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$SU(d)$-Biinvariant Random Walks on $SL(d,\mathbb{C})$ and their Euclidean Counterparts

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Abstract We establish a deformation isomorphism between the algebras of $SU(d)$-biinvariant compactly supported measures on $SL(d,\mathbb{C})$ and $SU(d)$-conjugation invariant measures on the Euclidean space H_d^0 of all Hermitian $d\times d$-matrices with trace 0. This isomorphism concisely explains a close connection between the spectral problem for sums of Hermititan matrices on one hand and the singular spectral problem for products of matrices from $SL(d,\mathbb{C})$ on the other, which has recently been observed by Klyachko [13]. From this deformation we further obtain an explicit, probability preserving and isometric isomorphism between the Banach algebra of bounded $SU(d)$-biinvariant measures on $SL(d,\mathbb{C})$ and a certain (non-invariant) subalgebra of the bounded signed measures on H_d^0. We demonstrate how this probability preserving isomorphism leads to limit theorems for the singular spectrum of $SU(d)$-biinvariant random walks on $SL(d,\mathbb{C})$ in a simple way. Our construction relies on deformations of hypergroup convolutions and will be carried out in the general setting of complex semisimple Lie groups.

Key words algebras of biinvariant measures · noncompact complex semisimple Lie groups · hermitian matrices · deformation of hypergroups · random walks · limit theorems.

Mathematics Subject Classifications (2000) Primary: 43A10 · Secondary: 43A85 · 60B15 · 43A62 · 22E46

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1. Introduction

The spectral problem for possible sums of two random Hermitian matrices with given spectra had been a long-standing problem, formulated as Horn's conjecture, until it was completely solved only recently by Klyachko, Knutson and Tao [12, 15]. In [13], Klyachko then observed a close connection between $SU(d)$- biinvariant random walks on $SL(d,\mathbb{C})$ and random walks on the additive group H_d^0 of all Hermitian $d\times d$-matrices with trace 0 whose transition probabilities are conjugation-invariant under $SU(d)$. He used this connection to reduce the description of the possible singular spectra of products of random matrices from $SL(d,\mathbb{C})$ with given singular spectra to the spectral problem for sums of Hermitian matrices. Basically, Klyachko's observation is a connection between the convolution algebras of the Gelfand pairs $(SL(d,\mathbb{C}),SU(d))$ and $(SU(d)\ltimes H_d^0,SU(d))$. It is closely related to a similar correspondence between the convolutions of conjugation-invariant measures on a compact Lie group (here $SU(d)$) on one hand and Ad-invariant measures on its Lie algebra (here iH_d^0) on the other, in terms of the so-called wrapping map (see [4] and Remark 3.1). Klyachko noticed in [13], but did not explain that his connection can be well expressed in terms of hypergroups; his proof goes via random walks in the group $SU(d)$ and relies on various identities between the spherical functions of $(SL(d,\mathbb{C}),SU(d))$, the characters of $SU(d)$ and the Euclidean group H_d^0, as well as Poisson's summation formula.

The main purpose of the present paper is to clarify and simplify Klyachko's approach [13] by using so-called deformations of hypergroup convolutions by positive semicharacters as introduced in [17, 18]. Our description in particular implies that the Banach algebra $M_b(SL(d,\mathbb{C})\|SU(d))$ of $SU(d)$-biinvariant signed measures on $SL(d,\mathbb{C})$ with total variation norm is isometrically isomorphic to a certain Banach subalgebra of $M_b(H_d^0)$, whereby probability measures are being preserved. We finally show how this leads to new proofs for (known) limit theorems for $SU(d)$-biinvariant random walks on $SL(d,\mathbb{C})$. The explicit construction of this isomorphism runs as follows:

(1) $SU(d)$ acts on H_d^0 by conjugation as a group of orthogonal transformations. The space $(H_d^0)^{SU(d)}$ of all orbits can be identified with

$$C:=\{x\in\mathbb{R}^d:\ x_1\geq x_2\geq\ldots\geq x_d,\ \sum_i x_i=0\},$$

where x represents the ordered eigenvalues of a matrix in H_d^0. The Banach algebra $M_b^{SU(d)}(H_d^0)$ of all $SU(d)$-invariant bounded measures on H_d^0 can be identified with the Banach algebra $(M_b(C),*)$ of the associated orbit hypergroup $C\simeq(H_d^0)^{SU(d)}$.

(2) The polar decomposition of $SL(d,\mathbb{C})$ shows that the double coset space $SL(d,\mathbb{C})//SU(d)$ may also be identified with C where now for $x\in C$, e^x corresponds to the singular spectrum $\sigma_{sing}(A)=\sigma(\sqrt{AA^*})$ of some $A\in SL(d,\mathbb{C})$. This leads to a canonical Banach algebra isomorphism between $M_b(SL(d,\mathbb{C})\|SU(d))$ and the Banach algebra $(M_b(C),\bullet)$ of the double coset hypergroup $C\simeq SL(d,\mathbb{C})//SU(d)$.

(3) The characters of the hypergroups $(C,\bullet)$ and $(C,*)$ are spherical functions of the corresponding symmetric spaces, and it is well-known that as functions on C, they only differ by a known factor $J^{-1/2}(x)>0$ (see [9]). We prove that such

a connection between the characters of two hypergroup structures on C implies that their convolutions are related by a so-called hypergroup deformation. In particular, the supports of the convolution products of two point measures are the same in both hypergroups. This immediately implies the equivalence of the two spectral problems for Hermitian and unitary matrices, the main result of Klyachko [13]. The deformation isomorphism, however, is not isometric and not probability preserving. To achieve this, a final correction is needed:

(4) Instead of looking at the Banach algebra $M_b^{SU(d)}(H_d^0)$ of all $SU(d)$-invariant measures on H_d^0, we take a suitable exponential function $e_\rho\colon H_d^0 \to]0,\infty[$ and observe that the norm-closure of

$$\{e_\rho \cdot \mu:\ \mu \in M_b^{SU(d)}(H_d^0),\ \operatorname{supp}\mu \text{ compact}\}$$

is a Banach algebra which turns out to be isometrically isomorphic to $(M_b(C), \bullet)$.

Putting all steps together one obtains the claimed probability preserving isometric isomorphism. For $d=2$ this construction reflects the known close connection between the so-called Naimark hypergroup $SL(2,\mathbb{C})//SU(2)$ and the Bessel–Kingman hypergroup $(\mathbb{R}^3)^{SO(3)}$; see [1, 17].

All results will be derived in the general setting of a complex connected semisimple Lie group G with finite center and maximal compact subgroup K; in the special situation above, $(G,K) = (SL(d,\mathbb{C}), SU(d))$. For further results concerning the associated hypergroup convolutions considered in this paper we refer to [5] and [6].

The paper is organized as follows: In Section 2 we collect some relevant facts on commutative hypergroups and their deformations. In Section 3 we then use this deformation to show how the Banach algebras of all K-biinvariant bounded complex measures on G appear as subalgebras of the Banach algebra of bounded complex measures on some Euclidean space. The final section is devoted to probabilistic applications.

2. Deformations of Commutative Hypergroups

In this section we give a quick introduction to hypergroups. We in particular prove a general result on deformations of hypergroup convolutions in Proposition 2.6 which is crucial for step (3) of our construction. Moreover, step (4) will be explained.

We first fix some notations. For a locally compact Hausdorff space X, $M^+(X)$ denotes the space of all positive Radon measures on X, and $M_b(X)$ the Banach space of all bounded regular complex Borel measures with the total variation norm. Moreover, $M^1(X) \subset M_b(X)$ is the set of all probability measures, $M_c(X) \subset M_b(X)$ the set of all measures with compact support, and δ_x the point measure in $x \in X$. The spaces $C(X) \supset C_b(X) \supset C_0(X) \supset C_c(X)$ of continuous functions are defined as usual. For details on the following we refer to [1] and [11].

2.1. Commutative Hypergroups

A hypergroup $(X,*)$ consists of a locally compact Hausdorff space K and a convolution $*$ on $M_b(X)$ such that $(M_b(X),*)$ becomes a Banach algebra, where $*$ is weakly continuous and probability preserving and preserves compact supports of measures. Moreover, there exists an identity $e \in X$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for $x \in X$, as well as a continuous involution $x \mapsto \overline{x}$ on X such that for $x, y \in X$, $e \in supp(\delta_x * \delta_y)$ is equivalent to $x = \overline{y}$, and $\delta_{\overline{x}} * \delta_{\overline{y}} = (\delta_y * \delta_x)^-$. Here for $\mu \in M_b(X)$, the measure μ^- is given by $\mu^-(A) = \mu(A^-)$ for Borel sets $A \subset X$.

A hypergroup $(X, *)$ is called commutative if and only if so is the convolution $*$. Hence, for a commutative hypergroup $(X, *)$ the triple $(M_b(X), *, .^-)$ is a commutative Banach-$*$-algebra with identity δ_e.

EXAMPLES 2.2.

(i) If G is a locally compact group, then $(G, *)$ is a hypergroup with the usual group convolution $*$.

(ii) Let K be a compact subgroup of a locally compact group G. Then

$$M_b(G\|K) := \{\mu \in M_b(G) :\ \delta_x * \mu * \delta_y = \mu\ \forall\ x, y \in K\}$$

is a Banach-$*$-subalgebra of $M_b(G)$ with identity dk where $dk \in M^1(G)$ is the normalized Haar measure of K embedded into G. Moreover, the double coset space $G//K := \{KxK :\ x \in G\}$ is a locally compact Hausdorff space, and the canonical projection $p : G \to G//K$ induces a probability preserving, isometric isomorphism $p : M_b(G\|K) \to M_b(G//K)$ of Banach spaces by taking images of measures. The transport of the convolution on $M_b(G\|K)$ to $M_b(G//K)$ via p leads to a hypergroup structure $(G//K, *)$ with identity $K \in G//K$ and involution $(KxK)^- := Kx^{-1}K$, and p becomes a probability preserving, isometric isomorphism of Banach-$*$-algebras.

(iii) Let $(V, \langle\, .\, , .\, \rangle)$ be a finite-dimensional Euclidean vector space and $K \subset O(V)$ a compact subgroup of the orthogonal group of V, acting continuously on V. For $\mu \in M_b(V)$, denote the image measure of μ under $k \in K$ by $k(\mu)$. Then the space of K-invariant measures

$$M_b^K(V) := \{\mu \in M_b(V) :\ k(\mu) = \mu\ \forall\ k \in K\}$$

is a Banach-$*$-subalgebra of $M_b(V)$ (with the group convolution), with identity δ_0. Moreover, the space $V^K := \{K.x :\ x \in V\}$ of all K-orbits in V is locally compact, and the canonical projection $p : V \to V^K$ induces a probability preserving, isometric isomorphism $p : M_b^K(V) \to M_b(V^K)$ of Banach spaces and an associated so-called orbit hypergroup $(V^K, *)$ such that p becomes a probability preserving, isometric isomorphism of Banach-$*$-algebras. The involution on $(V^K, *)$ is given by $\overline{K.x} = -K.x$.

We next collect some data of a commutative hypergroup $(X, *)$. By a result of R. Spector, there exists a (up to normalization) unique Haar measure $\omega \in M^+(X)$ which is characterized by $\omega(f) = \omega(f_x)$ for all $f \in C_c(X)$ and $x \in X$, where we use the notation

$$f_x(y) := f(x*y) := \int_X f\, d(\delta_x * \delta_y).$$

Similar to the dual of a locally compact abelian group, one defines the spaces

(a) $\chi(X) := \{\alpha \in C(X) :\ \alpha \neq 0,\ \alpha(x{*}y) = \alpha(x)\alpha(y)\ \forall\ x, y \in X\}$;
(b) $X^* := \{\alpha \in \chi(X) :\ \alpha(\overline{x}) = \overline{\alpha(x)}\ \forall\ x \in X\}$;
(c) $\widehat{X} := X^* \cap C_b(X)$.

The elements of X^* and $\widehat{X}$ are called semicharacters and characters, respectively. All spaces above are locally compact Hausdorff spaces w.r.t. the topology of compact-uniform convergence.

The Fourier transform on $L^1(X, \omega)$ is defined by

$$\widehat{f}(\alpha) := \int_X f(x)\overline{\alpha(x)}\, d\omega(x),\ \ \alpha \in \widehat{X}.$$

Similar, the Fourier–Stieltjes transform of $\mu \in M_b(X)$ is defined by $\widehat{\mu}(\alpha) := \int_X \overline{\alpha(x)} d\mu(x),\ \ \alpha \in \widehat{X}$. Both transforms are injective, c.f. [11]. In the following, we consider different hypergroup convolutions on X, and we write $\chi(X, *)$, ω_* etc. in order to specify the relevant convolution.

EXAMPLES 2.3.

(i) Assume that in the situation of 2.2(ii), $G//K$ is commutative. Then a K-biinvariant function $\varphi \in C(G)$ with $\varphi(e) = 1$ is by definition a spherical function of (G, K) if

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)\, dk \qquad \forall\ g, h \in G.$$

Multiplicative functions $\alpha \in \chi(G//K)$ are in one-to-one correspondence with spherical functions on G via $\alpha \mapsto \alpha \circ p$ for the projection $p : G \to G//K$. In this way, the Fourier(-Stieltjes) transform on $G//K$ corresponds to the spherical Fourier(-Stieltjes) transform.

(ii) In the situation of example 2.2(iii), the functions

$$\alpha_\lambda(K.x) = \int_K e^{i\langle\lambda,\, k.x\rangle} dk \quad (x \in V)$$

are continuous multiplicative functions of the orbit hypergroup $(V^K, *)$ for $\lambda \in V_{\mathbb{C}}$, the complexification of V, where $\alpha_\lambda \equiv \alpha_\mu$ holds if and only if $K.\lambda = K.\mu$. It is also well-known (see [11]) that $\widehat{V^K} = \{\alpha_\lambda :\ \lambda \in V\}$.

In [17], positive semicharacters were used to construct deformed hypergroup convolutions. More precisely, the following was proven there:

PROPOSITION 2.4. *Let $\alpha_0 \in X^*$ be a positive semicharacter on the commutative hypergroup $(X, *)$, i.e., $\alpha_0(x) > 0$ for $x \in X$. Then the convolution*

$$\delta_x \bullet \delta_y := \frac{1}{\alpha_0(x)\alpha_0(y)} \cdot \alpha_0(\delta_x {*} \delta_y) \qquad (x, y \in X)$$

extends uniquely to a bilinear, associative, probability preserving, and weakly continuous convolution $\bullet$ *on* $M_b(X)$. *Moreover,* $(X, \bullet)$ *becomes a commutative hypergroup with the identity and involution of* $(X, *)$. *For* $\mu, \nu \in M_c(X)$, *one has*

$$\alpha_0\mu \bullet \alpha_0\nu = \alpha_0(\mu * \nu). \tag{1}$$

Note that by Equation (1), the mapping $\mu \mapsto \alpha_0\mu$ establishes a canonical algebra isomorphism between $(M_c(X), *)$ and $(M_c(X), \bullet)$ which usually – when α_0 is unbounded – cannot be extended to $M_b(X)$. The hypergroup $(X, \bullet)$ is called the deformation of $(X, *)$ w.r.t. α_0. Clearly, many data of $(X, \bullet)$ can be expressed in terms of α_0 and corresponding data of $(X, *)$.

PROPOSITION 2.5. *In the above setting, we have*

(i) $\omega_\bullet := \alpha_0^2\omega_*$ *is a Haar measure of* $(X, \bullet)$.
(ii) The mapping $M_{\alpha_0}: \alpha \mapsto \alpha/\alpha_0$ *is a homeomorphism (w.r.t. the compact-uniform topology) between* $(X, *)^*$ *and* $(X, \bullet)^*$; *and also between* $\chi(X, *)$ *and* $\chi(X, \bullet)$.

Proof. For (i) and the first part of (ii) see [17]; the second part of (ii) is analogous. □

We next turn to the following converse statement; it will be crucial for this paper:

PROPOSITION 2.6. *Let* $(X, *)$ *and* $(X, \bullet)$ *be commutative hypergroups on* X. *Assume there is a positive semicharacter* α_0 *of* $(X, *)$ *such that the spaces of multiplicative continuous functions for* $(X, *)$ *and* $(X, \bullet)$ *are related via*

$$\chi(X, \bullet) = \left\{\frac{\alpha}{\alpha_0} : \alpha \in \chi(X, *)\right\}.$$

Then $(X, \bullet)$ *is the deformation of* $(X, *)$ *w.r.t.* α_0.

Proof. Let $(X, \circ)$ denote the deformation of $(X, *)$ via α_0. Take $\beta \in (X, \bullet)^\wedge$. Then by our assumption and the proposition above, β is multiplicative w.r.t. $\circ$ as well, and the Fourier–Stieltjes transforms of $\delta_x \circ \delta_y$ and $\delta_x \bullet \delta_y$ w.r.t. $(X, \bullet)$ satisfy

$$\begin{aligned}(\delta_x \circ \delta_y)^\wedge(\beta) &= \int_X \overline{\beta(z)}d(\delta_x \circ \delta_y)(z) = \overline{\beta(x \circ y)} \\ &= \overline{\beta(x)} \cdot \overline{\beta(y)} = (\delta_x \bullet \delta_y)^\wedge(\beta).\end{aligned}$$

By the injectivity of the Fourier–Stieltjes-transform on $M_b(X)$, we obtain $\delta_x \circ \delta_y = \delta_x \bullet \delta_y$. Thus the convolutions of $(X, \bullet)$ and $(X, \circ)$ coincide, and so do the involutions, because they are uniquely determined by the convolutions. □

EXAMPLE 2.7. It is well-known (see [1, 9, 11, 16]) that the double coset hypergroup $SL(2,\mathbb{C})//SU(2)$ may be realized as hypergroup $(X=[0,\infty[,\bullet)$ with multiplicative functions

$$\beta_\lambda(x):=\frac{\sin\lambda x}{\lambda\cdot\sinh x}\qquad(\lambda\in\mathbb{C}).$$

Via the correspondence of β_λ with λ, we have $\chi(X,\bullet)\simeq\mathbb{C}$. On the other hand, the orbit hypergroup $(\mathbb{R}^3)^{SO(3)}$ may be realized as the Bessel–Kingman hypergroup $(X=[0,\infty[,*)$ with multiplicative functions

$$\alpha_\lambda(x)=\frac{\sin\lambda x}{\lambda x}\qquad(\lambda\in\mathbb{C});$$

see [1, 11]. Clearly, $\alpha_i(x)=(\sinh x)/x$ is a positive semicharacter on $(X,*)$ and $\beta_\lambda=\alpha_\lambda/\alpha_i$ for $\lambda\in\mathbb{C}$. Proposition 2.6 implies the known fact that $(X,\bullet)$ is a deformation of $(X,*)$, c.f. [17].

We next consider further examples which explain step (4) in the introduction; they are similar to a construction in [18]. Let $(V,\langle\,.\,,\,.\,\rangle)$ be a Euclidean vector space of finite dimension n, $K\subset O(V)$ a compact subgroup of the orthogonal group of V, and $(V^K,*)$ the associated orbit hypergroup. Fix $\rho\in V$ with $-\rho\in K.\rho$, and consider the exponential

$$e_\rho(x):=e^{\langle\rho,x\rangle}$$

on V. Let further

$$M_c^{\rho,K}(V):=\{e_\rho\mu:\ \mu\in M_c(V)\ \ K-\text{invariant}\}.$$

The multiplicativity of e_ρ yields that $e_\rho\mu*e_\rho\nu=e_\rho(\mu*\nu)$, where $*$ denotes the group convolution on V. Hence $M_c^{\rho,K}(V)$ is a subalgebra of the Banach-$*$-algebra $M_b(V)$, and its norm-closure

$$M_b^{\rho,K}(V):=\overline{M_c^{\rho,K}(V)}$$

is a Banach subalgebra of $M_b(V)$. Notice that for $\rho\neq0$ this algebra is not closed under the involution on $M_b(V)$; for instance, the n-dimensional normal distribution $N_{\rho,I}$ with density $(2\pi)^{-n/2}e^{-|x-\rho|^2/2}$ is contained in $M_b^{\rho,K}(V)$ while this is not the case for $N_{\rho,I}^*=N_{-\rho,I}$. Nevertheless, we prove that $M_b^{\rho,K}(V)$ is isometrically isomorphic as a Banach algebra to the Banach algebra of measures of a suitable deformation of the orbit hypergroup $(V^K,*)$. More precisely:

PROPOSITION 2.8. *Let $\rho\in V$ with $-\rho\in K.\rho$ and define*

$$\alpha_0(K.x):=\int_K e_\rho(k.x)\,dk\quad(x\in V).$$

Then the following hold:

- *(i)* α_0 *is a positive semicharacter on* $(V^K, *)$.
- *(ii)* *If* $(V^K, \bullet)$ *is the deformation of* $(V^K, *)$ *w.r.t.* α_0, *then the canonical projection* $p : V \to V^K$ *induces (by taking image measures) a probability preserving isometric isomorphism of Banach algebras from* $M_b^{\rho,K}(V)$ *onto* $M_b(V^K, \bullet)$.

Proof.

(i) α_0 is obviously continuous and positive. For $x, y \in V$, we further have

$$\begin{aligned}\alpha_0(K.x * K.y) &= \int_{V^K} \alpha_0(K.z)\, d(\delta_{K.x} * \delta_{K.y})(K.z) \\ &= \int_K \int_K \int_K e_\rho(k.(k_1.x + k_2.y))\, dk dk_1 dk_2 \\ &= \int_K \left(\int_K e_\rho(kk_1.x)\, dk_1 \right) \cdot \left(\int_K e_\rho(kk_2.y)\, dk_2 \right) dk \\ &= \alpha_0(K.x)\, \alpha_0(K.y).\end{aligned}$$

Moreover, by the condition on ρ above and the properties of the Haar measure of K,

$$\alpha_0(\overline{K.x}) = \alpha_0(-K.x) = \int_K e^{\langle -\rho, k.x \rangle}\, dk = \int_K e^{\langle \rho, k.x \rangle}\, dk = \alpha_0(K.x).$$

Therefore, α_0 is a positive semicharacter on $(V^K, *)$.

(ii) Consider the diagram

$$\begin{array}{ccc} M_c^{0,K}(V) & \xrightarrow{\mu \;\mapsto\; e_\rho \mu} & M_c^{\rho,K}(V) \\ \downarrow & & \downarrow \\ (M_c(V^K), *) & \xrightarrow{\nu \;\mapsto\; \alpha_0 \nu} & (M_c(V^K), \bullet) \end{array}$$

where M_c always stands for a space of measures with compact support, and the vertical mappings are obtained by taking image measures w.r.t. p. The mapping p restricted to $M_c^{0,K}(V)$ and $M_c^{\rho,K}(V)$ is probability preserving and isometric. Moreover, for $\mu \in M_c^{0,K}(V)$ and $f \in C_c(V^K)$ we obtain by the K-invariance of μ and the definition of α_0,

$$\begin{aligned}\int f\, dp(e_\rho \mu) &= \int_V (f \circ p)\, e_\rho\, d\mu \\ &= \int_K \int_V (f \circ p)(k.x) e_\rho(k.x)\, d\mu(x)\, dk \\ &= \int_V (f \circ p)(x) \cdot \left(\int_K e_\rho(k.x)\, dk \right) d\mu(x) \\ &= \int_V (f \circ p)(\alpha_0 \circ p)\, d\mu = \int f \alpha_0\, dp(\mu),\end{aligned}$$

which proves that the diagram commutes. Therefore, as both horizontal mappings and the left vertical mapping are algebra isomorphisms, the right vertical mapping is also a probability preserving isometric isomorphism of Banach algebras from $M_c^{\rho,K}(V)$ onto $M_c(V^K, \bullet)$. A continuity and density argument shows that the Banach algebras $M_b^{\rho,K}(V)$ and $M_b(V^K, \bullet)$ are isomorphic via the probability preserving mapping p. □

3. Biinvariant Measures on Complex Noncompact Semisimple Lie Groups

We here identify the Banach algebra of all bounded Borel measures on a connected semisimple noncompact Lie group with finite center, which are biinvariant under a maximal compact subgroup, as a Banach algebra of measures in some Euclidean setting. For the general background we refer to [9, 10].

Let G be a complex, noncompact connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Consider the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra of G, and choose a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$. K acts on $\mathfrak{p}$ via the adjoint representation as a group of orthogonal transformations with respect to the Killing form as scalar product. Let further W be the Weyl group of K, which acts on $\mathfrak{a}$ as a finite reflection group, with root system $\sum \subset \mathfrak{a}$. Here and lateron, $\mathfrak{a}$ is always identified with its dual $\mathfrak{a}^*$ via the Killing form, which we denote by $\langle\,.\,,\,.\,\rangle$. We fix some Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and denote the associated system of positive roots by $\sum^+$. The closed chamber $C := \overline{\mathfrak{a}_+}$ is a fundamental domain for the action of W on $\mathfrak{a}$. Later on we shall need the half sum of roots,

$$\rho := \sum_{\alpha \in \Sigma^+} \alpha \in \mathfrak{a}_+ \,.$$

We now identify C with the orbit hypergroup $(\mathfrak{p}^K, *)$ where each K-orbit in $\mathfrak{p}$ corresponds to its unique representative in $C \subset \mathfrak{p}$. Then in view of example 2.3(ii) above and Prop. IV.4.8 of [9], the multiplicative continuous functions of $(C, *)$, considered as K-invariant functions on $\mathfrak{p}$, are given by

$$\psi_\lambda(x) = \int_K e^{i\langle \lambda,\, k.x\rangle} dk \quad (x \in \mathfrak{p}) \tag{2}$$

where λ runs through the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$. Moreover, $\psi_\lambda \equiv \psi_\mu$ iff λ and μ are in the same W-orbit. This is a special case of Harish–Chandra's integral formula for the spherical functions of a Cartan motion group. According to Theorem II.5.35 and Cor. II.5.36 of [9], they can also be written as

$$\psi_\lambda(x) = \frac{\pi(\rho)}{\pi(x)\pi(i\lambda)} \sum_{w \in W} (\det w) e^{i\langle \lambda,\, w.x\rangle} \tag{3}$$

with the fundamental alternating polynomial

$$\pi(\lambda) = \prod_{\alpha \in \Sigma^+} \langle \alpha, \lambda \rangle.$$

On the other hand, C can be identified with $G//K$ where $x \in C$ corresponds to the double coset $K(e^x)K$. According to this identification and the explicit formula for the spherical functions in Theorem IV.5.7 of [9], the multiplicative continuous functions on the commutative double coset hypergroup $(G//K, \bullet) = (C, \bullet)$ are (as functions on $\mathfrak{a}$) given by

$$\varphi_\lambda(x) = \frac{\pi(\rho) \sum_{w \in W} (\det w) e^{i\langle \lambda, wx \rangle}}{\pi(i\lambda) \sum_{w \in W} (\det w) e^{\langle \rho, wx \rangle}} \quad (x \in \mathfrak{a}) \tag{4}$$

with $\lambda \in \mathfrak{a}_{\mathbb{C}}$. Thus in particular,

$$\varphi_\lambda(x) = \frac{\psi_\lambda(x)}{\psi_{-i\rho}(x)} \qquad \forall\, x \in \mathfrak{a},\ \lambda \in \mathfrak{a}_{\mathbb{C}}. \tag{5}$$

Notice that $\psi_{-i\rho}$ is a positive semicharacter of $(C, *)$. By Weyl's formula ([9], Prop. I.5.15.),

$$\psi_{-i\rho}(x) = \prod_{\alpha \in \Sigma^+} \frac{\sinh\langle \alpha, x \rangle}{\langle \alpha, x \rangle}.$$

The square of $\psi_{-i\rho}$ (called $J(x)$ in [9]) determines the ratio of the volume elements in $\mathfrak{p}$ and G/K. Proposition 2.6 and Eq. (5) show that $(G//K, \bullet) = (C, \bullet)$ is the deformation of the orbit hypergroup $(\mathfrak{p}^K, *) = (C, *)$ via $\psi_{-i\rho}$. Moreover, we have $\psi_{-i\rho}(x) = \int_K e_\rho(k.x)dk$ with the half sum $\rho \in C$ and $e_\rho(x) = e^{\langle \rho, x \rangle}$. As the condition $-\rho \in K.\rho$ is satisfied, Proposition 2.8 further implies that the Banach algebra of measures of $(G//K, \bullet) = (C, \bullet)$ can be identified with $M_b^{\rho,K}(\mathfrak{p})$, which is the closure of

$$\{e_\rho \mu :\ \mu \in M_c(\mathfrak{p})\ \ K - \text{invariant}\}.$$

The claimed isomorphism between $M_b^{\rho,K}(\mathfrak{p})$ and $M_b(\mathfrak{p}^K, \bullet) \simeq (M_b(C), \bullet) \simeq M_b(G||K)$ is now given by taking image measures w.r.t. the canonical projection $\mathfrak{p} \mapsto \mathfrak{p}^K$.

Remark 3.1. The algebra isomorphism $M_c(\mathfrak{p}^K, *) \to M_c(G//K, \bullet), \mu \mapsto \alpha_0 \mu$ with the semicharacter $\alpha_0 = \psi_{-i\rho}$ is closely related to the so-called wrapping map for the compact Lie group K, see [4]. In fact, as G is complex, we have $\mathfrak{p} = i\mathfrak{k}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Thus $K \cong (K \times K)/K$ (on which K acts by conjugation) is the dual symmetric space of G/K. The wrapping map $\Phi : M_b(\mathfrak{k}) \to M_b(K)$ is defined by

$$\Phi(\mu)(f) := \mu(j\widetilde{f}), \quad f \in C(K)$$

where $\widetilde{f}(x) = f(\exp x)$ and $j : \mathfrak{k} \to \mathbb{R}$ is the K-invariant extension of

$$j(x) = \prod_{\alpha \in \Sigma^+} \frac{\sin\langle \alpha, x \rangle}{\langle \alpha, x \rangle} = \psi_{-i\rho}(ix), \quad x \in \mathfrak{a}.$$

Notice that $\|\Phi(\mu)\| \leq \|\mu\|$. As shown in [4], Φ is an algebra homomorphism from K-invariant measures in $M_b(\mathfrak{k})$ to conjugation-invariant measures in $M_b(K)$. The proof thereof is based on Weyl's integration formula and Kirillov's character formula for compact groups. In contrast to our situation, Φ does not preserve positivity and can therefore not be associated with a hypergroup deformation.

EXAMPLE 3.2. The group $K = SU(d)$ is a maximal compact subgroup of the connected semisimple Lie group $G = SL(d, \mathbb{C})$. In the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we obtain $\mathfrak{p}$ as the additive group H_d^0 of all Hermitian $d \times d$-matrices with trace 0, on which $SU(d)$ acts by conjugation. Moreover, $\mathfrak{a}$ consists of all real diagonal matrices with trace 0 and will be identified with

$$\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_i x_i = 0\}$$

on which the Weyl group acts as the symmetric group S_d as usual. We thus may take

$$C := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \geq x_2 \geq \ldots \geq x_d, \sum_i x_i = 0\}.$$

This set parametrizes the possible spectra of matrices from H_d^0. A system of positive roots corresponding to C is $\Sigma^+ = \{e_i - e_j : 1 \leq i < j \leq d\}$ where $e_1, \ldots, e_d$ denotes the standard basis of $\mathbb{R}^d$. The root system is of type A_{d-1}. In order to describe the probability preserving isometric isomorphism for this example explicitly, we realize the canonical projections $q: SL(d,\mathbb{C}) \rightarrow SL(d,\mathbb{C})//SU(d) \simeq \mathbb{C}$ and $p: H_d^0 \rightarrow (H_d^0)^{SU(d)} \simeq C$ as follows: For $A \in H_d^0$, define $p(A) := \sigma(A) \in C$ as the tuple of eigenvalues of A, ordered by size. For $B \in SL(d, \mathbb{C})$ define $q(B)$ as the element $x \in C$ such that the singular spectrum $\sigma_{sing}(B) = \sigma(\sqrt{BB^*})$ of B is $e^x := (e^{x_1} \geq \ldots \geq e^{x_d})$. We have

$$\rho = \sum_{\alpha \in \Sigma^+} \alpha = (d-1, d-3, d-5, \ldots, -d+3, -d+1) \in \mathfrak{a}_+$$

which implies that the Banach algebra $M_b^{\rho,K}(\mathfrak{p})$ above may be described as the closure of

$$\{e_\rho \mu : \mu \in M_c(H_d^0) \ SU(d) - \text{invariant}\},$$

with

$$e_\rho(A) = \exp\big(tr[A \cdot \text{diag}(d-1, d-3, d-5, \ldots, -d+3, -d+1)]\big).$$

As discussed above, the mappings p, q induce (by taking images of measures) probability preserving isometric isomorphisms from the Banach algebras $M_b^{\rho,K}(\mathfrak{p})$ and $M_b(SL(d,\mathbb{C})||SU(d))$ onto $(M_b(C), \bullet)$ for the double coset convolution $\bullet$ on $C \simeq SL(d,\mathbb{C})//SU(d)$.

Let us finally come back to the two spectral problems studied by Klyachko [13]. The hypergroup deformation between the two convolutions $*$ and $\bullet$ on C gives the natural explanation for the close connection of these problems: First, for fixed $x_1, x_2 \in C$, the probability measure $\delta_{x_1} * \delta_{x_2}$ is the distribution of possible spectra of sums $A_1 + A_2 \in H_d^0$ where the A_i run through all matrices from H_d^0 with $\sigma(A_i) = x_i$ $(i = 1, 2)$. On the other hand, $d(\delta_{x_1} \bullet \delta_{x_2})(y)$ describes the distribution of possible singular spectra e^y of products $B_1 B_2 \in SL(d, \mathbb{C})$ where the B_i run through all matrices from $SL(d,\mathbb{C})$ with given singular spectra e^{x_i}. As $\text{supp}(\delta_{x_1} \bullet \delta_{x_2}) = \text{supp}(\delta_{x_1} * \delta_{x_2})$ we obtain:

COROLLARY 3.3. *(c.f. [13], Theorem B) For elements $x_1, x_2, y \in C$ the following are equivalent:*

(i) There exist matrices $A_1, A_2 \in H_d^0$ with given spectra $\sigma(A_i) = x_i$ such that $y = \sigma(A_1 + A_2)$.
(ii) There exist matrices $B_i \in SL(d,\mathbb{C})$ with given singular spectra $\sigma_{sing}(B_i) = e^{x_i}$ such that $e^y = \sigma_{sing}(B_1 B_2)$.

We now briefly discuss the further classical series of complex, noncompact connected simple Lie groups with finite center (c.f. Appendix C in [14].)

3.3. The Further Classical Series of Simple Groups

(1) **The B_n-case**. For $n \geq 2$ consider $G = SO(2n+1,\mathbb{C})$ with the maximal compact subgroup $K = SO(2n+1)$. In this case $\mathfrak{a}$ may be identified with $\mathbb{R}^n$ with standard basis $e_1, \ldots, e_n$, and we may choose
$$C = \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$
and $\Sigma^+ = \{e_i \pm e_j : 1 \leq i \leq j \leq n\} \cup \{e_i : 1 \leq i \leq n\}$The Weyl group W is isomorphic to the semidirect product $S_n \ltimes \mathbb{Z}_2^n$, and $\rho = (2n-1, 2n-3, \ldots, 1)$.

(2) **The C_n-case**. For $n \geq 3$ consider $G = Sp(n,\mathbb{C})$ with the maximal compact subgroup $K = Sp(2n+1)$. In this case, again $\mathfrak{a} = \mathbb{R}^n$ with C and W as in the B_n-case. A positive root system is
$$\sum\nolimits^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\},$$
and $\rho = (2n, 2n-2, \ldots, 2)$. Comparing this with the B_n-case, we see from Equation (3) that the spherical functions ψ_λ of the orbit hypergroups $\mathfrak{p}^K$ are the same in both cases. The preceding results on hypergroup deformations therefore imply that the double coset hypergroups $Sp(n,\mathbb{C})//Sp(2n+1)$ and $SO(2n+1,\mathbb{C})//SO(2n+1)$ are deformations of each other w.r.t. certain positive semicharacters.

(3) **The D_n-case**. For $n \geq 4$ consider $G = SO(2n,\mathbb{C})$ with the maximal compact subgroup $K = SO(2n)$. In this case $\mathfrak{a} = \mathbb{R}^n$ with $C = \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq |x_n|\}$ and $\{\Sigma^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\}$. Thus $\rho = (2n-2, 2n-4, \ldots, 2, 0)$.

4. Limit Theorems for Biinvariant Random Walks

In this final section we use our previous results to translate limit theorems for random walks on $\mathfrak{p}$ into corresponding results for K-biinvariant random walks on G. We first sketch the method.

4.1. Description of the Method

Let $(Z_n)_{n\geq 1}$ be a sequence on independent G-valued random variables with distributions $\mu_n \in M^1(G||K)$ $(n \geq 1)$. Then $(S_n := Z_1 Z_2 \ldots Z_n)_{n\geq 0}$ with $S_0 := e$ forms a

K-biinvariant random walk on G. Using the canonical projection $q : G \mapsto G//K \simeq C$ and the associated isomorphism $q : M_b(G||K) \mapsto (M_b(C), \bullet)$ of Banach algebras, we see that $(q(S_n)))_{n\geq 0}$ is a Markov process on C with initial distribution δ_0 and transition probabilities

$$P(q(S_n) \in A \mid q(S_{n-1}) = x) \; = \; (\delta_x \bullet q(\mu_n))(A)$$

for $n \geq 1$, $x \in C$, and Borel sets $A \subset C$. This means that $(q(S_n)))_{n\geq 0}$ is a random walk on the hypergroup $(C, \bullet)$ with transition probabilities $(q(\mu_n))_{n\geq 1} \subset M^1(C)$. Clearly, $q(S_n)$ has distribution

$$q(\mu_1) \bullet \ldots \bullet q(\mu_n) = q(\mu_1 * \ldots * \mu_n).$$

On the other hand, when considering the canonical projection $p : \mathfrak{p} \to \mathfrak{p}^K \simeq C$, we find unique probability measures $(\nu_n)_{n\geq 1} \in M_b^{K,\rho}(\mathfrak{p})$ with $p(\nu_n) = q(\mu_n)$ for $n \in \mathbb{N}$. If $(X_n)_{n\geq 1}$ is a sequence on independent $\mathfrak{p}$-valued random variables with distributions $(\nu_n)_{n\geq 1}$, we get a random walk $(T_n := \sum_{k=0}^{n} X_k)_{n\geq 0}$ on $\mathfrak{p}$ whose projection $(p(T_n))_{n\geq 0}$ is a random walk on $(C, \bullet)$ with the same transition probabilities as $(q(S_n))_{n\geq 0}$, i.e., $(p(T_n))_{n\geq 0}$ and $(q(S_n))_{n\geq 0}$ have the same finite dimensional distributions and therefore admit the same limit theorems. Hence, limit theorems for the random walk $(T_n)_{n\geq 0}$ on $\mathfrak{p}$ may be translated into limit theorems for $(q(S_n))_{n\geq 0}$.

Here is an example. Assume that the $\mu_n = \mu$ are independent of n, and that for the associated ν on $\mathfrak{p}$ the first moment vector

$$m := \int_{\mathfrak{p}} x \, d\nu(x) \in \mathfrak{p}$$

exists. Then by Kolmogorov's strong law, $T_n/n \to m$ almost surely as $n \to \infty$. In other words, $T_n = nm + o(n)$ a.s.. As $p : \mathfrak{p} \to \mathfrak{p}^K \simeq C$ is homogeneous of degree 1 and contractive (see below), we obtain $p(T_n) = np(m) + o(n)$ and thus $p(T_n)/n \to p(m)$ a.s.. Therefore, $q(S_n)/n \to p(m)$ a.s. for $n \to \infty$.

This strong law for $(q(S_n))_{n\geq 0}$ has the computational drawback that $p(m)$ is described in terms of $\nu \in M_b^{K,\rho}(\mathfrak{p})$. To overcome this, we introduce a suitable modified moment function $m_1 : C \to C$ such that for all probability measures $\nu \in M_b^{K,\rho}(\mathfrak{p})$ having first moments,

$$p(m) = p\left(\int_{\mathfrak{p}} x \, d\nu(x)\right) = \int_C m_1 \, dq(\mu). \tag{6}$$

We shall show that m_1 is determined uniquely by (6) and give an explicit formula. This allows to compute the limiting constant $p(m)$ directly on C.

Let us now go into details. As indicated above, we need the following properties of p:

LEMMA 4.2. *Assume $\mathfrak{p}$ and $\mathfrak{a}$ carry the Euclidean norm given by the Killing form.*

(i) p is homogeneous of degree 1, i.e., $p(tx) = tp(x)$ for all $t > 0$ and $x \in \mathfrak{p}$.
(ii) $dist\,(K.x, K.y) = \|p(x) - p(y)\|$ for all $x, y \in \mathfrak{p}$.

Proof. Part (a) is obvious. For part (ii), see [9], Prop. I.5.18. □

We next define m_1. Motivated by (6) for $\nu \in M_b^{\rho,K}(\mathfrak{p})$ with $p(\nu) = \delta_x$ for $x \in C$, we put

$$m_1(x) := \frac{1}{\psi_{-i\rho}(x)} \int_K k.x \cdot e^{\langle k.x,\rho\rangle}\, dk \quad (x \in \mathfrak{p}). \tag{7}$$

Notice here that $\psi_{-i\rho}(x) = \int_K e^{\langle k.x,\rho\rangle}\, dk$. The function m_1 is obviously K-invariant, continuous, and satisfies $\|m_1(x)\| \le \|x\|$ for $x \in \mathfrak{p}$ with respect to the norm induced by the Killing form.

PROPOSITION 4.3.

(i) *We have $m_1(x) \in C$ for all $x \in \mathfrak{p}$.*
(ii) *For all $x \in C$,*

$$m_1(x) = \frac{\sum_{w\in W} (\det w) e^{\langle wx,\rho\rangle} wx}{\sum_{w\in W} (\det g) e^{\langle wx,\rho\rangle}} - \sum_{\alpha\in\Sigma^+} \frac{\alpha}{\langle \alpha, \rho\rangle}.$$

Proof. We first check that $m_1(x) \in \mathfrak{a}$ for $x \in \mathfrak{p}$. The definition of m_1 and the Harish–Chandra formula (2) show that for $\zeta \in \mathfrak{p}$,

$$\langle m_1(x), \zeta\rangle = \frac{1}{i\psi_{-i\rho}(x)} \cdot \partial_\zeta \psi_\lambda(x)|_{\lambda=-i\rho} \tag{8}$$

where ∂_ζ is the derivative in direction ζ w.r.t. λ. The open Weyl chamber $\mathfrak{a}_+$ corresponding to C is an orthogonal transversal manifold for the adjoint action of K on $\mathfrak{p}$ ([9], Ch.II, 3.4.(vi)). Therefore the orthogonal complement $\mathfrak{a}^\perp$ of $\mathfrak{a}$ in $\mathfrak{p}$ coincides with the tangent space of the orbit $K.\rho$ in ρ. As $\lambda \mapsto \psi_{-i\lambda}(x)$ is constant on K-orbits, $\partial_\zeta\psi_\lambda(x)|_{\lambda=-i\rho} = 0$ for $\zeta \in \mathfrak{a}^\perp$. Hence $m_1(x) \in (a^\perp)^\perp = \mathfrak{a}$. In order to check $m_1(x) \in C$, we recall from Ch. 3 of [7] that

$$\begin{aligned} C &= \{x \in \mathfrak{a} :\ d(x,\rho) \le d(x, w\rho) \quad \forall\, w \in W\} \\ &= \{x \in \mathfrak{a} :\ \langle x,\rho\rangle \ge \langle x, w\rho\rangle \quad \forall\, w \in W\}. \end{aligned} \tag{9}$$

On the other hand, an elementary rearrangement inequality (Theorem 368 of [8]) shows that for $z \in \mathfrak{p}$ and $w' \in W$,

$$\sum_{w\in W} \langle z,\, w\rho\rangle\, e^{\langle z,\, w\rho\rangle} \ge \sum_{w\in W} \langle z,\, ww'\rho\rangle\, e^{\langle z, w\rho\rangle}.$$

As $\psi_{-i\rho} > 0$, we obtain for $w' \in W$ that

$$\begin{aligned} \langle m_1(x), \rho\rangle &= \frac{1}{\psi_{-i\rho}(x)} \int_K \langle k.x, \rho\rangle\, e^{\langle k.x,\rho\rangle} dk \\ &= \frac{1}{|W|\,\psi_{-i\rho}(x)} \int_K \sum_{w\in W} \langle k.x,\, w\rho\rangle\, e^{\langle k.x,\, w\rho\rangle} dk \\ &\ge \frac{1}{|W|\,\psi_{-i\rho}(x)} \int_K \sum_{w\in W} \langle k.x,\, ww'\rho\rangle\, e^{\langle k.x, w\rho\rangle} dk \\ &= \langle m_1(x), w'\rho\rangle. \end{aligned}$$

Equation (9) now shows that $m_1(x) \in C$. This completes the proof of Part (i). Part (ii) follows from Eqs. (8) and (3) for the spherical functions ψ_λ on $\mathfrak{a}$. □

MOMENTS 4.4. Let $\nu \in M_b^{K,\rho}(\mathfrak{p})$ be a probability measure and $r > 0$. We say that ν admits r-th moments if $\int_\mathfrak{p} \|x\|^r > d\nu(x) < \infty$, or equivalently, if $\int_\mathfrak{p} |\langle \xi, x\rangle|^r \, d\nu(x) < \infty$ for all $\xi \in \mathfrak{p}$. This condition can be translated into a corresponding condition for $p(\nu) \in M^1(C)$. In fact, as K acts on $\mathfrak{p}$ as a group of orthogonal transformations,

$$\int_\mathfrak{p} \|x\|^r \, d\nu(x) = \int_\mathfrak{p} \|p(x)\|^r \, d\nu(x) = \int_C \|y\|^r \, dp(\nu)(y) \in [0, \infty].$$

Therefore, $\nu \in M_b^{K,\rho}(\mathfrak{p})$ admits r-th moments if and only if $p(\nu)$ admits r-th moments. Proposition 4.3 and the estimate $\|m_1(x)\| \le \|x\|$ for $x \in \mathfrak{p}$ show that that this condition for $r \ge 1$ implies that the modified moment vector $\int_C m_1(y) \, dp(\nu)(y) \in C$ exists. Moreover, as

$$\int_\mathfrak{p} x \, d\nu(x) = \int_C \int_K k.x \, \frac{e^{\langle k.x, \rho\rangle}}{\psi_{-i\rho}(x)} \, dk \, dp(\nu)(x),$$

we obtain from Propos. 4.3 that $\int_\mathfrak{p} x \, d\nu(x) \in C$ and hence, as claimed,

$$p\Big(\int_\mathfrak{p} x \, d\nu(x)\Big) = \int_\mathfrak{p} x \, d\nu(x) = \int_C m_1 \, dp(\nu) \in C.$$

As an application, we derive a strong law of large numbers of Marcinkiewicz–Zygmund:

THEOREM 4.5. *Let $r \in]0,2[$ and $\mu \in M^1(G||K)$ such that $q(\mu) \in M^1(C)$ admits r-th moments. Let $(Z_n)_{n\ge1}$ be a sequence of i.i.d. G-valued μ-distributed random variables. Then*

$$\frac{1}{n^{1/r}} \Big(q(Z_1 \cdot Z_2 \cdots Z_n) - n \cdot c \Big) \longrightarrow 0 \quad a.s.$$

for $n \to \infty$, with $c = \int_C m_1 dq(\mu)$ in case $r \in [1,2[$, while $c \in C$ is arbitrary for $r \in]0,1[$.

Proof. Let $r \in [1,2[$. By Section 4.4, the r-th moment of the associated $\nu \in M_b^{K,\rho}(\mathfrak{p})$ with $p(\nu) = q(\mu)$ exists. Let $(X_n)_{n\ge1}$ be i.i.d. $\mathfrak{p}$-valued random variables with distribution ν, and $(T_n = X_1 + \ldots + X_n)_{n\ge0}$ the associated random walk on $\mathfrak{p}$. The classical Marcinkiewicz–Zygmund law (Theorem 5.2.2 of [3]) yields that for all $\xi \in \mathfrak{p}$,

$$n^{-1/r} \big(\langle \xi, T_n \rangle - n \int_\mathfrak{p} \langle \xi, x \rangle \, d\nu(x) \big) \to 0 \quad a.s.$$

as $n \to \infty$. But this means that $T_n - n \int_\mathfrak{p} x \, d\nu(x) = o(n^{1/r})$ a.s. and hence, by 4.3 (i) and (6),

$$p(T_n) - n \int_C m_1 dq(\mu) = p(T_n) - np\big(\int_\mathfrak{p} x \, d\nu(x) \big) = o\big(n^{1/r}\big) \qquad a.s..$$

As $(q(Z_1Z_2\ldots Z_n))_{n\geq 0}$ and $p(T_n)_{n\geq 0}$ have the same finite-dimensional distributions, the claim follows. The case $r\in]0,1[$ is similar. □

For $(G,K)=(SL(d,\mathbb{C}),SU(d))$, the mapping q is given by $q(A):=(\ln a_1(A),\ldots,\ln a_d(A))$ where $a_1(A)\geq a_2(A)\geq\ldots\geq a_d(A)>0$ are the eigenvalues of $\sqrt{AA^*}$. We therefore obtain

COROLLARY 4.6. *Let $r\in[1,2[$ and $\mu\in M^1(SL(d,\mathbb{C})||SU(d))$ such that its projection $q(\mu)\in M^1(C)$ admits r-th moments. Then, for each sequence $(Z_n)_{n\geq 1}$ of i.i.d. $SL(d,\mathbb{C})$-valued and μ-distributed random variables,*

$$\frac{1}{n^{1/r}}\Big((\ln a_1(Z_1\cdot Z_2\cdots Z_n),\ldots,\ln a_d(Z_1\cdot Z_2\cdots Z_n)) \\ -n\cdot\int_C m_1\,dq(\mu)\Big)\longrightarrow 0 \quad a.s..$$

Remark 4.7. The strong laws 4.5 and 4.6 are in principle well-known especially for $r=1$; see for instance [2] and references cited therein. Nevertheless our approach may be of some interest, because it uses the close connection between (biinvariant) random walks on G and those on $\mathfrak{p}$ in a simple explicit way.

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