\beta^i := \alpha\delta^{-1}\eta^i \in S$ such that the matrix $A = (A_{ip})_{0\le i\le r-1,\, p\in L/I}$ with entries

$$A_{ip} = \beta^i(\xi+p)\delta(\xi+p) = \alpha(\xi)\eta^i(\xi+p)$$

is almost everywhere unitary. In particular, $\beta^0\delta = G$. We then define $\psi^i \in L^2(\overline{\mathfrak{a}_+},\omega)$ via its Hankel transform by

$$\mathcal{H}\psi^i(2\xi) := \beta^i(\xi)\delta(\xi)\mathcal{H}\phi(\xi).$$

□

5. Construction of radial scaling functions

It remains to discuss how a radial MRA can be actually obtained, i.e. how candidates for radial scaling functions can be found. As our lattice is $I = 2\pi\mathbb{Z}^n$, we can tile $\mathbb{R}^n$ with copies of $D = [0,2\pi[^n$ along periods of $e^{i\langle\xi,\cdot\rangle}$, which allows us to interlock Proposition 3.2 with its classical analogue for the Euclidean Fourier transform $\widehat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle\xi,x\rangle}dx$ on $L^2(\mathbb{R}^n)$.

Theorem 5.1. *Suppose $\phi_{\mathfrak{a}} \in L^2(\mathbb{R}^n)$ is a classical scaling function for a dyadic MRA in $L^2(\mathbb{R}^n)$ which is $\mathcal{S}_n$-invariant and such that its classical Fourier transform $\widehat{\phi_{\mathfrak{a}}}$ is continuous in 0 and satisfies $\widehat{\phi_{\mathfrak{a}}} \in L^2(\overline{\mathfrak{a}_+},\omega)$. Then*

$$\mathcal{H}\phi(\xi) := (2\pi)^{\frac{n}{2}} e^{-i\frac{n-1}{2}\langle\xi,\underline{1}\rangle}\widehat{\phi_{\mathfrak{a}}}(\xi) \tag{5.1}$$

defines a radial scaling function $\phi \in L^2(\overline{\mathfrak{a}_+},\omega)$, i.e. a scaling function for a radial MRA in Herm(n).

Conversely, if $\phi \in L^2(\overline{\mathfrak{a}_+},\omega)$ is a radial scaling function such that $\mathcal{H}\phi \in L^2(\mathbb{R}^n)$ and $\mathcal{H}\phi$ is continuous at 0, then the function $\phi_{\mathfrak{a}}$ defined by Eq. (5.1) is a classical scaling function on $\mathbb{R}^n$ which is $\mathcal{S}_n$-invariant.

Moreover, ϕ is an orthonormal scaling function if and only if $\phi_{\mathfrak{a}}$ is an orthonormal classical scaling function.

Proof. Suppose $\phi_\mathfrak{a}$ is a classical scaling function which is $\mathcal{S}_n$-invariant and such that $\widehat{\phi}_\mathfrak{a}$ is continuous in 0. Note that the $\mathcal{S}_n$-invariance implies that $\widehat{\phi}_\mathfrak{a}$ is $\mathcal{S}_n$-invariant as well, and thus definition (5.1) is meaningful. According to Propos. 5.7. in [Woj97] there are constants $0 < A \leq B < \infty$ such that

$$\frac{A}{(2\pi)^n} \leq \sum_{l\in\mathbb{Z}^n} |\widehat{\phi}_a(\xi + 2\pi l)|^2 \leq \frac{B}{(2\pi)^n} \quad \text{a.e..} \tag{5.2}$$

Moreover, $\phi_\mathfrak{a}$ is an orthonormal (classical) scaling function if and only if $A = B = 1$. Since $I = 2\pi\mathbb{Z}^n$, formula (5.2) can be written as

$$A \leq \sum_{q\in I} |\mathcal{H}\phi(\xi + q)|^2 \leq B \quad \text{a.e..}$$

So we can invoke Proposition 3.2 and Corollary 3.3 to see that $B_\phi := \{M_\lambda T^{(\lambda)}\phi : \lambda \in P_+\}$ forms a Riesz basis of $V_0 := \overline{\operatorname{span} B_\phi}$, and that this basis is orthonormal if and only if $\phi_\mathfrak{a}$ is orthonormal. Further, by Lemma 5.8 in [Woj97] there exists a $2\pi\mathbb{Z}^n$-periodic function m on $\mathbb{R}^n$ with $m|_D \in L^2(D)$ and such that $\widehat{\phi}_\mathfrak{a}(2\xi) = m(\xi)\widehat{\phi}_\mathfrak{a}(\xi)$. Since $\widehat{\phi}_\mathfrak{a}$ was assumed to be $\mathcal{S}_n$-invariant, m has to be $\mathcal{S}_n$-invariant as well. We again introduce the phase factor $\alpha(\xi) = e^{-i\frac{n-1}{2}\langle\xi,\underline{1}\rangle}$ and define $\gamma := \alpha\delta^{-1}m \in S$. We obtain

$$\begin{aligned}\Delta(2\xi)\mathcal{H}\phi(2\xi) &= \Delta(2\xi)\alpha(2\xi)(2\pi)^{\frac{n}{2}} m(\xi)\widehat{\phi}_\mathfrak{a}(\xi)\\ &= \frac{\Delta(2\xi)\alpha(2\xi)\alpha(\xi)^{-1}m(\xi)}{\Delta(\xi)}\Delta(\xi)\mathcal{H}\phi(\xi)\\ &= \gamma(\xi)\Delta(\xi)\mathcal{H}\phi(\xi),\end{aligned}$$

which is just the two-scale relation from Proposition 3.7. As $\widehat{\phi}_\mathfrak{a}$ is continuous in 0, we get $\widehat{\phi}_a(0) \neq 0$ and thus $\mathcal{H}\phi(0) \neq 0$, see [Dau92], Remark 3 on p. 144. Indeed, we have seen this reasoning already in the proof of Theorem 3.8. Hence, the conditions of Theorem 3.8 are satisfied and we obtain that ϕ is indeed a radial scaling function.

Conversely, suppose that $\mathcal{H}\phi \in L^2(\mathbb{R}^n)$. We proceed as before and note that

$$m(\xi) = \gamma(\xi)\delta(\xi)\alpha(\xi)^{-1} = \alpha(-\xi)G(\xi).$$

By Lemma 3.9, this shows that $m \in L^2(D)$. □

As an example, we consider the radial analogue of the Shannon wavelets. Again, we work with the phase factor $\alpha(\xi) := e^{-i\frac{n-1}{2}\langle\xi,\underline{1}\rangle}$. Consider $\phi_\mathfrak{a}$ defined via its classical Fourier transform $\widehat{\phi}_\mathfrak{a} = (2n)^{-n/2}\chi_{[-\pi,\pi]^n}$. Then by [Woj97], Thm. 2.13, Prop. 5.7 and Lem. 5.8, $\phi_\mathfrak{a}$ is a classical orthonormal scaling function which satisfies the conditions of our Theorem 5.1. Therefore $\mathcal{H}\phi(\xi) := \alpha(\xi)\chi_{[-\pi,\pi]^n}(\xi)$ defines an orthonormal radial scaling function ϕ. Note that

$$\mathcal{H}\phi(2\xi) = \alpha(\xi)\chi_Q(\xi)\mathcal{H}\phi(\xi) = \alpha(2\xi)\chi_{Q\,\cap\,[-\pi,\pi]^n}(\xi) \tag{5.3}$$

with the union of cubes

$$Q := Q^0 := \bigcup_{l\in\mathbb{Z}^n} (2\pi l + [-\pi/2, \pi/2]^n).$$

Now define functions $\beta^i = (\alpha\delta)^{-1}\chi_{Q^i} \in S$ with

$$Q^i = q^i + Q,$$

where q^i runs through the non-trivial representatives of L/I, i.e. through the set $\{0,\pi\}^n \setminus \{(0,\dots,0)\}$. Then in view of Theorem 4.3, one obtains an orthonormal radial wavelet basis (ψ^i) for $\mathrm{Herm}(n)$ by defining

$$\mathcal{H}\psi^i(2\xi) \coloneqq \beta^i(\xi)\delta(\xi)\mathcal{H}\phi(\xi) = \chi_{Q^i\cap[-\pi,\pi]^n}(\xi). \tag{5.4}$$

$Q^0 \cap [-\pi,\pi]^2 = [-\pi/2,\pi/2]^2$ $\quad Q^1 \cap [-\pi,\pi]^2$ with $q^1 = (0,\pi)$ $\quad Q^2 \cap [-\pi,\pi]^2$ with $q^2 = (\pi,0)$ $\quad Q^3 \cap [-\pi,\pi]^2$ with $q^3 = (\pi,\pi)$

FIGURE 1. The radial Shannon wavelets in the rank $n = 2$ case. The sets $Q^i \cap [-\pi,\pi]^2$ correspond to the scaling function ϕ $(i = 0)$ and the three wavelets $\psi^i (i = 1,\dots,3)$ via Eq. (5.3) and Eq. (5.4).

6. Appendix: Generalizations

Many of the previous results can be generalized in terms of Lie theory. Instead of the action of $U(n)$ on $\mathrm{Herm}(n)$ by conjugation we may consider an arbitrary compact connected Lie group K which acts on $\mathfrak{p} = i\mathrm{Lie}(K) \subseteq \mathrm{Lie}(K)_{\mathbb{C}}$ via the adjoint representation as a group of orthogonal transformations with respect to some K-invariant inner product on $\mathfrak{p}$. First, we replace $\mathbb{R}^n$ by a maximal abelian subspace $\mathfrak{a} = i\mathrm{Lie}(T) \subseteq \mathfrak{p}$ coming from a maximal torus $T \subseteq K$ and $\mathbb{R}^n_0$ by $\mathfrak{a}_0$, the span of the roots of K. The Weyl integration formula holds in this generality (cf. [GV88], Eq. (2.4.22)). Further, $\overline{\mathfrak{a}_+}$ is now the positive Weyl chamber associated with a set of simple positive roots $\alpha_1,\dots,\alpha_{\dim\mathfrak{a}_0}$. As usual, we identify $\mathfrak{a}$ with $\mathfrak{a}^*$ using a K-invariant inner product on $\mathfrak{p}$. The associated Weyl group W replaces the symmetric group $\mathcal{S}_n$ and π becomes the usual alternating polynomial $\pi = \prod_{i=1}^q \alpha_i$ with respect to the positive roots $\alpha_1,\dots,\alpha_q$. Again, we define a generalized translation by averaging the translation along adjoint orbits of K in $\mathfrak{p}$.

The Weyl character formula yields a natural choice for the trigonometric polynomials S_λ, which are indexed by the set of dominant weights P_+ of K with respect to T. On the other hand the Harish-Chandra-Integral ([HC57]) allows us to express the spherical functions $J(\,.\,,iy)$ of $K \ltimes \mathfrak{p}/K$ in a similar fashion. Thus interlocking these two allows for nice properties of the S_λ with respect to the generalized translation. More precisely, we obtain

$$S_\lambda(x) = \frac{1}{i^q\sqrt{|W|}\Delta(x)}\sum_{w\in W}\varepsilon(w)e^{i(\lambda+\rho)(wx)} = M_\lambda\frac{\pi(x)}{\Delta(x)}J(x,i(\lambda+\rho)) \tag{6.1}$$

as a generalization of Eq. (2.6). Here, $s_\lambda \coloneqq i^q\sqrt{|W|}S_\lambda$ is the trigonometric polynomial associated to the highest weight λ in the Weyl character formula ([Kna96], Thm. 5.113), Δ denotes the Weyl denominator and $M_\lambda = \mathrm{const.}\cdot\pi(\lambda+\rho)$ is a suitable normalization constant determined by the Harish-Chandra-Integral (see for example [McS21], Eq. 3.3). Again, the Peter-Weyl theorem and the Schur orthogonality relations ensure that we obtain an orthonormal basis.

Using a structure theorem for compact connected Lie groups ([Pro07], Ch. 10, §7.2, Thm. 4), we can decompose $K = (L \times H)/C$ with L compact, connected and semisimple, a torus H and a finite subgroup C of the center of $L \times H$ intersecting H trivially. This allows us to reduce to the semisimple case as in the proof of Proposition 2.3 and apply the results of [GS02] to find the density of the translation. Note that $\mathfrak{a}_0$ is the orthogonal complement of $i\mathrm{Lie}(H)$ in $\mathfrak{a}$ as $L, L\times H$ and K all have the same roots. We now take $I = i\ker(\exp\colon \mathrm{Lie}(T) \to T)$ as the integral lattice of T and replace D by a fundamental domain of the torus $\mathfrak{a}/I$. After adjusting these notations, the arguments up until Theorem 4.3 will still work. A small caveat is to see that the Weyl group acts on the integral lattice in order to see for example the $\mathcal{S}_n$-invariance of P_ϕ in Proposition 3.2. However, employing the analytic Weyl group $W \cong N_K(T)/T$, we immediatly verify

$$\exp(\mathrm{Ad}(k)x) = k\exp(x)k^{-1} = e$$

for $w = \mathrm{Ad}(k) \in W$ and $x \in \ker(\exp \mathrm{Lie}(T) \to T)$. After Section 4, however, the geometry will not be as compatible as in the case $K = U(n)$. For example, it is not clear how to obtain radial scaling functions as the lattice I will generally not behave well with the periodicy of $e^{i\langle \xi,\cdot\rangle}$, which was crucial in the proof of Theorem 5.1.

In the case where $K = SU(2)$ acts on SHerm(2) by conjugation, this generalization reduces to the situation of [RR03], which can be seen as follows: We identify $\mathfrak{a} = \mathrm{SHerm}(2) \cong \mathbb{R}^2_0 \cong \mathbb{R}$, so that $W = \{\pm 1\}, \pi(t) = 2t, \Delta(t) = e^{it} - e^{-it}$ and $\overline{\mathfrak{a}_+} = [0,\infty[$. Using these identifications and the fact that K is of rank 1, the function T is simply the indicator function $\chi_{[0,\infty[}$. Thus Proposition 2.3 reduces to

$$\begin{aligned}(\delta_r * \delta_s)(f) &= \frac{1}{2rs}\int_0^\infty f(t)t \sum_{i,j\in\{0,1\}} (-1)^i(-1)^j T((-1)^i r + (-1)^j s - t)\, dt \\ &= \frac{1}{2rs}\int_{|r-s|}^{r+s} f(t)t\, dt,\end{aligned}$$

which agrees with formula (4.1) of [RR03]. A direct calculation shows that $P_+ = \frac{1}{2}\mathbb{N}_0$, so that $S_\lambda(t) = \sqrt{2}\Delta(t)^{-1}\sin((k+1)t)$ for $\lambda = \frac{1}{2}k \in P_+$. This agrees, up to the factor $\Delta(t)^{-1}$, with the definition in [RR03], §4. In fact all our statements in this paper, such as for example the definition of the set S, are modified by this factor. This is a technical modification to accomodate for the fact that for non-semisimple Lie groups the Weyl vector ρ is not necessarily contained in the weight lattice P. By re-introducing the factor Δ^{-1} we compensate a possible loss of I-periodicity of the functions ΔS_λ.

It is a specialty of this rank 1 situation that the weight lattice and the integral lattice align nicely, which made the explicit construction of orthonormal wavelets in [RR03] work.

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