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# Radial Multiresolution in Dimension Three

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**Abstract.** We present a construction of a wavelet-type orthonormal basis for the space of radial $L^2$-functions in $\mathbf{R}^3$ via the concept of a radial multiresolution analysis. The elements of the basis are obtained from a single radial wavelet by usual dilations and generalized translations. Hereby the generalized translation reveals the group convolution of radial functions in $\mathbf{R}^3$. We provide a simple way to construct a radial scaling function and a radial wavelet from an even classical scaling function on $\mathbf{R}$. Furthermore, decomposition and reconstruction algorithms are formulated.

## 1. Introduction

Standard approaches in multivariate wavelet analysis are based on the construction of multiresolution analyses and wavelet bases from affine transformations of a finite set of basis functions, called multiwavelets (see, e.g., [15], [2]). The translations are taken from a lattice subgroup $\Gamma$ of $(\mathbf{R}^d, +)$ and the dilations are given by the integer powers of an expansive matrix which leaves $\Gamma$ invariant. The number of multiwavelets needed to obtain a full basis of $L^2(\mathbf{R}^d)$ depends on the determinant of the dilation matrix and is in general larger than 1 for $d > 1$.

However, if one restricts to the analysis of radially symmetric functions only, it suggests itself to exploit this symmetry in the construction of corresponding wavelet transforms in order to reduce the high amount of computational effort. A purely radial setting would, for example, naturally occur when separating variables in polar coordinates, and treating the spherical and radial parts separately. There is a broad literature dealing with wavelet analysis and multiresolution on spheres, see, e.g., [1], [9] and the references therein. In the radial case, it is not difficult to establish a continuous wavelet analysis based on the convolution structure of radial functions or measures instead of the usual translation in $\mathbf{R}^d$. Radial convolution structures are special cases (for half-integer indices) of so-called Bessel–Kingman hypergroups (see [4], [14]), and a continuous wavelet analysis can in fact be developed in this general setting, see, e.g., [18], [12], and [17]. Essentially the same concept underlies the approach of Epperson and Frazier [7], where radial wavelet expansions in $\mathbf{R}^d$ are constructed which are based on sampling lattices with the spatial

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discretization determined by the positive zeros of related Bessel functions of the first kind. The spatial lattice is equidistant only in the special cases $d = 1$ and $d = 3$. This can be seen as an obstruction against a multiscale approach to radial wavelets in arbitrary dimensions.

As to the author's knowledge, radial multiresolution analyses have in fact not been considered up to now, and there seems to be no general rigorous approach available for the construction of orthogonal radial wavelet bases in arbitrary dimension. This problem is closely related to the question of finding a Poisson summation formula compatible with the Bessel–Kingman translation, which still remains open. It is important to notice that the construction of [7] relies on the requirement that the involved radial wavelets are band-limited, i.e., their Fourier transforms have compact support. We also mention that in [10] a particular orthonormal basis which is adapted to a radial context is constructed on the basis of the classical Meyer wavelets.

In the present paper, we construct radial multiscale analyses and orthogonal radial wavelet bases in $\mathbf{R}^3$. In dimension 3, the algebraic structure of the radial convolution allows us to carry out the constructions along the same lines as in the well-known Euclidean setting. Hereby, the equidistance of the zeros of the corresponding Bessel function (which is simply a sinc-function) is of decisive importance. In order to motivate our approach and to make the intrinsic problems toward an extension to arbitrary dimensions more visible, we start the paper with a short account of the continuous wavelet transform for Bessel–Kingman hypergroups, and also present some material about Bessel frames. This in particular comprises the radial case in arbitrary dimension. The continuous transform is, up to normalization, the same as in [18], while the Bessel frames are constructed in the spirit of the radial wavelet bases in [7]. In particular, in the spatial discretization the zeros of associated Bessel functions occur in a natural way, and the wavelets are band-limited.

The radial analysis in $\mathbf{R}^3$ is then also carried out in the setting of the corresponding Bessel–Kingman hypergroup $H$ on $[0, \infty)$. Our concept of a radial multiresolution analysis (MRA) in $\mathbf{R}^3$ is in fact that of an MRA for the $L^2$-space $L^2(H)$ of this hypergroup. The scale spaces $(V_j)_{j \in \mathbf{Z}} \subset L^2(H)$ are obtained by dyadic dilations from $V_0$, which in turn is spanned by equidistant hypergroup translates of a fixed "radial" scaling function $\varphi \in L^2(H)$. It is characterized by a two-scale relation, but in contrast to classical MRAs, $\varphi$ itself is not contained in $V_0$, and the scale spaces are not shift-invariant (with respect to the hypergroup translation). Particular emphasis is put on the construction of orthogonal MRAs. Here periodicity arguments similar to the classical case are needed which would not be available in arbitrary dimensions. From a given orthogonal MRA we then derive an orthogonal wavelet basis for the underlying hypergroup. By construction, this "radial" basis has a direct interpretation as an orthogonal wavelet basis for the subspace of radial functions in $L^2(\mathbf{R}^3)$. We also provide a concise characterization of radial scaling functions in terms of even classical scaling functions on $\mathbf{R}$. This, in particular, implies that in contrast to the classical case, there do not exist any real-valued orthonormal radial scaling functions with compact support.

The paper is organized as follows: In Section 2, we recall basic facts from the analysis of radial functions in $\mathbf{R}^d$, explain the corresponding radial hypergroup convolution structure, and extend the setting to Bessel–Kingman hypergroups of arbitrary index. Section 3 contains a short account on the continuous wavelet transform based on the Bessel–

Kingman translation, as well as the construction of Bessel frames. In Section 4, radial multiresolution analyses in $\mathbf{R}^3$ and their scaling functions are introduced and discussed, while Section 5 is devoted to the construction of orthogonal radial wavelet bases. The connection between radial scaling functions in $\mathbf{R}^3$ and classical scaling functions on $\mathbf{R}$ is established in Section 6. Finally, in Section 7 decomposition and reconstruction algorithms are discussed.

## 2. Radial Analysis and Bessel–Kingman Hypergroups

Suppose $F \in L^2(\mathbf{R}^d)$ is radial, i.e., $F(Ax) = F(x)$ a.e. for all $A \in SO(d)$. Then there is a unique $f \in L^2(\mathbf{R}_+, \omega_{d/2-1})$ such that $F(x) = f(|x|)$, where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^d$ and for $\alpha \geq -\frac{1}{2}$, the measure $\omega_\alpha$ on $\mathbf{R}_+ = [0, \infty)$ is defined by

$$d\omega_\alpha(r) = (2^\alpha \Gamma(\alpha+1))^{-1} r^{2\alpha+1}\, dr.$$

Its normalization implies that $\|F\|_2 = \|f\|_{2,\,\omega_{d/2-1}}$, where $\|\cdot\|_2$ is taken with respect to the normalized Lebesgue measure $(2\pi)^{-d/2}\,dx$ on $\mathbf{R}^d$. On $L^2(\mathbf{R}_+, \omega_\alpha)$ the Hankel transform of index $\alpha$ is defined by

$$\widehat{f}^{\,\alpha}(\lambda) = \int_0^\infty j_\alpha(\lambda r) f(r)\, d\omega_\alpha(r)$$

with the normalized Bessel function

$$j_\alpha(z) = \Gamma(\alpha+1)(z/2)^{-\alpha} J_\alpha(z), \qquad J_\alpha(z) = \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n+\alpha}}{n!\,\Gamma(n+\alpha+1)}.$$

There is a Plancherel theorem for the Hankel transform, which states that $f \mapsto \widehat{f}^{\,\alpha}$ establishes a self-inverse, isometric isomorphism of $L^2(\mathbf{R}_+, \omega_\alpha)$. If $F \in L^2(\mathbf{R}^d)$ is radial with $F(x) = f(|x|)$, then a short calculation shows that its Plancherel transform

$$\mathcal{F}(F)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} F(x) e^{-i\langle x,\xi\rangle}\, dx$$

is again radial with $\mathcal{F}(F)(\xi) = \widehat{f}^{\,(d/2-1)}(|\xi|)$. This is due to the fact that

$$j_{d/2-1}(|z|) = \int_{S^{d-1}} e^{-i\langle z,\xi\rangle}\, d\sigma(\xi), \tag{2.1}$$

where $d\sigma$ denotes the spherical surface measure normalized according to $d\sigma(S^{d-1}) = 1$. In contrast, the usual group translates $x \mapsto F(x+y)$, $y \in \mathbf{R}^d$, will no longer be radial (apart from trivial cases). However, we observe that the spherical means

$$M_r F(x) := \int_{S^{d-1}} F(x + r\xi)\, d\sigma(\xi), \qquad r \in \mathbf{R}_+,$$

of $F$ are again radial. Moreover, $\|M_r F\|_2 \leq \|F\|_2$. Thus $M_r$ induces a norm-decreasing linear mapping

$$T_r : L^2(\mathbf{R}_+, \omega_{d/2-1}) \to L^2(\mathbf{R}_+, \omega_{d/2-1}), \qquad T_r f(|x|) := M_r F(x),$$

where $f$ and $F$ are related as above. Put $\alpha = d/2 - 1$. Then a short calculation in polar coordinates gives

(2.2)
$$T_r f(s) = C_\alpha \int_0^\pi f(\sqrt{r^2 + s^2 - 2rs\cos\varphi})\sin^{2\alpha}\varphi\, d\varphi \qquad \text{with } C_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}.$$

This defines a norm-decreasing generalized translation on $L^2(\mathbf{R}_+, \omega_\alpha)$ not only for $\alpha = d/2 - 1$, but also for general $\alpha \geq -\frac{1}{2}$. Having harmonic analysis in mind, we are thus led to introduce a corresponding measure algebra on $\mathbf{R}_+$: For $r, s \in \mathbf{R}_+$ we define a probability measure $\delta_r *_\alpha \delta_s$ on $\mathbf{R}_+$ by

$$(2.3) \quad \delta_r *_\alpha \delta_s(f) := C_\alpha \int_0^\pi f\big(\sqrt{r^2 + s^2 - 2rs\cos\varphi}\big)\sin^{2\alpha}\varphi\, d\varphi, \qquad f \in C_c(\mathbf{R}_+).$$

($C_c(\mathbf{R}_+)$ denotes the space of continuous, compactly supported functions on $\mathbf{R}_+$.) The convolution (2.3) of point measures extends uniquely to a bilinear, commutative, associative, and weakly continuous convolution on the space $M_b(\mathbf{R}_+)$ of regular bounded Borel measures on $\mathbf{R}_+$. It is probability preserving and makes $M_b(\mathbf{R}_+)$ a commutative Banach-*-algebra with respect to total variation norm, with neutral element $\delta_0$ and the mapping $\mu \mapsto \overline{\mu}$ as involution. The pair $(\mathbf{R}_+, *_\alpha)$ is called the Bessel–Kingman hypergroup of index $\alpha$. We write $H_\alpha$ instead of $\mathbf{R}_+$ when putting emphasis on the specific convolution structure. Generally speaking, a hypergroup is a locally compact Hausdorff space together with a weakly continuous and probability preserving convolution of regular bounded Borel measures generalizing the measure algebra of a locally compact group; it also has a unit and an involution substituting the group inverse. In particular, every locally compact group is also a hypergroup. There is a well-established harmonic analysis for commutative hypergroups which, in the special case of $H_{d/2-1}$, reflects the harmonic analysis of radial functions (and measures) in $\mathbf{R}^d$. We refer the reader to [13] or [4] for a general background on hypergroups, including [14] for the Bessel–Kingman case. Let us mention only some aspects which are of importance in our context: The measure $\omega_\alpha$ is a Haar measure for $H_\alpha$, i.e., it satisfies

$$\int_0^\infty T_s f\, d\omega_\alpha = \int_0^\infty f\, d\omega_\alpha \qquad \text{for all } f \in C_c(\mathbf{R}_+).$$

Up to a constant factor, $\omega_\alpha$ is the unique positive Radon measure on $\mathbf{R}_+$ with this property. Moreover,

$$(2.4) \qquad \int_0^\infty (T_s f) g\, d\omega_\alpha = \int_0^\infty f(T_s g)\, d\omega_\alpha \qquad \text{for } s \in \mathbf{R}_+,$$

whenever both integrals exist. The Bessel functions satisfy the product formula

$$\delta_r *_\alpha \delta_s(j_\alpha) = j_\alpha(r) j_\alpha(s) \qquad \text{for all } r, s \in \mathbf{R}_+,$$

see [19, 11.4]. For half-integers $\alpha$, this is easily deduced from (2.1). In fact, the functions $r \mapsto j_\alpha(\lambda r)$, $\lambda \in \mathbf{R}_+$, are exactly those which are bounded and multiplicative with respect to $*_\alpha$. They constitute the so-called dual space of the hypergroup $H_\alpha$. In this way, the Hankel transform $f \mapsto \widehat{f}^{\alpha}$ on $L^2(\mathbf{R}_+, \omega_\alpha)$ can be interpreted as a Plancherel transform for $H_\alpha$. For abbreviation, we put $L^p(H_\alpha) := L^p(\mathbf{R}_+, \omega_\alpha)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$ the scalar product and norm in $L^2(H_\alpha)$. It follows easily from (2.4) that, for $f \in L^2(H_\alpha)$,

$$\widehat{T_r f}^{\alpha}(\lambda) = j_\alpha(\lambda r)\widehat{f}^{\alpha}(\lambda). \tag{2.5}$$

## 3. Continuous Wavelet Transform and Frames for Bessel–Kingman Hypergroups

In order to put the multiresolution approach in the following sections into a suitable framework, we continue with a short account on the continuous wavelet transform and wavelet frames for Bessel–Kingman hypergroups. This in particular includes a continuous radial wavelet transform and radial wavelet frames in arbitrary dimensions.

### 3.1. *The Continuous Wavelet Transform*

The following construction is essentially the same as in [18], only with a different normalization of the dilation operators and the resulting wavelet transform. (In contrast to [18], we choose dilations to be unitary, see below.) We shall therefore be brief in our presentation, and refer the reader to [18, Sect. 6.III] for further details.

Let $B(L^2(H_\alpha))$ denote the space of continuous linear operators on $L^2(H_\alpha)$. Besides the translation operators $T_r \in B(L^2(H_\alpha))$ introduced in Section 2, we consider the dilations

$$D_a f(r) := \frac{1}{a^{\alpha+1}} f\left(\frac{r}{a}\right), \qquad a > 0,$$

which are obviously unitary in $L^2(H_\alpha)$. Notice also that

$$\widehat{(D_a f)}^{\alpha} = D_{1/a}\widehat{f}^{\alpha}. \tag{3.1}$$

We define

$$\pi : H_\alpha \times (0, \infty) \to B(L^2(H_\alpha)), \qquad \pi(r, a) := T_r D_a.$$

It is easily checked that $\pi$ is continuous with respect to the weak operator topology on $B(L^2(H_\alpha))$, see [18, Prop. 6.III.7].

**Definition 3.1.** A function $g \in L^2(H_\alpha)$ is called admissible, if

$$C_g := \int_0^\infty |\widehat{g}^{\alpha}(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

For abbreviation, put

$$d\widetilde{\omega}_\alpha(r, a) := \frac{1}{a^{2\alpha+3}} da\, d\omega_\alpha(r).$$

The following is a reformulation of [18, Theorem 6.III.1] in terms of our notation:

**Theorem 3.2** (Plancherel Theorem). *If* $g \in L^2(H_\alpha)$ *is admissible, then, for all* $f \in L^2(H_\alpha)$,

$$\int_{H_\alpha \times (0,\infty)} |\langle f, \pi(r,a)g\rangle|^2 \, d\widetilde{\omega}_\alpha(r,a) \;=\; C_g \cdot \|f\|_2^2 .$$

Polarization further implies for admissible $g_1,\ g_2$ and arbitrary $f_1,\ f_2 \in L^2(H_\alpha)$ the orthogonality relation

$$\int_{H_\alpha \times (0,\infty)} \langle f_1, \pi(r,a)g_1\rangle \overline{\langle f_2, \pi(r,a)g_2\rangle}\, d\widetilde{\omega}_\alpha(r,a) \;=\; \langle f_1, f_2\rangle \cdot \int_0^\infty \overline{\widehat{g}_1^\alpha(\lambda)}\, \widehat{g}_2^\alpha(\lambda) \frac{d\lambda}{\lambda} .$$

**Definition 3.3.** Let $g \in L^2(H_\alpha)$ be admissible. The mapping

$$\Psi_g : L^2(H_\alpha) \;\to\; L^2(H_\alpha \times (0,\infty), \widetilde{\omega}_\alpha), \qquad \Psi_g f(r,a) := \langle f, \pi(r,a)g\rangle,$$

is called the wavelet transform on $H_\alpha$ with analyzing wavelet $g$. For $\alpha = d/2 - 1$, it coincides with the continuous wavelet transform on $\mathbf{R}^d$ of a radial function $f$ with radial wavelet $g$, see [17].

Some types of inversion formulas for this transform can be found in [18].

### 3.2. *Bessel Frames*

Let us now turn to possible discretizations. In order to obtain discrete versions of the usual wavelet transform on $\mathbf{R}$, it is standard to use sampling lattices of the type $\{(nba^k, a^k),\ k, n \in \mathbf{Z}\}$ with constants $a > 1$, $b > 0$. Here the discretization of the translation parameter is in accordance with the related group structure of $\mathbf{R}$ and is therefore (in general) not appropriate for radial wavelet analysis. Following Epperson and Frazier [7], we propose lattices where the discretization of the translation parameter involves the positive zeros $0 < \nu_{\alpha,1} < \nu_{\alpha,2} < \cdots$ of the Bessel function $j_\alpha$. By a result of McMahon, these are asymptotically distributed according to

$$\nu_{\alpha,n} = \left(n + \frac{\alpha}{2} - \tfrac{1}{4}\right)\pi \;+\; O\left(\frac{1}{n}\right).$$

A standard lattice in $H_\alpha \times (0,\infty)$ is given by

$$\{(\nu_{\alpha,n}\, ba^k\,, a^k),\ k \in \mathbf{Z},\ n \in \mathbf{N}\} \qquad (a > 1,\ b > 0).$$

The "almost orthogonal" radial wavelet expansions of Epperson and Frazier [7] are based on this type of sampling lattice (with $a = 2$). In the following result, the discretization of the dilation parameter is still rather arbitrary.

**Theorem 3.4.** *Let* $Q$ *be a countable subset of* $(0,\infty)$ *and let* $g \in L^2(H_\alpha)$. *Assume that* $\operatorname{supp} \widehat{g}^\alpha$ *is contained in* $[0, l]$ *for some* $l > 0$, *i.e.,* $\widehat{g}^\alpha = 0$ *a.e. on* $(l, \infty)$, *and that there* *exist constants* $A, B > 0$ *such that*

$$A \leq \sum_{q\in Q} |\widehat{g}^{\alpha}(q\lambda)|^2 \leq B \qquad \text{for almost all } \lambda \in \mathbf{R}_+ .$$

*For* $n \in \mathbf{Z}$ *and* $q \in Q$ *define "wavelets"* $g_{n,q} \in L^2(H_\alpha)$ *by*

$$g_{n,q} := M_n^{\alpha} \cdot T_{r_n q} D_q(g) = M_n^{\alpha} \pi(r_n q, q) g,$$

*where*

$$r_n := \frac{1}{l} \nu_{\alpha,n} \qquad \text{and} \qquad M_n^{\alpha} = \frac{2^{(1-\alpha)/2}\, \nu_{\alpha,n}^{\alpha}}{\sqrt{\Gamma(\alpha+1)}\, |J_{\alpha+1}(\nu_{\alpha,n})|} .$$

*Then the set* $\{g_{n,q} : n \in \mathbf{N},\ q \in Q\}$ *is a frame for* $L^2(H_\alpha)$ *with frame bounds* $Al^{2\alpha+2}$ *and* $Bl^{2\alpha+2}$. *This means that, for* $f \in L^2(H_\alpha)$,

$$Al^{2\alpha+2} \cdot \|f\|_2^2 \leq \sum_{q\in Q}\sum_{n\in\mathbf{N}} |\langle g_{n,q}, f\rangle|^2 \leq Bl^{2\alpha+2} \cdot \|f\|_2^2.$$

**Proof.** The decisive point in the proof is the fact that the normalized Fourier–Bessel functions

$$\rho_n^{\alpha}(\lambda) := M_n^{\alpha} j_\alpha(\nu_{\alpha,n}\lambda), \qquad n \in \mathbf{N},$$

form an orthonormal basis of the Hilbert space $X_\alpha := L^2([0,1], \omega_\alpha|_{[0,1]})$; see, e.g., Erdélyi et al. [8]. Using (3.1) and (2.5), we write

$$\widehat{g}_{n,q}^{\alpha}(\lambda) = M_n^{\alpha} j_\alpha(r_n q \lambda) D_{1/q}\, \widehat{g}^{\alpha}(\lambda) = \rho_n^{\alpha}\left(\frac{q}{l}\lambda\right) D_{1/q}\, \widehat{g}^{\alpha}(\lambda).$$

By the Plancherel theorem for the Hankel transform, we obtain

$$\begin{aligned}\langle g_{n,q}, f\rangle &= \langle \widehat{g}_{n,q}^{\alpha}, \widehat{f}^{\alpha}\rangle = \int_0^{\infty} \rho_n^{\alpha}\left(\frac{q}{l}\lambda\right) D_{1/q}\, \widehat{g}^{\alpha}(\lambda)\, \overline{\widehat{f}^{\alpha}(\lambda)}\, d\omega_\alpha(\lambda) \\ &= \int_0^{\infty} \rho_n^{\alpha}\, D_{1/l}(\widehat{g}^{\alpha}) D_{q/l}(\overline{\widehat{f}^{\alpha}})\, d\omega_\alpha = \langle D_{1/l}(\widehat{g}^{\alpha}) D_{q/l} l(\overline{\widehat{f}^{\alpha}}), \rho_n^{\alpha}\rangle_{X_\alpha},\end{aligned}$$

where for the last identity we used that the support of $D_{1/l}(\widehat{g}^{\alpha})$ is contained in $[0,1]$. Parseval's identity for $X_\alpha$ now yields

$$\begin{aligned}\sum_{n\in\mathbf{N}} |\langle g_{n,q}, f\rangle|^2 &= \int_0^1 |D_{1/l}(\widehat{g}^{\alpha}) D_{q/l} l(\overline{\widehat{f}^{\alpha}})|^2\, d\omega_\alpha \\ &= l^{2\alpha+2} \int_0^{\infty} |\widehat{g}^{\alpha}(q\lambda)|^2\, |\widehat{f}^{\alpha}(\lambda)|^2\, d\omega_\alpha(\lambda).\end{aligned}$$

Hence,

$$\sum_{q\in Q}\sum_{n\in\mathbf{N}} |\langle g_{n,q}, f\rangle|^2 = l^{2\alpha+2} \int_0^{\infty} |\widehat{f}^{\alpha}(\lambda)|^2 \left(\sum_{q\in Q} |\widehat{g}^{\alpha}(q\lambda)|^2\right) d\omega_\alpha(\lambda).$$

This implies the assertion. ∎

The second condition of this theorem is rather implicit. Following, e.g., Bernier and Taylor [3], it is possible to obtain sufficient criteria which are easier to check by introducing the concept of "frame generators." But in order to stay concise, we restrict ourselves to the most interesting special case of a standard lattice as defined above. Here $Q = \{a^k,\ k \in \mathbf{Z}\}$ with $a > 1$.

**Proposition 3.5.** *Suppose that $g \in L^2(H_\alpha)$ satisfies the following conditions*:

(i) $\widehat{g}^\alpha$ *has compact support which is contained in the open interval* $(0, \infty)$;
(ii) $\operatorname{ess\,inf}\{|\widehat{g}^\alpha(\lambda)| : \lambda \in [a^n, a^{n+1}]\} \geq \sigma$ *for some* $n \in \mathbf{Z}$ *and* $\sigma > 0$;
(iii) $\tau := \|\widehat{g}^\alpha\|_\infty < \infty$.

*Then there exists a constant $M > 0$ such that*

$$\sigma^2 \leq \sum_{k\in\mathbf{Z}} |\widehat{g}^\alpha(a^k\lambda)|^2 \leq M\tau^2 \qquad \text{for almost all } \lambda \in \mathbf{R}_+ . \tag{3.2}$$

*Consequently, the set $\{g_{n,a^k} : n \in \mathbf{N},\ k \in \mathbf{Z}\}$ is a frame for $L^2(H_\alpha)$ with bounds $\sigma^2 l^{2\alpha+2}$ and $M\tau^2 l^{2\alpha+2}$.*

**Proof.** We use the arguments of [3, Sect. 4] in a simplified form which is adapted to our situation. Put $F := [1, a)$. Then the intervals $a^n F = [a^n, a^{n+1})$, $n \in \mathbf{Z}$, form a disjoint cover of $(0, \infty)$. As $T := \operatorname{supp} \widehat{g}^\alpha$ is compact in $(0, \infty)$, it is covered by finitely many of the $a^n F$. This implies that

$$M := \sup_{\lambda\in(0,\infty)} \sharp\{k \in \mathbf{Z} : \lambda \in a^{-k}T\}$$

is finite. By (iii), this gives the upper bound in (3.2). The lower bound follows from (ii) together with the fact that the $a^n F$ cover $(0, \infty)$. ■

## 4. Radial Multiresolution Analysis in $\mathbf{R}^3$

Radial analysis in $\mathbf{R}^3$ corresponds to the Bessel–Kingman hypergroup $H_\alpha$ with $\alpha = \frac{1}{2}$. For convenience we shall usually omit the subscript $\frac{1}{2}$ and put

$$d\omega(r) := d\omega_{1/2}(r) = \sqrt{\frac{2}{\pi}}\, r^2\, dr, \qquad \widehat{f} := \widehat{f}^{1/2}, \qquad j(r) := j_{1/2}(r) = \frac{\sin r}{r}.$$

We further write $H$ instead of $H_{1/2}$ and denote by $\langle\cdot\,,\,\cdot\rangle$ and $\|\cdot\|_2$ the scalar product and norm in $L^2(H) = L^2(H, \omega)$, respectively. Notice that the Bessel function $j$ is even on $\mathbf{R}$. Hence it is natural to assume the Hankel transform

$$\widehat{f}(\lambda) = \int_0^\infty j(\lambda r) f(r)\, d\omega(r)$$

of $f \in L^2(H)$ to be continued to an even function on $\mathbf{R}$ as well. We shall always do this throughout the paper. We also mention that by a change of variables, the generalized translation (2.2) on $H$ can be written in the simple form

$$T_r f(s) = \frac{1}{2rs} \int_{|r-s|}^{r+s} f(t)t\, dt. \tag{4.1}$$

The nonnegative zeros of the Bessel function $j$ are given by

$$t_k := k\pi, \qquad k \in \mathbf{N},$$

and the normalized Fourier–Bessel functions

$$\rho_k(r) := M_k\, j(t_k r) \qquad \text{with } M_k = 2^{1/4}\pi^{5/4}k \tag{4.2}$$

form an orthonormal basis of the Hilbert space $L^2([0,1], \omega|_{[0,1]})$. This is equivalent to the obvious fact that the functions

$$s_k(r) := (2/\pi)^{1/4}\, r\rho_k(r) = \sqrt{2}\sin(k\pi r), \qquad k \in \mathbf{N},$$

are an orthonormal basis for $L^2[0,1] := L^2([0,1], dr)$. It will be of importance in the following that the $s_k$ are 2-periodic.

Let us come to the definition of a radial multiresolution analysis (MRA) for $\mathbf{R}^3$, i.e., for the Bessel–Kingman hypergroup $H$. It is close to the well-known definition of Mallat [16] for $\mathbf{R}$. For convenience, we introduce the notation

$$T^{(k)} := T_{t_k} = T_{k\pi} \qquad (k \in \mathbf{N}).$$

If $f \in L^2(H)$, then according to (2.5),

$$(M_k T^{(k)} f)^\wedge(\lambda) = \rho_k(\lambda)\widehat{f}(\lambda) = \left(\frac{\pi}{2}\right)^{1/4} \frac{s_k(\lambda)}{\lambda}\, \widehat{f}(\lambda). \tag{4.3}$$

**Definition 4.1** (Radial Multiresolution Analysis). A radial MRA for $\mathbf{R}^3$ is a sequence $\{V_j\}_{j\in\mathbf{Z}}$ of closed linear subspaces of $L^2(H)$ such that:

(1) $V_j \subseteq V_{j+1}$ for all $j \in \mathbf{Z}$;
(2) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$;
(3) $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(H)$;
(4) $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$; and
(5) there exists a function $\varphi \in L^2(H)$ such that

$$B_\varphi := \{M_k T^{(k)}\varphi : k \in \mathbf{N}\}$$

is a Riesz basis of $V_0$, i.e., span $B_\varphi$ is dense in $V_0$ and there exist constants $A, B > 0$ such that

$$A\|\alpha\|_2^2 \le \left\| \sum_{k=1}^{\infty} \alpha_k M_k T^{(k)}\varphi \right\|_2^2 \le B\|\alpha\|_2^2$$

for all $\alpha = (\alpha_k)_{k\in\mathbf{N}} \in l^2(\mathbf{N})$; here $\|\alpha\|_2 = (\sum_{k=1}^{\infty} |\alpha_k|^2)^{1/2}$.

The function $\varphi$ in (5) is called a scaling function for the MRA $\{V_j\}$. We remark explicitly that in contrast to the classical case, $\varphi$ itself is not contained in $V_0$, and $V_0$ is not shift invariant; in fact, if $f \in V_0$, then $T^{(k)} f \notin V_0$ for all $k$. This will be shown in Corollary 4.6 below.

Our first aim is to determine an orthonormal basis for $V_0$ from its Riesz basis, i.e., a function $\varphi^* \in L^2(H)$ such that $B_{\varphi^*}$ constitutes an orthonormal basis for $V_0$. For $\varphi \in L^2(H)$ we define

$$P_\varphi(\lambda) := \sum_{n=-\infty}^{\infty} |\widehat{\varphi}(\lambda + 2n)|^2,$$

which is even and 2-periodic on $\mathbf{R}$.

**Proposition 4.2.** *Let $\varphi \in L^2(H)$ and $A, B > 0$. Then*

$$A\|\alpha\|_2^2 \le \left\| \sum_{k=1}^{\infty} \alpha_k M_k T^{(k)} \varphi \right\|_2^2 \le B\|\alpha\|_2^2 \qquad \text{for all } \alpha \in l^2(\mathbf{N}) \tag{4.4}$$

*if and only if*

$$A \le P_\varphi(\lambda) \le B \qquad \text{for almost all } \lambda \in \mathbf{R}. \tag{4.5}$$

**Proof.** Let $\alpha \in l^2(\mathbf{N})$ be an arbitrary finite sequence. Define

$$\widetilde{\alpha} := \sum_{k=1}^{\infty} \alpha_k s_k \in L^2[0, 1]. \tag{4.6}$$

We may regard $\widetilde{\alpha}$ as an odd, 2-periodic function on $\mathbf{R}$. By the Plancherel theorem for the Hankel transform and (4.3),

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \alpha_k M_k T^{(k)} \varphi \right\|_2^2 &= \left\| \sum_{k=1}^{\infty} \alpha_k \rho_k \widehat{\varphi} \right\|_2^2 = \int_0^\infty \left| \sum_{k=1}^{\infty} \alpha_k s_k(\lambda) \right|^2 |\widehat{\varphi}(\lambda)|^2 d\lambda \\ &= \frac{1}{2} \int_{-\infty}^{\infty} |\widetilde{\alpha}(\lambda)|^2 |\widehat{\varphi}(\lambda)|^2 \, d\lambda = \frac{1}{2} \int_{-1}^{1} |\widetilde{\alpha}(\lambda)|^2 P_\varphi(\lambda) \, d\lambda \\ &= \int_0^1 |\widetilde{\alpha}(\lambda)|^2 P_\varphi(\lambda) \, d\lambda. \end{aligned}$$

The $s_k$ forming an orthonormal basis of $L^2[0, 1]$, we have $\|\alpha\|_2 = \|\widetilde{\alpha}\|_{L^2[0,1]}$. As the finite sequences form a dense subspace of $l^2(\mathbf{N})$, this implies the assertion. ∎

With $A = B = 1$ we immediately obtain

**Corollary 4.3.** *For $\varphi \in L^2(H)$ the following statements are equivalent*:

(1) *the set* $B_\varphi = \{M_k T^{(k)} \varphi : k \in \mathbf{N}\}$ *is orthonormal in* $L^2(H)$;
(2) $P_\varphi = 1$ *a.e.*

For $\varphi \in L^2(H)$, put

$$V_\varphi := \overline{\text{span}\, B_\varphi}\,,$$

the closure being taken in $L^2(H)$. The set $B_\varphi$ is a Riesz basis of $V_\varphi$ if and only if there exist constants $A, B > 0$ such that the equivalent conditions of Proposition 4.2 are satisfied. This will be a standard requirement in the sequel, and we therefore introduce a separate notation:

**Definition 4.4.** A function $\varphi \in L^2(H)$ satisfies condition (RB) if $B_\varphi$ is a Riesz basis of $V_\varphi$.

As before, we shall often consider functions from $L^2[0, 1]$ as odd, 2-periodic functions on $\mathbf{R}$. We therefore define

$$S := \{\alpha : \mathbf{R} \to \mathbf{C},\ \alpha|_{[0,1]} \in L^2[0, 1],\ \alpha(-x) = -\alpha(x),\ \alpha(x+2) = \alpha(x) \text{ for almost all } x\}.$$

$S$ is a Hilbert space with norm $\|\cdot\|_{L^2[0,1]}$.

**Lemma 4.5.** *Let $\varphi \in L^2(H)$ satisfy* (RB). *Then for $f \in L^2(H)$ we have the equivalence*

$$f \in V_\varphi \quad \Leftrightarrow \quad \widehat{f}(\lambda) = \frac{\beta(\lambda)}{\lambda}\widehat{\varphi}(\lambda) \qquad \textit{with } \beta \in S.$$

*The function $f \in V_\varphi$ corresponding to $\beta = \sum_{k=1}^{\infty} \alpha_k s_k \in S$ with $(\alpha_k)_{k\in\mathbf{N}} \in l^2(\mathbf{N})$ is given by $f = (2/\pi)^{1/4} \sum_{k=1}^{\infty} \alpha_k M_k T^{(k)}\varphi$.*

**Proof.** By (4.3) we have

$$\frac{s_k(\lambda)}{\lambda}\widehat{\varphi}(\lambda) = \left(\frac{2}{\pi}\right)^{1/4} (M_k T^{(k)}\varphi)^\wedge(\lambda).$$

The translates $M_k T^{(k)}\varphi$ form a Riesz basis of $V_\varphi$, and hence the $(M_k T^{(k)}\varphi)^\wedge$ are a Riesz basis of $\widehat{V_\varphi}$. As $(s_k)_{k\in\mathbf{N}}$ is an orthonormal basis of $S$, this implies the assertion. ■

Lemma 4.5 is of particular interest when $\varphi$ is the scaling function of an MRA $\{V_j\}$. Then $V_0 = V_\varphi$, and we easily deduce the previously mentioned lack of shift-invariance:

**Corollary 4.6.** *Let $\{V_j\}_{j\in\mathbf{Z}}$ be a radial MRA. Then $f \in V_0$ implies that $T^{(k)}f = T_{t_k} f \notin V_0$ for all $k \in \mathbf{N}$. Similarly, $f \in V_j$ implies that $T_{2^{-j}t_k} f \notin V_j$ for all $k \in \mathbf{N}$.*

**Proof.** After rescaling it is enough to consider $V_0$. Recall that $M_k(T^{(k)}f)^\wedge = \rho_k \widehat{f}$. But if $\beta \in S$, then $\rho_k\beta \notin S$ for all $k$, because periodicity is lost. The characterization of $V_\varphi = V_0$ according to the previous lemma thus shows that for $f \in V_0$, $T^{(k)}f \notin V_0$. ■

In the situation of the lemma, we can easily determine an orthonormal basis of $V_\varphi$ by renormalization:

**Theorem 4.7** (Orthogonalization). *Suppose $\varphi \in L^2(H)$ satisfies condition* (RB). *Define $\varphi^* \in L^2(H)$ by its Hankel transform*

$$\widehat{\varphi^*} := \frac{\widehat{\varphi}}{\sqrt{P_\varphi}}. \tag{4.7}$$

*Then $B_{\varphi^*} = \{M_k T^{(k)} \varphi^* : k \in \mathbf{N}\}$ forms an orthonormal basis of $V_\varphi = V_{\varphi^*}$.*

If $\varphi$ is a scaling function of an MRA $\{V_j\}$, then $V_\varphi = V_{\varphi^*} = V_0$, and we call $\varphi^*$ an orthonormal scaling function for $\{V_j\}$.

**Proof.** By definition of $\varphi^*$ we have $P_{\varphi^*} = 1$ a.e. and hence $B_{\varphi^*}$ is orthonormal according to Corollary 4.3. It remains to prove that $V_{\varphi^*} = V_\varphi$. For this, we have to verify that $M_k T^{(k)} \varphi^* \in V_\varphi$ and $M_k T^{(k)} \varphi \in V_{\varphi^*}$ for all $k \in \mathbf{N}$. Employing Lemma 4.5, relation (4.3), and, finally, the relation $s_k(r) = (\pi/2)^{1/4} r \rho_k(r)$, one obtains that the above conditions are equivalent to

$$\frac{s_k}{\sqrt{P_\varphi}} \in S, \qquad s_k \sqrt{P_\varphi} \in S \qquad \text{for all } k \in \mathbf{N}.$$

But these conditions are obviously satisfied by our assumption on $P_\varphi$. ∎

Let us return to our definition of a radial MRA for $\mathbf{R}^3$. Suppose we start with a function $\varphi \in L^2(H)$ satisfying condition (RB) with Riesz constants $A, B > 0$. Define corresponding scale spaces $\{V_j\}_{j \in \mathbf{Z}}$ by

$$V_0 := V_\varphi, \qquad V_j := D_{2^{-j}} V_0,$$

where the dilation operator $D_a \in B(L^2(H))$ is defined as in Section 3. Then, in particular, the $V_j$ satisfy axiom (4) of Definition 4.1. Put further

$$\varphi_{j,k}(r) := D_{2^{-j}}(M_k T^{(k)} \varphi)(r) = 8^{j/2} M_k (T^{(k)} \varphi)(2^j r), \qquad j \in \mathbf{Z},\ k \in \mathbf{N}. \tag{4.8}$$

Then $\langle \varphi_{j,k}, \varphi_{j,l} \rangle = \langle \varphi_{0,k}, \varphi_{0,l} \rangle$ for all $j, k, l$. Thus the $\{\varphi_{j,k},\ k \in \mathbf{N}\}$ form a Riesz basis of $V_j$, with the same Riesz constants $A, B$ as for $j = 0$. In particular,

$$V_j = \overline{\operatorname{span}\{\varphi_{j,k}, k \in \mathbf{N}\}}.$$

Moreover, if $B_\varphi = \{\varphi_{0,k} : k \in \mathbf{N}\}$ is an orthonormal basis for $V_0$, then $\{\varphi_{j,k},\ k \in \mathbf{N}\}$ is an orthonormal basis of $V_j$.

Recall now axiom (1) of Definition 4.1, which requires that the $V_j$ are nested. As in the classical case, this condition can be reformulated in terms of a two-scale relation for $\varphi$:

**Proposition 4.8.** *For $\varphi$ with* (*RB*) *and $\{V_j\}_{j \in \mathbf{Z}}$ as above, the following statements are equivalent*:

(1) *$V_j \subseteq V_{j+1}$ for all $j \in \mathbf{Z}$.*

(2) $V_{-1} \subseteq V_0$.
(3) *There exists a function* $\gamma \in S$ *such that*

$$\sin(2\pi\lambda)\widehat{\varphi}(2\lambda) \;=\; \gamma(\lambda)\widehat{\varphi}(\lambda). \tag{4.9}$$

*In this case, the coefficients* $(h_k)_{k\in\mathbf{N}} \in l^2(\mathbf{N})$ *in the two-scale relation*

$$\varphi_{-1,1} \;=\; \sum_{k=1}^{\infty} h_k\, \varphi_{0,k} \tag{4.10}$$

*are the coefficients in the Fourier sine series of* $\gamma \in S$:

$$\gamma \;=\; \frac{1}{2}\sum_{k=1}^{\infty} h_k s_k.$$

**Proof.** Rescaling by the factor $2^j$ shows that (1) and (2) are equivalent. For (2), we need at least $\varphi_{-1,1} \in V_0$. According to Lemma 4.5 this is equivalent to the existence of a function $\beta \in S$ such that, for almost all $\lambda$,

$$\sqrt{8}M_1\, j(2\pi\lambda)\widehat{\varphi}(2\lambda) \;=\; \widehat{\varphi_{-1,1}}(\lambda) \;=\; \frac{\beta(\lambda)}{\lambda}\widehat{\varphi}(\lambda). \tag{4.11}$$

Moreover, if $\varphi_{-1,1}$ has the expansion (4.10), then $\beta = (\pi/2)^{1/4}\sum_{k=1}^{\infty} h_k s_k$. In turn, equation (4.11) is equivalent to relation (4.9) with $\gamma = (8\pi)^{-1/4}\beta$. This gives the stated connection between $\gamma$ and $\varphi_{-1,1}$. It remains to show that $\varphi_{-1,1} \in V_0$ (or, equivalently, relation (4.9)) already implies that $\varphi_{-1,k} \in V_0$ for all $k \in \mathbf{N}$. As above, the latter is equivalent to

$$\sin(2k\pi\lambda)\widehat{\varphi}(2\lambda) \;=\; k\gamma_k(\lambda)\widehat{\varphi}(\lambda) \tag{4.12}$$

with $\gamma_k \in S$. The relation between $\gamma_k$ and $\varphi_{-1,k}$ is now given by

$$\gamma_k \;=\; \frac{1}{2k}\sum_{l=1}^{\infty} h_l^{(k)} s_k, \qquad \varphi_{-1,k} \;=\; \sum_{l=1}^{\infty} h_l^{(k)}\varphi_{0,l}.$$

Comparison of (4.9) with (4.12) yields

$$\gamma_k(\lambda) \;=\; \gamma(\lambda)\frac{\sin(2k\pi\lambda)}{k\sin(2\pi\lambda)} \;=\; \gamma(\lambda)U_{k-1}(\cos 2\pi\lambda) \tag{4.13}$$

where

$$U_k(x) = \frac{\sin(k+1)t}{(k+1)\sin t}, \qquad x = \cos t,$$

denotes the $k$th Chebychev polynomial of the second kind, normalized such that $U_k(1) = 1$. Thus given $\gamma \in S$, we define

$$\gamma_k(\lambda) := \gamma(\lambda)U_{k-1}(\cos 2\pi\lambda).$$

As $U_{k-1}$ is bounded on $[-1, 1]$, $\gamma_k$ is contained in $S$ as well, and hence $\varphi_{-1,k} \in V_0$. ∎

Let us now consider the remaining axioms (2) and (3) of a radial MRA.

**Theorem 4.9.** *Let $\varphi \in L^2(H)$ satisfy condition* (RB) *and assume that the scale spaces*

$$V_j \;=\; \overline{\operatorname{span}\{\varphi_{j,k} : k \in \mathbf{N}\}}, \qquad j \in \mathbf{Z},$$

*satisfy $V_{-1} \subseteq V_0$. Suppose further that $|\widehat{\varphi}\,|$ is continuous at* 0. *Then $\{V_j\}_{j\in\mathbf{Z}}$ is a radial MRA if and only if $\widehat{\varphi}(0) \neq 0$. Moreover, $\varphi$ is an orthonormal scaling function if and only if $|\widehat{\varphi}(0)| = 1$.*

We remark that continuity of $\widehat{\varphi}$ at 0 (even on $\mathbf{R}_+$) is, for example, guaranteed if $\varphi \in L^2(H) \cap L^1(H)$.

**Proof.** We have to check axioms (2) and (3). This may be done by slight modifications of standard arguments in the affine case. The condition on $\widehat{\varphi}$ at 0 will be needed only for (3). We define an orthonormal scaling function $\varphi^*$ according to Theorem 4.7. The orthogonal projection $P_j$ of $L^2(H)$ onto $V_j$ is then given by

$$P_j f \;=\; \sum_{k=1}^{\infty} \langle f, \varphi_{j,k}^* \rangle \varphi_{j,k}^*$$

where the $\varphi_{j,k}^*$ are defined as in (4.8). For (2), we need to show that $\lim_{j\to-\infty} \|P_j f\|_2 = 0$ for all $f \in L^2(H)$. Since functions with compact support are dense in $L^2(H)$, we may assume that supp$f$ is contained in a compact interval $[0, R]$. Parseval's equation then implies

$$\begin{aligned} \|P_j f\|_2^2 \;&=\; \sum_{k=1}^{\infty} |\langle f, \varphi_{j,k}^*\rangle|^2 \;\le\; \sum_{k=1}^{\infty} \left| \int_0^R f(r)\overline{\varphi_{j,k}^*(r)}\, d\omega(r) \right|^2 \\ &\le\; \|f\|_2^2 \cdot \int_0^R \sum_{k=1}^{\infty} |\varphi_{j,k}^*(r)|^2\, d\omega(r) \;=\; \|f\|_2^2 \cdot \int_0^{2^j R} \sum_{k=1}^{\infty} |\varphi_{0,k}^*(r)|^2\, d\omega(r). \end{aligned}$$

Using the explicit formula (4.1) for the hypergroup translation in $H$ we further deduce

$$\sum_{k=1}^{\infty} |\varphi_{0,k}^*(r)|^2 \;=\; \sum_{k=1}^{\infty} |M_k T^{(k)} \varphi^*(r)|^2 \;=\; \sum_{k=1}^{\infty} \left| \frac{(2\pi)^{1/4}}{2r} \int_{|k\pi - r|}^{k\pi + r} \varphi^*(t) t\, dt \right|^2.$$

Now assume that $j$ is sufficiently small so that $2^j R < \pi/2$. Then, for $r \in [0, 2^j R\,]$, the integration domains $[k\pi - r, k\pi + r]$ do not overlap and we obtain

$$\sum_{k=1}^{\infty} |\varphi_{0,k}^*(r)|^2 \;\le\; \frac{C}{r} \sum_{k=1}^{\infty} \int_{k\pi - r}^{k\pi + r} |\varphi^*(t)|^2 t^2\, dt \;\le\; \frac{C'}{r} \|\varphi^*\|_2^2$$

with suitable constants $C, C' > 0$ independent of $j$. Hence, for $j$ sufficiently small,

$$\|P_j f\|_2^2 \;\le\; C'' \int_0^{2^j R} \frac{1}{r} d\omega(r),$$

which tends to 0 as $j \to -\infty$. This proves (2).

As to (3), suppose first that $\widehat{\varphi}(0) \neq 0$ and let $h \in (\bigcup_{j=-\infty}^{\infty} V_j)^{\perp}$, i.e., $P_j h = 0$ for all $j \in \mathbf{Z}$. We claim that $h = 0$. Indeed, for $\varepsilon > 0$ there exists a function $f \in L^2(H)$ such that the support of its Hankel transform $\widehat{f}$ is compact and $\|f - h\|_2 \leq \varepsilon$. This implies

$$\|P_j f\|_2 = \|P_j(f - h)\|_2 \leq \varepsilon \qquad \text{for all } j \in \mathbf{Z}.$$

By the Riesz basis assumption on $\varphi$, we further have

$$A \sum_{k=1}^{\infty} |\langle f, \varphi_{j,k}\rangle|^2 \leq \|P_j f\|_2^2 \leq B \sum_{k=1}^{\infty} |\langle f, \varphi_{j,k}\rangle|^2, \tag{4.14}$$

see Lemma 2.7 in [20]. Further, if $\operatorname{supp} \widehat{f} \in [0, R]$, then

$$\langle f, \varphi_{j,k}\rangle = \langle \widehat{f}, \widehat{\varphi}_{j,k}\rangle = \int_0^R \widehat{f}(\lambda) \rho_k^{(j)}(\lambda) \overline{\widehat{\varphi}(2^{-j}\lambda)}\, d\omega(\lambda)$$

where

$$\rho_k^{(j)} := D_{2^j} \rho_k.$$

Note that the functions $\{\rho_k^{(j)}, k \in \mathbf{N}\}$ form an orthonormal basis of $L^2([0, 2^j], \omega|_{[0,2^j]}) =: X_j$. Suppose now that $j$ is sufficiently large, i.e., $2^j \geq R$. Then

$$\langle f, \varphi_{j,k}\rangle = \langle \widehat{f}\, \overline{\widehat{\varphi}(2^{-j}\cdot)}, \rho_k^{(j)}\rangle_{X_j}.$$

Thus by Parseval's equation for $X_j$,

$$\sum_{k=1}^{\infty} |\langle f, \varphi_{j,k}\rangle|^2 = \|\widehat{f}\, \overline{\widehat{\varphi}(2^{-j}\cdot)}\|_{X_j}^2 = \int_0^R |\widehat{f}(\lambda)|^2 |\widehat{\varphi}(2^{-j}\lambda)|^2\, d\omega(\lambda).$$

As $|\widehat{\varphi}|$ is assumed to be continuous in 0, the functions $\lambda \to |\widehat{\varphi}(2^{-j}\lambda)|$ converge to the constant $|\widehat{\varphi}(0)| > 0$ uniformly on $[0, R]$ as $j \to \infty$. Hence

$$\varepsilon \geq \limsup_{j\to\infty} \|P_j f\|_2 \geq \sqrt{A}\, |\widehat{\varphi}(0)|\, \|\widehat{f}\|_2 \geq \sqrt{A}\, |\widehat{\varphi}(0)|(\|h\|_2 - \varepsilon).$$

As $\varepsilon$ is arbitrarily small, this shows that $h = 0$ and hence axiom (3) is satisfied. Vice versa, axiom (3) implies that

$$\lim_{j\to\infty} P_j f = f \qquad \text{for all } f \in L^2(H).$$

If $\widehat{f}$ is compactly supported, then the same calculation as above shows that

$$\lim_{j\to\infty} \|P_j f\|_2 \leq \sqrt{B}\, |\widehat{\varphi}(0)|\, \|\widehat{f}\|_2$$

which enforces $\widehat{\varphi}(0) \neq 0$.

If the $\varphi_{j,k}$, $k \in \mathbf{N}$, are orthonormal then we may choose $A = B = 1$ in (4.14). The converse is also true. Indeed, by assumption the $\varphi_{j,k}$, $k \in \mathbf{N}$, form a Riesz basis of $V_j$, in particular an exact frame, whose frame operator is the identity if $A = B = 1$ (see, e.g., [11, Theorem 2.1.3]). It then follows from Corollary 2.1.7 in [11] that the $\varphi_{j,k}$, $k \in \mathbf{N}$, are orthonormal. Furthermore, in the case $A = B = 1$ we obtain (for $f$ as just before)

$$\lim_{j\to\infty} \|P_j f\|_2 \;=\; |\widehat{\varphi}(0)|\,\|\widehat{f}\|_2 \;=\; |\widehat{\varphi}(0)|\,\|f\|_2.$$

Thus (3) is satisfied exactly if $|\widehat{\varphi}(0)| = 1$. ■

Let us now write the two-scale relation (4.9) in a slightly different form, namely

$$\widehat{\varphi}(2\lambda) \;=\; G(\lambda)\,\widehat{\varphi}(\lambda) \tag{4.15}$$

with

$$G(\lambda) := \frac{\gamma(\lambda)}{\sin(2\pi\lambda)}.$$

The filter function $G$ is obviously 2-periodic and even. As for a classical MRA one proves the following:

**Lemma 4.10.** *Suppose that $\varphi \in L^2(H)$ is an orthonormal scaling function of a radial MRA. Then the associated filter function $G$ satisfies*

$$|G(\lambda)|^2 + |G(\lambda+1)|^2 \;=\; 1 \quad \text{a.e.} \tag{4.16}$$

*Consequently, $G$ is essentially bounded and contained in $L^2[0,1]$, which allows us to develop it into a cosine series,*

$$G(\lambda) \;=\; \sqrt{2}\sum_{n=0}^{\infty} g_n \cos(n\pi\lambda).$$

*If, in addition, $\varphi \in L^1(H)$, then (4.16) holds pointwise and*

$$G(0) = 1, \qquad G(1) = 0,$$

*which implies*

$$\sqrt{2}\sum_{n=0}^{\infty} g_n \;=\; 1, \qquad \sqrt{2}\sum_{n=0}^{\infty}(-1)^n g_n \;=\; 0.$$

**Proof.** In view of Corollary 4.3, we have

$$\begin{aligned}
1 \;&=\; \sum_{n=-\infty}^{\infty} |\widehat{\varphi}(\lambda+2n)|^2 \;=\; \sum_{n=-\infty}^{\infty} |G(\lambda/2+n)|^2\,|\widehat{\varphi}(\lambda/2+n)|^2\\
&=\; |G(\lambda/2)|^2 \sum_{n=-\infty}^{\infty} |\widehat{\varphi}(\lambda/2+2n)|^2 \;+\; |G(\lambda/2+1)|^2 \sum_{n=-\infty}^{\infty} |\widehat{\varphi}(\lambda/2+2n+1)|^2\\
&=\; |G(\lambda/2)|^2 \;+\; |G(\lambda/2+1)|^2
\end{aligned}$$

almost everywhere. If $\varphi \in L^1(H)$, then $\widehat{\varphi}$ is continuous and $\widehat{\varphi}(0) \neq 0$ by Theorem 4.9. Hence $G(0) = 1$ by (4.15) and $G(1) = 0$ is an immediate consequence of (4.16). ■

## 5. Orthogonal Radial Wavelets

In this section we construct wavelets for a given radial MRA $\{V_j\}_{j\in\mathbf{Z}}$ in $\mathbf{R}^3$ with **orthonormal** scaling function $\varphi$ and filter function $G$. As usual, the wavelet space $W_j$ is defined as the orthogonal complement of $V_j$ in $V_{j+1}$,

$$W_j := V_{j+1} \ominus V_j .$$

Thus $L^2(H)$ decomposes as an orthogonal Hilbert sum

$$L^2(H) = \bigoplus_{j=-\infty}^{\infty} W_j .$$

Recall the definition of $S$ in Section 4 and the characterization of $V_0 = V_\varphi$ according to Lemma 4.5,

$$f \in V_0 \quad \Leftrightarrow \quad \widehat{f}(\lambda) = \frac{\beta(\lambda)}{\lambda}\,\widehat{\varphi}(\lambda) \qquad \text{with } \beta \in S. \tag{5.1}$$

Define

$$S_0 := \{\alpha \in S : \alpha(\lambda+1) = -\alpha(\lambda) \text{ for almost all } \lambda\}$$

which is a closed subspace of $S$ with respect to $\|\,.\|_{L^2[0,1]}$. Then $W_{-1} = V_0 \ominus V_{-1}$ is characterized as follows:

**Proposition 5.1.**

(i) *Let $f \in L^2(H)$. Then*

$$f \in W_{-1} \quad \Leftrightarrow \quad \widehat{f}(\lambda) = \frac{\alpha(\lambda)}{\lambda}\,\overline{G(\lambda+1)}\,\widehat{\varphi}(\lambda) \qquad \textit{for some } \alpha \in S_0.$$

(ii) *The mapping $S_0 \to W_{-1}$, $\alpha \mapsto f_\alpha$ with*

$$\widehat{f_\alpha}(\lambda) := (2\pi)^{1/4}\,\frac{\alpha(\lambda)}{\lambda}\,\overline{G(\lambda+1)}\,\widehat{\varphi}(\lambda)$$

*is an isometric isomorphism.*

**Proof.** For (i), notice first that the Hankel transform is a unitary isomorphism of $L^2(H)$, so $\widehat{W}_{-1} = \widehat{V}_0 \ominus \widehat{V}_{-1}$. Rescaling of (5.1) by the factor 2 and relation (4.15) imply that $h \in L^2(H)$ is contained in $\widehat{V}_{-1}$ if and only if there exists some $\widetilde{\beta} \in S$ such that

$$h(\lambda) = \frac{\widetilde{\beta}(2\lambda)}{\lambda}\,\widehat{\varphi}(2\lambda) = \frac{\widetilde{\beta}(2\lambda)}{\lambda}\,G(\lambda)\,\widehat{\varphi}(\lambda).$$

Thus $\beta \in S$ corresponds to $f \in W_{-1}$ according to (5.1) if and only if

$$\int_0^\infty \frac{\widetilde{\beta}(2\lambda)}{\lambda}\,G(\lambda)\,\frac{\overline{\beta(\lambda)}}{\lambda}\,|\widehat{\varphi}(\lambda)|^2\,d\omega(\lambda) = 0 \qquad \text{for all } \widetilde{\beta} \in S.$$

Up to a constant factor, the integral on the left equals

$$\begin{aligned}
&\int_{-\infty}^{\infty} \widetilde{\beta}(2\lambda)G(\lambda)\,\overline{\beta(\lambda)}|\widehat{\varphi}(\lambda)|^2\,d\lambda \\
&\qquad = \sum_{n\in\mathbf{Z}}\int_0^1 \widetilde{\beta}(2\lambda)\overline{\beta(\lambda+2n)}G(\lambda+2n)\,|\widehat{\varphi}(\lambda+2n)|^2\,d\lambda \\
&\qquad\quad + \sum_{n\in\mathbf{Z}}\int_0^1 \widetilde{\beta}(2\lambda)\overline{\beta(\lambda+2n+1)}G(\lambda+2n+1)|\widehat{\varphi}(\lambda+2n+1)|^2 d\lambda \\
&\qquad = \int_0^1 \widetilde{\beta}(2\lambda)(\overline{\beta(\lambda)}G(\lambda)+\overline{\beta(\lambda+1)}G(\lambda+1))\,d\lambda,
\end{aligned}$$

where we used the periodicity and symmetry properties of $\beta$, $\widetilde{\beta}$, $G$ as well as Corollary 4.3. Since $\widetilde{\beta} \in S$ is arbitrary, we conclude that the vectors $(\beta(\lambda), \beta(\lambda+1))^T$ and $(G(\lambda), G(\lambda+1))^T$ must be orthogonal in $\mathbf{C}^2$ for almost all $\lambda$. This means that

$$\begin{pmatrix}\beta(\lambda)\\ \beta(\lambda+1)\end{pmatrix} = \alpha(\lambda)\begin{pmatrix}\overline{G(\lambda+1)}\\ -\overline{G(\lambda)}\end{pmatrix} \tag{5.2}$$

for some function $\alpha : [0,1] \to \mathbf{C}$. Taking the norm in $\mathbf{C}^2$ on both sides and using (4.15) yields

$$|\beta(\lambda)|^2 + |\beta(\lambda+1)|^2 = |\alpha(\lambda)|^2,$$

which by $\beta \in S$ implies that $\alpha$ belongs to $L^2[0,1]$. Since $\beta$ and $G$ are 2-periodic on $\mathbf{R}$, $\beta$ is odd and $G$ is even, an extension of $\alpha$ to $\mathbf{R}$ must be 2-periodic and odd; hence $\alpha \in S$ with $\beta(\lambda) = \alpha(\lambda)\overline{G(\lambda+1)}$. Using the 2-periodicity of $\beta$ and $G$ in the second component of (5.2), we further deduce that $\beta(\lambda) = -\alpha(\lambda+1)\overline{G(\lambda+1)}$ and therefore $\alpha(\lambda+1) = -\alpha(\lambda)$ apart from the zero-set of $G(\lambda+1)$. If $G(\lambda+1) = 0$ we simply define $\alpha(\lambda+1) := -\alpha(\lambda)$, which clearly does not affect the equation in (i). Moreover, in this case $\alpha(\lambda)$ is well defined, since $G(\lambda) = 1$ by (4.15) except on a null-set. Thus $\widehat{f}$ is of the claimed form. Conversely, if $\alpha \in S_0$, then $\beta(\lambda) := \alpha(\lambda)\overline{G(\lambda+1)} \in S$, and (5.2) is satisfied.

For the proof of (ii), we calculate

$$\begin{aligned}
\|f_\alpha\|_2^2 = \|\widehat{f_\alpha}\|_2^2 &= \sqrt{2\pi}\int_0^\infty \frac{|\alpha(\lambda)|^2}{\lambda^2}\,|G(\lambda+1)|^2|\widehat{\varphi}(\lambda)|^2\,d\omega(\lambda) \\
&= \int_{-\infty}^{\infty} |\alpha(\lambda)|^2|G(\lambda+1)|^2|\widehat{\varphi}(\lambda)|^2\,d\lambda = \int_{-1}^1 |\alpha(\lambda)|^2|G(\lambda+1)|^2\,d\lambda,
\end{aligned}$$

where we used that $\alpha$ and $G$ are 2-periodic and $\varphi$ is an orthonormal scaling function. By assumption on $\alpha$, we have $\alpha(\lambda-1) = \alpha(\lambda+1) = -\alpha(\lambda)$. Thus, by Lemma 4.10,

$$\begin{aligned}
\int_{-1}^1 |\alpha(\lambda)|^2|G(\lambda+1)|^2\,d\lambda &= \int_0^1 |\alpha(\lambda-1)|^2|G(\lambda)|^2\,d\lambda + \int_0^1 |\alpha(\lambda)|^2|G(\lambda+1)|^2\,d\lambda \\
&= \int_0^1 |\alpha(\lambda)|^2\,d\lambda = \|\alpha\|_2^2.
\end{aligned}$$

This proves (ii). ■

It is now easy to obtain an orthonormal basis of $W_{-1}$. Recall that the $s_k,\ k \in \mathbf{N}$, form an orthonormal basis of $S$. Moreover, the $s_{2k-1},\ k \in \mathbf{N}$, are an orthonormal basis of $S_0$. Thus by the previous result, the functions

$$f_k := f_{s_{2k-1}} \qquad (k \in \mathbf{N})$$

constitute an orthonormal basis of $W_{-1}$. Define $\psi \in L^2(H)$ by

$$\widehat{\psi}(2\lambda) = \overline{G(\lambda+1)}\,\widehat{\varphi}(\lambda). \tag{5.3}$$

Then, in view of (4.3),

$$f_k = \frac{M_{2k-1}}{2} T^{(2k-1)} D_2 \psi.$$

To obtain an orthonormal basis of $W_0$, we just have to rescale. Extending the notation $T^{(r)} := T_{\pi r}$ to $r \in \mathbf{N}/2$ and using the relation $D_a T_x = T_{ax} D_a$, we obtain that an orthonormal basis of $W_0$ is given by the functions

$$\psi_k := D_{1/2} f_k = \frac{M_{2k-1}}{2} T^{((2k-1)/2)} \psi, \qquad k \in \mathbf{N}.$$

We call $\psi$ a (basic) wavelet for the radial mutiresolution $(V_j)$.

**Definition 5.2.** For $j \in \mathbf{Z}$ and $k \in \mathbf{N}$, define the "radial" wavelets

$$\psi_{j,k}(r) := D_{2^{-j}} \psi_k(r) = 8^{j/2} \frac{M_{2k-1}}{2} T^{((2k-1)/2)} \psi(2^j r).$$

We have proven:

**Theorem 5.3.**

(i) *For each $j \in \mathbf{Z}$, the set $\{\psi_{j,k} : k \in \mathbf{N}\}$ constitutes an orthonormal basis of $W_j$.*
(ii) *The set $\{\psi_{j,k} : j \in \mathbf{Z},\ k \in \mathbf{N}\}$ is an orthonormal wavelet basis of $L^2(H)$.*

**Corollary 5.4.** *The functions*

$$\Psi_{j,k}(x) := \psi_{j,k}(|x|), \qquad x \in \mathbf{R}^3, \quad j \in \mathbf{Z}, \quad k \in \mathbf{N},$$

*form an orthonormal basis for the closed subspace $L^2_{\mathrm{rad}}(\mathbf{R}^3) := \{f \in L^2(\mathbf{R}^3) : f \text{ radial}\}$ of radial functions in $L^2(\mathbf{R}^3)$.*

## 6. Construction of Radial Scaling Functions and Wavelets

Yet, we do not have handsome criteria in order to decide whether a given function $\varphi \in L^2(H)$ is a radial scaling function, i.e., a scaling function for a radial MRA. The analogy of our constructions to those on the group $(\mathbf{R}, +)$, however, leads to the following close relationship:

**Theorem 6.1.** *Suppose* $\varphi_{\mathbf{R}}$ *is a classical scaling function on* $\mathbf{R}$ *which is even and such that its (classical) Fourier transform* $\mathcal{F}(\varphi_{\mathbf{R}})$ *is continuous at* 0 *and satisfies* $\mathcal{F}(\varphi_{\mathbf{R}}) \in L^2(H)$. *Define* $\varphi \in L^2(H)$ *via its Hankel transform,*

$$\widehat{\varphi}(\lambda) := \sqrt{2\pi}\, \mathcal{F}(\varphi_{\mathbf{R}})(\pi\lambda). \tag{6.1}$$

*Then* $\varphi$ *is a radial scaling function.*

*Conversely, if* $\varphi$ *is a scaling function for a radial MRA such that* $\widehat{\varphi}$ *is continuous at* 0, *then* $\widehat{\varphi} \in L^2(\mathbf{R})$ *and the function* $\varphi_{\mathbf{R}}$ *defined by* (6.1) (*where* $\widehat{\varphi}$ *is extended to an even function on* $\mathbf{R}$) *is a classical scaling function on* $\mathbf{R}$.

*Moreover,* $\varphi$ *is an* orthonormal *radial scaling function if and only if* $\varphi_{\mathbf{R}}$ *is an* orthonormal *classical scaling function*.

**Proof.** Let us start with the first assertion. As $\varphi_{\mathbf{R}}$ is a classical scaling function, we have, by eq. (5.3.2) in [6],

$$\frac{A}{2\pi} \leq \sum_{k\in\mathbf{Z}} |\mathcal{F}(\varphi_{\mathbf{R}})(\xi + 2\pi k)|^2 \leq \frac{B}{2\pi} \quad \text{a.e.}$$

with suitable constants $0 < A \leq B < \infty$. Moreover, $\varphi_{\mathbf{R}}$ is orthonormal if and only if $A = B = 1$. Since $\varphi_{\mathbf{R}}$ is assumed to be even, definition (6.1) is compatible with the even extension of $\widehat{\varphi}$. By Proposition 4.2, the set $\{M_k T^{(k)}\varphi,\ k \in \mathbf{N}\}$ forms a Riesz basis for $V_0 = \overline{\text{span}\{B_\varphi\}}$ which is an orthonormal basis if and only if $\varphi_{\mathbf{R}}$ is orthonormal; see Corollary 4.3. Moreover, by eq. (5.3.18) of [6], there exists a $2\pi$-periodic function $m_0 \in L^2([-\pi, \pi])$ such that $\mathcal{F}(\varphi_{\mathbf{R}})(\xi) = m_0(\xi/2)\mathcal{F}(\varphi_{\mathbf{R}})(\xi/2)$, and $m_0$ is necessarily even in our case. Hence, with $\gamma(\lambda) := m_0(\lambda\pi)\sin(2\pi\lambda)$ which clearly is contained in $S$, we have $\sin(2\pi\lambda)\widehat{\varphi}(2\lambda) = \gamma(\lambda)\widehat{\varphi}(\lambda)$. This is exactly the radial two-scale equation (4.9). As $\widehat{\varphi}$ is continuous in 0, the condition $\widehat{\varphi}(0) \neq 0$ of Theorem 4.9 is automatically satisfied (see, e.g., Remark 3 on p. 144 in [6]), and thus we finally obtain that $\varphi$ is a radial scaling function.

For the converse part notice first that continuity of $\widehat{\varphi}$ in 0 already implies that $\widehat{\varphi} \in L^2(\mathbf{R})$. We further proceed similarly as before, using Propositions 5.3.1 and 5.3.2 in [6] and the corresponding results of the present paper. Hereby, it is important to note that the filter function of $\varphi_{\mathbf{R}}$ is given by

$$m_0(\xi) = \frac{\gamma(\xi/\pi)}{\sin(2\xi)} = G(\xi/\pi).$$

By Lemma 4.10, $m_0$ is thus contained in $L^2([-\pi, \pi])$. ■

This theorem supplies a variety of radial scaling functions since there are many classical scaling functions on $\mathbf{R}$ which satisfy the assumptions of the theorem. However, as to orthonormal radial scaling functions with compact support, a famous theorem of Daubechies implies the following negative result.

**Corollary 6.2.** *There do not exist any real-valued orthonormal radial scaling functions with compact support.*

**Proof.** The proof of Theorem 8.1.4 in [6] shows that an even, real-valued and compactly supported scaling function is necessarily the Haar function $\chi_{[-1/2,1/2]}$, the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$. However, its Fourier transform $\mathcal{F}(\chi_{[-1/2,1/2]})(\xi) = \sqrt{2/\pi}\,\sin(\xi/2)/\xi$ is not contained in $L^2(H)$. ■

If $\varphi$ corresponds to an even classical scaling function $\varphi_{\mathbf{R}}$ according to Theorem 6.1, then the hypergroup translates $\varphi_{0,k} = M_k T_{k\pi}\varphi$ may be expressed according to the formula

$$\varphi_{0,k}(x) = \frac{1}{(2\pi)^{1/4}\,x}\left(\varphi_{\mathbf{R}}\left(\frac{x}{\pi} - k\right) - \varphi_{\mathbf{R}}\left(\frac{x}{\pi} + k\right)\right). \tag{6.2}$$

In fact, by the Plancherel theorem for the Hankel transform and (2.5),

$$T_r\varphi(s) = \int_0^\infty j(\lambda r)\,j(\lambda s)\widehat{\varphi}(\lambda)\,d\omega(\lambda) = \frac{1}{\sqrt{2\pi}\cdot rs}\int_{-\infty}^\infty \widehat{\varphi}(\lambda)\sin(s\lambda)\sin(r\lambda)\,d\lambda$$

for all $r, s \in \mathbf{R}_+$. Here $\widehat{\varphi}$ is as usual extended to an even function on $\mathbf{R}$. Using relation (6.1) as well as basic trigonometric identities and the Plancherel theorem for the classical Fourier transform, we can write

$$\begin{aligned}
T_r\varphi(s) &= \frac{1}{2\sqrt{2\pi}\cdot rs}\int_{-\infty}^\infty \widehat{\varphi}(\lambda)(\cos\lambda(r-s) - \cos\lambda(r+s))\,d\lambda \\
&= \frac{1}{2rs}\int_{-\infty}^\infty \mathcal{F}(\varphi_{\mathbf{R}})(\pi\lambda)(e^{i\lambda(r-s)} - e^{i\lambda(r+s)})\,d\lambda \\
&= \frac{1}{\sqrt{2\pi}\cdot rs}\left(\varphi_{\mathbf{R}}\left(\frac{r-s}{\pi}\right) - \varphi_{\mathbf{R}}\left(\frac{r+s}{\pi}\right)\right).
\end{aligned}$$

This implies (6.2).

As an example, we consider the radial analogue of the Shannon wavelets. We define the scaling function via its Hankel transform,

$$\widehat{\varphi}(\lambda) = \chi_{[0,1]}(\lambda), \qquad \varphi(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(x) - x\cos(x)}{x^3}.$$

Constructing the associated basic wavelet according to formula (5.3) yields (after a short calculation)

$$\hat{\psi}(\lambda) = \chi_{[1,2]}(\lambda), \qquad \psi(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(2x) - \sin(x) - 2x\cos(2x) + x\cos(x)}{x^3}.$$

The translates of the scaling function and the wavelet turn out to be

$$\begin{aligned}
\varphi_{0,k}(x) &= \frac{1}{(2\pi)^{1/4}x}\left(\frac{\sin(x-k\pi)}{x-k\pi} - \frac{\sin(x+k\pi)}{x+k\pi}\right), \qquad k \in \mathbf{N}, \\
\psi_{0,k}(x) &= \frac{1}{(2\pi)^{1/4}x}\left(p\left(x - \frac{2k-1}{2}\pi\right) - p\left(x + \frac{2k-1}{2}\pi\right)\right), \qquad k \in \mathbf{N},
\end{aligned}$$

with $p(x) = (\sin(2x) - \sin(x))/x$.

## 7. Algorithms

For the use of our radial multiresolution in applications we need to formulate decomposition and reconstruction algorithms. The first step in such an algorithm consists of projecting the function $f$ into a scale space $V_j$ for some suitable $j$. We obtain a representation

$$P_j f \;=\; \sum_{k=1}^{\infty} c_k^{(j)} \varphi_{j,k}.$$

We shall discuss below how to obtain an approximation of the coefficients $c_k^{(j)}$.

So from now on we assume that we have given a function $f \in V_j$ in terms of its coefficients $c_k^{(j)}$. The decomposition algorithm consists of decomposing $f$ into $V_{j-1}$ and $W_{j-1}$, i.e., of calculating the coefficients $c_k^{(j-1)}$ and $d_k^{(j-1)}$ in the representation

$$f \;=\; \sum_{k=1}^{\infty} c_k^{(j-1)} \varphi_{j-1,k} + \sum_{k=1}^{\infty} d_k^{(j-1)} \psi_{j-1,k}.$$

(Such a representation exists, since by construction $\{\varphi_{j-1,k}, \psi_{j-1,k} \;:\; k \in \mathbf{N}\}$ is also a basis of $V_j$.) A reconstruction algorithm determines the coefficients $c_k^{(j)}$ when $f$ is given in terms of $c_k^{(j-1)}$ and $d_k^{(j-1)}$, $k \in \mathbf{N}$.

We still assume that $\varphi$ is an orthonormal scaling function (and $\psi$ is hence an orthonormal wavelet). Let

$$q_\ell^{(k)} \;:=\; \langle \varphi_{1,k}, \varphi_{0,l} \rangle \;=\; \langle \varphi_{j,k}, \varphi_{j-1,l} \rangle, \qquad r_\ell^{(k)} \;:=\; \langle \varphi_{1,k}, \psi_{0,l} \rangle \;=\; \langle \varphi_{j,k}, \psi_{j-1,l} \rangle.$$

By using Hilbert space techniques—in particular, Parseval's equation—we obtain analogously as in standard wavelet theory the decomposition formulas

$$c_\ell^{(j-1)} \;=\; \sum_{k=1}^{\infty} c_k^{(j)} q_\ell^{(k)}, \qquad d_\ell^{(j-1)} \;=\; \sum_{k=1}^{\infty} c_k^{(j)} r_\ell^{(k)},$$

and the reconstruction formula

$$c_k^{(j)} \;=\; \sum_{\ell=1}^{\infty} c_\ell^{(j-1)} \overline{q_\ell^{(k)}} + \sum_{\ell=1}^{\infty} d_\ell^{(j-1)} \overline{r_\ell^{(k)}}.$$

It turns out that the coefficients $q_\ell^{(k)}$ and $r_\ell^{(k)}$ are determined in terms of the numbers $g_n$ in the cosine expansion of $G$, i.e., the coefficients in

$$G(\lambda) \;=\; \sqrt{2} \sum_{n=0}^{\infty} g_n \cos(n\pi\lambda).$$

**Theorem 7.1.** *For $\ell, k \in \mathbf{N}$ it holds that*

$$q_\ell^{(k)} \;=\; \begin{cases} \overline{g_{k-2\ell} - g_{2\ell+k}} & \textit{for } 2\ell < k, \\ \overline{2g_0 - g_{4\ell}} & \textit{for } 2\ell = k, \\ \overline{g_{2\ell-k} - g_{2\ell+k}} & \textit{for } 2\ell > k, \end{cases}$$

$$r_\ell^{(k)} \;=\; \begin{cases} (-1)^{k-1}(g_{k-2\ell+1} - g_{k+2\ell-1}) & \textit{for } 2\ell - 1 < k, \\ 2g_0 - g_{4\ell-2} & \textit{for } 2\ell - 1 = k, \\ (-1)^{k-1}(g_{2\ell-1-k} - g_{2\ell-1+k}) & \textit{for } 2\ell - 1 > k. \end{cases}$$

**Proof.** Using the Plancherel theorem, relation (4.15), and Corollary 4.3 we obtain

$$
\begin{aligned}
q_\ell^{(k)} &= \langle \varphi_{1,k}, \varphi_{0,\ell}\rangle = \langle \hat\varphi_{1,k}, \hat\varphi_{0,\ell}\rangle \\
&= 8^{-1/2}\int_0^\infty \rho_k(\lambda/2)\rho_\ell(\lambda)\overline{G(\lambda/2)}|\hat\varphi(\lambda/2)|^2\, d\omega(\lambda) \\
&= \sqrt{2}\int_0^1 s_k(\lambda)s_\ell(2\lambda)\overline{G(\lambda)}\sum_{n=-\infty}^{\infty}|\hat\varphi(\lambda+2n)|^2\, d\lambda \\
&= 4\sum_{n=0}^{\infty}\overline{g_n}\int_0^1 \sin(k\lambda\pi)\sin(2\ell\lambda\pi)\cos(n\pi\lambda)\, d\lambda.
\end{aligned}
\tag{7.1}
$$

An easy calculation using trigonometric identities shows that

$$
\int_0^1 \sin(k\lambda\pi)\sin(t\lambda\pi)\cos(n\pi\lambda)\, d\lambda = \begin{cases} \frac14(\delta_{n,|t-k|}-\delta_{n,t+k}) & \text{for } n>0,\\ \frac14(2\delta_{0,t-k}-\delta_{0,t+k}) & \text{for } n=0. \end{cases}
\tag{7.2}
$$

Setting $t = 2\ell$ and inserting into (7.1) yields the assertion for $q_\ell^{(k)}$. We proceed similarly for $r_\ell^{(k)}$:

$$
\begin{aligned}
r_\ell^{(j)} &= \langle \varphi_{1,k}, \psi_{0,\ell}\rangle = \sqrt{2}\int_0^1 s_k(\lambda)s_{2\ell-1}(\lambda)G(\lambda+1)\, d\lambda \\
&= 4\sum_{n=0}^{\infty} g_n\int_0^1 \sin(k\pi\lambda)\sin((2\ell-1)\pi\lambda)\cos(n\pi(\lambda+1))\, d\lambda \\
&= 4\sum_{n=0}^{\infty} g_n(-1)^n\int_0^1 \sin(k\pi\lambda)\sin((2\ell-1)\pi\lambda)\cos(n\pi\lambda)\, d\lambda.
\end{aligned}
$$

Setting $t = 2\ell - 1$ in (7.2) and inserting into the last expression gives the result for $r_\ell^{(k)}$. ∎

Let us consider the case where only finitely many coefficients $g_k$ are different from zero. Although this is not possible for real-valued orthonormal scaling functions this assumption makes it easier to compare the radial wavelet algorithm with the classical one. Of course, in applications one can only handle finitely many coefficients anyway. So let us assume $\operatorname{supp} g \subset [0, N]$, i.e., $g_k = 0$ for $k \notin \{0, \ldots, N\}$. Elementary considerations show the following. Leaving $k$ fixed yields

$$
\begin{aligned}
q_\ell^{(k)} &= 0 \quad \text{for } \ell \notin \left[\frac{k-N}{2}, \frac{k+N}{2}\right] && \text{if } k > N,\\
q_\ell^{(k)} &= 0 \quad \text{for } \ell \notin \left[1, \frac{k+N}{2}\right] && \text{if } k \le N,\\
r_\ell^{(k)} &= 0 \quad \text{for } \ell \notin \left[\frac{k-N+1}{2}, \frac{N+k+1}{2}\right] && \text{if } k > N+1,\\
r_\ell^{(k)} &= 0 \quad \text{for } \ell \notin \left[1, \frac{N+k+1}{2}\right] && \text{if } k \le N+1.
\end{aligned}
\tag{7.3}
$$

If $\ell$ is fixed, then

$$\begin{aligned}
q_\ell^{(k)} &= 0 \quad \text{for } k \notin [2\ell - N, 2\ell + N] && \text{if } 2\ell > N,\\
q_\ell^{(k)} &= 0 \quad \text{for } k \notin [1, N + 2\ell] && \text{if } 2\ell \le N,\\
r_\ell^{(k)} &= 0 \quad \text{for } k \notin [2\ell - 1 - N, 2\ell - 1 + N] && \text{if } 2\ell - 1 > N,\\
r_\ell^{(k)} &= 0 \quad \text{for } k \notin [1, 2\ell - 1 + N] && \text{if } 2\ell - 1 \le N.
\end{aligned}$$

With

$$h_k := \begin{cases} g_{|k|} & \text{for } 1 \le |k| \le N,\\ 2g_0 & \text{for } k = 0,\\ 0 & \text{otherwise,} \end{cases}$$

it holds that $G(\lambda) = (1/\sqrt{2}) \sum_{k=-N}^{N} h_k e^{ik\pi\lambda}$. Because of the conditions on it, $G$ is also the filter function for an ordinary MRA on $\mathbf{R}$ with coefficients $h_k$. Now, if $2\ell > N$ (resp., $2\ell - 1 > N$), then it is easy to see that

$$\begin{aligned}
q_\ell^{(2\ell+k)} &= \overline{h_k} && \text{for all } k \ge -2\ell + 1,\\
r_\ell^{(2\ell-1+k)} &= (-1)^k h_k && \text{for all } k \ge -2\ell + 2.
\end{aligned}$$

Similarly, if $k > N$, then

$$q_\ell^{(k)} = \overline{h_{k-2\ell}}, \qquad r_\ell^{(k)} = (-1)^{k+1} h_{k+1-2\ell} \qquad \text{for all } \ell \in \mathbf{N}.$$

Hence, for $2\ell > N + 1$, the decomposition formulas become

$$c_\ell^{(j-1)} = \sum_{k=1}^{\infty} c_k^{(j)} \overline{h_{k-2\ell}}, \qquad d_\ell^{(j-1)} = \sum_{k=1}^{\infty} c_k^{(j)} (-1)^{k+1} h_{k+1-2\ell},$$

and, for $k > N + 1$, the reconstruction formula is

$$c_k^{(j)} = \sum_{\ell=1}^{\infty} c_\ell^{(j-1)} h_{k-2\ell} + \sum_{\ell=1}^{\infty} d_\ell^{(j-1)} (-1)^{k+1} \overline{h_{k+1-2\ell}}.$$

These formulas are well known. Indeed, they are the decomposition and reconstruction formulas of the classical discrete wavelet transform. So our approach leads to the classical algorithm if we are far enough away from the origin. If we are close to the origin we have derived an algorithm to handle the boundary point 0.

Let us finally discuss how to obtain an approximation of the coefficients $c_k^{(j)}$ representing $P_j f$ when the function $f$ is given. We assume for the moment that the classical scaling function $\varphi_{\mathbf{R}}$ related to $\varphi$ as in Theorem 6.1 is interpolatory, i.e., $\varphi_{\mathbf{R}}(k) = c\delta_{0,k}$ for $k \in \mathbf{Z}$, $c \neq 0$. According to (6.2) we have

$$P_j f(x) = \sum_{k=1}^{\infty} c_k^{(j)} \frac{8^{j/2}}{(2\pi)^{1/4} 2^j x} \left( \varphi_{\mathbf{R}}\left( \frac{2^j x}{\pi} - k \right) - \varphi_{\mathbf{R}}\left( \frac{2^j x}{\pi} + k \right) \right).$$

Letting $x = \pi\ell 2^{-j}$, $\ell \in \mathbf{N}$, and reordering yields

$$\sum_{k=1}^{\infty} c_k^{(j)} \varphi_{\mathbf{R}}(\ell - k) \;=\; M_\ell D_{2^j}(P_j f)(\ell\pi) + \sum_{l=1}^{\infty} c_k^{(j)} \varphi_{\mathbf{R}}(\ell + k).$$

Using the interpolatory condition we obtain

$$c_\ell^{(j)} \;=\; c^{-1} M_\ell D_{2^j}(P_j f)(\ell\pi) \;\approx\; c^{-1} M_\ell D_{2^j}(f)(\ell\pi). \tag{7.4}$$

We note that by axiom (3) it is reasonable that $P_j f(x) \approx f(x)$ if $j$ is large enough. So we suggest using the formula

$$c_k^{(j)} \;=\; c^{-1} M_k D_{2^j}(f)(k\pi)$$

as a heuristic approximation. As in the classical case this should also be reasonable even if $\varphi_{\mathbf{R}}$ is not interpolatory.

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