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Markov Processes Related with Dunkl Operators

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Dunkl operators are parameterized differential-difference operators on $\mathbb{R}^N$ that are related to finite reflection groups. They can be regarded as a generalization of partial derivatives and play a major role in the study of Calogero–Moser–Sutherland-type quantum many-body systems. Dunkl operators lead to generalizations of various analytic structures, like the Laplace operator, the Fourier transform, Hermite polynomials, and the heat semigroup. In this paper we investigate some probabilistic aspects of this theory in a systematic way. For this, we introduce a concept of homogeneity of Markov processes on $\mathbb{R}^N$ that generalizes the classical notion of processes with independent, stationary increments to the Dunkl setting. This includes analogues of Brownian motion and Cauchy processes. The generalizations of Brownian motion have the càdlàg property and form, after symmetrization with respect to the underlying reflection groups, diffusions on the Weyl chambers. A major part of the paper is devoted to the concept of modified moments of probability measures on $\mathbb{R}^N$ in the Dunkl setting. This leads to several results for homogeneous Markov processes (in our extended setting), including martingale characterizations and limit theorems. Furthermore, relations to generalized Hermite polynomials, Appell systems, and Ornstein–Uhlenbeck processes are discussed.

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1. INTRODUCTION

Dunkl operators are differential-difference operators associated with a finite reflection group, acting on some Euclidean space. In recent years these operators and several generalizations have gained considerable interest in various fields of mathematics and physics. They provide a useful tool in the study of special functions associated with root systems (see, e.g., [D2, H, vD]), and they are closely related to certain representations of degenerate affine Hecke algebras (see [C, O2] and the recent survey [Ki]). Moreover, Dunkl operators are naturally involved in the algebraic description of certain exactly solvable quantum many-body systems of the Calogero−Moser−Sutherland-type (CMS); see [B-F1, B-F2, L-V, P]. We start with an example for motivation.

EXAMPLE 1.1. The quantum CMS-model of type A_{N-1} for N particles on the line $\mathbb{R}$, with two-body potentials of inverse-square type and with spin-exchange terms, is described by the Schrödinger operator

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{1\le i<j\le N} \frac{2k}{(x_i - x_j)^2} \cdot \big(k - s_{i,j}\big), \tag{1.1}$$

where k is a real coupling parameter and $s_{i,j}$ is the operator interchanging the coordinates x_i, x_j of a function $f: \mathbb{R}^N \to \mathbb{C}$ (see [P]). In the classical setting without spin exchange, one is mainly interested in the symmetric eigenfunctions of H, i.e., functions invariant under the natural action of the symmetric group S_N on $\mathbb{R}^N$. The restriction of H to symmetric functions is given by

$$H_{sym} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2k(k-1) \sum_{1\le i<j\le N} \frac{1}{(x_i - x_j)^2}. \tag{1.2}$$

While the explicit solvability of the latter system goes back to Calogero ([Ca]), it was only recently observed (see [P]) that H can be written as $H = \sum_{j=1}^{N} p_j^2$ with commuting momentum operators $p_j = -i\tilde{T}_j$, where the $\tilde{T}_j$ are the "differential-difference" operators,

$$\tilde{T}_j = \frac{\partial}{\partial x_j} - k \sum_{i\neq j} \frac{s_{i,j}}{x_j - x_i} \qquad (j = 1, \ldots, N), \tag{1.3}$$

which are densely defined on $L^2(\mathbb{R}^N, dx)$. These operators are singular at 0 and do not properly act on $C^\infty(\mathbb{R}^N)$ or polynomials. This can be overcome by switching to the weighted space $L^2(\mathbb{R}^N, w_k(x)\, dx)$ with

$$w_k(x) = \prod_{1\le i<j\le N} |x_i - x_j|^{2k}.$$

This means considering the images of $H, p_i, \tilde{T}_i$ on $L^2(\mathbb{R}^N, w_k(x)\,dx)$ with respect to the multiplication operator

$$R\colon L^2(\mathbb{R}^N, dx) \to L^2\big(\mathbb{R}^N, w_k(x)\,dx\big), \qquad f \mapsto w_k^{-1/2} f.$$

A short calculation yields

$$T_i := R\tilde{T}_i R^{-1} = \frac{\partial}{\partial x_i} + k \sum_{j \neq i} \frac{1 - s_{i,j}}{x_i - x_j}. \tag{1.4}$$

These operators map polynomials to polynomials and are homogeneous of degree -1 (with respect to the natural grading on polynomials). They were originally by Dunkl ([D1, D2, D3]) in connection with a generalization of classical spherical harmonics where the spherical surface measure on the $(N-1)$-dimensional unit sphere is modified by a weight that is invariant under some reflection group; in the A_{n-1} case it has the form of w_k above.

The origin of this paper is the observation that the "Dunkl Laplacian"

$$\Delta_k := \sum_{i=1}^{N} T_i^2 = -RHR^{-1} \tag{1.5}$$

is the generator of a strongly continuous Markovian semigroup of operators on $C_0(\mathbb{R}^N)$, where the associated kernels admit a structure similar to classical Gaussian densities on $\mathbb{R}^N$ (see [R2]):

$$e^{t\Delta_k} f(x) = c_k \int_{\mathbb{R}^N} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) \cdot f(y) w_k(y)\,dy$$
$$\big(t > 0,\ x \in \mathbb{R}^N,\ f \in C_0(\mathbb{R}^N)\big),$$

where c_k is a normalization constant, and the "Dunkl kernel" $K(x, y)$, which was first introduced in [D2], generalizes the exponential function $e^{\langle x, y \rangle}$ on $\mathbb{R}^N \times \mathbb{R}^N$. Markov processes associated with the semigroup $(e^{t\Delta_k})_{t \geq 0}$ may be regarded as a generalization of the classical Brownian motion on $\mathbb{R}^N$, even though the reflection parts in Δ_k (or H) require that these processes cannot have continuous paths for $k > 0$. The investigation of generalizations of Brownian motions on $\mathbb{R}^N$ of this type is the main purpose of this paper. Before going into details, we discuss some extensions of the example above:

(1) Essential parts of the theory of Dunkl operators work for arbitrary finite reflection groups W on $\mathbb{R}^N$ (see [D1, D2, D3, dJ, O1]). We shall establish almost all results of this paper in the general setting. On the

other hand, Dunkl operators are related to quantum many-body systems for specific reflection groups only, like the A_{N-1} case above. We mention the B_N cases belonging to the reflection groups $S_N \ltimes \{\pm 1\}^N$, where $\{\pm 1\}^N$ acts on $\mathbb{R}^N$ by sign changes of the coordinates; these cases can be related to $2N+1$-body systems on $\mathbb{R}$.

(2) It is possible to add an external harmonic potential $\omega^2|x|^2 = \omega^2(x_1^2 + \cdots + x_N^2)$ to the Hamiltonian H in (1.1). This case was first studied in [Su] and can be handled in the same way as the classical harmonic oscillator by using generalized Hermite functions (see [B-F1, B-F2, vD, R2], and Section 8 below).

(3) Like the CMS-particle systems on $\mathbb{R}$ above, there are completely integrable particle systems on the torus $T := \{z \in \mathbb{C}: |z| = 1\}$ with two-body potentials of inverse square type. The parameterization $z_j = e^{ix_j}$, $x_j \in \mathbb{R}$ leads to the symmetric Hamiltonian,

$$-\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + k(k-1) \sum_{1 \le i < j \le N} \frac{1}{2\sin^2\big((x_i - x_j)/2\big)}.$$

The solvability of this model was originally shown by Sutherland; a complete solution in operator form, based on Dunkl operators, has recently been worked out in [L-V].

We turn next to the content and the organization of this paper.

Section 2 mainly serves as an introduction to the Dunkl theory. We there recapitulate basic facts on reflection groups, root systems, and multiplicity functions; then the associated Dunkl operators, the Dunkl kernel (as a generalization of the exponential function), and the Dunkl transform (as a generalization of the Fourier transform) are introduced. Most results are known (see [D1, D2, D3, dJ, R3, O1]) and are presented for the convenience of the reader only. Formally new results are the injectivity of the Dunkl transform of measures and Lévy's continuity theorem. Moreover, based on a result of Xu [X], we identify the Dunkl transform of radially symmetric functions on $\mathbb{R}^N$ in terms of a classical Hankel transform on $[0, \infty[$.

Section 3 is devoted to Dunkl's Laplacian, which generates a one-parameter semigroup of Markov kernels on $\mathbb{R}^N$. This semigroup may be considered a generalization of the semigroup of a Brownian motion and will be called the k-Gaussian semigroup. k-Gaussian semigroups form the leitmotiv of this paper, and so we include details here, even though most parts of Section 3 are already contained in [R2].

In Section 4 we use the algebraic connections between k-Gaussian semigroups and the Dunkl transform to introduce the concept of k-invariant Markov kernels on $\mathbb{R}^N$. This generalizes the notion of translation-invariant Markov kernels on $\mathbb{R}^N$ (which can be recovered for $k = 0$); it allows us to define semigroups of k-invariant Markov kernels as well as the associated Markov processes, which will be also called k-invariant. In Section 4 we study in particular the generators of these semigroups and the associated negative definite functions. Moreover, we show that k-invariant Markov processes always admit càdlàg versions, i.e, there are versions of these processes having right-continuous paths with limits from the left almost everywhere. Moreover, k-Gaussian càdlàg processes have automatically continuous paths after symmetrization with respect to the underlying reflection group (as the generator is here a second-order differential operator).

Further examples of k-invariant Markov kernels are presented in Section 5. These examples will be constructed via subordination from the k-Gaussian kernels (see, for instance, [Be-Fo] for this principle). This leads, in particular, to a generalization of Cauchy kernels. In the end of Section 5 we apply the generalized Cauchy kernels to solve the Dirichlet-type problem $u_{tt} + \Delta_k u = 0$ with initial condition $u(0,.) = f$ on $[0,\infty[\times\mathbb{R}^N$, where Δ_k denotes the Dunkl Laplacian.

In Section 6 we turn to a different topic in probability theory. With the interpretation of the Dunkl kernel K as a generalized exponential function in mind, we use K to construct "exponential" martingales from k-invariant Markov processes; we show that these processes are determined uniquely by such martingale properties. This will lead to a characterization of k-invariant Markov processes on $\mathbb{R}^N$ as unique solutions of martingale problems in the sense of Stroock and Varadhan [S-V]. This section requires some knowledge from semimartingale integration (see [K-S, W-W, Wi]) and may be skipped at a first reading. Our approach is motivated by similar results for commutative hypergroups in [Re-V] and shows how the martingale results of Sections 7 and 8 for moment functions fit into a general theory.

In Section 7 we generalize the monomials y^ν ($y \in \mathbb{R}^N$, $\nu \in \mathbb{Z}_+^N$) and introduce so-called moment functions m_ν ($\nu \in \mathbb{Z}_+^N$) on $\mathbb{R}^N$ via the Dunkl kernel K by

$$K(x,y) = \sum_{\nu\in\mathbb{Z}_+^N} \frac{1}{\nu!} m_\nu(x) y^\nu,$$

where the multi-index notation $y^\nu := y_1^{\nu_1} y_2^{\nu_2} \cdots y_N^{\nu_N}$ and $\nu! := \nu_1! \cdot \nu_2! \cdots \nu_N!$ is used for $y \in \mathbb{R}^N$ and $\nu \in \mathbb{Z}_+^N$. The functions m_ν are homogeneous polynomials of degree $|\nu| := \nu_1 + \cdots + \nu_N$. They make it possible to define the νth modified moment of a probability measure μ on $\mathbb{R}^N$ by $m_\nu(\mu) := \int_{\mathbb{R}^N} m_\nu \, d\mu$ (whenever this exists). It will turn out that modified moments of k-invariant Markov kernels satisfy algebraic relations of binomial type that are well known for classical moments of classical convolutions of measures on $\mathbb{R}^N$. These algebraic relations will allow us to construct martingales from k-invariant Markov processes by using moments. Section 7 is motivated by corresponding results for commutative hypergroups in [Bl-He, Z] and references there.

In Section 8 we systematically study modified moments of higher order for k-Gaussian measures. Motivated by related concepts in non-Gaussian white-noise analysis (see [ADKS, F-S, B-K1, B-K2]), we introduce two systems, $(R_\nu)_{\nu \in \mathbb{Z}_+^N}$ and $(S_\nu)_{\nu \in \mathbb{Z}_+^N}$, of functions on $\mathbb{R} \times \mathbb{R}^N$ by

$$R_\nu(t, x) := e^{-t\Delta_k} m_\nu(x) \qquad \text{and} \qquad S_\nu(t, x) := e^{-t\Delta_k} x^\nu.$$

These functions are called Appell characters and cocharacters, respectively; they can also be characterized via their generating functions involving the Dunkl kernel K. It will turn out that for all $t > 0$, the systems $(R_\nu(t, .))_{\nu \in \mathbb{Z}_+^N}$ and $(S_\nu(t, .))_{\nu \in \mathbb{Z}_+^N}$ form a biorthogonal basis of $L^2(\mathbb{R}^N, P_t(0, .))$, where $P_t(0, .)$ denotes the k-Gaussian measure with mean 0 and "variance" parameter t. Based on this observation, we derive a generalization of a formula of Macdonald [M], which was proved earlier in [D2] by different methods; see also [R-V2]. The functions R_ν and S_ν are closely related to generalizations of heat polynomials and Hermite polynomials; the latter ones wee extensively investigated in [B-F1, B-F2, vD, L] for special cases and in [R2] in full generality. To stress the connection with generalized Hermite polynomials, we close Section 8 with a list of properties of nonsymmetric generalized Hermite polynomials and the Appell systems above.

Section 9 contains some limit theorems for k-invariant Markov processes that are well known in the classical case $k = 0$. This in particular includes a law of the iterated logarithm for k-Gaussian processes, a strong law of large numbers for general k-invariant processes in discrete time, and a transcience criterion. Parts of this section are motivated by similar results for one-dimensional hypergroups in [Bl-He] and references cited there.

The final section, 10, is devoted to a generalization of Ornstein–Uhlenbeck processes to the Dunkl setting. We introduce the associated semigroups of Markov kernels explicitly by using the k-Gaussian kernels of Section 3. As in the classical case, the generators of generalized Ornstein–Uhlenbeck semigroups are given by the Schrödinger-type operators $c\Delta_k - \alpha \sum_{l=1}^N x_l \partial_l$ with parameters $c, \alpha > 0$. This observation leads to the explicit solution of the equation $u_t = (c\Delta_k - \alpha \sum_{l=1}^N x_l \partial_l) u$ on $[0, \infty[\times \mathbb{R}^N$ with initial condition $u(0, .) = f$. We conclude Section 10 with some properties of generalized Ornstein–Uhlenbeck processes; in particular, the stationary ones are defined in terms of k-Gaussian processes in a pathwise way.

We finally point out that this paper contains only a selection of probabilistic aspects of the Dunkl theory, and that some topics are not covered completely. Some results could be considerably improved with some additional effort. On the other hand, it was our aim to make at least most parts of this paper accessible to a broader mathematical audience.

2. DUNKL OPERATORS AND THE DUNKL TRANSFORM

The purpose of this section is to establish some basic notations and collect some facts on Dunkl operators and the Dunkl transform that will be important later on. General references here are [D2, D3, dJ]; for basics on reflection groups and root systems, we refer the reader to [Hu].

2.1. *Reflection Groups, Root Systems, and Multiplicity Functions*

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^N$ orthogonal to α, i.e., $\sigma_\alpha(x) = x - (2\langle \alpha, x\rangle / |\alpha|^2)\alpha$, where $\langle .,. \rangle$ is the Euclidean scalar product on $\mathbb{R}^N$ and $|x| := \sqrt{\langle x, x\rangle}$. (On $\mathbb{C}^N, |\,.\,|$ also denotes the standard Hermitian norm, while $\langle z, w\rangle := z_1 w_1 + \cdots + z_N w_N$.)

A finite set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a root system if $R \cap \mathbb{R}\cdot\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α $(\alpha \in R)$ generate a finite group $W \subset O(N)$, the reflection group associated with R. All reflections in W correspond to suitable pairs of roots; see [Hu]. For a given $\beta \in \mathbb{R}^N \setminus \bigcup_{\alpha\in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R: \langle \alpha, \beta\rangle > 0\}$; then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$. We assume from now on with no loss of generality that the root system R is normalized in the sense that $|\alpha| = \sqrt{2}$ for all $\alpha \in R$.

A function $k: R \to \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W. For abbreviation, we introduce the index

$$\gamma := \gamma(k) := \sum_{\alpha\in R_+} k(\alpha). \tag{2.1}$$

Moreover, let w_k denote the weight function

$$w_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.2}$$

which is W-invariant and homogeneous of degree 2γ. We finally introduce the Mehta-type constant

$$c_k := \left(\int_{\mathbb{R}^N} e^{-|x|^2} w_k(x)\, dx \right)^{-1}, \tag{2.3}$$

which is known for all Coxeter groups W; see [D1, Me, O1]. We shall use the following further abbreviations: $\mathscr{P} = \mathbb{C}[\mathbb{R}^N]$ denotes the algebra of polynomial functions on $\mathbb{R}^N$, and $\mathscr{P}_n$ $(n \in \mathbb{Z}_+)$ the subspace of homogeneous polynomials of degree n. We use the standard multi-index notations; i.e., for multi-indices $\nu, \rho \in \mathbb{Z}_+^N$ we write

$$|\nu| := \nu_1 + \cdots + \nu_N, \qquad \nu! := \nu_1! \cdot \nu_2! \cdots \nu_N!,$$

$$\binom{\nu}{\rho} := \binom{\nu_1}{\rho_1}\binom{\nu_2}{\rho_2} \cdots \binom{\nu_N}{\rho_N},$$

as well as

$$x^\nu := x_1^{\nu_1} \cdots x_N^{\nu_N} \qquad \text{and} \qquad A^\nu := A_1^{\nu_1} \cdots A_N^{\nu_N}$$

for $x = \mathbb{R}^N$ and any family $A = (A_1, \ldots, A_N)$ of commuting operators on $\mathscr{P}$. Finally, we shall need the partial ordering $\leq$ on $\mathbb{Z}_+^N$, which is defined by $\rho \leq \nu :\Leftrightarrow \rho_i \leq \nu_i$ for $i = 1, \ldots, N$.

2.2. *Dunkl Operators*

The Dunkl operators T_i $(i = 1, \ldots, N)$ on $\mathbb{R}^N$ associated with the finite reflection group W and multiplicity function k are given by

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha)\, \alpha_i \cdot \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \qquad f \in C^1(\mathbb{R}^N); \tag{2.4}$$

here ∂_i denotes the ith partial derivative. In the case $k = 0$, the T_i reduce to the corresponding partial derivatives. In this paper, we shall assume throughout that $k \geq 0$ (i.e., all values of k are nonnegative). The T_i have the following fundamental properties (see [D1]):

(1) The set $\{T_i\}$ generates a commutative algebra of differential-difference operators on $\mathscr{P}$.

(2) Each T_i is homogeneous of degree -1 on $\mathscr{P}$; i.e., $T_i p \in \mathscr{P}_{n-1}$ for $p \in \mathscr{P}_n$.

(3) (Product rule) $T_i(fg) = (T_i f)g + f(T_i g)$ for $i = 1, \ldots, N$ and all $f, g \in C^1(\mathbb{R}^N)$ with g being W-invariant.

A major tool in this paper is the Dunkl kernel $K(x, y)$ on $\mathbb{R}^N \times \mathbb{R}^N$, which generalizes the usual exponential function $e^{\langle x, y\rangle}$. It was introduced in [D2] by means of an intertwining isomorphism V of $\mathscr{P}$ which is characterized by the properties

$$V(\mathscr{P}_n) = \mathscr{P}_n, \qquad V|_{\mathscr{P}_0} = id, \qquad \text{and} \qquad T_i V = V\partial_i \qquad (i = 1, \dots, N).$$

Some details about V and the T_i for the symmetric group S_N and the B_N case will be discussed in Example 7.1. For $n \in \mathbb{Z}_+$, set

$$K_n(x, y) := V\left(\frac{\langle ., y\rangle^n}{n!}\right)(x) \qquad (x, y \in \mathbb{R}^N).$$

Then $K_n(x, y) = K_n(y, x)$ and $|K_n(x, y)| \leq |x|^n |y|^n / n!$. The Dunkl kernel K is now defined as

$$K(x, y) := \sum_{n=0}^{\infty} K_n(x, y) \qquad \left(= V(e^{\langle ., y\rangle})(x)\right). \tag{2.5}$$

For $y \in \mathbb{R}^N$, $K(x, y)$ may be also characterized as the unique analytic solution of $T_i f = y_i f$ $(i = 1, \dots, N)$ with $f(0) = 1$; see [O1]. Moreover, the Dunkl kernel K has a unique holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$.

EXAMPLES 2.1. (1) If $k = 0$, then $K(z, w) = e^{\langle z, w\rangle}$ for $z, w \in \mathbb{C}^N$. (Note that $\langle ., .\rangle$ was defined to be bilinear on $\mathbb{C}^N$.)

(2) If $N = 1$ and $W = \mathbb{Z}_2$, then the multiplicity function is a single parameter $k \geq 0$, and the normalization constant is $c_k = \Gamma(k + 1/2)$. The Dunkl kernel is given by

$$K(z, w) = j_{k-1/2}(izw) + \frac{zw}{2k+1} j_{k+1/2}(izw), \qquad z, w \in \mathbb{C},$$

where for $\alpha \geq -1/2$, j_α is the normalized spherical Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\, \Gamma(n + \alpha + 1)}.$$

For details and related material see [D3, R1, Ros, Wa] and references cited there. This basic example is connected with the quantum harmonic oscillator of Yang [Y].

(3) If $N = 2$ and W is a dihedral group of odd order, then k is again a single parameter. For $k = 1$, the W-invariant part of the Dunkl kernel K is computed (up to some transformation) in [Be-Mo] in terms of Tchebychev polynomials.

For later references, we next list some further known properties of the Dunkl kernel K.

THEOREM 2.2. *Let* $z \in \mathbb{C}^N$, $x, y \in \mathbb{R}^N$.

(1) $K(z,w) = K(w,z)$, $K(z,0) = 1$ *and* $K(\lambda z, w) = K(z, \lambda w)$ *for* $\lambda \in \mathbb{C}$.

(2) *For all* $\nu \in \mathbb{Z}_+^N$, $|\partial_z^\nu K(x,z)| \le |x|^{|\nu|} \cdot e^{|x| \cdot |\mathrm{Re}\, z|}$. *In particular,* $|K(x,z)| \le e^{|x||z|}$ *and* $|K(ix,y)| \le 1$.

(3) $T_j^x K(x,y) = y_j K(x,y)$ *for* $j = 1, \ldots, N$; *here the superscript* x *indicates that the operators act with respect to the* x*-variable.*

(4) $K(-ix,y) = \overline{K(ix,y)}$ *and* $K(g(x), g(y)) = K(x,y)$ *for* $g \in W$.

(5) *For each* $x \in \mathbb{R}^N$ *there exists a unique probability measure* $\mu_x \in M^1(\mathbb{R}^N)$ *with*

$$\operatorname{supp} \mu_x \subset \{\xi \in \mathbb{R}^N : |\xi| \le |x|\} \qquad \textit{and}$$
$$\operatorname{supp} \mu_x \cap \{\xi \in \mathbb{R}^N : |\xi| = |x|\} \ne \emptyset$$

such that

$$K(x,z) = \int_{\mathbb{R}^N} e^{\langle \xi, z \rangle}\, d\mu_x(\xi) \qquad \textit{for all } z \in \mathbb{C}^N.$$

In particular, $K(x,y) > 0$ *for all* $x, y \in \mathbb{R}^N$.

Proof. Parts (1), (3), and (4) can easily be derived from the construction of K in Section 2.2; see [D2, D3]. Part (5) is shown in [R3], and Part (2) is a consequence of Part (5); see also [R3]. ∎

The generalized exponential function K gives rise to an integral transform, called the Dunkl transform on $\mathbb{R}^N$, which was introduced in [D3] and has been thoroughly studied in [dJ]. To emphasize the similarity with the classical Fourier transform, we use the following notion.

2.3. *The Dunkl Transform*

The Dunkl transform associated with W and $k \ge 0$ is given by

$$\hat{}: L^1(\mathbb{R}^N, w_k(x)\,dx) \to C_b(\mathbb{R}^N);$$
$$\hat{f}(y) := \int_{\mathbb{R}^N} f(x) K(-iy, x) w_k(x)\, dx \qquad (y \in \mathbb{R}^N).$$

The Dunkl transform of a function $f \in L^1(\mathbb{R}^N, w_k(x)\,dx)$ has the following basic properties:

(1) $\|\hat{f}\|_\infty \le \|f\|_{1, w_k(x)\,dx}$; see Theorem 2.2(2).

(2) If $f^-(x) := f(-x)$ and $(f_g)(x) := f(g(x))$ for $g \in W$, $x \in \mathbb{R}^N$, then $(f^-)^\wedge(y) = \overline{\hat{f}(y)}$ and $(f_g)^\wedge(y) = \hat{f}(g(y))$ for $y \in \mathbb{R}^N$; see Theorem 2.2(4).

The inverse Dunkl transformation is given by $\check{f}(y) = \hat{f}(-y)$ and has corresponding properties.

The results listed in the following theorem are proved in [D3, dJ]:

THEOREM 2.3. (1) *The Schwartz space $\mathscr{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$ is invariant under the Dunkl transform.*

(2) (Lemma of Riemann–Lebesgue) $(L^1(\mathbb{R}^N, w_k(x)\,dx))^\wedge$ *is a* $\|.\|_\infty$-*dense subspace of* $C_0(\mathbb{R}^N)$.

(3) (L^1-inversion) *For all* $f \in L^1(\mathbb{R}^N, w_k(x)\,dx)$ *with* $\hat{f} \in L^1(\mathbb{R}^N, w_k(x)\,dx)$,

$$f = 4^{-\gamma - N/2} c_k^2 \hat{f}^\vee \qquad a.e.$$

(3) (Plancherel's Theorem) *The renormalized Dunkl transform* $f \mapsto c_k 2^{-\gamma - N/2}\hat{f}$ *can be uniquely extended to an isometric isomorphism on* $L^2(\mathbb{R}^N, w_k(x)\,dx)$.

We next show that Dunkl transforms of radial functions (i.e., $SO(N)$-invariant functions) in $L^1(\mathbb{R}^N, w_k(x)\,dx)$ are again radial, an that Dunkl transforms can be computed via associated classical Hankel transforms. This result is not obvious, as the weight w_k is usually invariant only under the reflection group W. Our proof is based on the explicit integration of the operator V over spheres in [X]. Before doing this, we recapitulate some facts about Hankel transforms:

2.4. *The Hankel Transform*

For $\alpha \ge -1/2$, define the measure ω_α on $[0, \infty[$ by

$$d\omega_\alpha(r) = (2^\alpha \Gamma(\alpha + 1))^{-1} r^{2\alpha+1}\, dr.$$

The Hankel transform $\mathscr{H}^\alpha$ of order α on $L^1([0, \infty[, \omega_\alpha)$ is then defined by

$$(\mathscr{H}^\alpha f)(\lambda) = \int_0^\infty f(r) j_\alpha(\lambda r)\, d\omega_\alpha(r);$$

Here j_α is the normalized spherical Bessel function as defined in Example 2.1(2). The transform $\mathscr{H}^\alpha$ can be uniquely extended to an isometric isomorphism on $L^2([0, \infty[, \omega_\alpha)$.

Proposition 2.4. *Let W be a finite reflection group with multiplicity function k and index $\gamma = \sum_{\alpha \in R_+} k(\alpha)$. Then there is a one-to-one correspondence between the space of all radial functions f in $L^1(\mathbb{R}^N, w_k(x)\,dx)$ and the space of all functions $F \in L^1([0,\infty[, \omega_{\gamma+N/2-1})$ via*

$$f(x) = F(|x|) \qquad \text{for } x \in \mathbb{R}^N.$$

Moreover, the Dunkl transform $\hat{f}$ of f is related to the Hankel transform $\mathscr{H}^{\gamma+N/2-1}F$ of F by

$$\hat{f}(y) = c_k^{-1} 2^{\gamma+N/2} \cdot (\mathscr{H}^{\gamma+N/2-1}F)(|y|) \qquad \text{for } y \in \mathbb{R}^N.$$

Proof. Let $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere with normalized surface measure $d\sigma$. Set

$$d_k := \int_{S^{N-1}} w_k(x)\, d\sigma(x) = \frac{1}{c_k} \cdot \frac{2}{\Gamma(\gamma + N/2)}.$$

Let f and F be related as in the proposition. Then the homogeneity of w_k leads to

$$\begin{aligned}\int_{\mathbb{R}^N} |f(x)|\, w_k(x)\, dx &= \int_0^\infty \left(\int_{S^{N-1}} w_k(ry)\, d\sigma(y) \right) |F(r)| r^{N-1}\, dr \\ &= d_k \int_0^\infty |F(r)| r^{2\gamma+N-1}\, dr.\end{aligned}$$

We now turn to the second statement. Corollary 2.2 of [X] states that for each polynomial p and $x \in \mathbb{R}^N$,

$$\int_{S^{N-1}} Vp(\langle x, .\rangle)(y) w_k(y)\, d\sigma(y) = d_k' \int_{-1}^1 p(t|x|)(1-t^2)^{\gamma+(N-3)/2}\, dt,$$

with some constant $d_k' > 0$ depending on k only. The series representation (2.5) of K and Mehler's formula ([Sz], (1.7.1.6)) lead to

$$\begin{aligned}\int_{S^{N-1}} K(ix, y) w_k(y)\, d\sigma(y) &= d_k' \int_{-1}^1 e^{it|x|}(1-t^2)^{\gamma+(N-3)/2}\, dt \\ &= d_k \cdot j_{\gamma+N/2-1}(|x|). \end{aligned} \tag{2.6}$$

Moreover, by Theorem 2.2(1) and the homogeneity of w_k,

$$\begin{aligned}\hat{f}(y) &= \int_{\mathbb{R}^N} F(|x|) K(-ix, y) w_k(x)\, dx \\ &= \int_0^\infty \left(\int_{S^{N-1}} K(-iry, z) w_k(z)\, d\sigma(z) \right) F(r) r^{2\gamma+N-1}\, dr.\end{aligned}$$

It follows from (2.6) that

$$\begin{aligned}\hat{f}(y) &= d_k \int_0^\infty j_{\gamma+N/2-1}(r|y|) F(r) r^{2\gamma+N-1}\, dr \\ &= c_k^{-1} 2^{\gamma+N/2} \cdot (\mathscr{H}^{\gamma+N/2-1} F)(|y|),\end{aligned}$$

which completes the proof. ▮

Proposition 2.4 leads to the following result, which will be needed in Section 9:

LEMMA 2.5. *For each compact set $L \subset \mathbb{R}^N$ there is a function $f \in C_c^+(\mathbb{R}^N)$ with $\hat{f} \in C_0^+(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, w_k(x)\,dx)$ such that $\hat{f} \geq 1$ on L.*

Proof. The Hankel transform $\mathscr{H}^\alpha$ $(\alpha \geq -1/2)$ can be regared as a Fourier transform for a suitable hypergroup structure on $[0, \infty[$, which is called a Bessel–Kingman hypergroup (see [Bl-He]). In particular, there is a positivity-preserving convolution $*_\alpha$ on $L^1([0, \infty[, \omega_\alpha)$ such that $\mathscr{H}^\alpha(f *_\alpha g) = \mathscr{H}^\alpha(f) \cdot \mathscr{H}^\alpha(g)$ for $f, g \in L^1([0, \infty[, \omega_\alpha)$. Now fix $g \in C_c^+([0, \infty[)$ with $\int_0^\infty g(x)\, d\omega_\alpha(x) > 0$. Then $g *_\alpha g \in C_c^+([0, \infty[)$ with $\mathscr{H}^\alpha(g *_\alpha g) = |\mathscr{H}^\alpha(g)|^2 \in C_0^+([0, \infty[) \cap L^1([0, \infty[, \omega_\alpha)$, and $|\mathscr{H}^\alpha(g)|^2 > 0$ on $[0, \epsilon]$ for a suitable $\epsilon > 0$. Suitable rescaling of g yields that for each compactum $R \subset [0, \infty[$ there is some g such that even $|\mathscr{H}^\alpha(g)|^2 \geq 1$ on R. The lemma now follows from Proposition 2.4 with $F := g *_\alpha g$. ▮

To use the Dunkl transform as a probabilistic tool, we next establish some further standard results like the uniqueness theorem and Lévy's continuity theorem. We denote the Banach space of all $\mathbb{C}$-valued, regular bounded Borel measures on $\mathbb{R}^N$ by $M_b(\mathbb{R}^N)$. Moreover, $M_b^+(\mathbb{R}^N)$ and $M^1(\mathbb{R}^N)$ are the subspaces consisting of all positive measures and probability measures, respectively. Moreover, we denote the σ-algebra of all Borel sets on $\mathbb{R}^N$ by $\mathscr{B}(\mathbb{R}^N)$. Weak convergence of measures means convergence with respect to the $\sigma(M_b(\mathbb{R}^N), C_b(\mathbb{R}^N))$-topology.

The Dunkl transform of $\mu \in M_b(\mathbb{R}^N)$ is given by $\hat{\mu}(y) := \int_{\mathbb{R}^N} K(-iy, x)\, d\mu(x)$ $(y \in \mathbb{R}^N)$.

THEOREM 2.6. (1) *If $\mu \in M_b(\mathbb{R}^N)$, then $\hat{\mu} \in C_b(\mathbb{R}^N)$ with $\|\hat{\mu}\|_\infty \leq \|\mu\|$.*

(2) *If $\mu \in M_b(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}^N, w_k(x)\,dx)$, then*

$$\int_{\mathbb{R}^N} \hat{\mu}(x) f(x) w_k(x)\, dx = \int_{\mathbb{R}^N} \hat{f}\, d\mu.$$

(3) *If $\mu \in M_b(\mathbb{R}^N)$ satisfies $\hat{\mu} = 0$, then $\mu = 0$.*

Proof. Part (1) follows readily from Theorem 2.2(2) and the dominated convergence theorem. Part (2) follows from Fubini's theorem, and Part (3) follows from Part (2) and the fact that $(L^1(\mathbb{R}^N, w_k(x)\,dx))^\wedge$ is $\|.\|_\infty$-dense in $C_0(\mathbb{R}^N)$. ∎

THEOREM 2.7. *Let* $(\mu_n)_{n\in\mathbb{N}} \subset M_b^+(\mathbb{R}^N)$.

(1) *If* $(\mu_n)_{n\in\mathbb{N}}$ *converges weakly to* $\mu \in M_b^+(\mathbb{R}^N)$, *then* $(\hat{\mu}_n)_{n\in\mathbb{N}}$ *converges to* $\hat{\mu}$ *uniformly on every compact subset of* $\mathbb{R}^N$.

(2) *If* $(\hat{\mu}_n)_{n\in\mathbb{N}}$ *converges pointwise to a complex-valued function* φ *on* $\mathbb{R}^N$ *that is continuous at* 0, *then there exists a unique* $\mu \in M_b^+(\mathbb{R}^N)$ *with* $\hat{\mu} = \varphi$, *and* $(\mu_n)_{n\in\mathbb{N}}$ *tends weakly to* μ.

Proof. As $(x, y) \mapsto K(ix, y)$ is analytic on $\mathbb{R}^N \times \mathbb{R}^N$, the mean value theorem ensures that for all $x \in \mathbb{R}^N$, $\epsilon > 0$ and all compacta $L \subset \mathbb{R}^N$, there exists an open neighborhood U of x with $|K(ix, y) - K(iz, y)| \le \epsilon$ for all $z \in U$, $y \in L$. The proof of part (1) can now be carried out exactly as in the classical group case; see, for instance, Theorem 23.8 in [Ba].

Moreover, Theorem 2.6 ensures that the proof of part (2) also carried over from the classical seting in [Ba] without changes. We omit the details. ∎

Remark 2.8. If a measure $\mu \in M_b(\mathbb{R}^N)$ is invariant under the action of the finite reflection group W, then its Dunkl transform is also W-invariant. On the other hand, it is not true that Dunkl transforms of radial measures are again radial; this is clear from Proposition 2.4 and the fact that the weight function w_k usually fails to be radial.

3. GENERALIZED LAPLACIANS AND HEAT KERNELS

3.1. *The Generalized Laplacian*

The generalized Laplacian Δ_k associated with some finite reflection group W on $\mathbb{R}^N$ and a multiplicity function $k \ge 0$ is defined by

$$\Delta_k f := \sum_{l=1}^{N} T_l^2 f = \Delta f + 2 \sum_{\alpha\in R_+} k(\alpha)\, \delta_\alpha f \qquad \big(f \in C^2(\mathbb{R}^N)\big),$$

with

$$\delta_\alpha f(x) = \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}.$$

It is shown in [R2] that the operator Δ_k on $C_0(\mathbb{R}^N)$ admits a closure (again denoted by Δ_k). This closure generates a positive, strongly continuous contraction semigroup $(e^{t\Delta_k})_{t\geq 0}$ on $C_0(\mathbb{R}^N)$, which is given explicitly in terms of the following generalized heat kernels.

3.2. *Generalized Heat Kernels*

The generalized heat kernel Γ_k is defined by

$$\Gamma_k(x,y,t) := \frac{c_k}{(4t)^{\gamma+N/2}} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right)$$

$$(x, y \in \mathbb{R}^n,\ t>0),$$

where c_k is given in (2.3). The heat kernel Γ_k has the following properties (see Lemma 4.5 in [R2]): Let $x, y, z \in \mathbb{R}^N$ and $t>0$. Then

(1) $\Gamma_k(x,y,t) = \Gamma_k(y,x,t)$

$$= \frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{-t|\xi|^2} K(ix,\xi)K(-iy,\xi)w_k(\xi)\, d\xi,$$

and, by the inversion formula, Theorem 2.3(3),

$$\Gamma_k(x,.,t)^\wedge(z) = e^{-t|z|^2}\cdot K(-ix,z).$$

(2) For fixed $y\in\mathbb{R}^N$, the function $u(x,t) := \Gamma_k(x,y,t)$ solves the generalized heat equation $\Delta_k u = u_t$ on $\mathbb{R}^N\times(0,\infty)$.

(3) $\int_{\mathbb{R}^N} \Gamma_k(x,y,t)w_k(x)\,dx = 1$ and

$$|\Gamma_k(x,y,t)| \leq \frac{M_k}{t^{\gamma+N/2}} e^{-(|x|-|y|)^2/4t}.$$

Moreover, the integral operators

$$H(t)f(x) := \int_{\mathbb{R}^N} \Gamma_k(x,y,t)f(y)w_k(y)\,dy \quad \text{for } t>0, \quad \text{and} \quad H(0)f := f,$$

have the following properties:

THEOREM 3.1. (1) *If $f\in C_0(\mathbb{R}^N)$, or if f is a polynomial, then $e^{t\Delta_k}f = H(t)f$ for $t\geq 0$.*

(2) *For each $f\in C_b(\mathbb{R}^N)$, the function $u(x,t) := H(t)f(x)$ is bounded and continuous on $\mathbb{R}^N\times[0,\infty[$ and solves the Cauchy problem $u_t = \Delta_k u$ on $\mathbb{R}^N\times[0,\infty[$ with $u(.,0)=f$.*

(3) *For each $f\in\mathscr{P}$, the function $u(x,t) := e^{t\Delta_k}f(x) = H(t)f(x)$ is a polynomial solution of the Cauchy problem $u_t = \Delta_k u$ on $\mathbb{R}^N\times[0,\infty[$ with*

$u(.,0)=f$; *moreover*, $\tilde{u}(x,t):=e^{-t\Delta_k}f(x)$ *solves* $\tilde{u}_t+\Delta_k\tilde{u}=0$ *on* $\mathbb{R}^N\times[0,\infty[$ *with* $\tilde{u}(.,0)=f$.

Proof. Part (1) for $f\in C_0(\mathbb{R}^N)$ and Part (2) are shown in Section 4 of [R2]. Notice further that $e^{c\Delta_k}$ ($c\in\mathbb{R}$) is well defined on $\mathscr{P}$. Proposition 2.1 of [D3] yields in our notation that

$$p(x)=\int_{\mathbb{R}^N}\Gamma_k(x,y,1/2)\big(e^{-\Delta_k/2}p\big)(y)w_k(y)\,dy \qquad \text{for } p\in\mathscr{P}. \quad (3.1)$$

From this, Part (1) for $t=1/2$ follows with $f=e^{-\Delta_k/2}p$. The general case $t>0$ follows by renormalization (see Lemma 2.1 of [R2]), and the case $t=0$ is trivial. This completes the proof of Part (1). Part (3) is also clear. ∎

The solutions for the polynomial initial value problems in Theorem 3.1 (3) will be studied in Section 8. In Section 5 we will show that suitably generalized Cauchy kernels (instead of generalized heat kernels) lead to solutions of the Dirichlet problem $u_{tt}+\Delta_k u=0$ on $\mathbb{R}^N\times[0,\infty[$ with given initial data for $t=0$. We turn next to a probabilistic interpretation of the generalized heat kernels.

3.3. *k-Gaussian Semigroups*

For $x\in\mathbb{R}^N$ and $A\in\mathscr{B}(\mathbb{R}^N)$ put

$$P_t^\Gamma(x,A):=\int_A\Gamma_k(x,y,t)w_k(y)\,dy \qquad (t>0)$$

and $P_0^\Gamma(x,A):=\epsilon_x(A)$; here ϵ_x denotes the point measure in $x\in\mathbb{R}^N$. We show that $(P_t^\Gamma)_{t\ge0}$ is a semigroup of Markov kernels on $\mathbb{R}^N$ in the following sense:

(1) Each P_t^Γ is a Markov kernel, and for all $s,t\ge0$, $x\in\mathbb{R}^N$, and $A\in\mathscr{B}(\mathbb{R}^N)$,

$$P_s^\Gamma\circ P_t^\Gamma(x,A):=\int_{\mathbb{R}^N}P_t^\Gamma(z,A)P_s^\Gamma(x,dz)=P_{s+t}^\Gamma(x,A).$$

(2) The mapping $[0,\infty[\,\to M^1(\mathbb{R}^N)$, $t\mapsto P_t^\Gamma(0,.)$, is weakly continuous.

Moreover, the semigroup has the following particular property:

(3) The Dunkl transforms of the probability measures $P_t^\Gamma(x,.)$ ($t\ge0$, $x\in\mathbb{R}^N$) satisfy

$$P_t^\Gamma(0,.)^\wedge(y)=e^{-t|y|^2} \quad\text{and}\quad P_t^\Gamma(x,.)^\wedge(y)=K(-ix,y)\cdot P_t^\Gamma(0,.)^\wedge(y)$$

for $y\in\mathbb{R}^N$.

For each constant $c > 0$, $(P_{ct}^{\Gamma})_{t \geq 0}$ is also a semigroup of Markov kernels with the properties (1)–(3) (the explicit formula for $P_t^{\Gamma}(0,.)^{\wedge}$ being modified in the obviuos way); these semigroups will be called *k-Gaussian semigroups* from now on.

Proof. Part (3) is clear from Section 3.2(1), and Part (2) follows from Part (3) and Theorem 2.7. Finally, Section 3.2 ensures that for $s, t \geq 0$ and $x, z \in \mathbb{R}^N$, each P_t^{Γ} is a Markov kernel with

$$\begin{aligned}\big(P_s^{\Gamma} \circ P_t^{\Gamma}(x,.)\big)^{\wedge}(z) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(-iy, z) P_t^{\Gamma}(w, dy) P_s^{\Gamma}(x, dw)\\ &= \int_{\mathbb{R}^N} e^{-t|z|^2} K(-iw, z) P_s^{\Gamma}(x, dw)\\ &= e^{-t|z|^2} \cdot e^{-s|z|^2} \cdot K(-ix, z) = P_{s+t}^{\Gamma}(x,.)^{\wedge}\end{aligned}$$

Theorem 2.6(3) implies that $P_s^{\Gamma} \circ P_t^{\Gamma} = P_{s+t}^{\Gamma}$, which completes the proof of Part (1). ∎

4. k-INVARIANT MARKOV KERNELS AND PROCESSES

In this section we first introduce general semigroups of Markov kernels that are consistent with respect to a given Dunkl transform. This consistency generalizes the classical notion of translation invariance of Markov kernels that can be recovered for $k = 0$. Examples of such semigroups are the k-Gaussian semigroups. Further examples will be studied in Section 5. For an elementary introduction to Markov kernels see, for instance, Section 36 of [Ba]. In the following, we fix a finite reflection group W with root system R and multiplicity function $k \geq 0$.

DEFINITION 4.1. A Markov kernel $P: \mathbb{R}^N \times \mathscr{B}(\mathbb{R}^N) \to [0,1]$ is called k-invariant if

$$P(x,.)^{\wedge}(y) = P(0,.)^{\wedge}(y) \cdot K(-ix, y) \qquad \text{for all } x, y \in \mathbb{R}^N$$

(note that we here regard $P(x,.)$ as a probability measure on $\mathbb{R}^N$).

If $k = 0$ and $\mu \in M^1(\mathbb{R}^N)$, then $P(x, A) := \epsilon_x * \mu(A)$ ($x \in \mathbb{R}^N$, $A \in \mathscr{B}(\mathbb{R}^N)$) defines a translation invariant Markov kernel. If $k \neq 0$, then there usually exists no associated k-invariant Markov kernel P with $P(0,.) = \mu$. For $N = 1$ and $k > 0$, there is a convolution of measures on $\mathbb{R}$ corresponding to the Dunkl transform that fails to be probability preserving; see [R1, Ros]. In general it is even unknown whether a convolution of bounded measures exists.

We next collect some basic properties of k-invariant Markov kernels.

LEMMA 4.2. *Let P and Q be k-invariant Markov kernels on $\mathbb{R}^N$. Then*

(1) $Pf(x) := \int_{\mathbb{R}^N} f(y)P(x,dy)$ *defines a continuous linear operator on* $C_0(\mathbb{R}^N)$.

(2) *The composition $P \circ Q$ defined by*

$$P \circ Q(x, A) = \int_{\mathbb{R}^N} Q(z, A)P(x, dz)$$

is a k-invariant Markov kernel on $\mathbb{R}^N$ with

$$((P \circ Q)(x,.))^\wedge(y) = Q(0,.)^\wedge(y) \cdot P(0,.)^\wedge(y) \cdot K(-ix, y) \quad \text{for } x, y \in \mathbb{R}^N. \tag{4.1}$$

(3) $|P(0,.)^\wedge(y)| \leq 1$ *for all* $y \in \mathbb{R}^N$.

Proof. (1) It suffices to check that $Pf \in C_0(\mathbb{R}^N)$ for all $f \in C_0(\mathbb{R}^N)$. Moreover, as $\mathscr{S}(\mathbb{R}^N)$ is $\|.\|_\infty$-dense in $C_0(\mathbb{R}^N)$, it suffices to do this for Schwartz functions $f \in \mathscr{S}(\mathbb{R}^N)$ only. In this case, $f = \hat{g}$ for some $g \in \mathscr{S}(\mathbb{R}^N)$ by Theorem 2.3(1). Hence, by Theorem 2.6,

$$\begin{aligned} Pf(x) &= \int_{\mathbb{R}^N} \hat{g}(y)P(x,dy) = \int_{\mathbb{R}^N} g(y) \cdot P(x,.)^\wedge(y) \cdot w_k(y)\, dy \\ &= \int_{\mathbb{R}^N} g(y) \cdot K(x, -iy) \cdot P(0,.)^\wedge(y) \cdot w_k(y)\, dy \\ &= \left(g \cdot P(0,.)^\wedge\right)^\wedge(x). \end{aligned}$$

As $g \cdot P(0,.)^\wedge \in L^1(\mathbb{R}^N, w_k)$, it follows from the Riemann–Lebesgue Lemma (Theorem 2.3(2)) that $Pf \in C_0(\mathbb{R}^N)$, which completes the proof.

(2) Clearly, Eq. (4.1) implies that $P \circ Q$ is k-invariant. To prove this equation, take $x, y \in \mathbb{R}^N$ and observe that the k-invariance of P and Q implies that

$$\begin{aligned} P \circ Q(x,.)^\wedge(y) &= \int_{\mathbb{R}^N} K(-iw, y)P \circ Q(x, dw) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(-iw, y)Q(z, dw)P(x, dz) \\ &= \int_{\mathbb{R}^N} Q(0,.)^\wedge(y) \cdot K(-iz, y)P(x, dz) \\ &= Q(0,.)^\wedge(y) \cdot P(0,.)^\wedge(y) \cdot K(-ix, y). \end{aligned}$$

(3) This follows from Theorem 2.2(2). ∎

We now turn to semigroups of k-invariant Markov kernels on $\mathbb{R}^N$.

DEIFNITION 4.3. A family $(P_t)_{t\geq 0}$ of k-invariant Markov kernels on $\mathbb{R}^N$ is called a semigroup of k-invariant Markov kernels, if the following statements hold:

(1) The kernels $(P_t)_{t\geq 0}$ form a semigoup, i.e., $P_s \circ P_t = P_{s+t}$ for $s, t \geq 0$.

(2) The mapping $[0, \infty[\to M^1(\mathbb{R}^N)$, $t \mapsto P_t(0, .)$, is weakly continuous.

We next collect some basic properties of semigroups of k-invariant Markov kernels that extend the case of translation-invariant Markov kernels on $\mathbb{R}^N$.

PROPOSITION 4.4. *Let $(P_t)_{t\geq 0}$ be a semigroup of k-invariant Markov kernels. Then*

(1) $P_t(0, .)^\wedge(y) \neq 0$ *for all* $y \in \mathbb{R}^N$ *and* $t \geq 0$.

(2) P_0 *is the trivial kernel* (*i.e.*, $P_0(x, .)$ *is the point measure* ϵ_x).

(3) *There is a unique function* $\varphi \in C(\mathbb{R}^N)$ *with* $P_t(0, .)^\wedge(y) = e^{-t\varphi(y)}$ *for* $t \geq 0$ *and* $y \in \mathbb{R}^N$. *The function* φ *satisfies* $\operatorname{Re} \varphi \geq 0$ *and*

$$\varphi(y) = \lim_{t\downarrow 0} \frac{1}{t}\big(1 - P_t(0, .)^\wedge(y)\big) \qquad \big(y \in \mathbb{R}^N\big).$$

The function φ *is called the negative definite function associated with* $(P_t)_{t\geq 0}$.

Proof. We first check part (2). As $P_0 \circ P_0 = P_0$, the continuous function $P_0(0, .)^\wedge$ only takes the values 0 and 1. As $P_0(0, .)^\wedge(0) = 1$, it follows that $P_0(0, .)^\wedge \equiv 1$. The injectivity of the Dunkl transform and the k-invariance of P_0 now ensure that P_0 is the trivial kernel.

To prove (1), assume that $P_t(0, .)^\wedge(y) = 0$ for some $y \in \mathbb{R}^N$ and $t > 0$. By (4.1), this would imply $P_t(0, .)^\wedge(y) = 0$ for all $t > 0$, in contradiction to Lévy's continuity theorem (Theorem 2.7) and $P_0(0, .)^\wedge \equiv 1$.

As $t \mapsto P_t(0, .)^\wedge(y)$ is continuous for all $y \in \mathbb{R}^N$, there exists $\varphi(y) \in \mathbb{C}$ with $e^{-t\varphi(y)} = P_t(0, .)^\wedge(y)$, where $\operatorname{Re} \varphi(y) \geq 0$ is a consequence of Lemma 4.2(3). The limit relation for $\varphi(y)$ is also clear. Finally, the continuity of φ follows from

$$\left(\int_0^\infty e^{-t}\mu_t \, dt\right)^\wedge (y) = \int_0^\infty e^{-t} e^{-t\varphi(y)} \, dt = (1 + \varphi(y))^{-1},$$

and the fact that $(\int_0^\infty e^{-t}\mu_t \, dt)^\wedge \in C_b(\mathbb{R}^N)$ holds by Theorem 2.6(1). ∎

PROPOSITION 4.5. *Each semigroup $(P_t)_{t\geq 0}$ of k-invariant Markov kernels on $\mathbb{R}^N$ has the following properties*:

(1) *For $f \in C_0(\mathbb{R}^N)$ and $t \geq 0$, $P_t f \in C_0(\mathbb{R}^N)$.*

(2) *If $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$ with $0 \leq f \leq 1$, then $0 \leq P_t f \leq 1$.*

(3) *$P_s \circ P_t = P_{s+t}$ for $s, t \geq 0$, and P_0 is the identity kernel.*

(4) *$\lim_{t\to 0}\|P_t f - f\|_\infty = 0$ for all $f \in C_0(\mathbb{R}^N)$.*

Proof. Part (1) follows from Lemma 4.2(1); (2) and (3) are clear (cf. Proposition 4.4(2)). To prove (4), take $f \in C_0(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. The k-invariance of P_t and Lévy's continuity theorem (Theorem 2.7) ensure that the mapping $t \mapsto P_t(x, .)$ is weakly continuous at $t = 0$, and hence $\lim_{t\to 0} P_t f(x) = f(x)$ for all $x \in \mathbb{R}^N$. A standard argument based on the resolvents of the semigroup $(P_t)_{t\geq 0}$ now yields that (4) holds (see, for example, p. 115 of [Wi]). ∎

4.1. *The Generator*

As a consequence of the above proposition, each semigroup $(P_t)_{t\geq 0}$ of k-invariant Markov kernels acts on $C_0(\mathbb{R}^N)$ as a strongly continuous positive contraction semigroup. Therefore, by a standard fact from Hille−Yosida theory, the generator

$$Lf := \lim_{t\to 0} \frac{1}{t}(P_t f - f)$$

has a $\|.\|_\infty$-dense domain in $C_0(\mathbb{R}^N)$. For later reference, we define the following extended domains of L:

$$D(L) := \left\{ f \in C(\mathbb{R}^N) : \frac{1}{t}(P_t f - f) \quad \text{converges uniformly on } \mathbb{R}^N \text{ for } t \to 0 \right\},$$

and

$$D_b(L) := D(L) \cap C_b(\mathbb{R}^N), \qquad D_0(L) := D(L) \cap C_0(\mathbb{R}^N).$$

Note that $D_0(L)$ is the domain of L on $C_0(\mathbb{R}^N)$, and that $Lf \in C_0(\mathbb{R}^N)$ for $f \in D_0(L)$.

We also remark that for $t > 0$ and $x, y \in \mathbb{R}^N$,

$$\begin{aligned} P_t K(., iy)(x) &= P_t(x, .)^\wedge(-y) = P_t(0, .)^\wedge(-y) K(x, iy) \\ &= P_t K(., iy)(0) K(x, iy), \end{aligned}$$

and hence

$$\lim_{t\to 0}\frac{1}{t}\big(P_tK(.,iy)(x) - K(.,iy)(x)\big)$$
$$= \lim_{t\to 0}\frac{1}{t}\big(P_tK(.,iy)(0) - 1\big)K(x,iy) = -\varphi(y)K(x,iy)$$

uniformly. This yields that $\{K(.,iy)\colon\ y\in\mathbb{R}^N\}\subset D_b(L)$ and $L(K(.,iy))(x) = -\varphi(y)K(x,iy)$.

We next introduce Markov processes on $\mathbb{R}^N$ associated with semigroups of k-invariant kernels.

DEFINITION 4.6. A Markov process $(X_t)_{t\geq 0}$ on $\mathbb{R}^N$ (with filtration $(\mathscr{F}_t)_{t\geq 0}$) is called k-invariant, if its transition probabilities satisfy

$$P\big(X_{s+t}\in A\mid X_s = x\big) = P_t(x,A)\qquad \big(s,t\geq 0,\, x\in\mathbb{R}^N,\, A\in\mathscr{B}(\mathbb{R}^N)\big)$$

for some semigroup $(P_t)_{t\geq 0}$ of k-invariant Markov kernels. This process is called k-Gaussian, if $(P_t)_{t\geq 0}$ is a k-Gaussian semigroup in the sense of Section 3.3.

Proposition 4.5 says that k-invariant Markov processes $(X_t)_{t\geq 0}$ on $\mathbb{R}^N$ have the so-called Feller–Dynkin property. The following theorem is therefore a consequence of a general theorem of Dynkin, Kinney, and Blumenthal (see, for instance, Theorem III.14.4 of [Wi]).

THEOREM 4.7. *Each k-invariant Markov process on $\mathbb{R}^N$ admits an equivalent càdlàg version, i.e., an equivalent Markov process with almost surely right-continuous paths and limits from the left.*

This result can be improved for k-Gaussian processes by using the fact that the associated generator $c\Delta_k$ $(c>0)$ is a differential-difference operator. We need some preparation. Let R be a root system of the finite reflection group W acting on $\mathbb{R}^N$. For each $\alpha\in R$ let H_α be the hyperplane orthogonal to α. Fix a Weyl chamber C of W, i.e, any connected component of $\mathbb{R}^N\setminus\bigcup_{\alpha\in R}H_\alpha$. The closure $\overline{C}$ of C in $\mathbb{R}^N$ is called a fundamental domain of W and can be identified with the space of all W-orbits on $\mathbb{R}^N$ in the obvious way (where the latter space carries the quotient topology); see Section 1.12 of [Hu]. Using the canonical projection

$$p\colon\mathbb{R}^N\to\mathbb{R}^N/W\cong\overline{C},\tag{4.2}$$

we first make the following observation:

LEMMA 4.8. (1) *Let P be a k-invariant Markov kernel on $\mathbb{R}^N$ with $P(0, A) = P(0, g(A))$ for all $g \in W$, $A \in \mathscr{B}(\mathbb{R}^N)$. Then P is W-invariant, i.e.,*

$$P(g(x), g(A)) = P(x, A) \qquad \text{for all } g \in W,\ x \in \mathbb{R}^N,\ A \in \mathscr{B}(\mathbb{R}^N).$$

(2) *Let $(X_t)_{t \geq 0}$ be a k-invariant Markov process on $\mathbb{R}^N$ such that its associated semigroup $(P_t)_{t \geq 0}$ of k-invariant Markov kernels is W-invariant. Then the projection $(p(X_t))_{t \geq 0}$ is a Markov process on $\mathbb{R}^N/W \cong \overline{C}$; the associated semigroup of Markov kernels is given by*

$$\tilde{P}_t(p(x), B) := P_t\big(x, p^{-1}(B)\big) \quad \text{for } x \in \mathbb{R}^N \text{ and } B \subset \mathbb{R}^N/W \quad \text{a Borel set}.$$

Proof. (1) The W-invariance of $P(0, .)$, together with the fact that $K(g(u), g(v)) = K(u, v)$ for all $u, v \in \mathbb{R}^N$ and $g \in W$ (Th. 2.2(4)), yields that $P(0, .)^\wedge(g(y)) = P(0, .)^\wedge(y)$ for all $g \in W$, $y \in \mathbb{R}^N$. Fix $g \in W$ and $x \in \mathbb{R}^N$. Using $P(g(x), g(.)) = g^{-1}(P(g(x), .))$, we conclude that

$$\begin{aligned}
P(g(x), g(.))^\wedge(y) &= \big(g^{-1}(P(g(x), .))\big)^\wedge(y) \\
&= \int_{\mathbb{R}^N} K(-ig^{-1}(z), y) P(g(x), dz) \\
&= \int_{\mathbb{R}^N} K(-iz, g(y)) P(g(x), dz) \\
&= K(-ig(x), g(y)) \cdot P(0, .)^\wedge(g(y)) \\
&= K(-ix, y) \cdot P(0, .)^\wedge(y) = P(x, .)^\wedge(y).
\end{aligned}$$

The injectivity of the Dunkl transform now completes the proof.

(2) The W-invariance of $(P_t)_{t \geq 0}$ ensures that $\tilde{P}_t$ is in fact well defined. Moreover, it is clear that $\tilde{P}_t(p(x), .)$ defines a probability measure on $\mathbb{R}^N/W \cong \overline{C}$ for each $x \in \mathbb{R}^N$, and that for each Borel set $B \subset \mathbb{R}^N/W$, the mapping $\mathbb{R}^N/W \cong \overline{C} \to [0, 1]$, $z \mapsto \tilde{P}_t(z, B)$ is Borel measurable. Now consider the canonical filtration $(\mathscr{F}_t)_{t \geq 0}$ associated with the process $(X_t)_{t \geq 0}$, and take $s, t \geq 0$ and a Borel set $B \subset \mathbb{R}^N/W$. Then

$$\begin{aligned}
\tilde{P}_t(p(X_s), B) &= P_t\big(X_s, p^{-1}(B)\big) = P\big(X_{s+t} \in p^{-1}(B) \mid \mathscr{F}_s\big) \\
&= P(p(X_{s+t}) \in B \mid \mathscr{F}_s)
\end{aligned}$$

almost surely; see, for instance, 42.3 in [Ba]. In particular, the conditional probability $P(p(X_{s+t}) \in B \mid \mathscr{F}_s)$ is $p(X_s)$-measurable. As the filtration $(\tilde{\mathscr{F}}_t)_{t \geq 0}$ associated with the process $(p(X_t))_{t \geq 0}$ satisfies $\tilde{\mathscr{F}}_t \subset \mathscr{F}_t$ for $t \geq 0$, it follows that

$$P\big(p(X_{s+t}) \in B \mid \tilde{\mathscr{F}}_s\big) = P\big(p(X_{s+t}) \in B \mid p(X_s)\big) = \tilde{P}_t\big(p(X_s), B\big)$$

almost surely, which completes the proof. ∎

Remark 4.9. The k-Gaussian kernels of Section 3 are obviously W-invariant. Moreover, all semigroups $(Q_t)_{t \geq 0}$ of k-invariant Markov kernels on $\mathbb{R}^N$ constructed via subordination (see Section 5 below) from a semigroup $(P_t)_{t \geq 0}$ of k- and W-invariant kernels are also W-invariant.

THEOREM 4.10. *Each k-Gaussian process on $\mathbb{R}^N$ admits an equivalent k-Gaussian process $(X_t)_{t \geq 0}$ with the càdlàg property. The projection $(p(X_t))_{t \geq 0}$ of this process on $\mathbb{R}^N/W \cong \overline{C}$ has almost surely continuous paths.*

Proof. In view of Theorem 4.7, it suffices to check that $(p(X_t))_{t \geq 0}$ is a.e. continuous. Our proof follows the exposition in Section III.28 of [Wi]. Remember that $(p(X_t))_{t \geq 0}$ is a Markov process on $\mathbb{R}^N/W \cong \overline{C}$ by Lemma 4.8. The generator of the associated semigroup $(\tilde{P}_t)_{t \geq 0}$ acting on $C_0(\overline{C})$ is given, up to a constant, by

$$L_k f = \Delta f + 2 \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} \tag{4.3}$$

with domain

$$C^2_{0,W}(\overline{C}) := \big\{ f \in C^2_0(\overline{C}) : \langle \nabla f(x), \alpha \rangle = 0 \text{ for } x \in H_\alpha \cap \overline{C},\ \alpha \in R_+ \big\} \tag{4.4}$$

(notice that L_k coincides with the restriction of the Dunkl Laplacian to W-invariant functions in $C^2_0(\mathbb{R}^N)$). The local form of L_k yields readily that

$$\lim_{t \downarrow 0} \frac{1}{t} \sup_{x \in K} \tilde{P}_t\big(x, \overline{C} \setminus U_\epsilon(x)\big) = 0 \tag{4.5}$$

for all compacta $K \subset \overline{C}$ and all ϵ-balls $U_\epsilon(x)$ around x with $\epsilon > 0$; see, for instance, Theorem 3.9′ of [Dy]. This implies that for all compacta $K \subset \overline{C}$

and all $\epsilon, u > 0$,

$$\lim_{n\to\infty} P\left(\bigcup_{k=0}^{n-1}\left\{\left|p\left(X_{ku/n}\right) - p\left(X_{(k+1)u/n}\right)\right| > \epsilon \quad\text{and}\quad X_s \in K \text{ for } s \in [0,u]\right\}\right) = 0.$$

Together with the càdlàg property of $(p(X_t))_{t\geq 0}$, this implies the almost sure continuity. ∎

In the end of this section we consider the projection $(|X_t|)_{t\geq 0}$ on $[0,\infty[$ for a k-Gaussian càdlàg process $(X_t)_{t\geq 0}$ on $\mathbb{R}^N$ with generator $\Delta_k/2$ (without loss of generality). The process $(|X_t|)_{t\geq 0}$ has continuous paths by Theorem 4.10. Moreover, we shall see that this process is a Bessel process of index $\alpha := \gamma + N/2 - 1 \geq -1/2$. For this, we first recapitulate that a Bessel process $(B_t^\alpha)_{t\geq 0}$ on $[0,\infty[$ of index $\alpha > -1$ is a Markov process with continuous paths and with the transition probabilities

$$P\left(B_{t+s}^\alpha \in A \mid B_s^\alpha = x\right) = P_t^\alpha(x, A)$$
$$:= \frac{2}{(2t)^{\alpha+1}\Gamma(\alpha+1)}\int_A j_\alpha(ixy/t)\, e^{-(x^2+y^2)/2t} y^{2\alpha+1}\, dy \tag{4.6}$$

for $t > 0$, $s, x \geq 0$, and $A \subset [0,\infty[$ a Borel set. We also notice here that the generator of the semigroup of Markov kernels $(P_t^\alpha)_{t\geq 0}$ on $[0,\infty[$ in (4.6) is given by the Bessel differential operator,

$$L^\alpha f := \frac{1}{2}\left(f'' + \frac{2\alpha+1}{x} f'\right) \qquad \left(f \in C^2([0,\infty[),\ f'(0) = 0\right). \tag{4.7}$$

For details on Bessel processes we refer to Section XI.1 of [Rev-Y]. In particular, the following result is well known for $k = 0$:

THEOREM 4.11. *If $(X_t)_{t\geq 0}$ is a k-Gaussian càdlàg process on $\mathbb{R}^N$ with generator $\Delta_k/2$, then $(|X_t|)_{t\geq 0}$ is a Bessel process of index $\gamma + N/2 - 1 \geq -1/2$.*

Proof. Let $x \in \mathbb{R}^N$, $0 \le a \le b$, and $A := \{z \in \mathbb{R}^N\colon a \le |z| \le b\}$. Then, by use of Eq. (2.6), the transition probabilities $P^{\Gamma}_{t/2}(x, A)$ of $(X_t)_{t \ge 0}$ satisfy

$$\begin{aligned} P^{\Gamma}_{t/2}(x, A) &= \int_A \Gamma_k(x, y, t/2) w_k(y)\, dy \\ &= \frac{c_k}{(2t)^{\gamma + N/2}} \int_A e^{-(|x|^2 + |y|^2)/2t} K(x/t, y) w_k(y)\, dy \\ &= \frac{c_k}{(2t)^{\gamma + N/2}} \int_a^b \left(\int_{S^{N-1}} K(x/t, rz) w_k(z)\, d\sigma(z) \right) \\ &\quad \times e^{-(|x|^2 + r^2)/2t} r^{2\gamma + N - 1}\, dr \\ &= \frac{2}{(2t)^{\gamma + N/2} \Gamma(\gamma + N/2)} \\ &\quad \times \int_a^b j_{\gamma + N/2 - 1}(i|x|r/t) e^{-(|x|^2 + r^2)/2t} r^{2\gamma + N - 1}\, dr. \end{aligned}$$

This in particular ensures that $P^{\Gamma}_{t/2}(x, A)$ depends on $|x|$ only, and the assertion follows readily. ∎

5. SUBORDINATION AND CAUCHY KERNELS

In this section we construct some examples of k-invariant Markov kernels from the k-Gaussian semigroup of Section 3.3 by subordination. This principle is based on convolution semigroups of probability measures on the group $(\mathbb{R}, +)$ that are supported by $[0, \infty[$; it leads from a given semigroup of kernels with certain algebraic properties to a new one with the same algebraic properties. In the related setting of translation invariant kernels on locally compact abelian groups, this principle is worked out very clearly in Section 9 of [Be-Fo]. The most prominent example of an underlying subordinating semigroup is given by the Poisson semigroup

$$\left(p_t := \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} \cdot \epsilon_k \right)_{t \ge 0}. \tag{5.1}$$

To describe the construction, fix a convolution semigroup $(\mu_t)_{t \ge 0}$ of probability measures on the group $(\mathbb{R}, +)$ (in the sense of [Be-Fo]), which is supported by $[0, \infty[$. The Laplace transforms $\mathscr{L}\mu_t(x) := \int_0^\infty e^{-xs}\, d\mu_t(s)$ $(\operatorname{Re} x \ge 0)$ of μ_t can be written as

$$\mathscr{L}\mu_t(x) = e^{-tf(x)} \qquad (t \ge 0, \operatorname{Re} x \ge 0),$$

with a unique function $f \in C([0,\infty[) \cap C^\infty(]0,\infty[)$. It is well known (see Section 9 of [Be-Fo]) that a function $f\colon [0,\infty[\to \mathbb{R}$ is a Bernstein function, i.e., $f \geq 0$ and $(-1)^n D^n f \leq 0$ for all $n \geq 1$, if and only if f is connected with some convolution semigroup $(\mu_t)_{t\geq 0}$ of probability measures on the group $(\mathbb{R}, +)$, supported by $[0,\infty[$, as described above. This leads to the following result:

PROPOSITION 5.1. *Let $(\mu_t)_{t\geq 0}$ and the Bernstein function f be related as above. If $(P_t)_{t\geq 0}$ is a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ with negative definite function φ, then*

$$Q_t(x, A) := \int_0^\infty P_s(x, A)\, d\mu_t(s)$$

defines a semigroup $(Q_t)_{t\geq 0}$ of k-invariant Markov kernels on $\mathbb{R}^N$ with negative definite function $f \circ \varphi$. We say that $(Q_t)_{t\geq 0}$ is obtained from $(P_t)_{t\geq 0}$ by subordination with respect to $(\mu_t)_{t\geq 0}$.

Proof. For each $t \geq 0$, the kernel Q_t is a Markov kernel on $\mathbb{R}^N$ with

$$\begin{aligned} Q_t(x,.)^\wedge(y) &= \int_0^\infty P_s(x,.)^\wedge(y)\, d\mu_t(s) \\ &= K(-ix, y) \cdot \int_0^\infty P_s(0,.)^\wedge(y)\, d\mu_t(s) \\ &= K(-ix, y) \cdot Q_t(0,.)^\wedge(y) \end{aligned}$$

and

$$\begin{aligned} Q_t(0,.)^\wedge(y) &= \int_0^\infty P_s(0,.)^\wedge(y)\, d\mu_t(s) = \int_0^\infty e^{-s\varphi(y)}\, d\mu_t(s) \\ &= (\mathscr{L}\mu_t)(\varphi(y)) = e^{-tf(\varphi(y))}. \end{aligned}$$

Together with Theorem 2.7 and the injectivity of the Dunkl transform, this immediately leads to the assertion. ∎

EXAMPLES 5.2. (1) The Poisson semigroup $(p_t)_{t\geq 0}$ of Eq. (5.1) is associated with the Bernstein function $f(x) = 1 - e^{-x}$. If P is any k-invariant Markov kernel on $\mathbb{R}^N$, then

$$Q_t(x, A) := \sum_{k=0}^{\infty} P^{(k)}(x, A) \cdot \frac{e^{-t} t^k}{k!},$$
$$\text{with } P^{(k)} = P \circ P^{(k-1)},\ P^{(0)}(x,.) = \epsilon_x,$$

defines a semigroup of k-invariant Markov kernels with negative definite function $\varphi(x) = 1 - P(0,.)^\wedge(x)$.

(2) The function $f(x) = x^\alpha$ is a Bernstein function for $\alpha \in [0,1]$. Using the negative definite function $x \mapsto |x|^2$ on $\mathbb{R}^N$ associated with the k-Gaussian kernels $(P_t^\Gamma)_{t\geq 0}$, we obtain from Proposition 5.1 and a time rescaling that for all $\alpha \in [0,1]$ and $c > 0$,

$$\varphi_{c,\alpha}(x) := c \cdot |x|^{2\alpha}$$

is the negative definite function of a semigroup of k-invariant Markov kernels.

(3) If one takes $\alpha = 1/2$ above, then the one-sided convolution semigroup $(\mu_t)_{t\geq 0}$ on $(\mathbb{R}, +)$ associated with the Bernstein function $f(x) = x^{1/2}$ is given by

$$d\mu_t(s) = \frac{1}{\sqrt{4\pi}} \cdot ts^{-3/2} e^{-t^2/4s}\, ds \qquad (s > 0)$$

(see Section 9 of [Be-Fo]). By Proposition 5.1, the kernels Q_t associated with the negative definite function $\varphi(x) = |x|$ are given by

$$Q_t(x, A) = \int_A c^k(x,y,t) w_k(y)\, dy,$$

with

$$\begin{aligned} c^k(x,y,t) &:= \int_0^\infty \Gamma_k(x,y,s)\, d\mu_t(s) \\ &= \frac{c_k t}{4^{\gamma+N/2}\sqrt{4\pi}} \int_0^\infty e^{-t^2/4s} s^{-(\gamma+(N+3)/2)} e^{-(|x|^2+|y|^2)/4s} \\ &\quad \times K\left(\frac{x}{2s}, y\right) ds, \end{aligned} \tag{5.2}$$

where the weight function w_k and the constants γ, c_k are given as in Section 2.1. In particular, for $x = 0$, $c^k(x,y,t)$ can be computed explicitly by using the substitution $r = (t^2 + |y|^2)/4s$ and the Gamma-integral:

$$\begin{aligned} c^k(0,y,t) &= \frac{c_k t}{4^{\gamma+N/2}\sqrt{4\pi}} \cdot \int_0^\infty s^{-(\gamma+(N+3)/2)} \cdot e^{-t^2/4s} \cdot e^{-|y|^2/4s}\, ds \\ &= \frac{c_k t}{4^{\gamma+N/2}\sqrt{4\pi}} \cdot \int_0^\infty e^{-r} r^{\gamma+(N-1)/2} \left(\frac{4}{t^2+|y|^2}\right)^{\gamma+(N+1)/2} dr \\ &= \frac{c_k \Gamma(\gamma + (N+1)/2)}{\sqrt{\pi}} \cdot \frac{t}{\left(t^2+|y|^2\right)^{\gamma+(N+1)/2}}. \end{aligned} \tag{5.3}$$

Having the classical case $k = 0$ in mind, we call the probability measures

$$dC_t^k(y) := c^k(0, y, t) w_k(y)\, dy \tag{5.4}$$

k-Cauchy distributions on $\mathbb{R}^N$, and c^k the k-Cauchy kernel. Moreover, a Markov process on $\mathbb{R}^N$ is called a k-Cauchy process, if its transition probabilities are associated with the semigroup $(Q_t)_{t \geq 0}$ of k-invariant Markov kernels above.

Proposition 5.1 has an interpretation on the level of k-invariant Markov processes. In fact, the following result is a special case of Section X.7 in [Fe]:

Remark 5.3. Let $(\mu_t)_{t \geq 0}$ be a convolution semigroup on $(\mathbb{R}, +)$ supported by $[0, \infty[$. Consider the associated càdlàg process $(T_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathscr{A}, P)$ with state space $[0, \infty[$ and with independent, stationary, and nonnegative increments. Assume also that $T_0 \equiv 0$. Moreover, let $(X_t)_{t \geq 0}$ be a k-invariant Markov process on the same space $(\Omega, \mathscr{A}, P)$, which is independent of $(T_t)_{t \geq 0}$ and also has the càdlàg property. Denote the semigroup of k-invariant kernels associated with $(X_t)_{t \geq 0}$ by $(P_t)_{t \geq 0}$. Then the stochastic process $(Y_t)_{t \geq 0}$ on $(\Omega, \mathscr{A}, P)$ with

$$Y_t(\omega) := X_{T_t(\omega)}(\omega) \qquad (t \geq 0,\ \omega \in \Omega)$$

is a k-invariant Markov process on $\mathbb{R}^N$ with the càdlàg property and with the same initial distribution as $(X_t)_{t \geq 0}$. Moreover, this process is associated with the semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ that is obtained from $(P_t)_{t \geq 0}$ by subordination with respect to $(\mu_t)_{t \geq 0}$.

EXAMPLE 5.4. Let $(\mathscr{F}_t)_{t \geq 0}$ be a filtration on some probability space $(\Omega, \mathscr{A}, P)$. Let $(B_t)_{t \geq 0}$ be a $(\mathscr{F}_t)_{t \geq 0}$-Brownian motion on $\mathbb{R}$ (in the classical sense and with continuous paths). It is well known (see, for instance, Section 50 of [Ba]) that the stopping times

$$T_t(\omega) := \inf\{\tau \geq 0 : B_\tau \geq t\} \qquad (t \geq 0)$$

form a process $(T_t)_{t \geq 0}$ on $[0, \infty[$ with $T_0 = 0$ and independent, stationary, and nonnegative increments; the associated convolution semigroup $(\mu_t)_{t \geq 0} \subset M^1([0, \infty[)$ is given by $\mu_0 = \epsilon_0$ and

$$d\mu_t(s) = \frac{t}{\sqrt{2\pi} \cdot s^{3/2}} \cdot e^{-t^2/2s}\, ds \qquad (s, t > 0).$$

Now let $(X_t)_{t \geq 0}$ be a k-Gaussian Markov process on $\mathbb{R}^N$ with filtration $(\mathscr{F}_t)_{t \geq 0}$. Assume that $(X_t)_{t \geq 0}$ has the càdlàg property and is independent of $(B_t)_{t \geq 0}$. Remark 5.3 and Example 5.2(3) show that

$$Y_t(\omega) := X_{T_t(\omega)}(\omega) \qquad (t \geq 0,\ \omega \in \Omega)$$

defines a k-Cauchy process $(Y_t)_{t \geq 0}$ on $(\Omega, \mathscr{A}, P)$ with values in $\mathbb{R}^N$.

We next turn to the connection between k-Cauchy kernels c^k and the Laplace-type equation $(\Delta_k + \partial_t^2)u = 0$ on the upper half space $\mathbb{R}^N \times [0,\infty[$. We start with the following observation:

LEMMA 5.5. *The k-Cauchy kernel c^k satisfies $(\Delta_k^y + \partial_t^2)c^k(x,y,t) = 0$ for $x, y \in \mathbb{R}^N$, $t > 0$.*

Proof. The negative definite function of the k-Cauchy semigroup is given by $\varphi(x) = |x|$ for $x \in \mathbb{R}^N$. Therefore, Proposition 5.1 and the inversion formula (Theorem 2.3(3)) imply that

$$c^k(x,y,t) = \frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N} K(-ix,z)K(iy,z)e^{-t|z|}w_k(z)\,dz. \tag{5.5}$$

The lemma now follows from $\Delta_k^y K(iy,z) = -|z|^2 \cdot K(iy,z)$ and interchanging $(\Delta_k^y + \partial_t^2)$ with the integration in (5.5). The latter is justified by the decay properties of the integrand and its derivatives, which allow application of the dominated convergence theorem. (Note that by Theorem 2.2(2), $|\partial_y^\nu K(iy,z)| \le |z|^{|\nu|}$ for all multi-indices ν.) ∎

THEOREM 5.6. *For each function $f \in C_b(\mathbb{R}^N)$, the function u given by*

$$u(x,t) = (Q_t f)(x) := \begin{cases} \int_{\mathbb{R}^N} c^k(x,y,t)f(y)w_k(y)\,dy & \text{for } t>0,\ x\in\mathbb{R}^N \\ f(x) & \text{for } t=0,\ x\in\mathbb{R}^N \end{cases} \tag{5.6}$$

is a $C_b(\mathbb{R}^N \times [0,\infty[) \cap C^2(\mathbb{R}^N \times]0,\infty[)$-solution of the Cauchy problem $u_{tt} + \Delta_k u = 0$ on $\mathbb{R}^N \times]0,\infty[$.

Proof. To check $u \in C^2(\mathbb{R}^N \times]0,\infty[)$ with $u_{tt} + \Delta_k u = 0$, we have only to make sure that the necessary differentiations of u may be carried out under the integral in (5.6). For this we again employ Theorem 2.2(2) together with the representation (5.5) of the Cauchy densities c^k. This ensures that the derivatives of c^k are sufficiently fast decaying, the dominated converge theorem now allows the necessary differentiations of u under the integral. Moreover, the positivity and normalization of c^k (see Example 5.2(3)) imply that u is bounded with $\|u\|_{\infty,\mathbb{R}^N\times[0,\infty[} \le \|f\|_{\infty,\mathbb{R}^N}$.

Finally, we have to show that $Q_t f(x) \to f(\xi)$ for $x \to \xi$ and $t \to 0$. Using representation (5.2) for c^k, we express Q_t by means of the heat operators $H(t)$ of Section 3.2 as $Q_t f(x) = \int_0^\infty H(s)f(x)\,d\mu_t(s)$, where $(\mu_t)_{t\ge 0}$ is the convolution semigroup of Example 5.2(3). Hence,

$$|Q_t f(x) - f(\xi)| \le \int_0^\infty |H(s)f(x) - f(\xi)|\,d\mu_t(s).$$

Now fix $\epsilon > 0$. The continuity of $(x, s) \mapsto H(s)f(x)$ on $\mathbb{R}^N \times [0, \infty[$ from Theorem 3.1(2) leads to some $\delta > 0$ with $|H(s)f(x) - f(\xi)| \le \epsilon$ for $0 < s < \delta$ and $|x - \xi| < \delta$. Hence, for $|x - \xi| < \delta$,

$$|Q_t f(x) - f(\xi)| \le \epsilon + \int_\delta^\infty |H(s)f(x) - f(\xi)|\, d\mu_t(s)$$
$$\le \epsilon + 2\|f\|_\infty \cdot \mu_t([\delta, \infty[).$$

As $\lim_{t \to 0} \mu_t([\delta, \infty[) = 0$, this completes the proof. ∎

6. MARTINGALE CHARACTERIZATIONS OF k-INVARIANT MARKOV PROCESSES

In this section we construct some martingales related with k-invariant Markov processes and show that k-invariant Markov processes appear as unique solutions of certain martingale problems in the sense of Stroock and Varadhan [S-V]. Parts of this section may be skipped by readers not familiar with stochastic integration.

Our first result is well known for processes with independent, stationary increments:

PROPOSITION 6.1. *Let $(P_t)_{t \ge 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$. Then an arbitrary stochastic process $(X_t)_{t \ge 0}$ on $\mathbb{R}^N$ is a Markov process related with $(P_t)_{t \ge 0}$ if and only if*

$$\left(\frac{1}{P_t(0, .)^\wedge(-\lambda)} \cdot K(X_t, i\lambda) \right)_{t \ge 0} \tag{6.1}$$

is a martingale for each $\lambda \in \mathbb{R}^N$.

Moreover, if $(X_t)_{t \ge 0}$ is a k-Gaussian process on $\mathbb{R}^N$ with negative definite function $\varphi(\lambda) = |\lambda|^2$ and $X_0 = 0$ almost surely, then $(K(X_t, \lambda) \cdot e^{-|\lambda|^2 t})_{t \ge 0}$ is a martingale for each $\lambda \in \mathbb{R}^N$.

Proof. Notice first that the semigroup $(P_t)_{t \ge 0}$ satisfies $P_t(x, .)^\wedge(y) \ne 0$ for all $t \ge 0$ and $x, y \in \mathbb{R}^N$; see Proposition 4.4(1). This ensures that the processes above are well defined. Let $(\Omega, \mathscr{A}, P)$ be the probability space on which the process $(X_t)_{t \ge 0}$ is defined.

To check the only-if past, take $s, t \geq 0$ and $\lambda \in \mathbb{R}^N$. Then for almost all $\omega \in \Omega$,

$$\begin{aligned} E\big(K(X_{s+t}, i\lambda)\big|\mathscr{F}_s\big)(\omega) &= E\big(K(X_{s+t}, i\lambda)\big|X_s\big)(\omega) \\ &= \int_{\mathbb{R}^N} K(x, i\lambda) P_t\big(X_s(\omega), dx\big) \\ &= \hat{\delta}_{X_s(\omega)}(-\lambda) \cdot P_t(0,.)^\wedge(-\lambda) \\ &= K\big(X_s(\omega), i\lambda\big) \cdot P_t(0,.)^\wedge(-\lambda). \end{aligned}$$

Hence, as $P_{s+t}(0,.)^\wedge = P_s(0,.)^\wedge \cdot P_t(0,.)^\wedge$, the process (6.1) is a martingale.

To check the if past, take again $s, t \geq 0$ and $\lambda \in \mathbb{R}^N$. Then, by our assumption,

$$E\big(K(X_{s+t}, i\lambda)\big|\mathscr{F}_s\big) = P_t(0,.)^\wedge(-\lambda) \cdot K(X_s, i\lambda) \qquad \text{a.s.}$$

Now take $F \in \mathscr{F}_s$ with $P(F) > 0$. Define the probability measure P_F on $(\Omega, \mathscr{A})$ by $P_F(A) := P(F)^{-1} \cdot P(A \cap F)$. The distributions $\mu_s^F, \mu_{s+t}^F \in M^1(\mathbb{R}^N)$ of X_s and X_{s+t}, respectively, under P_F satisfy

$$\begin{aligned} \big(\hat{\mu}_{s+t}^F\big)^\wedge(-\lambda) &= \int_{\mathbb{R}^N} K(y, i\lambda)\, d\mu_{s+t}^F(y) = \frac{1}{P(F)} \int_F K(X_{s+t}, i\lambda)\, dP \\ &= \frac{1}{P(F)} \int_F E\big(K(X_{s+t}, i\lambda)\big|\mathscr{F}_s\big)\, dP \\ &= \frac{1}{P(F)} \int_F P_t(0,.)^\wedge(-\lambda) \cdot K(X_s, i\lambda)\, dP \\ &= P_t(0,.)^\wedge(-\lambda) \cdot \big(\mu_s^F\big)^\wedge(-\lambda) = \big(P_t \circ \mu_s^F\big)^\wedge(-\lambda). \end{aligned}$$

As this holds for all $\lambda \in \mathbb{R}^N$, the injectivity (Theorem 2.6(3)) of the Dunkl tranform yields that $\mu_{s+t}^F = P_t \circ \mu_s^F$. Hence, for each Borel set $B \subset \mathbb{R}^N$ and each $F \in \mathscr{F}_s$,

$$\begin{aligned} \int_F 1_{\{X_{s+t} \in B\}}\, dP &= P\big(\{X_{s+t} \in B\} \cap F\big) = P(F) \cdot \mu_{s+t}^F(B) \\ &= P(F) \cdot \big(P_t \circ \mu_s^F\big)(B) = \int_F P_t\big(X_s(\omega), B\big)\, dP(\omega). \end{aligned}$$

As $\omega \mapsto P_t(X_s(\omega), B)$ is $\sigma(X_s)$-measurable, and as $\mathscr{F}_s \supset \sigma(X_s)$, we obtain that $P(X_{s+t} \in B \,|\, \mathscr{F}_s) = P(X_{s+t} \in B \,|\, X_s) = P_t(X_s, B)$ a.e. for all Borel sets $B \subset \mathbb{R}^N$. Hence, X is a k-invariant Markov process associated with $(P_t)_{t \geq 0}$, as claimed.

Finally, if $(X_t)_{t\geq 0}$ is k-Gaussian with $X_0 = 0$ a.s., then the random variables $K(X_t, \lambda)$ are integrable for all $t \geq 0$ and $\lambda \in \mathbb{R}^N$; see Theorem 2.2(2). The computation at the beginning of this proof (with $i\lambda$ instead of λ) yields the last statement of the proposition. ∎

We next employ the negative definite function φ of a semigroup $(P_t)_{t\geq 0}$ of k-invariant kernels to rewrite Proposition 6.1. The proof will be based on Ito's stochastic integration by parts; for stochastic integration with respect to semimartingales, see [K-S, W-W, Wi].

LEMMA 6.2. *Let $(P_t)_{t\geq 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ with negative definite function $\varphi \in C(\mathbb{R}^N)$, and let $(X_t)_{t\geq 0}$ be a càdlàg process on $\mathbb{R}^N$. Then, for each $\lambda \in \mathbb{R}^N$, the $\mathbb{C}$-valued process $((P_t(0,.)^\wedge(-\lambda))^{-1}. K(X_t, i\lambda))_{t\geq 0}$ is a martingale if and only if the process*

$$\left(X_t^\lambda := K(X_t, i\lambda) + \varphi(-\lambda)\cdot \int_0^t K(X_s, i\lambda)\, ds\right)_{t\geq 0}$$

is a martingale.

Proof. As $|K(x, i\lambda)| \leq 1$ for $x, \lambda \in \mathbb{R}^N$ by Theorem 2.2(2), both processes of the lemma are uniformly and L^2-bounded on compact time intervals. Hence each of them is a martingale if and only if it is a local L^2-martingale (see, for instance, Proposition 4.2.3 of [W-W]).

Assume now that $((P_t(0,.)^\wedge(-\lambda))^{-1}\cdot K(X_t, i\lambda))_{t\geq 0}$ is a martingale and hence a local L^2-martingale. Then $(K(X_t, i\lambda))_{t\geq 0}$ is a semimartingale. Integration by parts for $K(X_t, i\lambda)e^{t\varphi(-\lambda)}$, together with $[t, K(X_t, i\lambda)] = 0$ for the mutual variation (see Section 7.3 of [W-W]), implies that

$$d\big(K(X_t, i\lambda)e^{t\varphi(-\lambda)}\big) = e^{t\varphi(-\lambda)}\, dK(X_t, i\lambda) + K(X_{t-}, i\lambda)\, de^{t\varphi(-\lambda)}.$$

As $de^{t\varphi(-\lambda)} = \varphi(-\lambda)e^{t\varphi(-\lambda)}\, dt$ and $\int_0^t K(X_{s-}, i\lambda)\, ds = \int_0^t K(X_s, i\lambda)\, ds$ a.s., it follows that

$$d\big(K(X_t, i\lambda)e^{t\varphi(-\lambda)}\big) = e^{t\varphi(-\lambda)}\cdot \big(dK(X_t, i\lambda) + \varphi(-\lambda)K(X_t, i\lambda)\, dt\big). \tag{6.2}$$

Therefore,

$$dK(X_t, i\lambda) + \varphi(-\lambda)K(X_t, i\lambda)\, dt = e^{-t\varphi(-\lambda)}\cdot d\big(K(X_t, i\lambda)e^{\varphi t(-\lambda)}\big)$$

is the differential of a local L^2-martingale, as claimed.

Conversely, if $(X_t^\lambda)_{t\geq 0}$ is a martingale, then $(K(X_t, i\lambda))_{t\geq 0}$ is a semimartingale, and Eq. (6.2) holds also. Hence, $((P_t(0,.)^\wedge(-\lambda))^{-1}\cdot K(X_t, i\lambda))_{t\geq 0}$ is a local L^2-martingale, as claimed. ∎

We next present several equivalent characterizations of homogeneous Markov processes X on $\mathbb{R}^N$ associated with a specific semigroup $(P_t)_{t\geq 0}$ of k-invariant Markov kernels. In particular, such processes are unique solutions of martingale problems in the spirit of Stroock and Varadhan [S-V].

We introduce the following abbreviation: If X is a càdlàg process on $\mathbb{R}^N$, and if L is the generator of $(P_t)_{t\geq 0}$, then for $f \in D(L)$ we define the $\mathbb{C}$-valued process

$$\Pi_X^{L,f} = \left(\Pi_{X,t}^{L,f}\right)_{t\geq 0} = \left(f(X_t) - f(X_0) - \int_0^t L(f)(X_s)\, ds \right)_{t\geq 0}. \quad (6.3)$$

THEOREM 6.3. *Let $(P_t)_{t\geq 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ with negative definite function φ and generator L. Then the following statements are equivalent for each càdlàg process $X = (X_t)_{t\geq 0}$ on $\mathbb{R}^N$:*

(1) *X is a k-invariant Markov process associated with the semigroup $(P_t)_{t\geq 0}$.*

(2) *For each $\lambda \in \mathbb{R}^N$ the process $((P_t(0,.)^\wedge(-\lambda))^{-1} \cdot K(X_t, i\lambda))_{t\geq 0}$ is a martingale.*

(3) *$(K(X_t, i\lambda) + \varphi(-\lambda) \cdot \int_0^t K(X_s, i\lambda)\, ds)_{t\geq 0}$ is a martingale for each $\lambda \in \mathbb{R}^N$.*

(4) *$\Pi_X^{L, K(.,i\lambda)}$ is a martingale for each $\lambda \in \mathbb{R}^N$.*

(5) *$\Pi_X^{L,f}$ is a martingale for each $f \in D_b(L)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from Proposition 6.1 and Lemma 6.2; (3) $\Leftrightarrow$ (4) $\Leftarrow$ (5) is obvious, and (1) $\Rightarrow$ (5) is the well-known Dynkin formula; see Proposition 4.1.7 of [E-K]. ∎

The generator $c\Delta_k$ $(c > 0)$ of a k-Gaussian semigroup of Markov kernels is a second-order differential-difference operator; it is more convenient here to consider the spaces $C_0^2(\mathbb{R}^N)$, $C_b^2(\mathbb{R}^N)$ and $C^2(\mathbb{R}^N)$ instead of the abstract domains $D_0(\Delta_k)$, $D_b(\Delta_k)$, and $D(\Delta_k)$, respectively. Moreover, condition (5) can be improved by using test functions f which also depend on the time t (this generalization is in fact also possible in the setting of Theorem 6.3). To be precise, we take $f \in C^{2,1}(\mathbb{R}^N \times [0,\infty[)$ and a càdlàg process X on $\mathbb{R}^N$, and we define

$$\begin{aligned}\Pi_X^{c\Delta_k,f} &= \left(\Pi_{X,t}^{c\Delta_k,f}\right)_{t\geq 0} \\ &= \left(f(X_t,t) - f(X_0,0) - \int_0^t \left(\frac{\partial}{\partial s} + c\Delta_k \right)(f)(X_s,s)\, ds \right)_{t\geq 0}.\end{aligned}$$

Then, besides equivalences stated already in Theorem 6.3, we have the following:

THEOREM 6.4. *Let $(P_t)_{t\geq 0}$ be a semigroup of k-Gaussian Markov kernels on $\mathbb{R}^N$ with generator $c\Delta_k$, $c > 0$. Then the following statements are equivalent for any càdlàg process $(X_t)_{t\geq 0}$ on $\mathbb{R}^N$ whose radial component process $(|X_t|)_{t\geq 0}$ on $[0, \infty[$ is continuous*:

(1) *X is a k-Gaussian process associated with the semigroup $(P_t)_{t\geq 0}$.*

(5) *$\Pi_X^{c\Delta_k, f}$ is a martingale for each $f \in C_c^2(\mathbb{R}^N)$.*

(5′) *$\Pi_X^{c\Delta_k, f}$ is a martingale for each $f \in C_c^{2,1}(\mathbb{R}^N \times [0, \infty[)$.*

(6) *$\Pi_X^{c\Delta_k, f}$ is a local martingale for each $f \in C^2(\mathbb{R}^N)$.*

(6′) *$\Pi_X^{c\Delta_k, f}$ is a local martingale for each $f \in C^{2,1}(\mathbb{R}^N \times [0, \infty[)$.*

Proof. (6) ⇒ (1) ⇒ (5) follows from Theorem 6.3. Moreover, (1) ⇒ (5′) is known for arbitrary Feller–Dynkin processes (apply Proposition 4.1.7 of [E-K] to the Markov process $((X_t, t))_{t\geq 0}$ on $\mathbb{R}^N \times [0, \infty[$).

It remains to check (5) ⇒ (6) and (5′) ⇒ (6′). Here we check only (5′) ⇒ (6′), as both proofs are almost identical. For this take $f \in C^{2,1}(\mathbb{R}^N \times [0, \infty[)$. For each $m \in \mathbb{N}$ the exit times

$$T_m := \inf\left\{t \geq 0\colon |X_t|^2 + t^2 > m^2\right\}$$

of $((X_t, t))_{t\geq 0}$ from closed balls in $\mathbb{R}^{N+1}$ of radius m are stopping times. Now choose $f_m \in C_c^{2,1}(\mathbb{R}^N \times [0, \infty[)$, which are identically 1 on such balls. Then $f_m \cdot f \in C_c^{2,1}(\mathbb{R}^N \times [0, \infty[)$, and $(\Pi_{X, t\wedge T_m}^{c\Delta_k, f_m f})_{t\geq 0}$ is a uniformly integrable martingale by (5′). Moreover, as $(|X_t|)_{t\geq 0}$ is continuous,

$$\Pi_{X, t\wedge T_m}^{c\Delta_k, f_m f} = \Pi_{X, t\wedge T_m}^{c_\Delta k, f} \qquad \text{for all } t.$$

As $T_m \to \infty$ for $m \to \infty$, it follows that $\Pi_X^{c\Delta_k, f}$ is a local martingale. This proves (6′). ∎

The assertions (6) and (6′) in Theorem 6.4 are of particular interest for functions satisfying $(\partial/\partial s + \Delta_k) f = 0$. Examples of such functions will be provided by Appell characters in Theorem 8.2; see also the discussion in Remark 8.3(1).

We next present a probabilistic interpretation for Dirichlet problems in the Dunkl setting. For this, we say that a function $f \in C^2(U)$ is k-harmonic on some open W-invariant set $U \subset \mathbb{R}^N$ if $\Delta_k f = 0$ on U. The following theorem needs stronger assumptions than in the classical setting $k = 0$. This is due to the fact that k-Gaussian processes usually do not have a.s. continuous paths.

THEOREM 6.5. *Let $U \subset \mathbb{R}^N$ be open, bounded, and W-invariant. Let $h \in C^2(U) \cap C(\overline{U})$ be a solution of the k-Dirichlet problem $\Delta_k h = 0$ with*

$h = \varphi \in C(\partial U)$ on ∂U. Assume also that

(1) *h is W-invariant, or that*

(2) *there is an open set $V \supset \overline{U}$ such that h can be extended to a k-harmonic function on V.*

Then for all $x \in U$,

$$h(x) = E\big(h(B_T^x)\big),$$

where $(B_t^x)_{t \ge 0}$ is a k-Gaussian process on $\mathbb{R}^N$ starting in x, and $T :=$ $\inf\{t > 0: B_t \in \partial U\}$ is the entrance time of B_t^x at ∂U.

Proof. Assume first that condition (1) holds. Consider the open, W-invariant sets

$$U_n := \{y \in U: \operatorname{dist}(y, \partial U) < 1/n\} \qquad (n \ge 1)$$

and the entrance times $S_n := \inf\{t > 0: B_t \in U_n\}$. Then the W-invariance of U, U_n, and h; the continuity of h on $\overline{U}$, and Theorem 4.10 yield that $S_n \uparrow T$ and $h(B_{S_n}^x) \to \varphi(B_T^x)$ almost surely. Now choose a function $h_n \in C_c^2(\mathbb{R}^N)$ with $h_n = h$ on $\overline{U}_n$. Then, by Theorem 6.4(5), $(h(B_{t\wedge S_n}^x))_{t \ge 0}$ is a martingale. Thus,

$$h(x) = E\big(h(B_0^x)\big) = E\Big(h\Big(B_{S_n}^x\Big)\Big) \qquad \text{for } n \ge 1.$$

As $E(h(B_{S_n}^x)) \to E(h(B_T^x))$ by the dominated convergence theorem, the claim follows.

Assume now that condition (2) holds. Consider the open, W-invariant sets

$$V_n := \{y \in \mathbb{R}^N: \operatorname{dist}(y, U) < 1/n\} \supset \overline{U} \qquad (n \ge 1).$$

Then, for large n, $\overline{U} \subset V_n \subset \overline{V}_n \subset V$, and the stopping times $\tilde{S}_n := \inf\{t > 0: B_t \notin V_n\}$ satisfy $\tilde{S}_n \downarrow T$ and $h(B_{\tilde{S}_n}^x) \to \varphi(B_T^x)$ almost surely by the W-invariance of U and V_n, and by Theorem 4.10. The proof is now completed as above. ∎

7. MOMENT FUNCTIONS

The classical moments of probability measures on $\mathbb{R}$, or more generally on $\mathbb{R}^N$, have many applications to sums of independent random variables. The definition of classical moments is based on the monomials, or "moment functions,''

$$m_\nu : \mathbb{R}^N \mapsto \mathbb{R}, \quad x = (x_1, \dots, x_N) \mapsto x^\nu \qquad (\nu \in \mathbb{Z}_+^N). \tag{7.1}$$

We introduce modified moment functions for the Dunkl setting, which have applications to k-invariant Markov kernels and processes. Our approach is motivated by corresponding results for hypergroups in [Bl-He, Z]. Again, we fix a reflection group W with multiplicity function $k \geq 0$.

7.1. *Modified Moment Functions*

The Dunkl kernel $(x, y) \mapsto K(x, y)$ is analytic on $\mathbb{C}^{N \times N}$; see Section 2.2. Therefore, there exist unique analytic functions m_ν $(\nu \in \mathbb{Z}_+^N)$ on $\mathbb{C}^N$ with

$$K(x, y) = \sum_{\nu \in \mathbb{Z}_+^N} \frac{m_\nu(x)}{\nu!} y^\nu \qquad (x, y \in \mathbb{C}^N). \tag{7.2}$$

The restriction of m_ν to $\mathbb{R}^N$ is called the νth moment function on $\mathbb{R}^N$ (associated with W, k). The number $|\nu| := \nu_1 + \nu_2 + \cdots + \nu_N$ is called the degree of m_ν.

We denote the jth unit vector by $e_j \in \mathbb{Z}_+^N$. Hence, the moment functions of order 1 and 2 are given by m_{e_j} and $m_{e_j + e_k}$, respectively $(j, k = 1, \dots, N)$.

Moment functions have several useful descriptions: Clearly, m_ν can also be written as

$$m_\nu(x) = \left(\partial_y^\nu K(x, y)\right)\Big|_{y=0} = i^{|\nu|}\left(\partial_y^\nu K(x, -iy)\right)\Big|_{y=0}. \tag{7.3}$$

From the definition of K, which involves the intertwining isomorphism V on $\mathscr{P}$ (Section 2.2), we see that

$$m_\nu(x) = V(x^\nu) \qquad \text{for } \nu \in \mathbb{Z}_+^N. \tag{7.4}$$

In particular, for each $n \in \mathbb{Z}_+$ the moment functions m_ν with $|\nu| = n$ form a basis of the space $\mathscr{P}_n$ of all homogeneous polynomials of degree n.

EXAMPLES 7.1. (1) If $k = 0$, then $K(x, y) = e^{\langle x, y \rangle}$ and $m_\nu(x) = x^\nu$.

(2) If $N = 1$, $W = \mathbb{Z}_2$, and $k \geq 0$, then the explicit form of K in terms of Bessel functions (see Example 2.1(2)) implies that for $n \geq 0$,

$$m_{2n}(x) = \frac{\Gamma(k + 1/2)(2n)!}{\Gamma(n + k + 1/2)2^{2n} n!} x^{2n} \qquad \text{and}$$

$$m_{2n+1}(x) = \frac{\Gamma(k + 1/2)(2n + 1)!}{\Gamma(n + k + 3/2)2^{2n+1} n!} x^{2n+1}.$$

(3) The S_N case. For the symmetric group $W = S_N$ (acting on $\mathbb{R}^N$ by permuting the coordinates), the multiplicity function is a single parameter $k \geq 0$. The associated Dunkl operators are given by

$$T_i = \partial_i + k \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j} \qquad (i = 1, \dots, N),$$

where s_{ij} denotes the operator transposing the coordinates x_i and x_j. We now compute the moment functions of degree ≤ 2 by using the properties of the operator V in Section 2.2. To obtain the first moment function m_{e_l} for
$l \in \{1, \dots, N\}$, we write $Vx_l = \langle a, x \rangle$ with $a \in \mathbb{R}^N$. Now let $b := \sum_{i=1}^N a_i$ and use $V1 = 1$. Then, $T_i\langle a, x \rangle = T_i V x_l = V\partial_i x_l = \delta_{i,l}$ (= Kronecker δ); on the other hand,

$$T_i\langle a, x \rangle = a_i + k \sum_{j \neq i} \frac{a_i x_i + a_j x_j - a_j x_i - a_i x_j}{x_i - x_j} = a_i(1 + kN) - kb.$$

Hence, $a_l = (1 + kb)/(1 + kN)$ and $a_i = kb/(1 + kN)$ for $i \neq l$. Summation yields $b = 1$ and

$$m_{e_l}(x) = Vx_l = \frac{1}{1 + kN}\left(x_l + k \sum_{i=1}^N x_i\right). \tag{7.5}$$

A similar computation using (7.5) leads to

$$\begin{aligned} m_{2e_l}(x) = Vx_l^2 &= \frac{1}{(Nk + 1)(Nk + 2)}\left(\left(x_l + k\sum_{i=1}^N x_i\right)^2 + x_l^2 + k\sum_{i=1}^N x_i^2\right) \\ &= \frac{Nk + 1}{Nk + 2}\left(m_{e_l}(x)\right)^2 + \frac{1}{(Nk + 1)(Nk + 2)}\left(x_l^2 + k\sum_{i=1}^N x_i^2\right). \end{aligned} \tag{7.6}$$

In particular, the Cauchy–Schwarz inequality ensures that

$$m_{2e_l}(x) \geq m_{e_l}(x)^2 \geq 0 \qquad \text{for all } x \in \mathbb{R}^N. \tag{7.7}$$

In Proposition 7.2 we shall see that inequalities like (7.7) hold for general reflection groups W. We mention that all moment functions can be obtained as coefficients of some sophisticated power series, which makes their computation much easier; for details see [D4].

(4) The B_N case. Suppose that W is the Weyl group of type B_N, generated by sign changes and permutations. Here the multiplicity func-

tion is characterized by two parameters $k_0, k_1 \geq 0$, and the associated Dunkl operators are given by

$$T_i = \partial_i + k_1 \frac{1 - \sigma_i}{x_i} + k_0 \sum_{j \neq i} \left(\frac{1 - s_{i,j}}{x_i - x_j} + \frac{1 - \sigma_i \sigma_j s_{i,j}}{x_i + x_j} \right) \quad (i = 1, \ldots, N),$$

where the operator σ_i transfers the component x_i into $(-x_i)$, and $s_{i,j}$ is given as in the S_N case. From [D5] (or by explicit computation) one obtains that for $l, j \in \{1, \ldots, N\}$,

$$m_{e_l}(x) = Vx_l = \frac{x_l}{1 + 2k_1 + 2k_0(N-1)},$$

$$m_{e_l + e_j} = V(x_l x_j) = \frac{x_l x_j}{1 + 2k_1 + 2k_0(N-1)}, \qquad \text{for } l \neq j, \quad (7.8)$$

and

$$m_{2e_l}(x) = Vx_l^2 = \frac{x_l^2 + k_0 \sum_{i=1}^N x_i^2}{(1 + Nk_0)(1 + 2(N-1)k_0 + 2k_1)}. \qquad (7.9)$$

(5) It is no accident that in the B_N case $m_{e_l}(x) = Vx_l = cx_l$ holds with a constant $c = c(W, k)$. In fact, the natural action of any reflection group W on $\mathbb{R}^N$ is a group representation, and the intertwining operators $V = V(k)$ restricted to $\mathscr{P}_1 \cong \mathbb{R}^N$ are intertwining operators for this group representation; see Theorem 2.3 of Dunkl [D2]. Therefore, if W acts in an irreducible way on $\mathbb{R}^N$, then, by Schur's lemma, $V|_{\mathscr{P}_1}$ is a multiple of the identity.

We next collect some properties of moment functions. We mention that similar results are also available for hypergroups on $[0, \infty[$; see [Bl-He, Re-V, Z].

Proposition 7.2. *For all $x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}_+^N$, and $l \in \{1, \ldots, N\}$,*

(1) $T_l m_{\nu + e_l} = (\nu_l + 1) \cdot m_\nu$.

(2) $|m_\nu(x)| \leq |x|^{|\nu|}$ *and* $0 \leq m_\nu(x)^2 \leq m_{2\nu}(x)$.

(3) *Taylor formula*: *If $f \in C^n(\mathbb{R}^N)$ for $n \in \mathbb{N}$, then*

$$f(y) = \sum_{\nu \in \mathbb{Z}_+^N, |\nu| \leq n} \frac{m_\nu(y)}{\nu!} T^\nu f(0) + o(|y|^n) \qquad \text{for } y \to 0.$$

Moreover, if $f\colon \mathbb{C}^N \to \mathbb{C}$ *is analytic in a neighborhood of* 0, *then*

$$f(y) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \frac{m_\nu(y)}{\nu!} T^\nu f(0),$$

where the series $\sum_{n=0}^{\infty}$ *converges absolutely and uniformly in a neighborhood of* 0.

Proof. (1) The intertwining property of V in Section 2.2 and Eq. (7.4) yield

$$T_l m_{\nu+e_l} = T_l V x^{\nu+e_l} = V \partial_l x^{\nu+e_l} = (\nu_l + 1) \cdot V x^\nu = (\nu_l + 1) \cdot m_\nu .$$

(2) Theorem 2.2(5) and Eq. (7.3) imply that for each $x \in \mathbb{R}^N$ there exists a probability measure $\mu_x \in M^1(\mathbb{R}^N)$ with $\operatorname{supp} \mu_x \subset \{z \in \mathbb{R}^N\colon |z| \le |x|\}$ such that

$$m_\nu(x) = \int_{\mathbb{R}^N} y^\nu \, d\mu_x(y) \qquad \text{for all } \nu \in \mathbb{Z}_+^N \quad \text{and} \quad x \in \mathbb{R}^N;$$

see also [R3]. The first inequality of the lemma is now clear from the support properties of μ_x, while the second one follows from Jensen's inequality.

(3) Assume first that $f \in \mathscr{P}$ is a polynomial. As $V\mathscr{P}_n = \mathscr{P}_n$ and $V1 = 1$, we have $\partial^\nu f(0) = V \partial^\nu f(0) = T^\nu V f(0)$. Thus,

$$f(y) = \sum_\nu \frac{y^\nu}{\nu!} T^\nu V f(0) \qquad \text{and} \qquad (V^{-1} f)(y) = \sum_\nu \frac{y^\nu}{\nu!} T^\nu f(0),$$

which gives

$$f(y) = \sum_\nu \frac{m_\nu(y)}{\nu!} T^\nu f(0).$$

The assertions in (3) now follow from the corresponding results for the classical case. ∎

7.2. *Modified Moments of Probability Measures*

We say that the νth moment of a probability measure $\mu \in M^1(\mathbb{R}^N)$ exists if $m_\nu \in L^1(\mathbb{R}^N, \mu)$ holds; in this case, the νth moment of μ is defined as

$$m_\nu(\mu) := \int_{\mathbb{R}^N} m_\nu \, d\mu \qquad \left(\nu \in \mathbb{Z}_+^N\right). \tag{7.10}$$

For $N \geq 2$ it is not correct, even if $k = 0$, that the existence of the νth moment of μ implies the existence of the ρth moment for $\rho \leq \nu$ (which means that $\rho_i \leq \nu_i$ for all i). This and additional difficulties in the Dunkl setting for $k \neq 0$ oblige us to restrict our further attention to the spaces

$$M_n^1(\mathbb{R}^N) := \left\{ \mu \in M^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x^\rho| \, d\mu < \infty \text{ for all } \rho \in \mathbb{Z}_+^N \text{ with } |\rho| \leq n \right\} \qquad (n \geq 0).$$

The following criterion is slightly weaker than its classical counterpart; cf. 25.2 of [Ba]:

LEMMA 7.3. *For $n \in \mathbb{N}$ and $\mu \in M^1(\mathbb{R}^N)$, the following statements hold*:

(1) *$\mu \in M_n^1(\mathbb{R}^N)$ if and only if the νth modified moment of μ exists for all $\nu \in \mathbb{Z}_+^N$ with $|\nu| \leq n$.*

(2) *If one of the conditions in* (1) *holds, then $\hat{\mu} \in C^n(\mathbb{R}^N)$, with*

$$\partial^\nu \hat{\mu}(0) = (-i)^{|\nu|} \cdot m_\nu(\mu) \qquad \textit{for } |\nu| \leq n.$$

Proof. (1) follows from the properties of V. To prove (2), we use Theorem 2.2(2), which says that

$$\left| \partial_y^\nu K(x, iy) \right| \leq |x|^{|\nu|} \leq N^{|\nu|/2} \left(|x_1|^{|\nu|} + \cdots + |x_N|^{|\nu|} \right) \qquad \text{for all } \nu \in \mathbb{Z}_+^N,\ x, y \in \mathbb{R}^N.$$

This guarantees that $x \mapsto \partial_y^\nu K(x, iy)$ is μ-integrable for each $y \in \mathbb{R}^N$ and $|\nu| \leq n$ whenever one of the conditions in (1) holds. It is now standard to check inductively with the dominated convergence theorem that $\hat{\mu} \in C^n(\mathbb{R}^N)$, and the last identity follows from Eq. (7.3). ∎

LEMMA 7.4. *Let P, Q be k-invariant Markov kernels on $\mathbb{R}^N$ such that for all $x \in \mathbb{R}^N$, the measures $P(x,.)$, $Q(x,.)$, $Q \circ P(x,.)$ are contained in $M_n^1(\mathbb{R}^N)$. Then, for all $x \in \mathbb{R}^N$ and $\nu \in \mathbb{Z}_+^N$ with $|\nu| \leq n$,*

(1) $$m_\nu(P(x,.)) = \sum_{\rho \leq \nu} \binom{\nu}{\rho} m_\rho(P(0,.)) \cdot m_{\nu-\rho}(x),$$

and

(2) $$m_\nu(Q \circ P(x,.)) = \sum_{\rho \leq \nu} \binom{\nu}{\rho} m_\rho(P(x,.)) \cdot m_{\nu-\rho}(Q(0,.)).$$

Proof. (1) follows immediately from Lemma 7.3(2), the Leibniz rule for partial derivatives of products, and Eq. (7.3); in fact,

$$\begin{aligned} m_\nu(P(x,.)) &= i^{|\nu|}\cdot \partial_y^\nu\big(P(0,.)^\wedge(y)\cdot K(-ix,y)\big)\big|_{y=0} \\ &= \sum_{\rho\le\nu}\binom{\nu}{\rho} i^{|\rho|}\cdot \partial_y^\rho\big(P(0,.)^\wedge(y)\big)\big|_{y=0} \\ &\quad \times i^{|\nu|-|\rho|}\cdot \partial_y^{\nu-\rho}K(-ix,y)\big|_{y=0} \\ &= \sum_{\rho\le\nu}\binom{\nu}{\rho} m_\rho(P(0,.))\cdot m_{\nu-\rho}(x). \end{aligned}$$

Part (2) can be checked in the same way by using Lemma 4.2(2). ∎

The algebraic properties of moment functions in Lemma 7.4 can be used to construct martingales for k-invariant Markov processes on $\mathbb{R}^N$. For simplicity, we restrict our attention first to moments of degree at most 2. The following proposition is motivated by corresponding results for hypergroups in [Z, Bl-He]. We say from now on that an $\mathbb{R}^N$-valued random variable is L^p-integrable, whenever all of its components have this property.

PROPOSITION 7.5. *Let $(X_t)_{t\ge 0}$ be a $\mathbb{R}^N$-valued, k-invariant Markov process on some probability space $(\Omega,\mathscr{A},P)$ with filtration $(\mathscr{F}_t)_{t\ge 0}$. Let $(P_t)_{t\ge 0}$ be the associated semigroup of k-invariant Markov kernels with negative definite function φ. Then for all $l, j = 1,\dots,N$, the following conclusions hold*:

(1) *Assume that $X_0\in L^1(\Omega,\mathscr{A},P)$ and $P_t(x,.)\in M_1^1(\mathbb{R}^N)$ holds for all $t>0$ and $x\in\mathbb{R}^N$. Then $(m_{e_l}(X_t)-E(m_{e_l}(X_t)))_{t\ge 0}$ is a $(\mathscr{F}_t)_{t\ge 0}$-martingale with*

$$E\big(m_{e_l}(X_t)\big) = E\big(m_{e_l}(X_0)\big) - it\cdot\partial_l\varphi(0) \qquad \textit{for all } t\ge 0.$$

(2) *Assume that $X_0\in L^2(\Omega,\mathscr{A},P)$ and $P_t(x,.)\in M_2^1(\mathbb{R}^N)$ holds for all $t>0$ and $x\in\mathbb{R}^N$. Then the second-order moment function $m_{e_l+e_j}$ has the property that*

$$\begin{aligned} \Big(m_{e_l+e_j}(X_t) &- m_{e_l}(X_t)E\big(m_{e_j}(X_t)\big) - m_{e_j}(X_t)E\big(m_{e_l}(X_t)\big) \\ &+ E\big(m_{e_l}(X_t)\big)E\big(m_{e_j}(X_t)\big) - E\big(m_{e_l+e_j}(X_t)\big)\Big)_{t\ge 0} \end{aligned}$$

is a $(\mathscr{F}_t)_{t\geq 0}$*-martingale. Moreover, the "modified variances"*

$$V_k^l(X_t) := E\big(m_{2e_l}(X_t)\big) - E\big(m_{e_l}(X_t)\big)^2$$

satisfy

$$V_k^l(X_t) = V_k^l(X_0) + t\cdot \partial_l^2\varphi(0) \qquad \textit{for all } t\geq 0.$$

Proof. (1) By Proposition 4.4(3) we have $\varphi(0)=0$ and $P_t(0,.)^\wedge = e^{-t\varphi}$ for $t\geq 0$. Therefore, by Lemma 7.3,

$$m_{e_l}\big(P_t(0,.)\big) = i\cdot \partial_l\big(P_t(0,.)^\wedge\big)(0) = -it\,\partial_l\varphi(0).$$

Now take $s,t\geq 0$. Lemma 7.4(1) ensures that for almost all $\omega\in\Omega$,

$$\begin{aligned} E\big(m_{e_l}(X_{s+t})\,|\,\mathscr{F}_t\big)(\omega) &= \int_{\mathbb{R}^N} m_{e_l}\,dP_t(X_s(\omega),.)\\ &= m_{e_l}\big(P_t(0,.)\big) + m_{e_l}\big(X_s(\omega)\big)\\ &= -it\cdot\partial_l\varphi(0) + m_{e_l}\big(X_s(\omega)\big). \end{aligned}\tag{7.11}$$

If we take the usual expectation of both sides of (7.11) with $s=0$, then we obtain the formula for $E(m_{e_l}(X_t))$ in the proposition. Moreover, this formula, together with (7.11), readily yields that $(m_{e_l}(X_t) - E(m_{e_l}(X_t)))_{t\geq 0}$ is a martingale.

(2) Again, Proposition 4.4(3) and Lemma 7.3 yield that for $t\geq 0$,

$$\begin{aligned} m_{e_l+e_j}\big(P_t(0,.)\big) &= t\,\partial_l\partial_j\varphi(0) - t^2\cdot\partial_l\varphi(0)\,\partial_j\varphi(0)\\ &= t\,\partial_l\partial_j\varphi(0) + m_{e_l}\big(P_t(0,.)\big)m_{e_j}\big(P_t(0,.)\big). \end{aligned}$$

Now take $s,t\geq 0$. Lemma 7.4(2) and Part (1) imply that for almost all $\omega\in\Omega$,

$$\begin{aligned} &E\big(m_{e_l+e_j}(X_{s+t})\,|\,\mathscr{F}_t\big)(\omega)\\ &\quad= \int_{\mathbb{R}^N} m_{e_l+e_j}\,dP_t(X_s(\omega),.)\\ &\quad= m_{e_l+e_j}\big(P_t(0,.)\big) + m_{e_l+e_j}\big(X_s(\omega)\big) + m_{e_l}\big(P_t(0,.)\big)m_{e_j}\big(X_s(\omega)\big)\\ &\qquad\quad + m_{e_j}\big(P_t(0,.)\big)m_{e_l}\big(X_s(\omega)\big)\\ &\quad= t\,\partial_l\partial_j\varphi(0) - t^2\cdot\partial_l\varphi(0)\,\partial_j\varphi(0) + m_{e_l+e_j}\big(X_s(\omega)\big)\\ &\qquad\quad - it\cdot\Big[\partial_l\varphi(0)m_{e_j}\big(X_s(\omega)\big) + \partial_j\varphi(0)m_{e_l}\big(X_s(\omega)\big)\Big]. \end{aligned}$$

If we combine this with Eq. (7.11) and take the usual expectations of these equations, we obtain the martingale property claimed above. Finally, the formula for modified variances follows from that equation with $j = l$ and (7.11) again by taking the usual expectations. ∎

Remarks 7.6. (1) Let $(X_t)_{t\geq 0} = (X_t^1, \dots, X_t^N)_{t\geq 0}$ be a k-invariant Markov process as studied in Proposition 7.5(1). As the moment functions m_{e_l} $(l = 1, \dots, N)$ form a basis of the space $\mathscr{P}_1$ of all homogeneous polynomials of degree 1 by Section 7.1, all component processes $(X_t^l - E(X_t^l))_{t\geq 0}$ form martingales (under the corresponding integrability condition). This strict separation of the components is not usually possible for higher moments.

(2) There exists an obvious analogue of Proposition 7.5 for (not necessarily time-homogeneous) k-invariant Markov processes $(X_n)_{n\geq 0}$ in discrete time that are related to a sequence $(P_n)_{n\geq 0}$ of k-invariant Markov kernels by

$$P\big(X_n \in A \mid X_{n-1} = x\big) = P_n(x, A) \qquad \text{for } n \geq 1,\ x \in \mathbb{R}^N,\ A \in \mathscr{B}(\mathbb{R}^N). \tag{7.12}$$

In particular, the methods of the proof of Proposition 7.5 yield that under suitable moment conditions, the processes

$$\Big(m_{2e_l}(X_n) - 2m_{e_l}(X_n)\cdot E\big(m_{e_l}(X_n)\big) + 2\cdot E\big(m_{e_l}(X_n)\big)^2 - E\big(m_{2e_l}(X_n)\big)\Big)_{n\geq 0} \tag{7.13}$$

are martingales for $l = 1, \dots, N$. Moreover, the modified variances satisfy

$$V_k^l(X_n) = V_k^l(X_{n-1}) + m_{2e_l}\big(P_n(0,.)\big) - m_{e_l}\big(P_n(0,.)\big)^2. \tag{7.14}$$

(3) Let $n \geq 1$, and let P, Q be k-invariant Markov kernels with $P(0,.), Q(0,.) \in M_n^1(\mathbb{R}^N)$ (i.e., all moments up to degree n exist). It is our conjecture that then automatically also $Q \circ P(0,.) \in M_n^1(\mathbb{R}^N)$ and $P(x,.) \in M_n^1(\mathbb{R}^N)$ holds for all $x \in \mathbb{R}^N$. This conjecture is obviously true in the classical setting $k = 0$; it can also be checked for the one-dimensional examples discussed in Examples 2.1(2) and 7.1(2). The proof there is based on the knowledge of a convolution of measures in $M_b(\mathbb{R})$ with sufficiently nice properties; see [R1]. Clearly, this can also be extended to direct products of the one-dimensional case.

EXAMPLE 7.7 (k-Gaussian processes). Let $(X_t)_{t\geq 0}$ be a k-Gaussian process on $\mathbb{R}^N$ associated with the k-Gaussian semigroup $(P_t^\Gamma)_{t\geq 0}$ and starting in 0 at time 0; see Sections 3 and 4. In this case, all moments of $P_t^\Gamma(x,.)$ and X_t exist for all $x \in \mathbb{R}^N$ and $t \geq 0$. Therefore, all assumptions of the preceding results in Section 7 are satisfied. We can write

$$P_t^\Gamma(0,.)^\wedge(y) = e^{-t|y|^2} = \sum_{\nu \in \mathbb{Z}_+^N} \frac{(-t)^{|\nu|}}{\nu!} y^{2\nu} \qquad \text{for } t \geq 0,\ y \in \mathbb{R}^N.$$

Lemma 7.3(2) yields that

$$E\big(m_{2\nu}(X_t)\big) = m_{2\nu}\big(P_t^\Gamma(0,.)\big) = \frac{(2\nu)!}{\nu!} t^{|\nu|} \qquad \big(\nu \in \mathbb{Z}_+^N,\ t \geq 0\big), \quad (7.15)$$

as well as $E(m_\nu(X_t)) = 0$, whenever at least one component of ν is odd. In fact, this exactly extends the result known for classical N-dimensional normal distributions.

If we apply Proposition 7.5(1) and Remark 7.6(1), we conclude that the processes $(m_{e_l}(X_t))_{t \geq 0}$ as well as the coordinate processes $(X_t^l)_{t \geq 0}$ are martingales for $l \in \{1, \dots, N\}$. Moreover, Proposition 7.5(2), together with $E(m_{e_l}(X_t)) = 0$, yields that

$$\Big(m_{e_l+e_j}(X_t) - E\big(m_{e_l+e_j}(X_t)\big)\Big)_{t \geq 0}$$

is a martingale for all $l, j \in \{1, \dots, N\}$. As the moment functions $m_{e_j+e_l}$ form a basis of $\mathscr{P}_2$, it follows that for all $l, j \in \{1, \dots, N\}$, the processes

$$\big(X_t^l \cdot X_t^j - E\big(X_t^l \cdot X_t^j\big)\big)_{t \geq 0}$$

are martingales. For higher moments, results of this type will be more complicated; we deal with this problem in the next section.

Remark 7.8. As mentioned above, it is an interesting problem whether there exists a convolution $*$ on the Banach space $M_b(\mathbb{R}^N)$ associated with the Dunkl transform, i.e, with $(\mu * \nu)^\wedge = \hat{\mu} \cdot \hat{\nu}$ for all $\mu, \nu \in M_b(\mathbb{R}^N)$. If $*$ existed, then all k-invariant Markov kernels would satisfy $P(x,.) = \epsilon_x * P(0,.)$ for $x \in \mathbb{R}^N$. Moreover, by Lemma 7.4, the moment functions m_ν would satisfy

$$\epsilon_x * \epsilon_y(m_\nu) = \sum_{\rho \leq \nu} \binom{\nu}{\rho} m_\rho(x) m_{\nu-\rho}(y) \qquad \big(x, y \in \mathbb{R}^N,\ \nu \in \mathbb{Z}_+^N\big). \quad (7.16)$$

Linear extension of (7.16) would lead to $\epsilon_x * \epsilon_y$ for arbitrary polynomials in $\mathscr{P}$. On the other hand, it is possible to go the converse way and to define a weak form of a generalized translation via (7.16). This can be performed as in [Be] for the one-dimensional setting by using the estimation

$$\int_{\mathbb{R}^N} m_\nu^2 \, dP_t^\Gamma(0,.) \leq (4t)^{|\nu|} \cdot \nu! \qquad \left(t > 0,\, \nu \in \mathbb{Z}_+^N\right). \tag{7.17}$$

In fact, (7.17) is a consequence of Proposition 7.2(2), Eq. (7.15), and the relation $(2\nu)! \leq 4^{|\nu|} \cdot (\nu!)^2$.

8. APPELL CHARACTERS AND HERMITE POLYNOMIALS

Based on the moment funtions of the previous section and certain generating functions, we construct two systems $(R_\nu)_{\nu \in \mathbb{Z}_+^N}$ and $(S_\nu)_{\nu \in \mathbb{Z}_+^N}$ of functions on $\mathbb{R} \times \mathbb{R}^N$ associated with the k-Gaussian semigroup $(P_t^\Gamma)_{t \geq 0}$. These systems, called Appell characters and cocharacters, satisfy several useful algebraic relations. Among other results, we present a new proof for a generalized version of a formula of Macdonald [M], which is due to Dunkl [D3]. Our approach is motivated by related concepts in algebraic probability theory and white-noise analysis; see [ADKS, F-S, B-K1, B-K2], and references there. The notation of Appell (co-)characters has its origin in the umbral calculus; see [Rom]. Parts of this section are also published in [R-V2].

As several of these results can be obtained for more general k-invariant semigroups than just the Gaussian ones without additional effort, we start with some concepts in a general setting. Later on we shall restrict our attention to the k-Gaussian case only.

8.1. *Appell Characters*

For $n \geq 1$, let $(P_t)_{t \geq 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ such that $P_t(x,.) \in M_n^1(\mathbb{R}^N)$ holds for all $t \geq 0$ and $x \in \mathbb{R}^N$. Let φ be the associated negative definite function. We know from Lemma 7.3 and Proposition 4.4 that $P_t(0,.)^\wedge(y) = e^{-t\varphi(y)}$ is n times continuously partially differentiable with respect to y for all $t \geq 0$. Therefore,

$$y \mapsto \frac{K(x, -iy)}{P_t(0,.)^\wedge(y)} = K(x, -iy) \cdot e^{t\varphi(y)}$$

is n times continuously partially differentiable for all $t \geq 0$ and $x \in \mathbb{R}^N$. By Taylor's formula,

$$K(x, -iy) \cdot e^{t\varphi(y)} = \sum_{\nu \in \mathbb{Z}_+^N,\, |\nu| \leq n} \frac{(-iy)^\nu}{\nu!} R_\nu(t, x) + o(|y|^n) \qquad \text{for } y \to 0, \tag{8.1}$$

where the functions R_ν are determined uniquely and satisfy

$$\begin{aligned} R_\nu(t,x) &= i^{|\nu|}\,\partial_y^\nu\big(K(x,-iy)\cdot e^{t\varphi(y)}\big)\big|_{y=0} \\ &= i^{|\nu|}\sum_{\rho\in\mathbb{Z}_+^N,\ \rho\le\nu}\binom{\nu}{\rho}\partial_y^\rho(K(x,-iy))\big|_{y=0}\cdot\partial_y^{\nu-\rho}\big(e^{t\varphi(y)}\big)\big|_{y=0} \\ &= \sum_{\rho\in\mathbb{Z}_+^N,\ \rho\le\nu}\binom{\nu}{\rho}m_\rho(x)\cdot a_{\nu-\rho}^\varphi(t). \end{aligned} \tag{8.2}$$

For the last equation Eq. (7.3) was used, and

$$a_\lambda^\varphi(t) := i^{|\lambda|}\cdot\partial^\lambda\big(e^{t\varphi(y)}\big)\big|_{y=0} \qquad \big(\lambda\in\mathbb{Z}_+^N,\,|\lambda|\le n\big) \tag{8.3}$$

is a real-valued polynomial in $t\in\mathbb{R}$ of degree at most $|\lambda|$. Note that by Lemma 7.3(2), $a_\lambda^\varphi(-t) = m_\nu(P_t(0,.))\in\mathbb{R}$ for $t\ge 0$. Clearly, formulas (8.2) and (8.3) also make sense for all $t\in\mathbb{R}$. In summary, the functions R_ν ($\nu\in\mathbb{Z}_+^N$) are real polynomials in $(N+1)$ variables of degree $|\nu|$, and for each fixed $t\in\mathbb{R}$, $R_\nu(t,.)$ is a real polynomial of degree $|\nu|$. We call the polynomials R_ν the *Appell characters* associated with the semigroup $(P_t)_{t\ge 0}$ (where the reflection group W with multiplicity function k is assumed to be fixed).

LEMMA 8.1. *In the setting of Section* 8.1, *the following holds for all* $\nu\in\mathbb{Z}_+^N$ *with* $|\nu|\le n$:

(1) *Inversion formula*: *For all* $x\in\mathbb{R}^N$ *and* $t\in\mathbb{R}$,

$$m_\nu(x) = \sum_{\rho\in\mathbb{Z}_+^N,\ \rho\le\nu}\binom{\nu}{\rho}R_\rho(t,x)\cdot a_{\nu-\rho}^\varphi(-t).$$

(2) *For all* $t\in\mathbb{R}$ *and* $0\le l\le n$, *the family* $(R_\nu(t,.))_{\nu\in\mathbb{Z}_+^N,\,|\nu|\le l}$ *is a basis of the space* $\bigoplus_{j=0}^l\mathscr{P}_j$ *of all polynomials of degree at most* l.

(3) *For* $x\in\mathbb{R}^N$ *and* $t\ge 0$,

$$\int_{\mathbb{R}^N} R_\nu(t,y)\,dP_t(x,.)(y) = m_\nu(x).$$

(4) *For all* $x\in\mathbb{R}^N$, $t\in\mathbb{R}$, *and* $j\in\{1,\dots,N\}$, $T_jR_{\nu+e_j}(t,x) = (\nu_j+1)\cdot R_\nu(t,x)$; *here the Dunkl operator* T_j *acts with respect to the variable* x.

Proof. (1) Using (8.1) and (8.3), we obtain for y small enough that

$$\begin{aligned} K(x,-iy) &= e^{-t\varphi(y)}\cdot\left(e^{t\varphi(y)}K(x,-iy)\right) \\ &= \left(\sum_{|\lambda|\le n}\frac{a_\lambda^\varphi(-t)}{\lambda!}(-iy)^\lambda + o(|y|^n)\right) \\ &\quad\times\left(\sum_{|\rho|\le n}\frac{(-iy)^\rho}{\rho!}R_\rho(t,x) + o(|y|^n)\right) \\ &= \sum_{|\nu|\le n}\left(\sum_{\rho\le\nu}\binom{\nu}{\rho}a_{\nu-\rho}^\varphi(-t)R_\rho(t,x)\right)\frac{(-iy)^\nu}{\nu!} + o(|y|^n). \end{aligned}$$

A comparison of this expansion with Eq. (7.2) leads to Part (1).

(2) This follows from Part (1) of this lemma and the fact that $(m_\nu)_{|\nu|=l}$ is a basis of $\mathscr{P}_l$.

(3) Recall that $m_\lambda(P_t(0,.)) = a_\lambda^\varphi(-t)$. Thus Eq. (8.2) and Lemma 7.4(1) yield that

$$\begin{aligned} &\int_{\mathbb{R}^N}R_\nu(t,y)\,dP_t(x,.)(y) \\ &\quad= \sum_{\rho\le\nu}\binom{\nu}{\rho}a_{\nu-\rho}^\varphi(t)\cdot\int_{\mathbb{R}^N}m_\rho(y)\,dP_t(x,.)(y) \\ &\quad= \sum_{\rho\le\nu}\binom{\nu}{\rho}a_{\nu-\rho}^\varphi(t)\cdot\left(\sum_{\lambda\le\rho}\binom{\rho}{\lambda}m_\lambda(P_t(0,.))\cdot m_{\rho-\lambda}(x)\right) \\ &\quad= \sum_{\rho\le\nu}\binom{\nu}{\rho}a_{\nu-\rho}^\varphi(t)\cdot\left(\sum_{\lambda\le\rho}\binom{\rho}{\lambda}a_\lambda^\varphi(-t)\cdot m_{\rho-\lambda}(x)\right). \end{aligned}$$

The assertion now follows from Part (1).

(4) By Eq. (8.2) and Proposition 7.2(1),

$$\begin{aligned} T_jR_{\nu+e_j} &= \sum_{\rho\le\nu+e_j}\binom{\nu+e_j}{\rho}T_jm_\rho\cdot a_{\nu+e_j-\rho}^\varphi = \sum_{\rho\le\nu}\binom{\nu+e_j}{\rho+e_j}(\rho_j+1)m_\rho\cdot a_{\nu-\rho}^\varphi \\ &= (\nu_j+1)\cdot\sum_{\rho\le\nu}\binom{\nu}{\rho}m_\rho\cdot a_{\nu-\varphi}^\varphi = (\nu_j+1)\cdot R_\nu. \end{aligned}$$

∎

THEOREM 8.2. *Let $n \geq 1$ and suppose that $(P_t)_{t\geq 0}$ is a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ such that $P_t(x,.) \in M_n^1(\mathbb{R}^N)$ for all $t > 0$ and $x \in \mathbb{R}^N$. Let $(X_t)_{t\geq 0}$ be a k-invariant Markov process on $\mathbb{R}^N$ associated with $(P_t)_{t\geq 0}$. Then for each $\nu \in \mathbb{Z}_+^N$ with $|\nu| \leq n$, the process $(R_\nu(t, X_t))_{t\geq 0}$ is a martingale.*

Proof. We prove more generally that for all $y \in \mathbb{R}^N$ and $\nu \in \mathbb{Z}_+^N$ with $|\nu| \leq n$, the process

$$\left(W_t^{\nu,y} := \partial_y^\nu\left(K(X_t, -iy)\cdot e^{t\varphi(y)}\right)\right)_{t\geq 0} \tag{8.4}$$

is a martingale. The theorem then follows for $y = 0$; see Eq. (8.2).

The statement above will be proved by induction on $|\nu|$. In fact, the case $\nu = 0$ is shown in Proposition 6.1. Now take $j \in \{1, \dots, N\}$ and let $e_j \in \mathbb{Z}_+^N \subset \mathbb{R}^N$ be the jth unit vector. Assume that $(W_t^{\nu,y})_{t\geq 0}$ is a martingale for all $y \in \mathbb{R}^N$ and some $\nu \in \mathbb{Z}_+^N$ with $|\nu| \leq n$. To prove that $(W_t^{\nu+e_j,y})_{t\geq 0}$ is a martingale, we observe that for $t \geq 0$,

$$\lim_{h\to 0} \frac{1}{h}\left(W_t^{\nu,y} - W_t^{\nu,y+h\cdot e_j}\right) = W_t^{\nu+e_j,y} \qquad \text{pointwise}.$$

Moreover, by the mean value theorem, we find $r = r(\nu, X_t, y, h) \in [0, h]$ such that

$$\left|\frac{1}{h}\left(W_t^{\nu,y} - W_t^{\nu,y+h\cdot e_j}\right)\right| = |W_t^{\nu+e_j,y+r\cdot e_j}| \leq \sum_{s=1}^{|\nu|+1} d_s|X_t|^s \tag{8.5}$$

with bounded constants d_s for $h \in]0,1]$. In fact, the last inequality above follows from Eq. (8.4) and the estimations of Theorem 2.2(2). The integrability conditions on $(X_t)_{t\geq 0}$ and (8.5) ensure that the dominated convergence theorem may be applied to the limit above; hence,

$$\lim_{h\to 0}\left\|\frac{1}{h}\left(W_t^{\nu,y} - W_t^{\nu,y+h\cdot e_j}\right) - W_t^{\nu+e_j,y}\right\|_1 = 0 \qquad \text{for all } t \geq 0.$$

It follows for the filtration $(\mathcal{F}_t)_{t\geq 0}$ associated with the process $(X_t)_{t\geq 0}$ that for $s, t \geq 0$,

$$E\left(\frac{1}{h_s}\left(W_{s+t}^{\nu,y} - W_{s+t}^{\nu,y+h_s\cdot e_j}\right)\Big|\mathcal{F}_t\right) \to E\left(W_{s+t}^{\nu+e_j,y}\,|\mathcal{F}_t\right) \qquad \text{almost surely}$$

for any sequence $(h_s)_{s\geq 0}$ with $h_s \downarrow 0$. Hence, $(W_t^{\nu+e_j,y})_{t\geq 0}$ is a martingale. ∎

Remarks 8.3. (1) For $|\nu| = 1, 2$, the martingales $R_\nu(t, X_t)$ of Theorem 8.2 agree with the martingales of Proposition 7.5, i.e, Theorem 8.2 generalizes Proposition 7.5. Moreover, Proposition 8.7 below and Theorem 3.1(3) yield that the Appell characters R_ν^Γ for the k-Gaussian semigroup satisfy $(\partial_t + \Delta_k)R_\nu^\Gamma = 0$, reflecting the close connection between Theorems 8.2 and 6.4.

(2) There is a close connection between the Appell characters R_ν and Dunkl's intertwining operator V: Let $n \geq 0$, and let $(P_t^k)_{t\geq 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ such that $P_t^k(x, .) \in M_n^1(\mathbb{R}^N)$ for all $t \geq 0$ and $x \in \mathbb{R}^N$. Let R_ν^k be the associated Appell characters for $|\nu| \leq n$. By Theorem 2.2(5), there exist probability measures $\mu_x \in M^1(\mathbb{R}^N)$ such that the negative definite function φ associated with $(P_t^k)_{t\geq 0}$ satisfies

$$e^{-t\varphi(y)} = P_t^k(0, .)^\wedge(y) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{\langle z, -iy\rangle} \, d\mu_x(z) \, dP_t^k(0, .)(x)$$

for $t \geq 0$, $y \in \mathbb{R}^N$. Hence, the functions $e^{-t\varphi}$ are positive definite in the classical sense. Thus, by Bochner's theorem, there is a semigroup $(P_t^0)_{t\geq 0}$ of $(\mathbb{R}^N, +)$-translation invariant (or, 0-invariant) Markov kernels on $\mathbb{R}^N$. If the associated Appell characters are denoted by R^0, we obtain from (8.2) and (7.4) that

$$R_\nu^k(t, x) = \sum_{\rho\leq\nu} \binom{\nu}{\rho} m_\rho(x) a_{\nu-\rho}^\varphi(t) = \sum_{\rho\leq\nu} \binom{\nu}{\rho} (Vx^\rho) a_{\nu-\rho}^\varphi(t) = VR_\nu^0(t, x). \tag{8.6}$$

We restrict our attention to k-Gaussian semigroups from now on.

8.2. *Appell Characters of k-Gaussian Semigroups*

Let $(P_t^\Gamma)_{t\geq 0}$ be the k-Gaussian semigroup of Section 3, i.e.,

$$dP_t^\Gamma(x, .)(y) = \frac{c_k}{(4t)^{\gamma+N/2}} \cdot e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) \cdot w_k(y) \, dy.$$

Here all moments exist, and the Taylor expansion (8.1) becomes a power series. The coefficients $a_\nu^\Gamma(t)$ of the associated Appell characters R_ν^Γ satisfy $a_\nu^\Gamma(-t) = m_\nu(P_t^\Gamma(0, .))$ for $t \geq 0$. Equation (7.15) and analytic continuation show that for all $t \in \mathbb{R}$,

$$a_{2\nu}^\Gamma(t) = \frac{(2\nu)!}{\nu!} \cdot (-t)^{|\nu|} \quad \text{and} \quad a_\lambda^\Gamma(t) = 0 \qquad \text{otherwise},$$

i.e., if at least one component of $\lambda \in \mathbb{Z}_+^N$ is odd. Therefore,

$$R_\nu^\Gamma(t,x) = \sum_{\rho \in \mathbb{Z}_+^N,\, 2\rho \le \nu} \frac{\nu!}{(\nu - 2\rho)!\rho!} (-t)^{|\rho|} m_{\nu-2\rho}(x) \qquad \text{for } \nu \in \mathbb{Z}_+^N. \tag{8.7}$$

In particular, the homogeneity of the moment functions m_ν yields that

$$R_\nu^\Gamma(t,x) = \sqrt{t}^{|\nu|} \cdot R_\nu^\Gamma\left(1, x/\sqrt{t}\right) \qquad (x \in \mathbb{R}^N, t > 0). \tag{8.8}$$

EXAMPLES 8.4. (1) In the classical case $k = 0$ with $m_\nu(x) := x^\nu$, Eq. (8.7) leads to

$$R_\nu^\Gamma(t,x) = \sqrt{t}^{|\nu|} \cdot \tilde{H}_\nu\left(\frac{x}{2\sqrt{t}}\right) \qquad \left(x \in \mathbb{R}^N, \nu \in \mathbb{Z}_+^N, t \in \mathbb{R}\right) \tag{8.9}$$

where the $\tilde{H}_\nu$ are the classical, N-dimensional Hermite polynomials defined by

$$\tilde{H}_\nu(x) = \prod_{i=1}^N H_{\nu_i}(x_i) \qquad \text{with} \qquad H_n(y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j n!}{j!(n-2j)!} (2y)^{n-2j};$$

cf. Section 5.5 of [Sz] for the one-dimensional case.

(2) If $N = 1$, $W = \mathbb{Z}_2$, and $k \ge 0$, then the results of Example 7.1(2) lead to

$$R_{2n}^\Gamma(t,x) = (-1)^n 2^{2n} n!\, t^n L_n^{(k-1/2)}\left(x^2/4t\right)$$

and

$$R_{2n+1}^\Gamma(t,x) = (-1)^n 2^{2n+1} n!\, t^n x L_n^{(k+1/2)}\left(x^2/4t\right)$$

for $n \in \mathbb{Z}_+$, where the $L_n^{(\alpha)}$ are the Laguerre polynomials (see Section 5.1 of [Sz]) with

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \cdot \frac{d^n}{dx^n}\left(x^{n+\alpha} e^{-x}\right) = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}.$$

The polynomials $(R_n^\Gamma)_{n \ge 0}$ are called generalized Hermite polynomials (see, e.g., [Ros]). For each $t > 0$ the polynomials $(R_n^\Gamma(t,.))_{n \ge 0}$ are orthogonal with respect to the k-Gaussian measure

$$dP_t^\Gamma(0,.)(x) = \frac{\Gamma(k+1/2)}{(4t)^{k+1/2}} |x|^{2k} e^{-x^2/4t}\, dx.$$

An uninformed reader might suggest from these examples that k-Gaussian Appell characters are always orthogonal with respect to the k-Gaussian measure $P_t^\Gamma(0,.)$ for $t > 0$. We shall see, however, in Remark 8.6 that this is not correct in many cases. We therefore now introduce so-called Appell cocharacters, which turn out to form biorthogonal systems for the Appell characters.

8.3. *Appell Cocharacters of k-Gaussian Semigroups*

Denote the $P_t^\Gamma(0,.)$-density of the k-Gaussian measure $P_t^\Gamma(x,.)$ by $\theta_t(x,.)$ for $t > 0$. Then

$$\theta_t(x,y) := \frac{dP_t^\Gamma(x,.)(y)}{dP_t^\Gamma(0,.)(y)} = e^{-|x|^2/4t}K(x, y/2t)$$

$$= \sum_{n=0}^{\infty} \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!} S_\nu^\Gamma(t,y), \tag{8.10}$$

where, by Proposition 7.2(3), the coefficients S_ν are given by

$$S_\nu^\Gamma(t,y) = T_x^\nu\big(e^{-|x|^2/4t}K(x,y/2t)\big)\Big|_{x=0}.$$

Like the $R_\nu^\Gamma(t,.)$, the $S_\nu^\Gamma(t,.)$ are polynomials of degree $|\nu|$. The convergence of the series $\sum_{n=0}^{\infty}$ in (8.10) is normal on $\mathbb{C}^N \times \mathbb{C}^N$ by Proposition 3.8 of [R2]. The functions S_ν^Γ are called the *Appell cocharacters* of the k-Gaussian semigroup $(P_t^\Gamma)_{t\geq 0}$.

Using the homogeneity of m_ν, we obtain the following analogue of (8.8):

$$S_\nu^\Gamma(t,y) = \left(\frac{1}{\sqrt{t}}\right)^{|\nu|} \cdot S_\nu^\Gamma\big(1, y/\sqrt{t}\,\big) \qquad \big(y \in \mathbb{R}^N,\, t > 0\big). \tag{8.11}$$

A comparison of the homogeneous parts of degree n in the expansions (8.1) and (8.10) shows that the linear spaces generated by $(S_\nu^\Gamma(t,.))_{|\nu|=n}$ and $(R_\nu^\Gamma(t,.))_{|\nu|=n}$ are equal for each $t > 0$. Hence, by Lemma 8.1(2), $(S_\nu^\Gamma(t,.))_{|\nu|\leq n}$ is a basis of $\bigoplus_{j=0}^{n} \mathscr{P}_j$.

Appell characters and cocharacters are related by the following biorthogonality relation:

THEOREM 8.5. *Let $t > 0$, $\nu, \rho \in \mathbb{Z}_+^N$, and let $p \in \mathscr{P}$ be a polynomial of degree less than $|\nu|$. Then*

(1) $\displaystyle\int_{\mathbb{R}^N} R_\nu^\Gamma(t,y) \cdot S_\rho^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y) = \nu!\,\delta_{\nu,\rho}.$

(2) $\displaystyle\int_{\mathbb{R}^N} p(y) \cdot S_\nu^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y) = \int_{\mathbb{R}^N} p(y) \cdot R_\nu^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y)$
$= 0.$

Proof. We use the definition of θ_t and Lemma 8.1(3) and conclude that for $x \in \mathbb{R}^N$,

$$\begin{aligned} m_\nu(x) &= \int_{\mathbb{R}^N} R_\nu^\Gamma(t,y)\,\theta_t(x,y)\, dP_t^\Gamma(0,.)(y) \\ &= \int_{\mathbb{R}^N} \sum_{n=0}^{\infty} \sum_{|\rho|=n} R_\nu^\Gamma(t,y) S_\rho^\Gamma(t,y) \frac{m_\rho(x)}{\rho!}\, dP_t^\Gamma(0,.)(y) \\ &= \sum_{n=0}^{\infty} \sum_{|\rho|=n} \frac{m_\rho(x)}{\rho!} \int_{\mathbb{R}^N} R_\nu^\Gamma(t,y) S_\rho^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y), \quad (8.12) \end{aligned}$$

where we still have to justify that summation and integration commute. For this, we restrict our attention to the case $t = 1/4$, as the general case then follows by renormalization. We follow the proof of Proposition 3.8 of [R2] and decompose $\theta_{1/4}(x,y)$ into its x-homogeneous parts:

$$\theta_{1/4}(x,y) = \sum_{n=0}^{\infty} L_n(y,x) \quad \text{with} \quad L_n(y,x) = \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!} S_\nu^\Gamma(1/4, y).$$

The estimations of Theorem 2.2(2) imply that

$$\left| L_{2n}(y,x) \right| \le \frac{|x|^{2n}}{n!} \cdot \left(1 + 2|y|^2\right)^n \qquad \text{for } n \in \mathbb{Z}_+,$$

and a similar estimation holds for odd indices. (For details see the proof of 3.8 in [R2].) Therefore,

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}^N} |L_n(y,x)| R_\nu^\Gamma(1/4, y)\, dP_{1/4}^\Gamma(0,.)(y) < \infty.$$

The dominated convergence theorem now justifies the last step in Eq. (8.12) for $t = 1/4$, which yields Part (1) of the theorem. Part (2) follows from Part (1), together with Lemma 8.1(2) and the present section. ∎

Remark 8.6. The most important case of the biorthogonality in Theorem 8.5(1) occurs for $t = 1/2$. It shows that $(R_\nu^\Gamma(1/2,.))_{\nu \in \mathbb{Z}_+^N}$ is orthogonal with respect to the k-Gaussian measure $dP_{1/2}^\Gamma(0,.)$ if and only if $R_\nu^\Gamma(1/2,x) = c_\nu S_\nu^\Gamma(1/2,x)$ with suitable constants $c_\nu \in \mathbb{R}$. A comparison of the expansions (8.1) and (8.10) shows that this is equivalent to $m_\nu(x) = c_\nu x^\nu$. This is in fact obviously true for Examples 8.4. On the other hand, this is not correct for the S_N and the B_N cases of Examples 7.1 for $N \ge 3$.

The following result reflects the dual nature of k-Gaussian characters and cocharacters.

PROPOSITION 8.7. *Let $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, and $\nu \in \mathbb{Z}_+^N$. Then*

$$R_\nu^\Gamma(t,x) = e^{-t\Delta_k} m_\nu(x) \qquad \text{and} \qquad S_\nu^\Gamma(t,x) = \left(\frac{1}{2t}\right)^{|\nu|} e^{-t\Delta_k} x^\nu.$$

Proof. Lemma 8.1(3) and Theorem 3.1(1) yield that $e^{t\Delta_k} R_\nu^\Gamma(t,x) = m_\nu(x)$ for $t \geq 0$. This yields the first statement for $t \geq 0$. As both sides there are polynomials in t, this holds generally.

Let Δ_k^y be the k-Laplacian acting on the variable y, and let V_x be the intertwining operator acting on the variable x. Then

$$\begin{aligned} e^{t\Delta_k^y}\left(e^{-|x|^2/4t} K(x, y/2t)\right) &= e^{-|x|^2/4t} \cdot e^{|x|^2/4t} K(x, y/2t) \\ &= K(x, y/2t) = V_x(e^{\langle x, y/2t\rangle}). \end{aligned}$$

Now consider on both sides the homogeneous part W_n of degree n in the variable x. Using the left-hand side, we obtain from (8.10) that

$$W_n = e^{t\Delta_k^y}\left(\sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!} S_\nu^\Gamma(t,y)\right) = \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!} e^{t\Delta_k^y} S_\nu^\Gamma(t,y).$$

Moreover, using the right-hand side, we conclude from Section 2.2 and $V(x^\nu) = m_\nu(x)$ that

$$W_n = V_x\left(\sum_{|\nu|=n} \frac{x^\nu}{\nu!} (y/2t)^\nu\right) = \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!} (y/2t)^\nu.$$

A comparison of the corresponding coefficients leads to the second statement. ∎

We now combine Theorem 8.5 and Proposition 8.7 to rediscover a generalization of a formula of Macdonald [M] due to Dunkl [D2]; our proof is completely different from that in [D2]. We need the following notation: For a multiplicty function $k \geq 0$, the bilinear form $[.,.]_k$ on $\mathscr{P}$ is given by

$$[p,q]_k := (p(T)q)(0) \qquad \text{for } p, q \in \mathscr{P}.$$

COROLLARY 8.8. *For all $p, q \in \mathscr{P}$ and $t > 0$,*

$$[p, q]_k = \frac{1}{(2t)^{|\nu|}} \cdot \int_{\mathbb{R}^N} (e^{-t\Delta_k} p) \cdot (e^{-t\Delta_k} q)\, dP_t^\Gamma(0, .).$$

In particular, $[.,.]_k$ is positive-definite and symmetric on $\mathscr{P}$.

Proof. Let $t > 0$ and $\nu, \rho \in \mathbb{Z}_+^N$. Then, by Theorem 8.5(1) and Proposition 8.7,

$$\frac{1}{(2t)^{|\nu|}} \cdot \int_{\mathbb{R}^N} (e^{-t\Delta_k} x^\nu) \cdot \left(e^{-t\Delta_k} m_\rho\right) dP_t^\Gamma(0, .) = \nu! \cdot \delta_{\nu, \rho}.$$

On the other hand, as V acts on $\mathscr{P}$ in a homogeneous way:

$$\left[x^\nu, m_\rho\right]_k = \left(T^\nu m_\rho\right)(0) = (T^\nu V x^\rho)(0) = (V(\partial^\nu x^\rho))(0) = \nu! \cdot \delta_{\nu, \rho}.$$

This yields the first statement. The second statement is clear. ∎

We give a further application of Theorem 8.5 for $t = 1/2$. For this, we employ the adjoint operator T_j^* of the Dunkl operator T_j $(j = 1, \ldots, N)$ on $L^2(\mathbb{R}^N, dP_{1/2}^\Gamma(0, .))$, which is given by

$$T_j^* f(x) = x_j f(x) - T_j f(x) = -e^{|x|^2/2} \cdot T_j\left(e^{-|x|^2/2} f(x)\right) \qquad (f \in \mathscr{P}); \tag{8.13}$$

see Lemma 3.7 of [D2]. (The second equation is a consequence of the product rule (Section 2.2(3)).)

COROLLARY 8.9. *For all $\nu \in \mathbb{Z}_+^N$, $j = 1, \ldots, N$, $x \in \mathbb{R}^N$, and $t > 0$,*

(1) $S_{\nu+e_j}^\Gamma(1/2, x) = T_j^* S_\nu^\Gamma(1/2, x)$.

(2) *Rodriguez formula*: $S_\nu^\Gamma(t, x) = (-1)^{|\nu|} e^{|x|^2/4t} T^\nu (e^{-|x|^2/4t})$.

Proof. For simplicity, we suppress the time parameter $t = 1/2$ in Part (1). Theorem 8.5(1) and Lemma 8.1(4) yield that for all $\rho \in \mathbb{Z}_+^N$,

$$\int_{\mathbb{R}^N} R_{\rho+e_j}^\Gamma \cdot T_j^* S_\nu^\Gamma \, dP^\Gamma = \int_{\mathbb{R}^N} T_j R_{\rho+e_j}^\Gamma \cdot S_\nu^\Gamma \, dP^\Gamma = (\rho_j + 1) \int_{\mathbb{R}^N} R_\rho^\Gamma \cdot S_\nu^\Gamma \, dP^\Gamma$$
$$= \delta_{\rho,\nu} \cdot (\rho + e_j)! = \int_{\mathbb{R}^N} R_{\rho+e_j}^\Gamma \cdot S_{\nu+e_j}^\Gamma \, dP^\Gamma.$$

As $\mathscr{P}$ is dense in $L^2(\mathbb{R}^N, dP_{1/2}^\Gamma(0, .))$, Part (1) is clear. Part (2) for $t = 1/2$ follows now from (8.13), and the general case is a consequence of Eq. (8.11). ∎

Theorem 8.5 and orthogonalization within the spaces

$$V_n := e^{-\Delta_k/4}\mathscr{P}_n \subset \mathscr{P}$$

lead to systems of orthogonal polynomials on $\mathbb{R}^N$ with respect to $P^{\Gamma}_{1/4}(0,.)$. Such polynomials are called generalized Hermite polynomials; they are discussed in [B-F1, B-F2, vD, L, R2], and references cited therein. Here we discuss only some main features of these polynomials:

8.4. *Generalized Hermite Polynomials*

Let $\{\varphi_\nu,\ \nu \in \mathbb{Z}_+^N\}$ be an orthonormal basis of $(\mathscr{P}, [.,.]_k)$ with real coefficients and with $\varphi_\nu \in \mathscr{P}_{|\nu|}$. As $\mathscr{P} = \bigoplus_{n\geq 0} \mathscr{P}_n$ and $\mathscr{P}_n \perp \mathscr{P}_m$ for $n \neq m$, the φ_ν with $|\nu| = n$ can be constructed, for example, by Gram–Schmidt orthogonalization within $\mathscr{P}_n$ from an arbitrary ordered real-coefficient basis of $\mathscr{P}_n$. The generalized Hermite polynomials $\{H_\nu,\ \nu \in \mathbb{Z}_+^N\}$ associated with the basis $\{\varphi_\nu\}$ on $\mathbb{R}^N$ are defined by

$$H_\nu(x) := 2^{|\nu|} e^{-\Delta_k/4}\varphi_\nu(x) \in V_{|\nu|}. \tag{8.14}$$

By a standard argument, $\mathscr{P}$ is dense in $L^2(\mathbb{R}^N, dP^{\Gamma}_{1/4}(0,.)$ (see, for instance, [R3]). It thus follows from Corollary 8.8 that the $\{2^{-|\nu|/2}H_\nu,\ \nu \in \mathbb{Z}_+^N\}$ form an orthonormal basis of $L^2(\mathbb{R}^N, dP^{\Gamma}_{1/4}(0,.))$.

We conclude this section with a list of known properties of generalized Hermite polynomials and of k-Gaussian Appell characters and cocharacters.

PROPOSITION 8.10. *For all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^N$, $n \in \mathbb{N}$, and $\nu \in \mathbb{Z}_+^N$:*

(1) $R^{\Gamma}_\nu(t,x) = e^{-t\Delta_k} m_\nu(x)$ *and* $S^{\Gamma}_\nu(t,x) = \left(\dfrac{1}{2t}\right)^{|\nu|} e^{-t\Delta_k} x^\nu.$

(2) *Rodriguez formulas for R^{Γ}_ν and H_ν: Let $m_\nu(T)$ and $\varphi_\nu(T)$ denote the operators that are obtained from $m_\nu(x)$, $\varphi_\nu(x)$ by replacing the variables x_j by the Dunkl operators T_j. Then*

$$R^{\Gamma}_\nu(t,x) = (-2t)^{|\nu|} e^{|x|^2/4t} m_\nu(T) e^{-|x|^2/4t}$$

and

$$H_\nu(x) = (-1)^{|\nu|} e^{|x|^2} \varphi_\nu(T) e^{-|x|^2}.$$

(3) *Eigenfunctions of a CMS-type Schrödinger operator: The functions $R^{\Gamma}_\nu(t,.)$ and $S^{\Gamma}_\nu(t,.)$ satisfy*

$$\left(4t\Delta_k - 2\sum_{l=1}^N x_l \partial_l\right) f = -2|\nu| \cdot f.$$

Moreover, H_ν is a solution of this equation for $t = 1/4$.

(4) *The functions* $f(x) := e^{-|x|^2/8t} R_\nu^\Gamma(t,x)$ *and* $f(x) := e^{-|x|^2/8t} S_\nu(t,x)$ *satisfy*

$$\left(4t\Delta_k - |x|^2\right) f = -(2|\nu| + 2\gamma + N) f.$$

Moreover, the Hermite function $h_\nu(x) := e^{-|x|^2/2} H_\nu(x)$ *solves this equation for* $t = 1/4$.

(5) *Eigenfunctions of the Dunkl transform*: *If* $p \in V_n$ *and* $h(x) = e^{-|x|^2/2} p(x)$, *then* $\hat{h} = 2^{\gamma+N/2} c_k^{-1} (-i)^n h$.

(6) *Mehler formula*: *For all* $r \in \mathbb{C}$ *with* $|r| < 1$,

$$\sum_{\nu \in \mathbb{Z}_+^N} \frac{H_\nu(x) H_\nu(y)}{2^{|\nu|}} r^{|\nu|}$$

$$= \sum_{\nu \in \mathbb{Z}_+^N} \frac{R_\nu^\Gamma(1/4, x) S_\nu^\Gamma(1/4, y)}{\nu!} r^{|\nu|}$$

$$= \frac{1}{(1-r^2)^{\gamma+N/2}} \exp\left\{ -\frac{r^2\left(|x|^2 + |y|^2\right)}{1-r^2} \right\} K\left(\frac{2rx}{1-r^2}, y \right).$$

Proof. Part (1) is Proposition 8.7, and Part (2) follows from Corollary 8.9; for the Hermite polynomials, it is shown in [R2]. Part (3) follows for $t = 1/4$ from Part (1) and Theorem 3.4(1) of [R2]; renormalization then leads to the case $t > 0$, and the general case follows by analytic continuation. Part (4) follows in the same way from Theorem 3.4(2) of [R2]. Finally, Part (5) and Part (6) for the Hermite case are also shown in [R2]. The extension in (6) to Appell characters is a consequence of the identity

$$K(x,y) = \sum_\nu \varphi_\nu(x) \varphi_\nu(y) = \sum_\nu \frac{m_\nu(x) \cdot y^\nu}{\nu!},$$

which follows from Lemma 3.1 in [R2] and Eq. (7.2). The proof can now be completed in the same way as in [R2], by an obvious extension of Lemma 3.11 there; we omit the details. ∎

9. STRONG LAWS AND TRANSIENCE

In this section we present some limit theorems for k-invariant Markov processes on $\mathbb{R}^N$ in continuous and discrete time. Our first result is the iterated logarithm for k-Gaussian processes.

THEOREM 9.1. *Let $(X_t)_{t\geq 0}$ be a k-Gaussian, right-continuous process on $\mathbb{R}^N$ with $X_0 = 0$ a.e. and with generator $\Delta_k/2$. Then, almost surely,*

$$\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{2t\ln(\ln t)}} = 1 \qquad \textit{and} \qquad \limsup_{t\to 0} \frac{|X_t|}{\sqrt{2t\ln(\ln t^{-1})}} = 1.$$

Proof. As $(|X_t|)_{t\geq 0}$ is a Bessel process of index $\gamma + N/2 - 1 \geq -1/2$ Theorem 4.11, the assertion follows from a corresponding result for Bessel processes. This is known to specialists, but for the sake of completeness we include a proof: By the classical law of the iterated logarithm for Brownian motions on $\mathbb{R}^d$ (see Section 47 of [Ba]), Bessel processes $(B_t^\alpha)_{t\geq 0}$ of index $\alpha = d/2 - 1 \geq -1/2$ satisfy

$$\limsup_{t\to\infty} \frac{B_t^\alpha}{\sqrt{2t\ln(\ln t)}} = 1 \quad \text{and} \quad \limsup_{t\to 0} \frac{B_t^\alpha}{\sqrt{2t\ln(\ln t^{-1})}} = 1 \quad \text{a.e.}$$

To extend this to all $\alpha \geq -1/2$, one has to notice that Bessel processes exist for all $\alpha > -1$, and that for independent Bessel processes $(B_t^\alpha)_{t\geq 0}$ and $(B_t^\beta)_{t\geq 0}$ on a common probability space, $(((B_t^\alpha)^2 + (B_t^\beta)^2)^{1/2})_{t\geq 0}$ is again a Bessel process of index $\alpha + \beta + 1$; see Section XI.1 of [Rev-Y]. Hence, for all $\alpha \geq -1/2$ we can realize Bessel processes on a common probability space such that

$$B_t^{\lfloor 2\alpha \rfloor/2} \leq B_t^\alpha \leq B_t^{(\lfloor 2\alpha \rfloor + 1)/2}.$$

This completes the proof. ∎

We turn next to strong laws of large numbers for general k-invariant Markov processes. For simplicity, we restrict our attention to processes in discrete time with rather strong moment conditions. To describe the setting, we recapitulate that $M_2^1(\mathbb{R}^N)$ denotes the space of all probability measures on $\mathbb{R}$ having all moments up to order 2. Let $(P_n)_{n\geq 1}$ be a sequence of k-invariant Markov kernels on $\mathbb{R}^N$ such that $P_n(x, .) \in M_2^1(\mathbb{R}^N)$ holds for all $n \geq 1$ and $x \in \mathbb{R}^N$. Then a k-invariant (usually time-inhomogeneous) Markov process $(X_n)_{n\geq 0}$ on $\mathbb{R}^N$ associated with $(P_n)_{n\geq 1}$ satisfies

$$P\big(X_n \in A \mid X_{n-1} = x\big) = P_n(x, A) \qquad \text{for } n \geq 1,\ x \in \mathbb{R}^N,\ A \in \mathscr{B}(\mathbb{R}^N).$$

In this case, the following version of Kolmogorov's law of large numbers holds:

THEOREM 9.2. *Let $(X_n)_{n\geq 0}$ be a k-invariant Markov process on $\mathbb{R}^N$ with $X_0 = 0$ a.e. and associated with $(P_n)_{n\geq 1}$ as described above. Let $j \in$*

$\{1, \ldots, N\}$ *and* $(r_n)_{n\geq 1} \subset]0, \infty[$ *be a sequence with* $\lim_{n\to\infty} r_n = \infty$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{r_n}\Big(m_{2e_j}(P_n(0,.)) - \big(m_{e_j}(P_n(0,.))\big)^2\Big) < \infty. \tag{9.1}$$

Then, almost surely,

$$\lim_{t\to\infty} \frac{1}{\sqrt{r_n}}\Big(m_{e_j}(X_n) - E\big(m_{e_j}(X_n)\big)\Big) = 0.$$

Proof. Proposition 7.5 in the discrete-time setting (cf. Remark 7.6(2)) implies that

$$\Big(m_{2e_j}(X_n) - 2m_{e_j}(X_n)\cdot E\big(m_{e_j}(X_n)\big)$$
$$+2\cdot E\big(m_{e_j}(X_n)\big)^2 - E\big(m_{2e_j}(X_n)\big)\Big)_{n\geq 0}$$

is a martingale, and that for $n \geq 1$,

$$\Big[E\big(m_{2e_j}(X_n)\big) - E\big(m_{e_j}(X_n)\big)^2\Big] - \Big[E\big(m_{2e_j}(X_{n-1})\big) - E\big(m_{e_j}(X_{n-1})\big)^2\Big]$$
$$= m_{2e_j}(P_n(0,.)) - \big(m_{e_j}(P_n(0,.))\big)^2.$$

As $m_{2e_j} \geq (m_{e_j})^2$ by Proposition 7.2(2), it follows from Jensen's inequality that both sides above and all summands of (9.1) are nonnegative. In particular, we conclude that

$$\Big(Y_n := m_{2e_j}(X_n) - 2m_{e_j}(X_n)\cdot E\big(m_{e_j}(X_n)\big) + E\big(m_{e_j}(X_n)\big)^2\Big)_{n\geq 0}$$

is a nonnegative submartingale with

$$\sum_{n=1}^{\infty} \frac{1}{r_n} E(Y_n - Y_{n-1}) = \sum_{n=1}^{\infty} \frac{1}{r_n}\Big(m_{2e_j}(P_n(0,.)) - \big(m_{e_j}(P_n(0,.))\big)^2\Big) < \infty.$$

Chow's law of large numbers (see Corollary 3.3.4 of [St]) now yields that $\lim_{n\to\infty} Y_n/r_n = 0$ a.e.. As $m_{2e_j} \geq (m_{e_j})^2$ ensures that

$$\Big(m_{e_j}(X_n) - E\big(m_{e_j}(X_n)\big)\Big)^2 \leq Y_n,$$

the proof is complete. ∎

If assumption (9.1) in Theorem 9.2 holds for all $j = 1, \ldots, N$, then the conclusion of the theorem holds for all j, and as $\langle m_{e_1}(x), \ldots, m_{e_N}(x)\rangle = \langle x_1, \ldots, x_N\rangle$, it follows that

$$\lim_{n\to\infty} \frac{1}{\sqrt{r_n}}\big(X_n - E(X_n)\big) = 0 \qquad \text{a.e.,} \tag{9.2}$$

where the abbreviation $E(X_n) := (E(X_n^1), \ldots, E(X_n^N)) \in \mathbb{R}^N$ is used. In the time-homogeneous case (i.e., the P_n are independent of n and equal to some kernel P), we obtain, in particular, the following:

COROLLARY 9.3. *Let $(X_n)_{n\geq 0}$ be a k-invariant, time-homogeneous Markov process with $X_0 = 0$ and associated with some k-invariant Markov kernel P satisfying $P(0,.) \in M_2^1(\mathbb{R}^N)$. Then*

$$\lim_{n\to\infty} \frac{1}{n^\lambda}\big(X_n - E(X_n)\big) = 0 \qquad a.e. \textit{ for all } \lambda \in \left]1/2, 1\right].$$

In the end of this section, we turn to a transience criterion for semigroups of k-invariant Markov kernels on $\mathbb{R}^N$. We start with some preparatory notation:

DEFINITION 9.4. Let $(P_t)_{t\geq 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$.

(1) The semigroup $(P_t)_{t\geq 0}$ is said to be transient if

$$\kappa f(x) := \int_0^\infty P_t f(x)\, dt < \infty \qquad \text{for all } f \in C_c^+(\mathbb{R}^N), \quad x \in \mathbb{R}^N.$$

The positive kernel κ defined by this equation is called the potential (kernel) for $(P_t)_{t\geq 0}$.

(2) If $(P_t)_{t\geq 0}$ is not transient, then it is called recurrent.

(3) The resolvent measures $\rho_\lambda \in M_b^+(\mathbb{R}^N)$ of $(P_t)_{t\geq 0}$ are given by

$$\rho_\lambda = \int_0^\infty e^{-\lambda t} P_t(0,.)\, dt \qquad \text{for } \lambda > 0.$$

It is easily seen via monotone convergence that in each case the resolvents ρ_λ are related to the potential κ by $\kappa f(0) = \lim_{\lambda\to 0} \rho_\lambda(f)$ for $f \in C_c(\mathbb{R}^N)$.

Similar to the setting of locally compact abelian groups or commutative hypergroups (see [Be-Fo] and [Bl-He]), there exist transience criteria in terms of negative definite functions. We say from now on that a (measurable) function $g: \mathbb{R}^N \to \mathbb{C}$ is locally w_k-integrable, if $\int_L |g(x)| w_k(x)\, dx < \infty$

for all compacta $L \subset \mathbb{R}^N$. With this notation, we have the following theorem:

THEOREM 9.5. *Let $(P_t)_{t\geq 0}$ be a semigroup of k-invariant Markov kernels on $\mathbb{R}^N$ with negative definite function $\varphi \in C(\mathbb{R}^N)$.*

(1) *If $(P_t)_{t\geq 0}$ is transient, then $\mathrm{Re}(1/\varphi)$ is locally w_k-integrable.*

(2) *If $(1/\varphi)$ is locally w_k-integrable, then $(P_t)_{t\geq 0}$ is transient with $\|\kappa f\|_\infty < \infty$ for all $f \in C_c^+(\mathbb{R}^N)$.*

Proof. (1) Combining Lemma 4.2(3) and Proposition 4.4(3), we first observe that $\mathrm{Re}\,\varphi \geq 0$, and hence $\mathrm{Re}(1/\varphi) \geq 0$. For a fixed compactum $L \subset \mathbb{R}^N$, we choose $f \in C_c^+(\mathbb{R}^N)$ with $\hat{f} \in C_0^+(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, w_k(x)\,dx)$ and $\hat{f} \geq 1$ on L according to Lemma 2.5. Therefore, Fatou's lemma, Theorem 2.6(2), as well as

$$\hat{\rho}_\lambda(y) = \int_0^\infty P_t(0,.)^\wedge(y)e^{-t\lambda}\,dt = \int_0^\infty e^{-t(\varphi(y)+\lambda)}\,dt = \frac{1}{\lambda+\varphi(y)}, \quad (9.3)$$

lead to

$$\begin{aligned}\int_L \mathrm{Re}(1/\varphi(x))w_k(x)\,dx &= \int_L \lim_{\lambda\downarrow 0} \mathrm{Re}\left(\frac{1}{\lambda+\varphi(x)}\right)w_k(x)\,dx \\ &\leq \int_{\mathbb{R}^N} \liminf_{\lambda\downarrow 0} \mathrm{Re}\left(\frac{1}{\lambda+\varphi(x)}\right)\hat{f}(x)w_k(x)\,dx \\ &\leq \liminf_{\lambda\downarrow 0} \int_{\mathbb{R}^N} \hat{f}(x)\mathrm{Re}\left(\frac{1}{\lambda+\varphi(x)}\right)w_k(x)\,dx \\ &\leq \lim_{\lambda\downarrow 0} \frac{1}{2}\int_{\mathbb{R}^N} f^{\wedge\wedge}\,d(\rho_\lambda + \rho_\lambda^-) < \infty.\end{aligned}$$

Part (1) is now clear.

(2) Now assume that $1/\varphi$ is locally w_k-integrable. For $f \in C_c^+(\mathbb{R}^N)$ we find $g \in C_c^+(\mathbb{R}^N)$ with $f \leq \hat{g}$ on $\mathbb{R}^N$ by Lemma 2.5. Hence, Theorem 2.6(2), the dominated convergence theorem, and Theorem 2.2(2) imply that

for all $x \in \mathbb{R}^N$,

$$\begin{aligned}\kappa f(x) \le \kappa \hat{g}(x) &\le \limsup_{\lambda \downarrow 0} \int_0^\infty \int_{\mathbb{R}^N} \hat{g}(y)\, dP_t(x,.)(y) e^{-\lambda t}\, dt \\ &= \limsup_{\lambda \downarrow 0} \int_0^\infty \int_{\mathbb{R}^N} g(y) P_t(x,.)^\wedge(y) w_k(y)\, dy\, e^{-\lambda t}\, dt \\ &= \limsup_{\lambda \downarrow 0} \int_{\mathbb{R}^N} \frac{g(y)}{\lambda + \varphi(y)} K(-ix, y) w_k(y)\, dy \\ &= \int_{\mathbb{R}^N} \frac{g(y)}{\varphi(y)} K(-ix, y) w_k(y)\, dy \\ &\le \int_{\mathbb{R}^N} \frac{g(y)}{|\varphi(y)|} w_k(y)\, dy < \infty,\end{aligned}$$

which gives the transience as claimed. ∎

COROLLARY 9.6. *A semigroup of k-Gaussian kernels on $\mathbb{R}^N$ is transient if and only if $2\gamma + N > 2$.*

Proof. The negative definite function is here given by $\varphi(x) = |x|^2$. Hence, for $2\gamma + N > 2$,

$$\begin{aligned}\int_{|x|\le 1} \frac{1}{\varphi(x)} w_k(x)\, dx &= \int_{|x|\le 1} |x|^{-2} \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}\, dx \\ &\le C \int_{|x|\le 1} |x|^{2\gamma - 2}\, dx < \infty\end{aligned}$$

with a suitable constant C. On the other hand, if $2\gamma + N \le 2$, then either $N = 2$ with $\gamma = 0$ or $N = 1$ with $\gamma \le 1/2$, i.e., a classical two-dimensional recurrent Gaussian semigroup emerges, or, in the second case, one has $\int_{-1}^{1} x^{-2} w_k(x)\, dx = \int_{-1}^{1} x^{2\gamma-2}\, dx = \infty$, which also yields recurrence. ∎

The negative definite function of a k-Cauchy process is $\varphi(x) = c|x|$ for some $c > 0$; hence:

COROLLARY 9.7. *A k-Cauchy semigroup on $\mathbb{R}^N$ is recurrent if and only if $N = 1$ and $k = 0$.*

Remark 9.8. It is well known that a convolution semigroup on a locally compact abelian group is transient if and only if the associated negative definite function φ has the property that $\mathrm{Re}(1/\varphi)$ is locally integrable. We do not know whether this stronger result holds also in the Dunkl setting. On the other hand, Theorem 9.5 is completely sufficient in the context of this paper, as we here consider only examples of semigroups with real-valued negative definite functions.

10. GENERALIZED ORNSTEIN–UHLENBECK PROCESSES

In this section we show how the k-Gaussian processes and semigroups of Sections 3 and 4 lead to k-analogues of classical multidimensional Ornstein–Uhlenbeck processes and semigroups in a natural way. There are different approaches to classical Ornstein–Uhlenbeck processes:

(a) Ornstein–Uhlenbeck semigroups can be defined in terms of heat semigroups.

(b) The generators of the associated positive, strongly continuous contraction semigroups on $C_0(\mathbb{R}^N)$ are the "oscillator-operators" $c\Delta - \alpha\sum_{i=1}^N x_i\partial_i$ with parameters $c, \alpha > 0$.

(c) The pathwise definition of stationary Ornstein–Uhlenbeck is in terms of a Brownian motion on $\mathbb{R}^N$.

(d) Ornstein–Uhlenbeck processes are solutions of certain stochastic differential equations.

Because of technical difficulties in approach (d), we deal here only with aspects (a)–(c). We start with (a). As usual, let a reflection group W and multiplicity function $k \geq 0$ be fixed.

10.1. *k-Ornstein–Uhlenbeck Semigroups of Markov Kernels*

Let $(P_t^\Gamma)_{t\geq 0}$ be the k-Gaussian semigroup of Section 3.3. The k-Ornstein–Uhlenbeck Markov kernels $(P_t^O)_{t\geq 0}$ with parameters $c, \alpha > 0$ are defined by

$$P_t^O(x, A) := P^\Gamma_{(1-e^{-2\alpha t})\cdot c/2\alpha}(e^{-\alpha t}x, A) \qquad \left(x \in \mathbb{R}^N,\, A \in \mathscr{B}(\mathbb{R}^N)\right).$$

LEMMA 10.1. *For all parameters $\alpha, c > 0$, $(P_t^O)_{t\geq 0}$ is a semigroup of Markov kernels on $\mathbb{R}^N$.*

Proof. Obviously, each P_t^O is a Markov kernel. We know from Section 3.3 that the k-Gaussian Markov kernels $(P_t^\Gamma)_{t\geq 0}$ form a semigroup of Markov kernels with the homogeneity property

$$P_{r^2t}(rx, rA) = P_t(x, A) \qquad \text{for } x \in \mathbb{R}^N,\, A \in \mathscr{B}(\mathbb{R}^N),\, r, t > 0. \quad (10.1)$$

This implies that for all $s, t > 0$, $x \in \mathbb{R}^N$, and $A \in \mathscr{B}(\mathbb{R}^N)$,

$$\begin{aligned}
P_s^O \circ P_t^O(x, A) &= \int_{\mathbb{R}^N} P_t^O(z, A) P_s^O(x, dz) \\
&= \int_{\mathbb{R}^N} P^\Gamma_{(1-e^{-2\alpha t})\cdot c/2\alpha}(e^{-\alpha t}z, A) P^\Gamma_{(1-e^{-2\alpha s})\cdot c/2\alpha}(e^{-\alpha s}x, dz) \\
&= \int_{\mathbb{R}^N} P^\Gamma_{(1-e^{-2\alpha t})\cdot c/2\alpha}(y, A) P^\Gamma_{e^{-2\alpha t}(1-e^{-2\alpha s})\cdot c/2\alpha}(e^{-\alpha(s+t)}x, dy) \\
&= P^\Gamma_{(1-e^{-2\alpha t}+e^{-2\alpha t}-e^{-2\alpha(s+t)})\cdot c/2\alpha}(e^{-\alpha(s+t)}x, A) = P^O_{s+t}(x, A).
\end{aligned}$$

∎

PROPOSITION 10.2. *For all $c, \alpha > 0$, the integral operators*

$$H^O(t) f(x) := \int_{\mathbb{R}^N} f(y) P_t^O(x, dy) \qquad (t \geq 0)$$

form a positive, strongly continuous contraction semigroup on $C_0(\mathbb{R}^N)$ with generator

$$G^O f(x) := \left(c\Delta_k - \alpha \sum_{i=1}^N x_i \partial_i \right) f(x) \qquad \text{for } f \in C_c^2(\mathbb{R}^N).$$

Proof. By construction and by Lemma 10.1, $(H^O(t))_{t\geq 0}$ is a semigroup consisting of positive contraction operators on $C_0(\mathbb{R}^N)$. To see the strong continuity, we note that the operators $H^O(t)$ are related to the heat operators $H(t)$ of Section 3.2 by

$$H^O(t) f(x) = H\left(\frac{1 - e^{-2\alpha t}}{2\alpha} \cdot c \right) f(e^{-\alpha t} x)$$
$$\left(x \in \mathbb{R}^N, t \geq 0, f \in C_0(\mathbb{R}^N)\right), \quad (10.2)$$

and recall that the semigroup $(H(t))_{t\geq 0}$ is strongly continuous. Now fix $f \in C_0(\mathbb{R}^N)$. Then

$$\sup_{x\in\mathbb{R}^N} \left| f(e^{-\alpha t}x) - f(x) \right| \to 0 \qquad \text{for } t \downarrow 0,$$

and hence

$$\sup_{x\in\mathbb{R}^N} \left| H^O(t)f(x) - f(x) \right| = \sup_{x\in\mathbb{R}^N} \left| H\left(\frac{1-e^{-2\alpha t}}{2\alpha}\cdot c\right) f(e^{-\alpha t}x) - f(x) \right|$$
$$\leq \left\| H\left(\frac{1-e^{-2\alpha t}}{2\alpha}\cdot c\right) f - f \right\|_\infty + \sup_{x\in\mathbb{R}^N} \left| f(e^{-\alpha t}x) - f(x) \right| \to 0$$

for $t\downarrow 0$ as claimed. The proof of the generator formula is similar; we have

$$\frac{H^O(t)f(x) - f(x)}{t} = \frac{H(\tau)f(e^{-\alpha t}x) - f(e^{-\alpha t}x)}{\tau}\cdot\frac{\tau}{t} + \frac{f(e^{-\alpha t}x) - f(x)}{t}, \qquad (10.3)$$

with $\tau := c(1-e^{-2\alpha t})/(2\alpha)$. For $t\downarrow 0$,

$$\tau/t \to c \qquad \text{and} \qquad \big(H(\tau)f - f\big)/\tau \to \Delta_k f \text{ uniformly;}$$

it follows that the first summand in (10.3) tends to $c\Delta_k f$ uniformly for $t\downarrow 0$. Moreover, the mean value theorem ensures that there exists $\lambda_t \in [e^{-\alpha t}, 1]$ with

$$\frac{f(e^{-\alpha t}x) - f(x)}{t} = \langle \nabla f(\lambda_t x), x\rangle \cdot \frac{e^{-\alpha t}-1}{t},$$

which uniformly tends to $-\alpha\langle\nabla f(x), x\rangle$ for $t\downarrow 0$. This finishes the proof. ∎

PROPOSITION 10.3. *For each $f\in C_b(\mathbb{R}^N)$, the function*

$$u(x,t) := H^O(t)f(x) = \int_{\mathbb{R}^N} f(y)\,P_t^O(x,dy)$$

is a $C_b(\mathbb{R}^N\times[0,\infty[)$-solution of the Cauchy problem

$$u_t = \left(c\Delta_k - \alpha\textstyle\sum_{i=1}^N x_i\partial_i\right)u$$

on $\mathbb{R}^N\times\,]0,\infty[$ with initial data $u(x,0) = f(x)$ for $x\in\mathbb{R}^N$.

Proof. We write $u(x,t) = H(c\cdot(1-e^{-2\alpha t})/2\alpha)f(e^{-\alpha t}x)$ and recall from Section 3.2 that $w(t,x) := H(t)f(x)$ belongs to $C_b(\mathbb{R}^N\times[0,\infty[)$ and satisfies $\partial_t w = \Delta_k w$. Therefore,

$$\begin{aligned} u_t(x,t) &= c \cdot e^{-2\alpha t} w_t\left(e^{-\alpha t}x, \frac{1-e^{-2\alpha t}}{2\alpha} \cdot c\right) \\ &\quad + \sum_{i=1}^{N} w_{x_i}\left(e^{-\alpha t}x, \frac{1-e^{-2\alpha t}}{2\alpha} \cdot c\right) \cdot (-\alpha x_i) \cdot e^{-\alpha t} \\ &= c \cdot \Delta_k u(x,t) - \alpha \sum_{i=1}^{N} x_i u_{x_i}(x,t). \end{aligned} \tag{10.4}$$

∎

Remark 10.4. The Mehler formula (Proposition 8.10(6)) for generalized Hermite polynomials says that

$$\Gamma_k\left(e^{-2t}x, y, \frac{1-e^{-4t}}{4}\right) = c_k e^{-|y|^2} M(x, y, e^{-2t}) \qquad (t > 0,\ x, y \in \mathbb{R}^N),$$

with the Mehler kernel

$$M(x,y,r) := \sum_{\nu \in \mathbb{Z}_+^N} \frac{H_\nu(x) H_\nu(y)}{2^{|\nu|}} \cdot r^{|\nu|} \qquad (|r| < 1,\ x, y \in \mathbb{R}^N).$$

As the generalized Hermite polynomials satisfy $(\Delta_k - 2\sum_{i=1}^N x_i \partial_i) H_\nu = -2|\nu| \cdot H_\nu$ for $\nu \in \mathbb{Z}_+^N$ (see Proposition 8.10(3)), it follows that for each $y \in \mathbb{R}^N$, the function $\tilde{M}_y(x,t) := M(x,y,e^{-2t})$ satisfies $(\Delta_k - 2\sum_{i=1}^N x_i \partial_i)\tilde{M}_y = \partial_t \tilde{M}_y$. This leads to an alternative proof of Eq. (10.4) above.

10.2. *k-Ornstein–Uhlenbeck Processes*

A Markov process $(X_t)_{t \geq 0}$ on $\mathbb{R}^N$ is called a k-Ornstein–Uhlenbeck process (with parameters $\alpha, c > 0$) if its transition probabilities are given by the semigroup $(P_t^O)_{t \geq 0}$ of Section 10.1. Such a process is called a stationary k-Ornstein–Uhlenbeck process (with parameters $c, \alpha > 0$) if its initial distribution is given by

$$dP^\Gamma_{c/2\alpha}(0,.)(x) = \Gamma_k(0, x, c/2\alpha) \cdot w_k(x)\, dx.$$

This notion is justified by the fact that each stationary k-Ornstein–Uhlenbeck process $(X_t)_{t \geq 0}$ is stationary, i.e., $P^\Gamma_{c/2\alpha}(0,.)$ is the distribution of all X_t, $t \geq 0$. In fact, Eq. (10.1) shows that

$$\begin{aligned} P(X_t \in A) &= \int_{\mathbb{R}^N} P_t^O(x, A) P_{c/2\alpha}^{\Gamma}(0, dx) \\ &= \int_{\mathbb{R}^N} P_{(1-e^{-2\alpha t})\cdot c/2\alpha}^{\Gamma}(e^{-\alpha t}x, A) P_{c/2\alpha}^{\Gamma}(0, dx) \\ &= \int_{\mathbb{R}^N} P_{(1-e^{-2\alpha t})\cdot c/2\alpha}^{\Gamma}(x, A) P_{e^{-2\alpha t}\cdot c/2\alpha}^{\Gamma}(0, dx) \\ &= P_{(1-e^{-2\alpha t}+e^{-2\alpha t})\cdot c/2\alpha}^{\Gamma}(0, A) \\ &= P_{c/2\alpha}^{\Gamma}(0, A). \end{aligned}$$

The following result shows that stationary k-Ornstein–Uhlenbeck processes can be constructed directly from k-Gaussian processes:

LEMMA 10.5. *Let $(B_t)_{t \geq 0}$ be a k-Gaussian process on some probability space $(\Omega, \mathscr{A}, P)$ with values in $\mathbb{R}^N$. Assume that this process starts in $0 \in \mathbb{R}^N$ at time 0, and that it is associated with the k-Gaussian semigroup $(P_t^{\Gamma})_{t \geq 0}$ of Section 3.3. Then for any $\alpha, \beta > 0$,*

$$X_t(\omega) := e^{-\alpha t} \cdot B_{\beta \cdot \exp(2\alpha t)}(\omega) \qquad (\omega \in \Omega, t \geq 0)$$

defines a stationary k-Ornstein–Uhlenbeck process with parameters α and $c := 2\alpha\beta$.

Proof. Obviously, $(X_t)_{t \geq 0}$ is a Markov process with initial distribution $P_{\beta}^{\Gamma}(0, .) \in M^1(\mathbb{R}^N)$. Take $s, t \geq 0$, $x \in \mathbb{R}^N$, and $A \in \mathscr{B}(\mathbb{R}^N)$. The homogeneity $P_{r^2 t}^{\Gamma}(rx, rA) = P_t^{\Gamma}(x, A)$ yields that

$$\begin{aligned} P\big(X_{s+t} \in A \mid X_s = x\big) &= P\Big(B_{\beta \exp(2\alpha(s+t))} \in e^{\alpha(s+t)}A \mid B_{\beta \exp(2\alpha s)} = e^{\alpha s}x\Big) \\ &= P_{\beta(\exp(2\alpha(s+t)) - \exp(2\alpha s))}^{\Gamma}(e^{\alpha s}x, e^{\alpha(s+t)}A) \\ &= P_{\beta(1-\exp(-2\alpha t))}^{\Gamma}(e^{-\alpha t}x, A). \end{aligned}$$

A comparison with the kernels P_t^O of Section 10.1 completes the proof. ∎

If the process $(B_t)_{t \geq 0}$ in Lemma 10.5 has the continuity properties of Theorem 4.10, then the associated stationary k-Ornstein–Uhlenbeck process $(X_t)_{t \geq 0}$ also has these properties. Combining this with Proposition 10.2 and the methods in the proofs of Theorems 4.7 and 4.10, we obtain:

THEOREM 10.6. *Each k-Ornstein–Uhlenbeck process on $\mathbb{R}^N$ admits an equivalent k-Ornstein–Uhlenbeck process $(X_t)_{t \geq 0}$ such that $(X_t)_{t \geq 0}$ has the càdlàg property and that its projection $p(X_t))_{t \geq 0}$ on $\mathbb{R}^N/W \cong \overline{C}$ has almost surely continuous paths.*

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REFERENCES

[ADKS] S. Albeverio, Yu. L. Daletsky, Yu. G. Kondratiev, and L. Streit, Non-Gaussian infinite-dimensional analysis, *J. Funct. Anal.* **138** (1996), 311–350.

[B-F1] T. H. Baker and P. J. Forrester, The Calogero–Sutherland model and polynomials with prescribed symmetry, *Nuclear Phys. B* **492** (1997), 682–716.

[B-F2] T. H. Baker and P. J. Forrester, Non-symmetric Jack polynomials and integral kernels, *Duke Math. J.* to appear.

[Ba] H. Bauer, "Probability Theory," de Gruyter, Berlin, 1996.

[Be-Mo] Y. Berest, Y. Molchanov, Fundamental solutions for partial differential equations with reflection group invariance, *J. Math. Phys.* **36** (1995), 4324–4339.

[Be] Yu. M. Berezansky, Construction of the operators of generalized translation by Appell characters, *in* "Proceedings Devoted to S. G. Krein," World Scientific, Singapore, to appear.

[B-K1] Yu. M. Berezansky and Yu. G. Kondratiev, Biorthogonal systems in hypergroups: An extension of non-Gaussian analysis, *Methods Funct. Anal. Topology* **2** (1996), 1–50.

[B-K2] Yu. M. Berezansky and Yu. G. Kondratiev, Non-Gaussian analysis and hypergroups, *Funct. Anal. Appl.* **29** (1996), 188–191.

[Be-Fo] C. Berg and G. Forst, "Potential Theory on Locally Compact Abelian Groups," Springer-Verlag, Berlin/Heidelberg, 1975.

[Bl-He] W. R. Bloom and H. Heyer, "Harmonic Analysis of Probability Measures on Hypergroups," de Gruyter, Berlin, 1995.

[Ca] F. Calogero, Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.* **12** (1971), 419–436.

[C] I. Cherednik, A unification of the Knizhnik–Zamolodchikov equations and Dunkl operators via affine Hecke algebras, *Invent. Math.* **106** (1991), 411–432.

[vD] J. F. van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement, *Comm. Math. Phys.* **188** (1997), 467–497.

[D1] C. F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[D2] C. F. Dunkl, Integral kernels with reflection group invariance, *Can. J. Math.* **43** (1991), 1213–1227.

[D3] C. F. Dunkl, Hankel transforms associated to finite reflection groups, in "Proceedings of the Special Session on Hypergeometric Functions on Domains of Positivity, Jack Polynomials and Applications," Proceedings, Tampa 1991, *Contemp. Math.* **138** (1992), 123–138.

[D4] C. F. Dunkl, Intertwining operators and polynomials associated with the symmetric group, *Math. Nachr.* to appear.

[D5] C. F. Dunkl, Intertwining operators of type B_N, Technical report CRM-2380, Centre de recherches mathématiques, 1996.

[Dy] E. B. Dynkin, "Markov Processes I," Springer-Verlag, Berlin/Heidelberg, 1965.

[E-K] S. N. Ethier and T. G. Kurtz, "Markov Processes: Characterization and Convergence," Wiley, New York, 1986.

[F-S] Ph. Feinsilver and R. Schott, "Algebraic Structures and Operator Calculus," Vol. III, "Representations of Lie groups," Kluwer, Dordrecht, The Netherlands, 1996.

[Fe] W. Feller, "An Introduction to Probability Theory and Its Applications," Wiley, New York, 1965.

[H] G. J. Heckman, An elementary approach to the hypergeometric shift operators of Opdam, *Invent. Math.* **103** (1991), 341–350.

[Hu] J. E. Humphreys, "Reflection Groups and Coxeter Groups," Cambridge Univ. Press, Cambridge, England, 1990.

[dJ] M. F. E. de Jeu, The Dunkl transform, *Invent. Math.* **113** (1993), 147–162.

[K-O] S. Kamefuchi and Y. Ohnuki, "Quantum Field Theory and Parastatistics," Univ. of Tokyo Press, Tokyo, and Springer-Verlag, Berlin/New York, 1982.

[K-S] I. Karatzas and S. E. Shreve, "Brownian Motion and Stochastic Calculus," Springer-Verlag, Berlin/New York, 1985.

[Ki] A. A. Kirillov, Lectures on affine Hecke algebras and Macdonald conjectures, *Bull. Amer. Math. Soc.* **34** (1997), 251–292.

[L-V] L. Lapointe and L. Vinet, Exact operator solution of the Calogero–Sutherland model, *Comm. Math. Phys.* **178** (1996), 425–452.

[L] M. Lassalle, Polynômes de Hermite généralisés, *C. R. Acad. Sci. Paris* **313** (1991), 579–582.

[M] I. G. Macdonald, The volume of a compact Lie group, *Invent. Math.* **56** (1980), 93–95.

[Me] M. L. Mehta, "Random Matrices and Statistical Theory of Energy Levels," Academic Press, New York, 1967.

[O1] E. M. Opdam, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, *Compositio Math.* **85** (1993), 333–373.

[O2] E. M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, *Acta Math.* **175** (1995), 75–121.

[P] A. P. Polychronakos, Exchange operator formalism for integrable systems of particles, *Phys. Rev. Lett.* **69** (1992), 703–705.

[Re-V] C. Rentzsch and M. Voit, Homogeneous Markov processes and Gaussian processes on hypergroups, preprint, 1997.

[Rev-Y] D. Revuz and M. Yor, "Continuous Martingales and Brownian Motion," Springer-Verlag, Berlin/New York, 1991.

[Rom] S. Roman, "The Umbral Calculus," Academic Press, San Diego, 1984.

[R1] M. Rösler, Bessel-type signed hypergroups on $\mathbb{R}$, *in* "Probability Measures on Groups and Related Structures XI, Proceedings, Oberwolfach, 1994" (H. Heyer and A. Mukherjea, Eds.), World Scientific, Singapore, 1995.

[R2] M. Rösler, Generalized Hermite polynomials and the heat equation for Dunkl operators, *Comm. Math. Phys.* **192** (1998), 519–542.

[R3] M. Rösler, Positivity of Dunkl's intertwining operator, *Duke Math. J.*, to appear.

[R-V1] M. Rösler and M. Voit, An uncertainty principle for Hankel transforms, *Proc. Amer. Math. Soc.* to appear.

[R-V2] M. Rösler and M. Voit, Biorthogonal polynomials associated with reflection groups and a formula of Macdonald, *J. Comput. Appl. Math.*, to appear.

[Ros] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, *in* "Operator Theory: Advances and Applications," Vol. 73, pp. 369–396, Birkhäuser Verlag, Basel, 1994.

[St] W. F. Stout, "Almost Sure Convergence," Academic Press, San Diego, 1989.

[S-V] D. Stroock and S. R. S. Varadhan, "Multidimensional Diffusion Processes," Springer-Verlag, New York, 1979.

[Su] B. Sutherland, Quantum many-body problem in one dimension, I, II, *J. Math. Phys.* **12** (1971), 246–250.

[Sz] G. Szegö, "Orthogonal Polynomials," Am. Math. Soc., New York, 1959.

[V] M. Voit, A Lévy-characterization of one-dimensional diffusions, *Arch. Math.* **70** (1998), 235–238.

[Wa] S. Watanabe, Sobolev type theorems for an operator with singularity, *Proc. Amer. Math. Soc.* **125** (1997), 129–136.

[W-W] H. von Weizsäcker and G. Winkler, "Stochastic Integrals," Vieweg, Wiesbaden, 1990.

[Wi] D. Williams, "Diffusions, Markov Processes, and Martingales," Vol. I, Wiley, Chichester/New York, 1979.

[X] Y. Xu, Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups, *Proc. Amer. Math. Soc.* **125** (1997), 2963–2973.

[Y] L. M. Yang, A note on the quantum rule of the harmonic oscillator, *Phys. Rev.* **84** (1951), 788–790.

[Z] Hm. Zeuner, Moment functions and laws of large numbers on hypergroups, *Math. Z.* **211** (1992), 369–407.