Limit theorems for radial random walks on $p \times q$-matrices as p tends to infinity

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For a fixed probability measure $\nu \in M^1([0,\infty[)$ and any dimension $p \in \mathbb{N}$ there is a unique radial probability measure $\nu_p \in M^1(\mathbb{R}^p)$ with ν as its radial part. In this paper we study the limit behavior of $\|S_n^p\|_2$ for the associated radial random walks $(S_n)_{n\geq 0}$ on $\mathbb{R}^p$ whenever n, p tend to ∞ in some coupled way. In particular, weak and strong laws of large numbers as well as a large deviation principle are presented.

In fact, we shall derive these results in a higher rank setting, where $\mathbb{R}^p$ is replaced by the space of $p \times q$ matrices and $[0,\infty[$ by the cone Π_q of positive semidefinite matrices. All proofs are based on the fact that in this general setting the $(S_k^p)_{k\geq 0}$ form Markov chains on Π_q whose transition probabilities are given in terms Bessel functions J_μ of matrix argument with an index μ depending on p. The limit theorems then follow from new asymptotic results for the J_μ as $\mu \to \infty$.

KEYWORDS: Bessel functions of matrix argument, matrix cones, Bessel functions associated with root systems, asymptotics, radial random walks, laws of large numbers, large deviations.

1 Introduction

This paper has its origin in the following problem: Let $\nu \in M^1([0,\infty[)$ be a fixed probability measure. Then for each dimension $p \in \mathbb{N}$ there is a unique radial probability measure $\nu_p \in M^1(\mathbb{R}^p)$ with ν as its radial part, i.e., ν is the image of ν_p under the mapping $\varphi_p(x) := \|x\|$ where $\|.\|$ is the usual Euclidean norm. For each $p \in \mathbb{N}$ consider i.i.d. $\mathbb{R}^p$-valued random variables X_k^p, $k \in \mathbb{N}$, with law ν_p as well as the associated radial random walks

$$\big(S_n^p := \sum_{k=1}^{n} X_k^p\big)_{n\geq 0}$$

on $\mathbb{R}^p$. The aim is to find limit theorems for the $[0,\infty[$-valued random variables $\|S_k^{p_k}\|$ for for suitable sequences $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ of dimensions and suitable measures ν where we exclude the trivial case $\nu = \delta_0$.

To get a first feeling for possible results, assume that ν has second moment $\sigma^2(\nu) := \int_0^\infty r^2\, d\nu(r) \in]0,\infty[$. The classical central limit theorem (CLT) on $\mathbb{R}^p$ and the well-known relation between the standard normal distribution on $\mathbb{R}^p$ and the χ^2-distribution χ_p^2 with p degrees of freedom imply after some short computation that for fixed p and $k \to \infty$, the variables $\frac{p}{k\cdot\sigma^2(\nu)}\|S_k^p\|^2$ tend in distribution to χ_p^2. Moreover, as Z_p/p tends to 1 in probability for χ_p^2-distributed random variables Z_p, we obtain that $\|S_k^p\|/\sqrt{k} \to \sqrt{\sigma^2(\nu)}$ in probability if we first take $k \to \infty$ and then $p \to \infty$. We already observed in [V1] that this result remains correct for other combinations of $k, p \to \infty$:

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Theorem 1.1 *Assume that* $\nu \in M^1([0,\infty[)$ *has the second moment* $\sigma^2(\nu) \in]0,\infty[$. *Then for each sequence* $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *of dimensions with* $\lim_{k\to\infty} p_k = \infty$, $\|S_k^{p_k}\|/\sqrt{k} \to \sqrt{\sigma^2(\nu)}$ *in probability.*

We mention that explicit estimates for the CLT with an explicit dependence of the constants on the dimensions (see [B] and references cited in [BGPR]) even imply a CLT under the restriction that the dimensions p_k do not grow too fast in comparison with k. Also, there is a further CLT for fast growing dimensions p_k; see [V3].

One purpose of this paper is to prove an associated strong law. For simplicity we will assume that ν has a compact support.

Theorem 1.2 *Assume that* $\nu \in M^1([0,\infty[)$ *has compact support. Let* $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *and* $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *sequences of dimensions and time steps with the following properties:*

(1) $\lim_{k\to\infty} p_k/k^a = \infty$ *for all* $a \in \mathbb{N}$;
(2) $\lim_{k\to\infty} p_k/(n_k^2(\ln k)^2) = \infty$;
(3) $\lim_{k\to\infty} n_k/(\ln k)^2 = \infty$.

Then $\|S_{n_k}^{p_k}\|/\sqrt{n_k} \to \sqrt{\sigma^2(\nu)}$ *almost surely.*

For the case $n_k = k$, only condition (1) on the dimensions remains, i.e., the dimensions have to grow faster than any polynomial. Unfortunately, we are not able to get rid of this strong growth condition via our approach. We shall discuss the conditions also in Section 4 below. Besides these laws of large numbers we shall also derive a large deviation principle for $S_k^{p_k}$ in Section 5 under the condition that p_k grows fast enough.

Theorems 1.1 and 1.2 and, in part, also this large deviation principle will appear as special cases of extensions of these results in two directions in this paper.

The first extension concerns a higher rank setting. We consider the following geometric situation: For fixed dimensions $p, q \in \mathbb{N}$ let $M_{p,q} = M_{p,q}(\mathbb{F})$ denote the space of $p \times q$-matrices over one of the division algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. This is a Euclidean vector space of (real) dimension dpq with scalar product $\langle x, y\rangle = \Re tr(x^*y)$ where $x^* := \overline{x}^t$, $\Re t := \frac{1}{2}(t + \overline{t})$ is the real part of $t \in \mathbb{F}$, and tr is the trace in $M_q := M_{q,q}$. A measure on $M_{p,q}$ is called radial if it is invariant under the action of the unitary group $U_p = U_p(\mathbb{F})$ by left multiplication, $U_p \times M_{p,q} \to M_{p,q}, \quad (u, x) \mapsto ux$. This action is orthogonal w.r.t. the scalar product above, and, by uniqueness of the polar decomposition, two matrices $x, y \in M_{p,q}$ belong to the same U_p-orbit if and only if $x^*x = y^*y$. Thus the space $M_{p,q}^{U_p}$ of U_p-orbits in $M_{p,q}$ is naturally parameterized by the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$-matrices over $\mathbb{F}$. We identify $M_{p,q}^{U_p}$ with Π_q via $U_q x \simeq (x^*x)^{1/2}$, i.e., the canonical projection $M_{p,q} \to M_{p,q}^{U_p}$ will be realized as the mapping

$$\varphi_p : M_{p,q} \to \Pi_q, \quad x \mapsto (x^*x)^{1/2}.$$

The square root is used here in order to ensure for $q = 1$ and $\mathbb{F} = \mathbb{R}$ that $\Pi_1 = [0,\infty[$ and $\varphi_p(x) = \|x\|$, i.e. the setting above appears. By taking images of measures, the mapping φ_p induces a Banach space isomorphism between the space $M_b^{U_q}(M_{p,q})$ of all bounded radial Borel measures on $M_{p,q}$ and the space $M_b(\Pi_q)$ of bounded Borel measures on the cone Π_q. In particular, for each probability measure $\nu \in M^1(\Pi_q)$ there is a unique radial probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$. We shall say that $\nu \in M^1(\Pi_q)$ admits a second moment if $\int_{\Pi_q} \|s\|^2 \, d\nu(s) < \infty$ where again, $\|s\| = (trs^2)^{1/2}$ is the Hilbert-Schmidt norm. In this case, the second moment of ν is defined as the matrix-valued integral

$$\sigma^2(\nu) := \int_{\Pi_q} s^2 \, d\nu(s) \in \Pi_q.$$

With these notions, we shall derive the following generalizations of Theorems 1.1 and 1.2:

Theorem 1.3 *Let* $\nu \in M^1(\Pi_q)$ *be a probability measure with finite second moment* $\sigma^2(\nu) \in \Pi_q$. *For each dimension* $p \in \mathbb{N}$ *consider the unique* U_p*-invariant probability measure* $\nu_p \in M^1(M_{p,q})$ *with* $\varphi_p(\nu_p) = \nu$. *Furthermore, let* $(X_l^p)_{l\in\mathbb{N}}$ *be a sequence of i.i.d.* $M_{p,q}$*-valued random variables with law* ν_p. *Then for each sequence* $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *of dimensions with* $\lim_{k\to\infty} p_k = \infty$,

$$\frac{1}{\sqrt{k}}\varphi_{p_k}\Big(\sum_{l=1}^{k} X_l^{p_k}\Big) \to \sqrt{\sigma^2(\nu)} \in \Pi_q \quad \textit{in probability.}$$

Theorem 1.4 *Let $\nu \in M^1(\Pi_q)$ be a probability measure with compact support. For each dimension $p \in \mathbb{N}$ consider the unique U_p-invariant probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$. Furthermore, let $(X_l^p)_{l\in\mathbb{N}}$ be a sequence of i.i.d. $M_{p,q}$-valued random variables with law ν_p. Let $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ sequences of dimensions and time steps with the following properties:*

(1) $\lim_{k\to\infty} p_k/k^a = \infty$ *for all* $a \in \mathbb{N}$;
(2) $\lim_{k\to\infty} p_k/(n_k^2(\ln k)^2) = \infty$;
(3) $\lim_{k\to\infty} n_k/(\ln k)^2 = \infty$.

Then $\varphi_{p_k}\big(\sum_{l=1}^{n_k} X_l^{p_k}\big)/\sqrt{n_k}$ *tends to* $\sqrt{\sigma^2(\nu)} \in \Pi_q$ *almost surely.*

We next turn to a further generalization of these theorems. Consider again the Banach space isomorphism between $M_b^{U_q}(M_{p,q})$ and $M_b(\Pi_q)$. The usual group convolution on $M_{p,q}$ induces a Banach-$*$-algebra-structure on $M_b(\Pi_q)$ such that this isomorphism becomes a probability-preserving Banach-$*$-algebra isomorphism. The space Π_q together with this new convolution becomes a commutative orbit hypergroup; see [J] and [BH] for a general background and [R3] for our specific example. It follows from Eq. (3.5) and Corollary 3.2 of [R3] that in case $p \geq 2q$, the convolution product of two point measures on Π_q induced from $M_{p,q}$ is given by

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu}\int_{D_q} f\big(\sqrt{r^2+s^2+svr+rv^*s}\big)\,\Delta(I-vv^*)^{\mu-\rho}\,dv \tag{1.1}$$

with $\mu := pd/2$, $\rho := d\big(q-\frac{1}{2}\big)+1$,

$$D_q := \{v \in M_q : v^*v < I\}$$

(where $v^*v < I$ means that $I - v^*v$ is strictly positive definite), and with the normalization constant

$$\kappa_\mu := \int_{D_q} \Delta(I-v^*v)^{\mu-\rho}dv. \tag{1.2}$$

The convolution of arbitrary measures is just given by bilinear, weakly continuous extension.

It was observed in [R3] that Eq. (1.1) defines a commutative hypergroup actually for all indices $\mu \in \mathbb{R}$ with $\mu > \rho - 1$. In all cases, $0 \in \Pi_q$ is the identity of the hypergroup and the involution is given by the identity mapping. These hypergroup structures are closely related with a product formula for Bessel functions of index μ on the matrix cone Π_q and are therefore called Bessel hypergroups on Π_q. Indeed, the hypergroup characters are given in terms of matrix Bessel functions J_μ. We refer to the monograph [FK] for Bessel functions on cones, and to [R3] for the particular details. For general indices μ, the Bessel hypergroups on Π_q do not have a nice geometric (orbit) interpretation as in the cases $\mu = pd/2$ with integral p, but nevertheless the notion of random walks on these hypergroups is meaningful in the general cases just as well.

Definition 1.5 Fix $\mu > \rho - 1$ and a probability measure $\nu \in M^1(\Pi_q)$. A Bessel random walk $(S_n^\mu)_{n\geq 0}$ on Π_q of index μ and with law ν is a time-homogeneous Markov chain on Π_q with $S_0^\mu = 0$ and transition probability

$$P(S_{n+1}^\mu \in A | S_n^\mu = x) \;=\; (\delta_x *_\mu \nu)(A)$$

for $x \in \Pi_q$ and Borel sets $A \subset \Pi_q$.

This notion is quite common on hypergroups (see [BH]) and was already used in [K] for the one-dimensional case $q = 1$ and $\mathbb{F} = \mathbb{R}$. The notion has its origin in the following well-known fact for the orbit cases $\mu = pd/2$ with $p \in \mathbb{N}$: If we fix a radial measure $\nu_p \in M^1(M_{p,q})$ and consider a sequence of i.i.d. $M_{p,q}$-valued random variables $(X_l^p)_{l\in\mathbb{N}}$ with law ν_p, then $\big(\varphi_p\big(\sum_{l=1}^k X_l^p\big)\big)_{k\geq 0}$ is a random walk on Π_q of index μ with law $\varphi_p(\nu_p)$. Having this in mind, we can state generalizations of Theorems 1.3 and 1.4 for such random walks on Π_q for indices $\mu \to \infty$ and time steps $k \to \infty$. This will be done in Section 4 where we state and prove our results in this generality. The preceding limit results will then appear just as special cases.

The proofs of the limit results in Section 4 are as follows: As the characters of the Bessel hypergroups on Π_q can be expressed in terms of Bessel functions J_μ, the multidimensional Hankel transform on Π_q is just the hypergroup Fourier transform, and we can write down these transforms of the distributions of the S_n^μ. On the other hand, we derive uniform limit results for $J_\mu(\mu x)$ as $\mu \to \infty$ which implies that the Hankel transforms tend to Laplace transforms of these distributions, which leads to the stated limit theorems. We point out that direct proofs of Theorems 1.3 and 1.4 are precisely the same as in the slightly more general setting below.

The organization of this paper is as follows: In Section 2 we recapitulate some known results about Bessel functions and Bessel convolutions on matrix cones from [FK][FT][H], and [R3]. The central part of the paper is Section 3, where we present several uniform asymptotic results for $J_\mu(\mu x)$ as $\mu \to \infty$. Except for partial results proven by one of the authors already in [V1] for $q = 1$, these results seem to be new even in the one-variable case $q = 1$. This is surprising as in the classical monograph [W] a complete Chapter is devoted to $J_\mu(\mu x)$ with $\mu \to \infty$. Finally, the results of Section 3 are used as a basis for the proofs of the laws of large numbers in Section 5 and the large deviation principle in Section 6.

2 Bessel functions and Bessel hypergroups on matrix cones

In this section we collect some known facts about Bessel functions on matrix cones and the associated Bessel hypergroups. The material is mainly taken from [FK] and [R3]. We also refer to [H], [Di], and [FT].

2.1 Bessel functions associated with matrix cones

Let $\mathbb{F}$ be one of the real division algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. Denote the usual conjugation in $\mathbb{F}$ by $t \mapsto \overline{t}$, the real part of $t \in \mathbb{F}$ by $\Re t = \frac{1}{2}(t + \overline{t})$, and by $|t| = (t\overline{t})^{1/2}$ its norm.

For $p, q \in \mathbb{N}$ we denote by $M_{p,q} := M_{p,q}(\mathbb{F})$ the vector space of all $p \times q$-matrices over $\mathbb{F}$ and put $M_q := M_q(\mathbb{F}) := M_{q,q}(\mathbb{F})$ for abbreviation. Let further

$$H_q = H_q(\mathbb{F}) = \{x \in M_q(\mathbb{F}) : x = x^*\}$$

the space of Hermitian $q \times q$-matrices over $\mathbb{F}$. All these spaces are real Euclidean vector spaces with scalar product $\langle x, y\rangle := \Re tr(x^*y)$ and the associated norm $\|x\| = \langle x, x\rangle^{1/2}$. Here $x^* := \overline{x}^t$ and tr denotes the trace. The dimension of H_q is given by $dim_{\mathbb{R}} H_q := q + \frac{d}{2}q(q-1)$. Let further

$$\Pi_q := \{x^2 :\ x \in H_q\} = \{x^*x : x \in H_q\}$$

be the set of all positive semidefinite matrices in H_q, and Ω_q its topological interior which consists of all strictly positive definite matrices. Ω_q is a symmetric cone, i.e. an open convex cone which is self-dual and whose linear automorphism group acts transitively; see [FK] for details.

To define the Bessel functions associated with the symmetric cone Ω_q we first introduce their basic building blocks, the so-called spherical polynomials. These are just the polynomial spherical functions of Ω_q considered as a Riemannian symmetric space. They are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q) \in \mathbb{N}_0^q$ (we write $\lambda \geq 0$ for short) and are given by

$$\Phi_\lambda(x) = \int_{U_q} \Delta_\lambda(uxu^{-1})du, \quad x \in H_q$$

where du is the normalized Haar measure of U_q and Δ_λ is the power function

$$\Delta_\lambda(x) := \Delta_1(x)^{\lambda_1-\lambda_2}\Delta_2(x)^{\lambda_2-\lambda_3} \cdot \ldots \cdot \Delta_q(x)^{\lambda_q} \quad (x \in H_q).$$

The $\Delta_i(x)$ are the principal minors of the determinant $\Delta(x)$, see [FK] for details. There is a renormalization $Z_\lambda = c_\lambda \Phi_\lambda$ with constants $c_\lambda > 0$ depending on the underlying cone such that

$$(tr\, x)^k \ = \sum_{|\lambda|=k} Z_\lambda(x) \qquad \text{for} \quad k \geq 0; \tag{2.1}$$

see Section XI.5. of [FK] where these Z_λ are called zonal polynomials. By construction, the Z_λ are invariant under conjugation by U_q and thus depend only on the eigenvalues of their argument. More precisely, for $x \in H_q$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$, one has

$$Z_\lambda(x) = C_\lambda^\alpha(\xi) \quad \text{with} \quad \alpha = \frac{2}{d} \tag{2.2}$$

where the C^{α}_{λ} are the Jack polynomials of index α in a suitable normalization (c.f. [FK], [Ka], [R3]). The Jack polynomials C^{α}_{λ} are homogeneous of degree $|\lambda|$ and symmetric in their arguments.

The matrix Bessel functions associated with the cone Ω_q are defined as ${}_0F_1$-hypergeometric series in terms of the Z_λ, namely

$$J_\mu(x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(x), \tag{2.3}$$

where for $\lambda = (\lambda_1, \dots \lambda_q) \in \mathbb{N}_0^q$, the generalized Pochhammer symbol $(\mu)_\lambda$ is given by

$$(\mu)_\lambda = (\mu)^{2/d}_\lambda \quad \text{where } (\mu)^\alpha_\lambda := \prod_{j=1}^{q} \big(\mu - \frac{1}{\alpha}(j-1)\big)_{\lambda_j} \quad (\alpha > 0),$$

and $\mu \in \mathbb{C}$ is an index satisfying $(\mu)^\alpha_\lambda \neq 0$ for all $\lambda \geq 0$. If $q = 1$, then $\Pi_q = \mathbb{R}_+$ and the Bessel function $\mathcal{J}_\mu$ is independent of d with

$$J_\mu\big(\frac{x^2}{4}\big) = j_{\mu-1}(x)$$

where $j_\kappa(z) = {}_0F_1(\kappa+1; -z^2/4)$ is the usual modified Bessel function in one variable.

2.2 Bessel hypergroups on matrix cones

Hypergroups are convolution structures which generalize locally compact groups insofar as the convolution product of two point measures is in general not a point measure again, but just a probability measure on the underlying space. More precisely, a hypergroup $(X, *)$ is a locally compact Hausdorff space X together with a convolution $*$ on $M_b(X)$ (the regular bounded Borel measures on X), such that $(M_b(X), *)$ becomes a Banach algebra, where $*$ is weakly continuous, probability preserving and preserves compact supports of measures. Moreover, one requires an identity $e \in X$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for $x \in X$, as well as a continuous involution $x \mapsto \bar{x}$ on X such that for all $x, y \in X$, $e \in supp(\delta_x * \delta_y)$ is equivalent to $x = \bar{y}$, and $\delta_{\bar{x}} * \delta_{\bar{y}} = (\delta_y * \delta_x)^-$. Here for $\mu \in M_b(X)$, the measure μ^- is given by $\mu^-(A) = \mu(A^-)$ for Borel sets $A \subset X$. A hypergroup $(X, *)$ is called commutative if and only if so is the convolution $*$. Thus for a commutative hypergroup $(X, *)$, the measure space $M_b(X)$ becomes a commutative Banach-$*$-algebra with identity δ_e. Notice that due to its weak continuity, the convolution of measures on a hypergroup is uniquely determined by the convolution product of point measures.

On a commutative hypergroup $(X, *)$ there exists a (up to a multiplicative factor) unique Haar measure ω, i.e. ω is a positive Radon measure on X satisfying

$$\int_X \delta_x * \delta_y(f) d\omega(y) = \int_X f(y) d\omega(y) \quad \text{for all } x \in X,\ f \in C_c(X).$$

The decisive object for harmonic analysis on a commutative hypergroup is its dual space, which is defined by

$$\widehat{X} := \{\varphi \in C_b(X) : \varphi \neq 0,\ \varphi(\overline{x}) = \overline{\varphi(x)},\ \delta_x * \delta_y(\varphi) = \varphi(x)\varphi(y) \text{ for all } x, y \in X\}.$$

The elements of $\widehat{X}$ are also called characters. As in the case of LCA groups, the dual of a commutative hypergroup is a locally compact Hausdorff space with the topology of locally uniform convergence and can be identified with the symmetric spectrum of the convolution algebra $L^1(X, \omega)$.

The following theorem contains some of the main results of [R3].

Theorem 2.1 *Let $\mu \in \mathbb{R}$ with $\mu > \rho - 1$. Then*

(a) *The assignment*

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f\big(\sqrt{r^2 + s^2 + svr + rv^*s}\,\big)\, \Delta(I - vv^*)^{\mu-\rho}\, dv, \quad f \in C(\Pi_q) \tag{2.4}$$

with κ_μ as in (1.2), defines a commutative hypergroup structure on Π_q with neutral element $0 \in \Pi_q$ and the identity mapping as involution. The support of $\delta_r *_\mu \delta_s$ satisfies

$$supp(\delta_r *_\mu \delta_s) \subseteq \{t \in \Pi_q : \|t\| \leq \|r\| + \|s\|\}.$$

(b) *A Haar measure of the hypergroup $\Pi_{q,\mu} := (\Pi_q, *_\mu)$ is given by*

$$\omega_\mu(f) = \frac{\pi^{q\mu}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta(r)^\gamma dr$$

with $\gamma = \mu - \frac{d}{2}(q-1) - 1$.

(c) *The dual space of $\Pi_{q,\mu}$ is given by*

$$\widehat{\Pi_{q,\mu}} = \{\varphi_s : s \in \Pi_q\} \qquad \text{with} \qquad \varphi_s(r) := \mathcal{J}_\mu(\frac{1}{4} r s^2 r) = \varphi_r(s).$$

The hypergroup $\Pi_{q,\mu}$ is self-dual via the homeomorphism $s \mapsto \varphi_s$. Under this identification of $\widehat{\Pi_{q,\mu}}$ with $\Pi_{q,\mu}$, the Plancherel measure on $\Pi_{q,\mu}$ is $(2\pi)^{-2\mu q}\omega_\mu$.

3 Estimates for Bessel functions of large indices

We first recapitulate the following well known one-dimensional inequalities for the exponential function (see, for instance, Sections 3.6.2 and 3.6.3 of [Mi]):

$$(1 - z/r)^r \leq e^{-z} \quad \text{for} \quad r > 0, z \in \mathbb{R}, \tag{3.1}$$

$$0 \leq e^{-z} - (1 - z/r)^r \leq z^2 e^{-z}/r \quad \text{for} \quad r, z \in \mathbb{R},\ r \geq 1,\ |z| \leq r; \tag{3.2}$$

$$(1 + z/r)^r \leq e^z \leq (1 + z/r)^{r+z/2} \quad \text{for} \quad r, z > 0. \tag{3.3}$$

These results have the following matrix-valued extension:

Lemma 3.1 (1) *For all $\mu > 1$ and $v \in \sqrt{\mu} \cdot D_q \subset M_q$,*

$$0 \leq e^{-\langle v,v \rangle} - \Delta(I - \frac{1}{\mu} v^* v)^\mu \ \leq\ \frac{1}{\mu} tr((v^*v)^2) \cdot e^{-\langle v,v \rangle}.$$

(2) *For all $\mu > 0$ and $v \in M_q$,*

$$\Delta(I + \frac{1}{\mu} v^* v)^\mu \leq e^{\langle v,v \rangle} \leq \Delta(I + \frac{1}{\mu} v^* v)^{\mu + m/2}$$

where $m \geq 0$ is the maximal eigenvalue of v^*v.

Proof. (1) The positive semidefinite matrix v^*v may be written as

$$v^*v = u \cdot diag(a_1, a_2, \ldots, , a_q) \cdot u^*$$

with some $u \in U_q$ and the eigenvalues $a_1, \ldots, a_q \in [0, m]$ of v^*v. Then

$$\Delta(I - \frac{1}{\mu} v^* v)^\mu = \prod_{k=1}^q (1 - a_k/\mu)^\mu$$

and $\langle v, v \rangle = tr(v^*v) = a_1 + \ldots + a_q$. Using (3.2) and a telescope sum argument, we obtain

$$\begin{aligned} 0 &\leq e^{-\langle v,v \rangle} - \Delta(I - \frac{1}{\mu} v^* v)^\mu \\ &= \sum_{l=1}^q \Big[\prod_{k=1}^{l-1} (1 - a_k/\mu)^\mu \cdot \big(e^{-a_l} - (1 - a_l/\mu)^\mu\big) \cdot \prod_{k=l+1}^q e^{-a_k}\Big] \\ &\leq \frac{1}{\mu} \sum_{l=1}^q \Big[\prod_{k=1}^q e^{-a_k}\Big] a_l^2 \ =\ \frac{1}{\mu} tr((v^*v)^2) \cdot e^{-\langle v,v \rangle} \end{aligned}$$

as claimed.

(2) is proven in the same way by use of (3.3).

□

Our first estimate for the Bessel functions J_μ as $\mu \to \infty$ will be based on the following integral representation of J_μ for $\mu > \rho - 1$ (see Eq. (3.12) of [R3]):

$$J_\mu(x^*x) = \frac{1}{\kappa_\mu} \int_{D_q} e^{-2i\langle v,x\rangle} \Delta(I - v^*v)^{\mu-\rho} dv. \tag{3.4}$$

Proposition 3.2 *There exists a constant $C = C(q,d) > 0$ such that for $\mu > \rho - 1$ and all $x \in M_q$,*

$$\left| J_\mu(\mu x^*x) - e^{-\langle x,x\rangle} \right| \le C/\mu$$

and

$$|\mu^{dq^2/2}\kappa_\mu - \pi^{dq^2/2}| \le C/\mu.$$

P r o o f. In a first step we obtain for $x \in M_q$,

$$\begin{aligned} D &:= \Big| \int_{\sqrt{\mu}\cdot D_q} e^{-i\langle v,x\rangle} \Delta(I - \frac{1}{\mu} v^*v)^{\mu-\rho}\, dv - \int_{M_q} e^{-i\langle v,x\rangle} e^{-\langle v,v\rangle}\, dv \Big| \\ &\le \int_{M_q \setminus (\sqrt{\mu}\cdot D_q)} e^{-\langle v,v\rangle}\, dv \,+\, \int_{\sqrt{\mu}\cdot D_q} \Big(e^{-\langle v,v\rangle(1-\rho/\mu)} - \Delta\big(I - \frac{1-\rho/\mu}{\mu-\rho} v^*v\big)^{\mu-\rho} \Big)\, dv \\ &+ \int_{\sqrt{\mu}\cdot D_q} e^{-\langle v,v\rangle} \big(e^{\langle v,v\rangle \rho/\mu} - 1 \big)\, dv. \end{aligned}$$

By Lemma 3.1(1) for $\mu - \rho$ instead of μ and with the elementary estimate

$$e^z - 1 \le (e^\rho - 1)z \quad \text{for} \quad z \in [0,\rho]$$

we further obtain that for $\mu \ge 2\rho$,

$$\begin{aligned} D &\le \frac{C_1}{\mu} + \frac{1}{\mu-\rho} \int_{\sqrt{\mu}\cdot D_q} e^{-\langle v,v\rangle/2} tr((v^*v)^2)\, dv \,+\, \frac{(e^\rho - 1)\rho}{\mu} \int_{\sqrt{\mu}\cdot D_q} e^{-\langle v,v\rangle} \langle v,v\rangle\, dv \\ &\le \frac{C_2}{\mu} \end{aligned} \tag{3.5}$$

with suitable constants $C_1, C_2 > 0$. We next observe that $M_q \simeq \mathbb{R}^{dq^2}$ implies

$$\int_{M_q} e^{-i\langle v,x\rangle} e^{-\langle v,v\rangle}\, dv \;=\; \pi^{dq^2/2} \cdot e^{-\langle x,x\rangle/4}. \tag{3.6}$$

Moreover, replacing x by $(\sqrt{\mu}/2)\cdot x$ and v by $(1/\sqrt{\mu})\cdot v$ in integral representation (3.4), we obtain

$$J_\mu(\frac{\mu}{4} x^*x) \;=\; \frac{1}{\mu^{dq^2/2}\kappa_\mu} \int_{\sqrt{\mu}\cdot D_q} e^{-i\langle v,x\rangle} \Delta(I - \frac{1}{\mu} v^*v)^{\mu-\rho} dv. \tag{3.7}$$

We now conclude from (3.5), (3.6) and (3.7) that for $\mu \ge 2\rho$ and $x \in M_q$,

$$\left| \mu^{dq^2/2}\kappa_\mu \cdot J_\mu(\frac{\mu}{4} x^*x) - \pi^{dq^2/2} \cdot e^{-\langle x,x\rangle/4} \right| \le C_2/\mu.$$

For $x = 0$ we in particular observe that

$$|\mu^{dq^2/2}\kappa_\mu - \pi^{dq^2/2}| \le C_2/\mu. \tag{3.8}$$

As $|J_\mu(\frac{\mu}{4}x^*x)| \leq 1$ by (3.4), it follows for $\mu \geq 2\rho$ and $x \in M_q$ that

$$\begin{aligned}
&\big|J_\mu(\frac{\mu}{4}x^*x) - e^{-\langle x,x\rangle/4}\big| \\
&\leq |J_\mu(\frac{\mu}{4}x^*x)| \cdot \big|1 - \frac{\mu^{dq^2/2}\cdot\kappa_\mu}{\pi^{dq^2/2}}\big| + \frac{1}{\pi^{dq^2/2}}\big|\mu^{dq^2/2}\cdot\kappa_\mu J_\mu(\frac{\mu}{4}x^*x) - \pi^{dq^2/2}e^{-\langle x,x\rangle/4}\big| \\
&\leq C_3/\mu
\end{aligned}$$

with some constant $C_3 > 0$. Together with (3.8), this implies the proposition in case $\mu \geq 2\rho$. Within the range $\rho - 1 < \mu \leq 2\rho$, the proposition is immediate in view of the estimate $|J_\mu(\mu x^*x)| \leq 1$ for all $x \in M_q$. □

We next derive a variant of Proposition 3.2 which is based on the power series (2.3) and provides a good estimate for small arguments. We start with some basic inequalities for the zonal polynomials Z_λ:

Lemma 3.3 *For all partitions $\lambda \geq 0$ and and $y \in \Pi_q$, $|Z_\lambda(-y)| \leq Z_\lambda(y)$.*

Proof. We use the relation between the Z_λ and the Jack polynomials C_λ^α in Section 2 and the well-known fact that the C_λ^α are nonnegative linear combinations of monomials, see [KS]. This yields for the eigenvalues $\xi = (\xi_1, \ldots, \xi_q)$ of $-y$ and $|\xi| := (|\xi_1|, \ldots, |\xi_q|)$ of y that

$$|Z_\lambda(-y)| = |C_\lambda^\alpha(\xi)| \leq C_\lambda^\alpha(|\xi|) = Z_\lambda(y).$$

□

Lemma 3.4 *For all partitions $\lambda \geq 0$, $\mu > \rho - 1$ and $(\mu)_\lambda = (\mu)_\lambda^{2/d}$,*

$$\Big|1 - \frac{\mu^{|\lambda|}}{(\mu)_\lambda}\Big| \leq dq \cdot 2^{dq(q-1)/2} \cdot \frac{|\lambda|^2}{\mu}.$$

Proof. Consider $(\mu)_\lambda = \prod_{j=1}^q (\mu - \frac{d}{2}(j-1))_{\lambda_j}$. In this product, each factor can be estimated below by $\mu - \frac{d}{2}(q-1)$. Moreover, precisely

$$\big(0 + 1 + \ldots + (q-1)\big)\lceil d/2\rceil \;=\; \frac{(q-1)q}{2} \cdot \lceil d/2\rceil \;=:\; r$$

of these factors are smaller than μ. As $\mu > \rho - 1 = d(q - 1/2)$, this implies

$$(\mu)_\lambda \;\geq\; (\mu - \frac{d}{2}(q-1))^r \cdot \mu^{|\lambda|-r} \;\geq\; \big(\mu/2\big)^r \mu^{|\lambda|-r} \;\geq\; 2^{-dq(q-1)/2} \cdot \mu^{|\lambda|},$$

and thus

$$\mu^{|\lambda|}/(\mu)_\lambda \;\leq\; 2^{dq(q-1)/2}. \tag{3.9}$$

We now prove by induction on the length $k := |\lambda|$ that for $\mu > \rho - 1 = d(q - 1/2)$,

$$\Big|1 - \frac{\mu^{|\lambda|}}{(\mu)_\lambda}\Big| \leq \frac{dq}{2(\mu - d(q-1)/2)} \cdot 2^{dq(q-1)/2}|\lambda|^2 \tag{3.10}$$

which immediately implies the lemma. In fact, for $k = 0, 1$, the left hand side of (3.10) is equal to zero, while the right-hand side is nonnegative.

For the induction step, consider a partition λ of length $k \geq 2$. Then there is a partition $\tilde\lambda$ with $|\tilde\lambda| = k - 1$ for which there exists precisely one $j = 1, \ldots, q$ with $\lambda_j = \tilde\lambda_j + 1$ while all the other components are equal. Hence, if we assume the inequality to hold for $\tilde{\lambda}$ and use (3.9), we obtain

$$\begin{aligned}
\Big|1-\frac{\mu^k}{(\mu)_\lambda}\Big| &= |1-\frac{\mu^{k-1}}{(\mu)_{\tilde\lambda}}+\frac{\mu^{k-1}}{(\mu)_{\tilde\lambda}}-\frac{\mu^k}{(\mu)_\lambda}| \\
&\leq \frac{dq}{\mu-d(q-1)/2}\cdot 2^{dq(q-1)/2-1}\cdot(k-1)^2+\frac{\mu^{k-1}}{(\mu)_{\tilde\lambda}}\cdot\Big|1-\frac{\mu}{(\mu-d(j-1)/2)+\lambda_j-1}\Big| \\
&\leq \frac{dq}{\mu-d(q-1)/2}\cdot 2^{dq(q-1)/2-1}\cdot(k-1)^2+2^{dq(q-1)/2}\cdot\Big|\frac{-d(j-1)/2+\lambda_j-1}{(\mu-d(j-1)/2)+\lambda_j-1}\Big| \\
&\leq \frac{2^{dq(q-1)/2-1}}{\mu-d(q-1)/2}\cdot\Big(dq(k-1)^2+dq+2k-2\Big) \\
&\leq \frac{2^{dq(q-1)/2-1}}{\mu-d(q-1)/2}\cdot dqk^2
\end{aligned}$$

for $k \geq 2$. This completes the proof. □

Proposition 3.5 *There exists a constant $C = C(q,d) > 0$ such that for $\mu > 2\rho$ and $y \in \Pi_q$,*

$$\big|J_\mu(\mu y)-e^{-tr\,y}\big| \leq C\frac{(tr\,y)^2}{\mu}.$$

P r o o f. Using the power series (2.3) as well as (2.1) in terms of the homogeneous polynomials Z_λ, we obtain

$$J_\mu(\mu y)-e^{-tr\,y}=\sum_{\lambda\geq 0}\frac{1}{|\lambda|!}\Big(\frac{\mu^{|\lambda|}}{(\mu)_\lambda}-1\Big)\cdot Z_\lambda(-y).$$

As

$$(\mu)_{(1,0,\ldots,0)}=\mu,\quad (\mu)_{(2,0,\ldots,0)}=\mu(\mu+1),\quad (\mu)_{(1,1,0,\ldots,0)}=\mu(\mu-d/2),$$

we may write this expansion as

$$J_\mu(\mu y)-e^{-tr\,y}=R_2+R_3$$

with

$$R_2=\frac{1}{2}\Big(\Big(\frac{\mu^2}{\mu(\mu+1)}-1\Big)Z_{(2,0,\ldots,0)}(-y)+\Big(\frac{\mu^2}{\mu(\mu-d/2)}-1\Big)Z_{(1,1,0,\ldots,0)}(-y)\Big)$$

and

$$R_3=\sum_{k\geq 3}\frac{1}{k!}\sum_{|\lambda|=k}\Big(\frac{\mu^k}{(\mu)_\lambda}-1\Big)\cdot Z_\lambda(-y).$$

Recall from Lemma 3.3 that $|Z_\lambda(-y)| \leq Z_\lambda(y)$ and $Z_\lambda(y) \geq 0$. Hence Eq. (2.1) implies for $|\lambda| = 2$ that $|Z_\lambda(-y)| \leq (tr\,y)^2$. Therefore, $|R_2| \leq M_1\frac{(tr\,y)^2}{\mu}$ with a suitable constant $M_1 > 0$. Moreover, Lemmata 3.3 and 3.4 imply that

$$\begin{aligned}
|R_3| &\leq \sum_{k\geq 3}\frac{1}{k!}\sum_{|\lambda|=k}M_2\frac{k^2}{\mu}Z_\lambda(y) = \frac{M_2}{\mu}\sum_{k\geq 3}\frac{k^2}{k!}(tr\,y)^k \\
&\leq \frac{2M_2}{\mu}(tr\,y)^2\sum_{k\geq 1}\frac{1}{k!}(tr\,y)^k \leq \frac{2M_2}{\mu}(tr\,y)^2e^{tr\,y}
\end{aligned} \tag{3.11}$$

with a constant $M_2 > 0$. In summary we have

$$|J_\mu(\mu y)-e^{-tr\,y}| \leq M_3\cdot\frac{(tr\,y)^2}{\mu}\cdot(1+e^{tr\,y}).$$

Together with the estimate of Proposition 3.2 for large y, this yields the stated result. □

Summarizing Propositions 3.2 and 3.5, we obtain:

Theorem 3.6 *There exists a constant $C = C(q, d) > 0$ such that for $\mu > 2\rho$ and $y \in \Pi_q$,*

$$\left|J_\mu(\mu y) - e^{-tr\, y}\right| \le \frac{C}{\mu} \cdot min(1, (tr\, y)^2).$$

We next turn to an estimate for $J_\mu(-\mu y)$ with $y \in \Pi_q$. In order to simplify formulas, we replace the factor μ in the argument by $\mu - \rho$.

Proposition 3.7 *There exists a constant $C = C(q, d) > 0$ such that for $\mu > 2\rho$ and all $x \in M_q$,*

$$e^{\langle x,x\rangle}\Big(1 - \frac{C}{\mu}\|x\|^4 - H(x, \sqrt{\mu-\rho})\Big) \le J_\mu(-(\mu-\rho)x^*x) \le e^{\langle x,x\rangle}(1 + C/\mu)$$

where

$$H(x, r) := \int_{M_q \setminus rD_q} e^{-\|v-x\|^2}\, dv \qquad \text{for} \quad r > 0.$$

P r o o f. We first conclude (by analytic continuation) from integral representation (3.4) that

$$J_\mu(-(\mu-\rho)x^*x) = \frac{1}{(\mu-\rho)^{dq^2/2}\kappa_\mu} \int_{\sqrt{\mu-\rho}\cdot D_q} e^{2\langle v,x\rangle} \Delta(I - \frac{1}{\mu-\rho} v^*v)^{\mu-\rho} dv. \tag{3.12}$$

Moreover, Proposition 3.2 implies that

$$\left|\frac{1}{(\mu-\rho)^{dq^2/2}\kappa_\mu} - \pi^{-dq^2/2}\right| = O(1/\mu). \tag{3.13}$$

We next estimate the integral in (3.12). For this we use Lemma 3.1(1) and observe that

$$\begin{aligned}\int_{\sqrt{\mu-\rho}\cdot D_q} e^{2\langle v,x\rangle} \Delta(I - \frac{1}{\mu-\rho} v^*v)^{\mu-\rho}\, dv &\le \int_{M_q} e^{2\langle v,x\rangle} e^{-\langle v,v\rangle}\, dv \\ &= \pi^{dq^2/2} e^{\langle x,x\rangle}.\end{aligned}$$

Together with Eq. (3.12) and (3.13) this yields

$$J_\mu(-(\mu-\rho)x^*x) \le (\pi^{-dq^2/2} + O(1/\mu)) \cdot \pi^{dq^2/2} e^{\langle x,x\rangle}.$$

This proves the upper estimate as claimed.

For the lower estimate, we use Lemma 3.1(1) again. We obtain

$$\begin{aligned}\int_{\sqrt{\mu-\rho}\cdot D_q} e^{2\langle v,x\rangle} \Delta(I - \frac{1}{\mu-\rho} v^*v)^{\mu-\rho}\, dv &\ge \int_{\sqrt{\mu-\rho}\cdot D_q} e^{2\langle v,x\rangle - \langle v,v\rangle} \cdot \big(1 - \frac{1}{\mu-\rho} tr((v^*v)^2)\big)\, dv \\ &= \int_{M_q} e^{2\langle v,x\rangle - \langle v,v\rangle}\, dv - I_1 - \frac{1}{\mu-\rho} I_2 \\ &= \pi^{dq^2/2} e^{\langle x,x\rangle} - I_1 - \frac{1}{\mu-\rho} I_2\end{aligned}$$

with

$$I_1 := \int_{M_q \setminus \sqrt{\mu-\rho}\cdot D_q} e^{2\langle v,x\rangle - \langle v,v\rangle}\, dv$$

and

$$I_2 := \int_{\sqrt{\mu-\rho}\cdot D_q} e^{2\langle v,x\rangle - \langle v,v\rangle} \cdot tr((v^*v)^2)\, dv.$$

We have

$$I_1 = e^{\langle x,x\rangle} \cdot \int_{M_q \setminus \sqrt{\mu-\rho}\cdot D_q} e^{-\|v-x\|^2}\, dv = e^{\langle x,x\rangle} \cdot H(x, \sqrt{\mu-\rho})$$

and

$$I_2 = e^{\langle x,x\rangle} \cdot \int_{\sqrt{\mu-\rho}\cdot D_q} e^{-\|v-x\|^2} \cdot tr((v^*v)^2)\, dv = e^{\langle x,x\rangle} \cdot O(\|x\|^4),$$

which finally leads to the lower estimate. □

Remark 3.8 There is a close connection between the Bessel functions and convolutions on Π_q above and the theory of Dunkl operators associated with the root system B_q which is explained in [R3]. This connection and the asymptotic results above lead easily to asymptotic results for Dunkl kernels of type B_q. Details will be explained elsewhere.

4 Laws of large numbers

Let $\nu \in M^1(\Pi_q)$ be a probability measure and $\mu > \rho - 1 = d(q-1/2)$ a fixed index. We say that a time-homogeneous Markov chain $(S_k^\mu)_{k\geq 0}$ on Π_q is a Bessel-type random walk on Π_q of index μ with law ν if $S_0^\mu = 0$ and if its transition probability is given by

$$P(S_{k+1}^\mu \in A | S_k^\mu = x) \;=\; (\delta_x *_\mu \nu)(A)$$

for all $k \in \mathbb{N}_0$, $x \in \Pi_q$ and Borel sets $A \subseteq \Pi_q$. It is easily checked by induction on k that the distribution of S_k^μ is just the k-fold convolution power $\nu^{(k,\mu)} = \nu *_\mu \nu *_\mu \ldots *_\mu \nu$ of ν with respect to the Bessel convolution of index μ. As announced in the introduction, we are interested in limit theorems for the random variables S_k^μ as $k, \mu \to \infty$. Our first result in this direction is the following weak law of large numbers:

Theorem 4.1 *Let $\nu \in M^1(\Pi_q)$ be a probability measure with finite second moment*

$$\sigma^2(\nu) := \int_{\Pi_q} s^2 \, d\nu(s) \in \Pi_q,$$

and let $(\mu_k)_{k\in\mathbb{N}} \subset]\rho-1,\infty[$ be an arbitrary sequence of indices with $\lim_{k\to\infty}\mu_k = \infty$. Let $S_k^{\mu_k}$ be the k-th member of the Bessel-type random walk of index μ_k with law ν. Then

$$\frac{1}{\sqrt{k}} S_k^{\mu_k} \to \sqrt{\sigma^2(\nu)}$$

in probability as $k \to \infty$.

This first main result has the following consequence which was stated as Theorem 1.3 in the introduction:

Corollary 4.2 *Let $\nu \in M^1(\Pi_q)$ be a probability measure with finite second moment $\sigma^2(\nu) \in \Pi_q$. For each dimension $p \in \mathbb{N}$ consider the unique U_p-invariant probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$ where $\varphi_p : M_{p,q} \to \Pi_q$, $x \mapsto (x^*x)^{1/2}$, is the canonical projection. Let further $(X_l^p)_{l\in\mathbb{N}}$ be a sequence of i.i.d. $M_{p,q}$-valued random variables with law ν_p. Then for each sequence $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ of dimensions with $\lim_{k\to\infty} p_k = \infty$, the Π_q-valued random variables*

$$\frac{1}{\sqrt{k}} \varphi_{p_k}\Big(\sum_{l=1}^k X_l^{p_k}\Big)$$

tend in probability to the constant $\sqrt{\sigma^2(\nu)}$.

Proof. This is clear from Theorem 4.1 because $\big(\varphi_p\big(\sum_{l=1}^k X_l^p\big)\big)_{k\geq 0}$ is a Bessel-type random walk on Π_q with index $\mu = pd/2$. □

The proof of Theorem 4.1 relies on estimates for matrix Bessel functions from the preceding section and on standard properties of the Laplace transform on matrix cones. These properties are likely to be known but we include them for the reader's convenience.

Recall that the Laplace transform $L\nu \in C_b(\Pi_q)$ of a measure $\nu \in M^1(\Pi_q)$ is defined by

$$L\nu(x) = \int_{\Pi_q} e^{-\langle x,y\rangle} d\nu(y), \quad x \in \Pi_q.$$

The Laplace transform on the cone Π_q satisfies the following Levy-type continuity theorem.

Proposition 4.3 *For probability measures ν, $(\nu_k)_{k\geq 1} \in M^1(\Pi_q)$ the following statements are equivalent:*

(1) $\nu_k \to \nu$ *weakly.*

(2) $L\nu_k(x) \to L\nu(x)$ *for all* $x \in \Pi_q$.

(3) $L\nu_k(x) \to L\nu(x)$ *for all* $x \in \Omega_q$.

P r o o f. $(1) \Longrightarrow (2) \Longrightarrow (3)$ is obvious. For $(3) \Longrightarrow (1)$ observe that for $x \in \Omega_q$, the exponential function $e_x(y) := e^{-\langle x,y\rangle}$ is contained in $C_0(\Pi_q)$, i.e. it vanishes at infinity. Moreover, the linear span of $\{e_x,\, x \in \Omega_q\}$ is a $\|.\|_\infty$-dense subspace of $C_0(\Pi_q)$ by the Stone-Weierstrass theorem. It follows from (3) and a 3ε-argument that $\int f\, d\nu_k \to \int f\, d\nu$ for all $f \in C_0(\Pi_q)$ which implies (1). □

The following result can be readily derived from the dominated convergence theorem as in the classical setting:

Lemma 4.4 *Let $\nu \in M^1(\Pi_q)$ be a probability measure which admits r-th moments for $r \in \mathbb{N}$, i.e., $\int_{\Pi_q} \|y\|^r d\nu(r) < \infty$. Then $L\nu$ is r-times continuously differentiable on Π_q.*

Using the Taylor formula at $0 \in \Pi_q$, we in particular obtain:

Corollary 4.5 *Let $\nu \in M^1(\Pi_q)$ with finite second moment $\sigma^2(\nu) \in \Pi_q$. Then*

$$\int_{\Pi_q} e^{-\langle sx,sx\rangle}\, d\nu(s) = 1 - tr(x\sigma^2(\nu)x) + o(\|x\|^2) \quad \text{as } x \to 0 \text{ in } \Pi_q. \tag{4.1}$$

Moreover, if $\nu \in M^1(\Pi_q)$ admits fourth moments, then even $O(\|x\|^4)$ is true instead of $o(\|x\|^2)$ in relation (4.1).

The following result is a variant of the preceding corollary:

Lemma 4.6 *Let $\nu \in M^1(\Pi_q)$ with finite second moment $\sigma^2(\nu) \in \Pi_q$, and let $(\mu_k)_{k\geq 1} \subset]0,\infty[$ be as in Theorem 4.1. Then for each $x \in \Pi_q$,*

$$\int_{\Pi_q} J_{\mu_k}\big(\frac{\mu_k}{k} xs^2x\big)\, d\nu(s) = 1 - \frac{1}{k} tr\big(x\sigma^2(\nu)x\big) \; + \; o(\|x\|^2/k) \quad \text{as } k \to \infty.$$

Moreover, if $\nu \in M^1(\Pi_q)$ admits fourth moments, then the error term $o(1/k)$ can be replaced by

$$O\big(\|x\|^4/k^2 + \|x\|^2/(k\mu_k)\big).$$

P r o o f. We first conclude from Theorem 3.6 that for $y \in \Pi_q$,

$$\big|J_{\mu_k}(\mu_k y) - e^{-tr\, y}\big| \leq \frac{c}{\mu_k}\, tr\, y.$$

Therefore

$$\begin{aligned}\int_{\Pi_q} \Big|J_{\mu_k}\Big(\frac{\mu_k}{k} xs^2x\Big) - e^{-\frac{1}{k}tr(xs^2x)}\Big|\, d\nu(s) &\leq \frac{c}{k\mu_k} \int_{\Pi_q} tr(xs^2x)\, d\nu(s) \\ &\leq \frac{c\|x\|^2}{k\mu_k} \int_{\Pi_q} \|s\|^2\, d\nu(s) \\ &\leq \frac{\tilde{c}\|x\|^2}{k\mu_k}\end{aligned}$$

with suitable constants $c, \tilde{c} > 0$. On the other hand, we conclude from Corollary 4.5 that

$$\int_{\Pi_q} e^{-\frac{1}{k}tr(xs^2x)}\, d\nu(s) = 1 - \frac{1}{k} tr(x\sigma^2(\nu)x) + o(\frac{\|x\|^2}{k})$$

which yields the first claim. The second statement follows readily from the second statement in Corollary 4.5. □

Proof of Theorem 4.1. Let $\nu^{(k,\mu_k)}$ be the k-fold Bessel convolution power of ν with index μ_k. Then $\nu^{(k,\mu_k)}$ is the distribution of the random variable $S_k^{\mu_k}$. Being hypergroup characters, the matrix Bessel functions $s \mapsto J_\mu(xs^2x)$ are multiplicative w.r.t. the Bessel convolution of index μ. Together with the preceding lemma this implies that

$$\begin{aligned}\lim_{k\to\infty} \int_{\Pi_q} J_{\mu_k}\Big(\frac{\mu_k}{k}xs^2x\Big)\, d\nu^{(k,\mu_k)}(s)&\\ &= \lim_{k\to\infty}\Big(\int_{\Pi_q} J_{\mu_k}\Big(\frac{\mu_k}{k}xs^2x\Big)\, d\nu(s)\Big)^k\\ &= \lim_{k\to\infty}\Big(1-\frac{1}{k}tr(x\sigma^2(\nu)x)+o(1/k)\Big)^k\\ &= e^{-tr(x\sigma^2(\nu)x)} =: A(x). \end{aligned} \tag{4.2}$$

We thus conclude from Proposition 3.2 that

$$\lim_{k\to\infty}\int_{\Pi_q} e^{-\frac{1}{k}tr(xs^2x)}\, d\nu^{(k,\mu_k)}(s) = \lim_{k\to\infty}\int_{\Pi_q} J_{\mu_k}\Big(\frac{\mu_k}{k}xs^2x)\, d\nu^{(k,\mu_k)}(s) = A(x)$$

for $x \in \Pi_q$. From this we conclude (after a quadratic transformation of the argument) that the Laplace transforms of the distributions of $(S_k^{\mu_k})^2/k$ tend to the Laplace transform of the point measure $\delta_{\sigma^2(\nu)}$ on Π_q as $k \to \infty$. The theorem now follows from Proposition 4.3. □

We next turn to a strong law of large numbers which generalizes Theorems 1.2 and 1.4.

Theorem 4.7 *Let $\nu \in M^1(\Pi_q)$ be a probability measure with compact support. Let $(\mu_k)_{k\in\mathbb{N}} \subset]\rho-1,\infty[$ be an arbitrary sequence of indices and $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ a sequence of time steps with the following properties:*

(1) $\lim_{k\to\infty} \mu_k/k^a = \infty$ *for all* $a \in \mathbb{N}$;
(2) $\lim_{k\to\infty} \mu_k/(n_k^2(\ln k)^2) = \infty$;
(3) $\lim_{k\to\infty} n_k/(\ln k)^2 = \infty$.

Let $S_{n_k}^{\mu_k}$ be the n_k-th member of the Bessel-type random walk of index μ_k with law ν. Then,

$$\frac{1}{\sqrt{n_k}} S_{n_k}^{\mu_k} \to \sqrt{\sigma^2(\nu)}$$

for $k \to \infty$ almost surely.

As for the WLLN in the beginning of this section, this theorem immediately implies Theorems 1.2 and 1.4.

Recall that the dimension of H_q as a real vector space is given by $n = q + \frac{d}{2}q(q-1)$. The proof of Theorem 4.7 relies on the following elementary observation:

Lemma 4.8 *There exist matrices $b_1, \ldots, b_n \in \Pi_q$ such that for all $a \in \Pi_q$ and sequences $(a_k)_{k\in\mathbb{N}} \subset \Pi_q$ we have $a_k \to a$ if and only if $\langle b_j, a_k\rangle \to \langle b_j, a\rangle$ for all $j = 1, \ldots, n$.*

Proof. If $b_1, \ldots, b_n$ is any $\mathbb{R}$-basis of the vector space H_q of Hermitian matrices with dimension $n = q + q(q-1)d/2$, then obviously $a_k \to a$ if and only if $\langle b_j, a_k\rangle \to \langle b_j, a\rangle$ for all $j = 1, \ldots, n$. On the other hand, we can find a basis consisting of elements from Π_q. For instance, we may take the q diagonal matrices of the form $diag(0, \ldots, 0, 1, 0, \ldots, 0)$ together with the matrices of the form

$$I + \frac{1}{2}\big(le_{i,j} + l^* e_{j,i}\big) \in \Pi_q \tag{4.3}$$

for $1 \le i < j \le q$ and the $l \in \mathbb{F}$ with $|l| = 1$ forming an $\mathbb{R}$-basis of $\mathbb{F}$ where the $e_{i,j}$ are the elementary matrices with 1 in the (i,j)-coordinate and 0 otherwise. Notice that these matrices are positive definite by the Gershgorin criterion. □

Proof of Theorem 4.7. Let μ_k and n_k be given as in the theorem. By Lemma 4.8, it suffices to prove that for each $c \in \Pi_q$,

$$\frac{1}{n_k}\langle c^2, (S_{n_k}^{\mu_k})^2\rangle \to \langle c^2, \sigma^2(\nu)\rangle \qquad \text{almost surely.} \tag{4.4}$$

For this we shall prove for each $\varepsilon > 0$ that

$$P\Big(\frac{1}{n_k}\langle c^2, (S_{n_k}^{\mu_k})^2\rangle \geq \langle c^2, \sigma^2(\nu)\rangle + \varepsilon\Big) = O(1/k^2) \tag{4.5}$$

and

$$P\Big(\frac{1}{n_k}\langle c^2, (S_{n_k}^{\mu_k})^2\rangle \leq \langle c^2, \sigma^2(\nu)\rangle - \varepsilon\Big) = O(1/k^2). \tag{4.6}$$

Relation (4.4) then follows immediately from the Borel-Cantelli lemma.

We first turn to the proof of relation (4.6). Here we proceed as in the beginning of the proof of Lemma 4.6 and conclude from Theorem 3.6 that

$$\begin{aligned} E\Big(e^{-\frac{2\ln k}{\varepsilon\cdot n_k}\cdot tr(c^2(S_{n_k}^{\mu_k})^2)}\Big) &= \int_{\Pi_q} e^{-\frac{2\ln k}{\varepsilon\cdot n_k}\cdot tr(cs^2c)}\, d\nu^{(n_k,\mu_k)}(s) \\ &= \int_{\Pi_q} J_{\mu_k}\Big(\frac{2\mu_k \ln k}{\varepsilon\cdot n_k}\cdot cs^2c\Big)\, d\nu^{(n_k,\mu_k)}(s) + O\Big(\frac{1}{\mu_k}\Big). \\ &= \Big(\int_{\Pi_q} J_{\mu_k}\Big(\frac{2\mu_k \ln k}{\varepsilon\cdot n_k}\cdot cs^2c\Big)\, d\nu(s)\Big)^{n_k} + O\Big(\frac{1}{\mu_k}\Big). \end{aligned} \tag{4.7}$$

Moreover, using the stronger statement of Lemma 4.6, Eq. (3.1), and the assumptions (1) and (3) of the theorem, we obtain

$$\begin{aligned} &\Big(\int_{\Pi_q} J_{\mu_k}\big(\frac{2\mu_k \ln k}{\varepsilon\cdot n_k}\cdot cs^2c\big)\, d\nu(s)\Big)^{n_k} \\ &\quad = \Big(1 - \frac{2\ln k}{\varepsilon\cdot n_k}\cdot tr(c\sigma^2(\nu)c) + O\Big(\frac{(\ln k)^2}{n_k^2} + \frac{\ln k}{n_k\mu_k}\Big)\Big)^{n_k} \\ &\quad \leq e^{-\frac{2\ln k}{\varepsilon}\cdot tr(c\sigma^2(\nu)c)}\cdot O(1) \end{aligned} \tag{4.8}$$

The Markov inequality and estimates (4.7), (4.8) now lead to

$$\begin{aligned} &P\Big(\frac{1}{n_k}\langle c^2, (S_{n_k}^{\mu_k})^2\rangle \leq \langle c^2, \sigma^2(\nu)\rangle - \varepsilon\Big) \\ &\quad = P\Big(e^{-\frac{2\ln k}{\varepsilon\cdot n_k}\cdot tr(c^2(S_{n_k}^{\mu_k})^2)} \geq e^{-\frac{2\ln k}{\varepsilon}(tr(c\sigma^2(\nu)c)-\varepsilon)}\Big) \\ &\quad \leq \frac{1}{e^{-\frac{2\ln k}{\varepsilon}(tr(c\sigma^2(\nu)c)-\varepsilon)}}\cdot E\big(e^{-\frac{2\ln k}{\varepsilon\cdot n_k}\cdot tr(c^2(S_{n_k}^{\mu_k})^2)}\big) \\ &\quad \leq e^{\frac{2\ln k}{\varepsilon}tr(c\sigma^2(\nu)c)}e^{-2\ln k}\Big(e^{-\frac{2\ln k}{\varepsilon}tr(c\sigma^2(\nu)c)}\cdot O(1) + O\Big(\frac{1}{\mu_k}\Big)\Big) \\ &\quad \leq O\Big(\frac{1}{k^2}\Big) + O\Big(\frac{k^a}{\mu_k}\Big) \end{aligned}$$

with a suitable constant $a = a(\varepsilon, c) > 0$. Condition (1) of the theorem now completes the proof of (4.6).

We now turn to the proof of relation (4.5). Assume that $supp\,\nu \subset \{x \in \Pi_q :\ \|x\|_2 \leq M\}$ holds for a suitable constant $M > 0$. Then by the support properties of the Bessel convolution on Π_q, we have for all $k \in \mathbb{N}$

$$supp\,\nu^{(n_k,\mu_k)} \subset \{x \in \Pi_q :\ \|x\|_2 \leq n_k M\}. \tag{4.9}$$

We now consider the function H and the constant $C > 0$ of Proposition 3.7. We conclude from Eq. (4.9) and condition (2) of the theorem that for all sequences $s_k \in supp\, \nu^{(n_k,\mu_k)}$,

$$\frac{(\ln k)^2 \|s_k\|_2^4}{\mu_k n_k^2} \to 0 \qquad \text{and} \qquad \frac{1}{\sqrt{\mu_k - \rho}} \cdot \sqrt{\frac{\ln k}{n_k}} \cdot s_k \to 0.$$

Thus by the definition of H we have for each $c \in \Pi_q$

$$H\Big(\sqrt{\frac{2\ln k}{\varepsilon n_k}} \cdot c s_k, \sqrt{\mu_k - \rho}\Big) \to 0$$

and

$$R_k(s) := \Big(1 - \frac{4C}{\varepsilon^2} \cdot \frac{(\ln k)^2 \|s\|_2^4}{\mu_k n_k^2} - H\Big(\sqrt{\frac{2\ln k}{\varepsilon n_k}} \cdot c s_k, \sqrt{\mu_k - \rho}\Big)\Big)^{-1}$$

remains bounded as $k \to \infty$ and $s \in supp\, \nu^{(n_k,\mu_k)}$. This fact together with the estimates of Proposition 3.7 and conditions (2) and (3) of the theorem imply that

$$\begin{aligned}
E\Big(e^{\frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(c(S_{n_k}^{\mu_k})^2 c)}\Big) &= \int_{\Pi_q} e^{\frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(cs^2c)}\, d\nu^{(n_k,\mu_k)}(s) \\
&\le \int_{\Pi_q} J_{\mu_k}\Big(-(\mu_k - \rho)\frac{2\ln k}{\varepsilon \cdot n_k} \cdot cs^2c\Big)\, d\nu^{(n_k,\mu_k)}(s) \cdot O(1) \\
&= \Big(\int_{\Pi_q} J_{\mu_k}\Big((\mu_k - \rho)\frac{2\ln k}{\varepsilon \cdot n_k} \cdot cs^2c\Big)\, d\nu(s)\Big)^{n_k} \cdot O(1) \\
&\le \Big(\int_{\Pi_q} e^{\frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(cs^2c)}\, d\nu(s)\Big)^{n_k} \cdot \big(1 + C/\mu_k\big)^{n_k} \cdot O(1) \\
&\le \Big(\int_{\Pi_q} \Big(1 + \frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(cs^2c) + O((\ln k/n_k)^2)\Big)\, d\nu(s)\Big)^{n_k} e^{Cn_k/\mu_k} \cdot O(1) \\
&= \Big(1 + \frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(c\sigma^2(\nu)c) + O((\ln k/n_k)^2)\Big)^{n_k} \cdot O(1) \\
&\le e^{\frac{2\ln k}{\varepsilon} \cdot tr(c\sigma^2(\nu)c)} \cdot O(1).
\end{aligned} \tag{4.10}$$

Employing again the Markov inequality we thus obtain

$$\begin{aligned}
&P\Big(\frac{1}{n_k}\langle c^2, (S_{n_k}^{\mu_k})^2\rangle \ge \langle c^2, \sigma^2(\nu)\rangle + \varepsilon\Big) \\
&\quad = P\Big(e^{\frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(c(S_{n_k}^{\mu_k})^2 c)} \ge e^{\frac{2\ln k}{\varepsilon}(tr(c\sigma^2(\nu)c) + \varepsilon)}\Big) \\
&\quad \le \frac{1}{e^{\frac{2\ln k}{\varepsilon}(tr(c\sigma^2(\nu)c) + \varepsilon)}} \cdot E\big(e^{\frac{2\ln k}{\varepsilon \cdot n_k} \cdot tr(c(S_{n_k}^{\mu_k})^2 c)}\big) \\
&\quad \le e^{-2\ln k} \cdot O(1) \quad = O(1/k^2)
\end{aligned}$$

as claimed. This proves Eq. (4.5) and completes the proof of the theorem. □

Remarks 4.9 (1) Let us briefly comment on the conditions of Theorem 4.7. The most interesting case appears for $n_k = k$, where only the growth condition (1) on the indices μ_k and the compact support condition for ν remain. Condition (1) is the essential condition in the end of the proof of Eq. (4.6), and we see no possibility to weaken this one. On the other hand, the compact support of ν has been used mainly in order to derive estimate (4.10) in a smooth way. We expect that here somewhat more involved estimations (for example, by using Hölders inequality in between) might also lead to (4.10) under weaker conditions on the support of ν. It is however clear that any proof along our approach will need that square-exponential moments of ν exist, i.e. $\int_{\Pi_q} e^{tr(cs^2c)}\, d\nu(s) < \infty$ for all $c \in \Pi_q$.

(2) We expect that there exist also central limit theorems associated with the laws of large numbers above. In particular, the convergence of χ^2-distributions to normal distributions for $q=1$ and convergence of Wishart distributions to multidimensional normal distributions for $q \geq 2$ suggest that in a CLT normal distributions appear as limits after taking squares after suitable renormalizations.

(3) Let us briefly return to the case $q=1$ discussed in Theorems 1.1 and 1.2. In this context one might ask for limit theorems for series of random walks on series of two-point homogeneous spaces where the number of steps and the dimensions of these spaces tend to infinity. For spheres and projective spaces over $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, central limit theorems were given in [V2] and references cited therein. It should also be interesting to study the non-compact cases, i.e. random walks on hyperbolic spaces.

(4) As explained in Section 4, there is a close connection between Bessel convolutions on the matrix cones Π_q and the theory of Dunkl operators on a B_q-Weyl chamber in $\mathbb{R}^q$ for certain indices. It is clear that we may project Theorems 4.1 and 4.7 to these particular cases. We do not state this result separately. Under the hypothesis that Dunkl operators are related to commutative hypergroups on Weyl chambers for all root systems and all positive multiplicities (see [R2]), it will become an interesting question in Dunkl theory whether there exist laws of large numbers for random walks on Weyl chambers similar to Theorems 4.1 and 4.7 when the multiplicities of Dunkl theory tend to infinity.

5 A large deviation principle

In this section we derive a large deviation principle (LDP) for $q=1$ and $\mathbb{F}=\mathbb{R}$ which fits to the laws of large numbers given in Theorems 1.1 and 1.2. Before going into details we explain the restriction $q=1$. Our proof of a LDP will be based on the limits

$$E\Big(e^{\langle c,(S_{n_k}^{\mu_k})^2\rangle}\Big) \qquad \text{for} \quad k\to\infty \quad \text{and all} \quad c\in H_q \tag{5.1}$$

(in the notion of the preceding section) together with a standard result from LDP theory (see e.g. Theorem II.6.1 of Ellis [E]) which states that suitable convergence of Laplace transforms implies a LDP. Unfortunately we can prove this convergence only for matrices of the form $\pm c\in H_q$ with $c\in\Pi_q$, as our convergence proofs depend on estimates for the Bessel functions J_μ which were derived in Section 3 from the integral representation (3.4) which is not available for arbitrary matrices $c\in H_q$ for $q\geq 2$. We therefore restrict our attention to $q=1$ and consider the Bessel-type random walks $(S_k^\mu)_{k\geq 0}$ on $[0,\infty[=\Pi_1$ of indices μ with fixed law $\nu\in M^1([0,\infty[)$.

Proposition 5.1 *Let $\nu\in M^1(\Pi_q)$ be a probability measure with compact support. Let $(\mu_k)_{k\in\mathbb{N}}\subset]1/2,\infty[$ be a sequence of indices and $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ a sequence of time steps with $n_k\to\infty$ and $\lim_{k\to\infty} e^{an_k}/\mu_k=0$ for all $a>0$. Then, $c_k(t):=\frac{1}{n_k}\ln E(e^{t(S_{n_k}^{\mu_k})^2})$ converges for $t\in\mathbb{R}$ and $k\to\infty$ to*

$$c(t):=\ln\Big(\int_0^\infty e^{ts^2}\,d\nu(s)\Big).$$

Proof. We proceed as the proof of Theorem 4.7. The case $t=0$ is trivial. Now let $t>0$ and put $h(t):=\int_0^\infty e^{ts^2}\,d\nu(s)$. Using Theorem 3.6 twice, we obtain that

$$\begin{aligned}
c_k(-t) &= \frac{1}{n_k}\ln\Big(\int_0^\infty e^{-ts^2}\,d\nu^{(n_k,\mu_k)}(s)\Big)\\
&= \frac{1}{n_k}\ln\Big(\int_0^\infty J_{\mu_k}(\mu_k ts^2)\,d\nu^{(n_k,\mu_k)}(s)+O(1/\mu_k)\Big)\\
&= \frac{1}{n_k}\ln\Big(\Big(\int_0^\infty J_{\mu_k}(\mu_k ts^2)\,d\nu(s)\Big)^{n_k}+O(1/\mu_k)\Big)\\
&= \frac{1}{n_k}\ln\Big((h(-t)+O(1/\mu_k))^{n_k}+O(1/\mu_k)\Big)\\
&= \ln\Big(h(-t)+O(1/\mu_k)\Big)+\frac{1}{n_k}\ln\Big(1+\frac{1}{(h(-t)+O(1/\mu_k))^{n_k}\mu_k}\Big) \quad\to\quad c(-t)
\end{aligned} \tag{5.2}$$

by the convergence conditions of the theorem. Furthermore, we obtain from the estimations in Proposition 3.7 and with the notions there that

$$\begin{aligned}\int_0^\infty e^{ts^2}\, d\nu^{(n_k,\mu_k)}(s) &\geq (1+O(1/\mu_k))^{-1}\cdot \int_0^\infty J_{\mu_k}(-(\mu_k-3/2)ts^2)\, d\nu^{(n_k,\mu_k)}(s)\\ &= (1+O(1/\mu_k))^{-1}\cdot \Big(\int_0^\infty J_{\mu_k}(-(\mu_k-3/2)ts^2)\, d\nu(s)\Big)^{n_k}\\ &\geq \frac{1}{1+O(1/\mu_k)}\Big(\int_0^\infty e^{ts^2}\Big[1-Cs^4t^2/\mu_k - H(s\sqrt{t},\sqrt{\mu_k-3/2})\Big]\, d\nu(s)\Big)^{n_k}.\end{aligned}$$

As $[\ldots]\to 1$ uniformly on the compact set $supp\nu$, it follows readily that $\liminf c_k(t)\geq c(t)$. Finally, Proposition 3.7, $supp\,\nu^{(n_k,\mu_k)}\subset[0,Mn_k]$ for a suitable $M>0$, and the convergence condition of the theorem imply

$$\begin{aligned}\int_0^\infty e^{ts^2}\, d\nu^{(n_k,\mu_k)}(s) &\leq \int_0^\infty \frac{J_{\mu_k}(-(\mu_k-3/2)ts^2)}{1-Cs^4t^2/\mu_k - H(s\sqrt{t},\sqrt{\mu_k-3/2})}\, d\nu^{(n_k,\mu_k)}(s)\\ &\leq (1+o(1/n_k))\int_0^\infty J_{\mu_k}(-(\mu_k-3/2)ts^2)\, d\nu^{(n_k,\mu_k)}(s)\\ &= (1+o(1/n_k))\Big(\int_0^\infty J_{\mu_k}(-(\mu_k-3/2)ts^2)\, d\nu\Big)^{n_k}\\ &\leq (1+o(1/n_k))(1+O(1/\mu_k))^{n_k}h(t)^{n_k}\end{aligned}$$

and thus $\limsup c_k(t)\leq c(t)$. In summary, $c_k(t)\to c(t)$ for $t>0$ which completes the proof. □

Notice that the very strong convergence condition in the proposition was needed in the end of (5.2) only where $h(-t)<1$ may become arbitrarily small. This is caused by the fact that the difference estimation in Theorem 3.6 does not fit well to the "multiplicative" structure of LDPs. For all other estimates in the proof above much weaker polynomial convergence conditions are sufficient.

We here notice that the free energy function c of Proposition 5.1 is precisely the same as for the classical LDP of Cramer for sums of i.i.d. random variables on $[0,\infty[$ with common law ν (see e.g. Ch. II.4 of [E]). Moreover, Proposition 5.1 together with Theorem II.6.1 of Ellis [E] immediately imply that in the setting of Proposition 5.1, the distributions of the random variables $(S_{n_k}^{\mu_k})^2$ have the large deviation property with scaling parameters n_k and the rate function

$$I(s):=\sup_{t\in\mathbb{R}}(st-c(t))\qquad (s\in\mathbb{R})$$

in the sense of Definition II.3.1 of [E]. We skip the details here.

Remark 5.2 If the conditions of Proposition 5.1 are satisfied, we obtain that the free energy function c is differentiable on $\mathbb{R}$ with $c'(0)=\sigma^2(\nu)>0$. Theorems II.6.3 and II.6.4 of [E] now imply that (after taking square roots) $S_{n_k}^{\mu_k}/\sqrt{n_k}$ converges to $\sqrt{\sigma^2(\nu)}$ almost surely. Notice that this strong law of large numbers (SLLN) holds under conditions which are slightly different from those in Theorem 4.7 for $q=1$. This SLLN can be also derived directly for arbitrary $q\geq 1$ similar to the proof of Theorem 4.7. As the conditions concerning the parameters μ_k are extremely strong here, we omit details.

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