SONINE FORMULAS AND INTERTWINING OPERATORS IN DUNKL THEORY

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ABSTRACT. Let V_k denote Dunkl's intertwining operator associated with some root system R and multiplicity function k. For two multiplicities k, k' on R, we study the operator $V_{k',k} = V_{k'} \circ V_k^{-1}$, which intertwines the Dunkl operators for multiplicity k with those for multiplicity k'. While it is well-known that the operator V_k is positive for nonnegative k, it has been a long-standing conjecture that its generalizations $V_{k',k}$ are also positive if $k' \geq k \geq 0$, which is known to be true in rank one. In this paper, we disprove this conjecture by constructing examples for root system B_n with multiplicites $k' \geq k \geq 0$ for which $V_{k',k}$ is not positive. This matter is closely related to the existence of integral representations of Sonine type between the Dunkl kernels and Bessel functions associated with the relevant multiplicities. In our examples, such Sonine formulas do not exist. As a consequence, we obtain necessary conditions on Sonine-type integral formulas for Heckman-Opdam hypergeometric functions of type BC_n as well as conditions on the existence of positive branching coefficients between systems of multivariable Jacobi polynomials.

1. INTRODUCTION

In the theory of rational Dunkl operators initiated by C.F. Dunkl in [D1, D2], the intertwining operator plays a significant role. This operator intertwines Dunkl operators with the usual partial derivatives on some Euclidan space. To become more precise, let R be a (not necessarily crystallographic) root system in a finite-dimensional Euclidean space $(\mathfrak{a}, \langle \, . \, , \, . \, \rangle)$ with finite Coxeter group W, and fix a W-invariant function $k : R \to \mathbb{C}$ (called multiplicity function) with $\operatorname{Re} k \geq 0$. Denote by $\{T_\xi(k), \xi \in \mathfrak{a}\}$ the associated commuting family of rational Dunkl operators. The intertwining operator V_k is then characterized as the unique isomorphism on the vector space $\mathcal{P} = \mathbb{C}[a]$ of polynomial functions on $\mathfrak{a}$ which preserves the degree of homogeneity and satisfies

$$V_k(1) = 1, \quad T_\xi(k)V_k = V_k \partial_\xi \quad \text{ for all } \xi \in \mathfrak{a};$$

c.f. [DJO]. The Dunkl kernel E_k associated with R and k, which solves the joint eigenvalue problem for the $T_\xi(k)$ and generalizes the usual exponential kernel, can be represented by means of the intertwiner V_k as $E_k(x, z) = V_k\big(e^{\langle \, \cdot \, , z \rangle}\big)(x)$ for all $x \in \mathfrak{a}$ and $z \in \mathfrak{a}_\mathbb{C}$, where $\mathfrak{a}_\mathbb{C}$ denotes the complexification of $\mathfrak{a}$.

For nonnegative multiplicities $k \geq 0$, it was shown in [R1] that V_k is positive on $\mathcal{P}$, i.e. for $p \in \mathcal{P}$ with $p \geq 0$ on $\mathfrak{a}$, it follows that $V_k p \geq 0$ on $\mathfrak{a}$. Further, for each

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$x \in \mathfrak{a}$ there exists a unique probability measure μ_x^k on $\mathfrak{a}$ such that

$$E_k(x, z) = \int_{\mathfrak{a}} e^{\langle \xi, z \rangle} d\mu_x^k(\xi), \quad \forall \, x \in \mathfrak{a}, z \in \mathfrak{a}_{\mathbb{C}}. \tag{1.1}$$

The representing measure μ_x^k is compactly supported with $\operatorname{supp} \mu_x^k \subseteq \operatorname{co}(W.x)$, the convex hull of the W-orbit of x. Formula (1.1) generalizes the Harish-Chandra integral representation for the spherical functions of a symmetric space of Euclidean type. Indeed, for certain half-integer valued multiplicities k, the Bessel functions

$$J_k(x, z) = \frac{1}{|W|} \sum_{w \in W} E_k(wx, z), \quad z \in \mathfrak{a}_{\mathbb{C}},$$

can be interpreted as the spherical functions of a Cartan motion group, where R and k are determined by the root space data of the underlying symmetric space, see [O1, dJ2] for details. In these geometric cases, the integral formula for J_k obtained from (1.1) by taking W-means is a direct consequence of the Harish-Chandra formula together with Kostant's convexity theorem [Hel, Propos. IV.4.8 and Theorem IV.10.2].

In this paper, we shall consider two multiplicities k, k' on R with $k' \geq k \geq 0$ (i.e., $k'(\alpha) \geq k(\alpha) \geq 0 \,\forall \alpha \in R$) and study the operator

$$V_{k',k} := V_{k'} \circ V_k^{-1}.$$

Notice that $V_{k',0} = V_{k'}$. The operator $V_{k',k}$ intertwines the Dunkl operators with multiplicities k and k',

$$T_\xi(k')V_{k',k} = V_{k',k} \, T_\xi(k) \quad \text{for all } \xi \in \mathfrak{a}.$$

It has been a long-standing conjecture that $V_{k',k}$ is also positive on polynomials, which is (as will be explained in Section 2) equivalent to the statement that for each $x \in \mathfrak{a}$, there exists a compactly supported probability measure $\mu_x^{k',k}$ on $\mathfrak{a}$ such that

$$E_{k'}(x, z) = \int_{\mathfrak{a}} E_k(\xi, z) \, d\mu_x^{k',k}(\xi) \quad \text{for all } z \in \mathfrak{a}_{\mathbb{C}}. \tag{1.2}$$

Note that (1.2) implies an analogous formula for the Bessel function:

$$J_{k'}(x, z) = \int_{\mathfrak{a}} J_k(\xi, z) \, d\widetilde{\mu}_x^{k',k}(\xi) \quad (z \in \mathfrak{a}_{\mathbb{C}}) \tag{1.3}$$

with some W-invariant probability measures $\widetilde{\mu}_x^{k',k}$.

In the rank-one case with $R = \{\pm 1\} \subset \mathbb{R}$, one has $J_k(x, y) = j_{k-1/2}(ixy)$ with the (modified) one-variable Bessel function

$$j_\alpha(z) = {}_0F_1(\alpha + 1; -z^2/4) \quad (\alpha \in \mathbb{C} \setminus \{-1, -2, \ldots\}). \tag{1.4}$$

In this case, formula (1.3) is just the classical Sonine formula ([A2, formula (3.4)]):

$$j_{\alpha+\beta}(z) = 2 \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta)} \int_0^1 j_\alpha(zx) x^{2\alpha+1} (1 - x^2)^{\beta-1} dx \tag{1.5}$$

for all $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\beta > 0$.

In the rank-one case also the operator $V_{k',k}$ with $k' > k \geq 0$ is known to be positive. Indeed, Y. Xu obtained in [X] an explicit positive integral representation for $V_{k',k}$ which leads to a positive Sonine-type representation for the rank-one Dunkl kernel, see Remark 2.8 for details.

In the present paper, we shall construct examples which reveal that the above positivity conjecture is not true in general. Our examples are related to root system

$$B_n = \{\pm e_i,\ \pm e_i \pm e_j,\ 1 \le i < j \le n\} \subset \mathbb{R}^n$$

with $n \ge 2$, where multiplicities are denoted as $k = (k_1, k_2)$, with k_1 and k_2 the values of k on e_i and $e_i \pm e_j$, respectively. We prove that for $k = (k_1, k_2)$ with $k_1 \ge 0, k_2 > 0$ and $k' = k'(h) = (k_1 + h, k_2)$ with $h > -k_1$, the Bessel function $J_{k'(h)}^B$ of type B_n has no positive Sonine representation with respect to J_k^B unless h is contained in the set

$$\Sigma(k_2) :=]k_2(n-1), \infty[\cup (\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{Z}_+);\quad \mathbb{Z}_+ = \{0, 1, 2, \ldots\}.$$

This implies that the intertwining operator $V_{k'(h),k}$ is not positive unless $h \in \Sigma(k_2)$. More generally, we shall consider also complex multiplicities and obtain similar conditions for Sonine representations with complex bounded Radon measures.

The proof of our main result, which is contained in Corollary 3.6, is based on the fact that the Bessel function of type B_n can be expressed as a multivariable $_0F_1$-hypergeometric function in the sense of [K] (see also [BF]). Via Kadell's [Ka] generalization of the Selberg integral one obtains an explicit Sonine formula for this hypergeometric function and therefore also for the Bessel function $J_{k'(h)}^B$ in terms of J_k^B within the range $\operatorname{Re} h > k_2(n-1)$. This explicit formula allows a distributional extension to a larger range of the parameter h, which is based on results of [dJ2] for the intertwiner V_k. Employing arguments of Sokal [So] for the characterization of Riesz distributions on symmetric cones, we then obtain necessary conditions on h under which our distributional Sonine formulas can actually be given by positive or complex measures. Indeed, the set $\Sigma(k_2)$ is similar to the so-called Wallach set, which describes those Riesz distributions of a symmetric cone which are actually positive measures. We also mention that for Bessel functions on symmetric cones, Sonine formulas were recently studied in [RV2].

Our results concerning Sonine formulas in the rational Dunkl setting are contained in Section 3, which is preceded by preperations for intertwining operators in Section 2. In Section 4, we apply the results from Section 3 to the trigonometric theory of Heckman, Opdam and Cherednik (see [HS, O2]) which generalizes the spherical harmonic analysis on Riemannian symmetric spaces of the non-compact and compact type. We shall use a well-known contraction procedure from the trigonometric to the rational case in order to derive necessary conditions on the existence of Sonine-type integral representations between hypergeometric functions and Heckman-Opdam polynomials (also called Jacobi polynomials) associated with root system BC_n as well as the positivity of branching coefficients between two such polynomial systems with different multiplicities. These results are complemented by motivating examples in rank one and the case of symmetric spaces. Let us mention that in geometric cases, branching rules and Sonine-type formulas for Bessel functions were recently also studied in [HZ] in connection with the geometry of moment mappings.

2. Intertwining operators and Sonine formula for Dunkl kernels

We start with some background and notation in rational Dunkl theory supplementing the material in the introduction. For more information, the reader is referred to [dJ1, O1, DJO, dJ2, DX] and the references cited there. Again, R is a

root system in a finite-dimensional Euclidean space $(\mathfrak{a}, \langle\,.\,,\,.\,\rangle)$ and $W = W(R)$ be the associated finite Coxeter group. We assume in this section that R is reduced, but not necessarily crystallographic. Let $\mathcal{K} = \{k : R \to \mathbb{C} \,:\, k \text{ is } W\text{-invariant}\}$ denote the space of multiplicity functions on R. For two multiplicities $k, k' \in \mathcal{K}$ we write $k' \geq k$ ($\operatorname{Re} k' \geq \operatorname{Re} k$) if $k'(\alpha) \geq k(\alpha)$ ($\operatorname{Re} k'(\alpha) \geq \operatorname{Re} k(\alpha)$) for all $\alpha \in R$. The Dunkl operators associated with R and $k \in \mathcal{K}$ are given by

$$T_\xi(k) = \partial_\xi + \frac{1}{2} \sum_{\alpha \in R} k(\alpha)\langle\alpha, \xi\rangle \frac{1}{\langle\alpha, .\rangle}(1 - \sigma_\alpha), \quad \xi \in \mathfrak{a}$$

where the action of W on functions $f : \mathfrak{a} \to \mathbb{C}$ is given by $w.f(x) = f(w^{-1}x)$. It was shown in [D1] that the $T_\xi(k)$, $\xi \in \mathfrak{a}$ commute. A multiplicity k is called regular if the joint kernel of the $T_\xi(k)$, considered as linear operators on $\mathcal{P} = \mathbb{C}[\mathfrak{a}]$, consists of the constants only. This is equivalent to the existence of a (necessarily unique) intertwining operator V_k as described in the introduction. The set $\mathcal{K}^{reg}$ of regular multiplicities is open in $\mathcal{K}$ and contains the set $\{k \in \mathcal{K} : \operatorname{Re} k \geq 0\}$, see [DJO]. Moreover, for each $k \in \mathcal{K}^{reg}$ and $y \in \mathfrak{a}_\mathbb{C}$, there exists a unique solution $f = E_k(\,.\,,y)$ of the joint eigenvalue problem

$$T_\xi(k)f = \langle\xi, y\rangle f \quad \forall\, \xi \in \mathfrak{a}, \; f(0) = 1.$$

The function E_k is called the Dunkl kernel. The mapping $(k, x, y) \mapsto E_k(x, y)$ is analytic on $\mathcal{K}^{reg} \times \mathfrak{a}_\mathbb{C} \times \mathfrak{a}_\mathbb{C}$ and satisfies $E_k(x, y) = E_k(y, x)$ as well as

$$E_k(\lambda x, y) = E_k(x, \lambda y), \quad E_k(wx, wy) = E_k(x, y) \quad (\lambda \in \mathbb{C}, w \in W).$$

We shall from now on always assume that $\operatorname{Re} k \geq 0$. In this case, the following estimate for the Dunkl kernel is due to [dJ1]:

$$|E_k(x, z)| \leq \sqrt{|W|}\, e^{\max_{w \in W}\langle wx, \operatorname{Re} z\rangle} \quad \forall\, x \in \mathfrak{a}, z \in \mathfrak{a}_\mathbb{C}. \tag{2.1}$$

Denote by $\mathcal{E}(\mathfrak{a})$ the space $C^\infty(\mathfrak{a})$ of smooth functions on $\mathfrak{a}$, equipped with its usual Fréchet space topology. According to [dJ2], the operator V_k (uniquely) extends to a homeomorphism of $\mathcal{E}(\mathfrak{a})$ retaining the intertwining property. Thus

$$E_k(x, z) = V_k\big(e^{\langle\,\cdot\,,z\rangle}\big)(x), \quad \forall\, x \in \mathfrak{a}, z \in \mathfrak{a}_\mathbb{C}.$$

We next recapitulate some facts about the Dunkl transfrom from [dJ1]. Consider the (complex-valued) W-invariant weight

$$\omega_k(x) = \prod_{\alpha \in R} |\langle\alpha, x\rangle|^{k(\alpha)}.$$

The Dunkl transform associated with R and k on $L^1(\mathfrak{a}, |\omega_k|)$ is defined by

$$\widehat{f}^k(\xi) = \int_\mathfrak{a} f(x)E_k(x, -i\xi)\omega_k(\xi)d\xi, \quad \xi \in \mathfrak{a}.$$

The Dunkl transform $\mathcal{D}_k : f \mapsto \widehat{f}^k$ is a homeomorphism of the Schwartz space $\mathcal{S}(\mathfrak{a})$ with inverse

$$\mathcal{D}_k^{-1}f(x) = \frac{1}{c_k^2}\mathcal{D}_k f(-x), \quad c_k = \int_\mathfrak{a} e^{-|x|^2/2}\omega_k(x)dx.$$

Notice that $c_k \neq 0$ by [dJ1, Cor. 4.17]. Dunkl operators act continuously on $\mathcal{S}(\mathfrak{a})$ and therefore also on the space $\mathcal{S}'(\mathfrak{a})$ of tempered distributions on $\mathfrak{a}$, via

$$\langle T_\xi(k)u, \varphi\rangle := -\langle u, T_\xi(k)\varphi\rangle, \quad u \in \mathcal{S}'(\mathfrak{a}), \; \varphi \in \mathcal{S}(\mathfrak{a}). \tag{2.2}$$

Moreover, the Dunkl transform extends to a homeomorphism $u \mapsto \widehat{u}^k$ of $\mathcal{S}'(\mathfrak{a})$ by

$$\langle \widehat{u}^k, \varphi \rangle := \langle u, \widehat{\varphi}^k \rangle, \quad \varphi \in \mathcal{S}(\mathfrak{a}).$$

For $R > 0$ let $B_R(0) := \{x \in \mathfrak{a} : |x| < R\}$ and $\overline{B}_R(0) := \{x \in \mathfrak{a} : |x| \le R\}$, where $|\,.\,|$ denotes the norm associated with the given inner product. We shall use the following facts concerning the intertwiner V_k.

Proposition 2.1. [dJ2, Theorem 5.1]

(1) If $\varphi \in \mathcal{E}(\mathfrak{a})$ vanishes on $B_R(0)$, then also $V_k\varphi$ and $V_k^{-1}\varphi$ vanish on $B_R(0)$.

(2) Let $\varphi \in \mathcal{S}(\mathfrak{a})$. Then for all $x \in \mathfrak{a}$,

$\quad$ (a) $V_k\varphi(x) = \dfrac{c_k^2}{c_0^2}\mathcal{D}_k^{-1}(\omega_k^{-1}\mathcal{D}_0)\varphi(x) = \dfrac{1}{c_0^2}\displaystyle\int_{\mathfrak{a}} \widehat{\varphi}^0(\xi) E_k(ix, \xi)\, d\xi.$

$\quad$ (b) $V_k^{-1}\varphi(x) = \dfrac{c_0^2}{c_k^2}\mathcal{D}_0^{-1}(\omega_k\mathcal{D}_k)\varphi(x) = \dfrac{1}{c_k^2}\displaystyle\int_{\mathfrak{a}} \widehat{\varphi}^k(\xi) e^{i\langle x, \xi \rangle}\, \omega_k(\xi)d\xi.$

For an open subset $\Omega \subseteq \mathfrak{a}$ we denote by $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ the set of test functions and by $\mathcal{D}'(\Omega)$ the set of distributions on Ω. Recall that the topological dual $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$ coincides with the set of compactly supported distributions on Ω, and that compactly supported distributions on $\mathfrak{a}$ are tempered.

Definition 2.2. We define the Dunkl-Laplace transform of $u \in \mathcal{E}'(\mathfrak{a})$ by

$$\mathcal{L}_k u : \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}, \; \mathcal{L}_k u(z) := \langle u(x), E_k(x, -z) \rangle,$$

where the notion $u(x)$ indicates that u acts on functions of the variable x.

As in the classical case, we have the following fact for compactly supported distributions, c.f. also [BSO].

Lemma 2.3. Let $u \in \mathcal{E}'(\mathfrak{a})$. Then $\mathcal{L}_k u$ is analytic on $\mathfrak{a}_{\mathbb{C}}$, and the Dunkl transform $\widehat{u}^k$ is a regular tempered distribution given by $\mathcal{L}_k u$ in the sense that

$$\langle \widehat{u}^k, \varphi \rangle = \int_{\mathfrak{a}} \varphi(\xi)\mathcal{L}_k u(i\xi)\, \omega_k(\xi)d\xi, \quad \varphi \in \mathcal{S}(\mathfrak{a}). \tag{2.3}$$

Proof. This is the same as in [Hö, Theorem 7.1.14] for the classical case. We briefly note the steps: According to [Hö, Theorem 2.1.3] $\mathcal{L}_k u$ is smooth on $\mathfrak{a}_{\mathbb{C}}$ and differentiations with respect to z may be taken in the argument $E_k(x, -z)$. As this kernel is analytic in z, the same follows for $\mathcal{L}_k u(z)$. For the proof of (2.3), it suffices to consider $\varphi \in \mathcal{D}(\mathfrak{a})$. By the Fubini theorem for compactly supported distributions we obtain

$$\langle u, \widehat{\varphi}^k \rangle = \Big\langle u(x), \int_{\mathfrak{a}} \varphi(\xi) E_k(-ix, \xi) w_k(\xi)d\xi \Big\rangle$$

$$= \big\langle u(x) \otimes \varphi(\xi)\omega_k(\xi), E_k(-ix, \xi) \big\rangle = \int_{\mathfrak{a}} \varphi(\xi)\langle u(x), E_k(-ix, \xi) \rangle\, \omega_k(\xi)d\xi.$$

This implies the assertion. $\qquad\square$

Lemma 2.3 implies injectivity of the Dunkl Laplace transform:

Corollary 2.4. Let $u \in \mathcal{E}'(\mathfrak{a})$.

(1) If $\mathcal{L}_k u = 0$, then $u = 0$.

(2) *Suppose that $m \in M_b(\mathfrak{a})$ is a complex bounded Radon measure satisfying*

$$\mathcal{L}_k u(i\xi) = \int_{\mathfrak{a}} E_k(x, -i\xi) dm(x) \quad \text{for all } \xi \in \mathfrak{a}.$$

Then $m = u$.

Proof. (1) is obvious by the above Lemma, because the Dunkl transform is a homeomorphism of $\mathcal{S}'(\mathfrak{a})$.

(2) Consider m as a tempered distribution on $\mathfrak{a}$. By Lemma 2.3 and our assumption we obtain for test functions $\varphi \in \mathcal{S}(\mathfrak{a})$,

$$\langle \widehat{u}^k, \varphi \rangle = \int_{\mathfrak{a}} \varphi(\xi) \Big(\int_{\mathfrak{a}} E_k(x, -i\xi) dm(x) \Big) \omega_k(\xi) d\xi = \langle \widehat{m}^k, \varphi \rangle.$$

Thus $\widehat{m}^k = \widehat{u}^k$ which implies $m = u$ by the injectivity of the Dunkl transform on $\mathcal{S}'(\mathfrak{a})$. $\qquad\square$

Consider now a fixed root system $R \subset \mathfrak{a}$ with two multiplicities k, k' satisfying $\operatorname{Re} k \geq 0, \operatorname{Re} k' \geq 0$. Then the operator

$$V_{k',k} := V_{k'} \circ V_k^{-1}$$

is a topological isomorphism of $\mathcal{E}(\mathfrak{a})$ and intertwines the Dunkl operators associated with multiplicities k and k',

$$T_\xi(k') V_{k',k} = V_{k',k} T_\xi(k) \quad \text{for all } \xi \in \mathfrak{a}.$$

Note that for all $x \in \mathfrak{a}$ and $z \in \mathfrak{a}_{\mathbb{C}}$,

$$E_{k'}(x, z) = V_{k',k} \big(E_k(\,.\,, z) \big)(x). \tag{2.4}$$

For fixed $x \in \mathfrak{a}$ the assignment $\langle u_x^{k',k}, \varphi \rangle := V_{k',k}\, \varphi(x)$ defines a compactly supported distribution $u_x^{k',k} \in \mathcal{E}'(\mathfrak{a})$ satisfying

$$\langle u_x^{k',k}, E_k(\,.\,, z) \rangle = E_{k'}(x, z). \tag{2.5}$$

Lemma 2.5. *The support of $u_x^{k',k}$ is contained in the closed ball $\overline{B}_{|x|}(0)$.*

Proof. Let $\varphi \in \mathcal{D}(\mathfrak{a})$ with $\operatorname{supp} \varphi \cap \overline{B}_{|x|}(0) = \emptyset$. Then by Proposition 2.1, $(V_{k'} \circ V_k^{-1})(\varphi)$ vanishes on $B_{|x|}(0)$ and therefore $\langle u_x^{k',k}, \varphi \rangle = 0$. $\qquad\square$

Lemma 2.6. *For $\operatorname{Re} k \geq 0$, $\operatorname{Re} k' \geq 0$ the following are equivalent.*
  (1) *$V_{k',k}$ is positive on $\mathcal{P}$, i.e. $V_{k',k}\, p \geq 0$ on $\mathfrak{a}$ for all $p \in \mathcal{P}$ with $p \geq 0$ on $\mathfrak{a}$.*
  (2) *$V_{k',k}$ is positive on $\mathcal{E}(\mathfrak{a})$, i.e. $V_{k',k}\, f \geq 0$ for all $f \in \mathcal{E}(\mathfrak{a})$ with $f \geq 0$.*
  (3) *For each $x \in \mathfrak{a}$ there exists a probability measure $\mu_x^{k',k} \in M^1(\mathfrak{a})$ such that*

$$E_{k'}(x, iy) = \int_{\mathfrak{a}} E_k(\xi, iy) d\mu_x^{k',k}(\xi) \quad \forall y \in \mathfrak{a}. \tag{2.6}$$

In this case, the representing measure $\mu_x^{k',k}$ is unique and compactly supported.

Proof. (1) $\Rightarrow$ (2) Let $f \in \mathcal{E}(\mathfrak{a})$ with $f > 0$ on $\mathfrak{a}$. As $\mathcal{P}$ is dense in $\mathcal{E}(\mathfrak{a})$ (see [Tr, Chap.15]), there exists a sequence (p_n) consisting of real-valued polynomials $p_n \in \mathcal{P}$ such that $p_n \to \sqrt{f}$ in $\mathcal{E}(\mathfrak{a})$. Then the polynomials $q_n := p_n^2$ are nonnegative on $\mathfrak{a}$ and $q_n \to f$ in $\mathcal{E}(\mathfrak{a})$. It follows that for alll $x \in \mathfrak{a}$,

$$V_{k',k} f(x) = \langle u_x^{k',k}, f \rangle = \lim_{n \to \infty} \langle u_x^{k',k}, q_n \rangle = \lim_{n \to \infty} V_{k',k}\, q_n(x) \geq 0.$$

A simple approximation argument, using $V_{k',k}1 = 1$, implies the assertion.

(2) $\Rightarrow$ (3) By assumption, the distribution $u_x^{k',k}$ is positive and therefore given by a compactly supported positive Radon measure ([Hö, Theorem 2.1.7]). Denoting this measure by $\mu_x^{k',k}$, we obtain from (2.4) that

$$E_{k'}(x,z) = \int_{\mathfrak{a}} E_k(\xi,z)d\mu_x^{k',k}(\xi), \quad z \in \mathfrak{a}_{\mathbb{C}}.$$

As $E_k(\xi,0) = 1$, evaluation at $z = 0$ shows that $\mu_x^{k',k}$ is a probability measure.

(3) $\Rightarrow$ (1) In view of formula (2.5), Corollary 2.4(2) implies that $u_x^{k',k} = \mu_x^{k',k}$. In particular, $\mu_x^{k',k}$ is compactly supported and uniquely determined by (2.6). We claim that $V_{k',k}$ acts on $\mathcal{P}$ by

$$V_{k',k}\,p(x) = \int_{\mathfrak{a}} p(\xi)\,d\mu_x^{k',k}(\xi). \tag{2.7}$$

After replacing p by $V_k p$, it suffices to prove that

$$V_{k'}p(x) = \int_{\mathfrak{a}} V_k p(\xi)\,d\mu_x^{k',k}(\xi) \quad \forall p \in \mathcal{P}.$$

But homogeneous expansion in (2.6) gives

$$\sum_{n=0}^{\infty} \frac{1}{n!} V_{k'}(\langle\,.\,,iy\rangle^n)(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathfrak{a}} V_k(\langle\,.\,,iy\rangle^n)(\xi)d\mu_x^{k',k}(\xi).$$

Comparison of the homogeneous parts in y shows that

$$V_{k'}(\langle\,.\,,y\rangle^n)(x) = \int_{\mathfrak{a}} V_k(\langle\,.\,,y\rangle^n)(\xi)d\mu_x^{k',k}(\xi)$$

for all $y \in \mathfrak{a}$ and $n \in \mathbb{Z}_+$. This implies the assertion. $\qquad\square$

The following analyticity result will be important in the next section.

Lemma 2.7. *Let $\varphi \in \mathcal{S}(\mathfrak{a})$. Then for fixed $x \in \mathfrak{a}$ and $k \in \mathcal{K}$ with $\mathrm{Re}\,k \geq 0$, the mapping $k' \mapsto V_{k',k}\,\varphi(x)$ is analytic on $\{k' \in \mathcal{K} : \mathrm{Re}\,k' > 0\}$.*

Proof. By Proposition 2.1,

$$V_{k',k}\,\varphi(x) = \frac{c_{k'}^2}{c_k^2}\mathcal{D}_{k'}^{-1}\big(\omega_k\omega_{k'}^{-1}\mathcal{D}_k\big)\varphi(x) = \frac{1}{c_k^2}\int_{\mathfrak{a}} \widehat{\varphi}^k(\xi)E_{k'}(ix,\xi)\omega_k(\xi)d\xi. \tag{2.8}$$

As $\widehat{\varphi}^k$ belongs to $\mathcal{S}(\mathfrak{a})$ and $k' \to E_{k'}(ix,\xi)$ is analytic on $\{k' : \mathrm{Re}\,k' > 0\}$ with $|E_{k'}^B(ix,\xi)| \leq \sqrt{|W|}$, it follows by standard arguments (dominated convergence and Morera's theorem) that the integral in (2.8) depends analytically on k' with $\mathrm{Re}\,k' > 0$. $\qquad\square$

Remark 2.8. In the rank one case, Y. Xu derived in [X, Lemma 2.1] for $k' > k > 0$ the explicit formula

$$V_{k',k}f(x) = \frac{\Gamma(k'+1/2)}{\Gamma(k'-k)\Gamma(k+1/2)}\int_{-1}^{1} f(xt)|t|^{2k}(1+t)(1-t^2)^{k'-k-1}dt.$$

This operator and the associated Sonine type integral representation for the rank one Dunkl kernel,

$$E_{k'}(x,z) = \frac{\Gamma(k'+1/2)}{\Gamma(k'-k)\Gamma(k+1/2)}\int_{-1}^{1} E_k(xt,z)|t|^{2k}(1+t)(1-t^2)^{k'-k-1}dt$$

were further studied in [Sol].

3. The Sonine formula for Bessel functions of type B_n

In this section, we consider the Dunkl kernel E_k^B and the Bessel function J_k^B associated with root system

$$B_n = \{\pm e_i, 1 \le i \le n\} \cup \{\pm e_i \pm e_j, \, 1 \le i < j \le n\} \subset \mathbb{R}^n,$$

where $\mathbb{R}^n$ is equipped with its usual inner product. The associated reflection group is the hyperoctahedral group $W(B_n) = S_n \ltimes \mathbb{Z}_2^n$, and the multiplicity is of the form $k = (k_1, k_2)$ where k_1 and k_2 denote the value on the roots $\pm e_i$ and $\pm e_i \pm e_j$ respectively. We shall derive an explicit Sonine formula for the Bessel function J_k^B at the reference point $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, extend it in a distributional sense to larger classes of multiplicities and construct counterexamples where the associated Bessel functions have no Sonine formula. Of decisive importance for our calculations is the well-known fact that J_k^B can be expressed in terms of a certain multivariable hypergeometric function. To recall this, we need some further notation.

Fix some index $\alpha > 0$. For partitions $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n, \lambda_1 \ge \ldots \ge \lambda_n$ (for short, $\lambda \ge 0$) we denote by C_λ^α the Jack polynomials of index α in n variables (c.f. [Sta]), normalized such that

$$(z_1 + \ldots + z_n)^m = \sum_{|\lambda| = m} C_\lambda^\alpha(z) \quad \text{for all } m \in \mathbb{Z}_+.$$

Following the notation of [K] and [BF], we define for $\mu \in \mathbb{C}$ with $\mathrm{Re}\, \mu > \frac{1}{\alpha}(n-1)$ the hypergeometric function

$${}_0F_1^\alpha(\mu; z, w) := \sum_{\lambda \ge 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!} \cdot \frac{C_\lambda^\alpha(z) C_\lambda^\alpha(w)}{C_\lambda^\alpha(\mathbf{1})} \quad (z, w \in \mathbb{C}^n)$$

with the generalized Pochhammer symbol

$$(\mu)_\lambda^\alpha := \prod_{j=1}^n \big(\mu - \frac{1}{\alpha}(j-1)\big)_{\lambda_j}.$$

In the one-dimensional case $n = 1$, the Jack polynomials are independent of α and given by $C_\lambda^\alpha(z) = z^\lambda$, $\lambda \in \mathbb{Z}_+$. Thus

$${}_0F_1^\alpha(\mu; -\frac{z^2}{4}, 1) = j_{\mu-1}(z).$$

In the general case, the Bessel function J_k^B is expressed in terms of ${}_0F_1^\alpha$ as follows:

Proposition 3.1. *Let* $k = (k_1, k_2)$ *with* $\mathrm{Re}\, k_1 \ge 0$ *and* $k_2 > 0$. *Then*

$$J_k^B(z, w) = {}_0F_1^\alpha\big(\mu; \frac{z^2}{2}, \frac{w^2}{2}\big) = {}_0F_1^\alpha\big(\mu; \frac{z^2}{4}, w^2\big) \quad (z, w \in \mathbb{C}^n),$$

where $\alpha = \frac{1}{k_2}$ *and* $\mu = \mu(k) := k_1 + k_2(n-1) + \frac{1}{2}$.

Proof. See [R2, Propos. 4.5] and [BF, Section 6] for real $k_1 \ge 0$. The general case follows by analytic continuation. $\square$

The key to the subsequent Sonine type integral representation for the Bessel function J_k^B is Kadell's generalization of the Selberg integral. For parameters $\kappa, \mu, \nu \in \mathbb{C}$

with $\operatorname{Re}\kappa \geq 0$ and $\operatorname{Re}\mu, \operatorname{Re}\nu > \operatorname{Re}\kappa(n-1)$, the Selberg integral is given by

$$\int_{]0,1[^n} \prod_{j=1}^n x_j^{\mu-\kappa(n-1)-1}(1-x_j)^{\nu-\kappa(n-1)-1} \prod_{1\leq i<j\leq n} |x_i - x_j|^{2\kappa} dx$$

$$= \prod_{j=1}^n \frac{\Gamma(1+\kappa j)}{\Gamma(1+\kappa)} \cdot \prod_{j=1}^n \frac{\Gamma(\mu-\kappa(j-1))\Gamma(\nu-\kappa(j-1))}{\Gamma(\mu+\nu-\kappa(j-1))} := I_n(\kappa,\mu,\nu).$$

(see e.g [FW]). With the normalized Selberg density

$$s_{\mu,\nu}^\kappa(x) := \frac{1}{I_n(\kappa,\mu,\nu)} \cdot \prod_{j=1}^n x_j^{\mu-\kappa(n-1)-1}(1-x_j)^{\nu-\kappa(n-1)-1} \prod_{1\leq i<j\leq n} |x_i - x_j|^{2\kappa},$$

Kadell's [Ka] generalization of the Selberg integral (c.f. also [FW, (2.46)]) reads

$$\int_{]0,1[^n} \frac{C_\lambda^\alpha(x)}{C_\lambda^\alpha(\mathbf{1})} s_{\mu,\nu}^{1/\alpha}(x)dx = \frac{(\mu)_\lambda^\alpha}{(\mu+\nu)_\lambda^\alpha}. \tag{3.1}$$

Formula (3.1) implies that for $z \in \mathbb{C}^n$ and $\mu,\nu \in \mathbb{C}$ with $\operatorname{Re}\mu, \operatorname{Re}\nu > \frac{1}{\alpha}(n-1)$,

$$_0F_1^\alpha(\mu+\nu; z, \mathbf{1}) = \int_{]0,1[^n} {}_0F_1^\alpha(\mu; z, x) s_{\mu,\nu}^{1/\alpha}(x)dx. \tag{3.2}$$

This is a Sonine formula for $_0F_1^\alpha$; in case $n=1$ it reduces to the classical Sonine integral (1.5) for one-variable Bessel functions.

The Sonine formula (3.2) translates to the Bessel function of type B_n as follows: Let $k = (k_1, k_2)$ with $\operatorname{Re}k_1 \geq 0$ and $k_2 > 0$. For $h \in \mathbb{C}$ put

$$k'(h) := (k_1 + h, k_2).$$

Then for $h \in \mathbb{C}$ with $\operatorname{Re}h > k_2(n-1)$ and all $z \in \mathbb{C}^n$,

$$J_{k'(h)}^B(z, \mathbf{1}) = \int_{]0,1[^n} J_k^B(z, x) f_{k,h}(x)dx \tag{3.3}$$

with the density

$$f_{k,h}(x) = \frac{2^n}{I_n(k_2, \mu(k), h)} \prod_{j=1}^n (x_j^2)^{k_1}(1-x_j^2)^{h-k_2(n-1)-1} \prod_{i<j} |x_i^2 - x_j^2|^{2k_2}$$

and with $\mu(k)$ as in Proposition 3.1. Note that $f_{k,h}$ is $W(B_n)$-invariant, and therefore

$$J_{k'(h)}^B(z, \mathbf{1}) = \int_{]0,1[^n} E_k^B(z, x) f_{k,h}(x)dx. \tag{3.4}$$

We extend $f_{k,h}$ by zero to a measurable function on $\mathbb{R}^n$. For $\operatorname{Re}h > k_2(n-1)$ we have $f_{k,h} \in L_{loc}^1(\mathbb{R}^n)$, which corresponds to a complex Radon measure

$$d\rho_{k,h}(x) = f_{k,h}(x)dx.$$

Now recall from Section 2 the distributions $u_x^{k',k} \in \mathcal{E}'(\mathbb{R}^n)$ defined by $\langle u_x^{k',k}, \varphi \rangle = V_{k',k}\varphi(x)$.

Definition 3.2. Consider $k = (k_1, k_2)$ on root system B_n with $\operatorname{Re}k_1 \geq 0$ and $k_2 > 0$ as above. For $h \in \mathbb{C}$ with $\operatorname{Re}h \geq -\operatorname{Re}k_1$ denote by $S_{k,h} \in \mathcal{E}'(\mathbb{R}^n)$ the $W(B_n)$-mean of $u_{\mathbf{1}}^{k'(h),k}$, i.e.

$$\langle S_{k,h}, \varphi \rangle := \frac{1}{2^n n!} \sum_{w \in W(B_n)} \langle u_{\mathbf{1}}^{k'(h),k}, w.\varphi \rangle.$$

According to Lemma 2.5, $S_{k,h}$ is supported in the Euclidean ball $B_{\sqrt{n}}(0)$, and it is $W(B_n)$-invariant. Thus in view of (2.5), we have the following distributional extension of the Sonine formula (3.3):

$$J^B_{k'(h)}(z, \mathbf{1}) = \langle S_{k,h}, J^B_k(z, \,.\,)\rangle = \langle S_{k,h}, E^B_k(z, \,.\,)\rangle, \quad z \in \mathbb{C}^n. \qquad (3.5)$$

The next result is a simple criterion for the existence of a Sonine-type integral representation for the Bessel function of type B_n.

Proposition 3.3. *Let $k = (k_1, k_2)$ with $k_2 > 0$ and $\operatorname{Re} k_1 \geq 0$. Then for $\operatorname{Re} h \geq -\operatorname{Re} k_1$ the following are equivalent:*

(1) *The distribution $S_{k,h}$ is a complex (positive) measure.*
(2) *There exists a bounded complex (positive) Radon measure $m \in M_b(\mathbb{R}^n)$ such that the following Sonine formula holds:*

$$J^B_{k'(h)}(i\xi, \mathbf{1}) = \int_{\mathbb{R}^n} J^B_k(i\xi, x)\, dm(x) \quad \text{for all } \xi \in \mathbb{R}^n.$$

In this case, the measure m in (2) is unique and given by $m = S_{k,h}$.

Proof. The implication $(1) \Rightarrow (2)$ is immediate from identity (3.5). For the converse direction, note first that we may assume that m is $W(B_n)$-invariant. Thus from (3.5), we obtain that

$$\langle S_{k,h}, E^B_k(i\xi, \,.\,)\rangle = \langle m, J^B_k(i\xi, \,.\,)\rangle = \langle m, E^B_k(i\xi, \,.\,)\rangle \quad \text{for all } \xi \in \mathbb{R}^n.$$

Corollary 2.4(2) for the Laplace transform now implies that $m = S_{k,h}$. $\qquad \square$

Identity (3.5) together with (3.4) and the injectivity of the Dunkl Laplace transform (Corollary 2.4) imply that for $\operatorname{Re} h > k_2(n-1)$,

$$S_{k,h} = \rho_{k,h}.$$

We are interested to know for which range of h the distribution $S_{k,h}$ is actually a complex Radon measure, i.e. of order zero. The following useful observation of Sokal [So, Lemmata 2.1, 2.2 and Proposition 2.3]) will provide a necessary condition.

Lemma 3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $D \subseteq \mathbb{C}$ open and connected. Suppose that*

$$F : \Omega \times D \to \mathbb{C}, \quad (x, \lambda) \to f_\lambda(x) := F(x, \lambda)$$

is a continuous function such that $F(x, .)$ is analytic on D for each $x \in \Omega$. Extend f_λ by zero to all of $\mathbb{R}^n$ and define $u_\lambda \in \mathcal{D}'(\Omega)$ by

$$\langle u_\lambda, \varphi \rangle = \int_\Omega \varphi(x) f_\lambda(x)\, dx.$$

Then the following hold:

(1) *The map $\lambda \mapsto u_\lambda$, $D \to \mathcal{D}'(\Omega)$ is weakly analytic, which means that $\lambda \mapsto \langle u_\lambda, \varphi \rangle$ is analytic for all $\varphi \in \mathcal{D}(\Omega)$.*
(2) *Let $D_0 \subseteq D$ be a nonempty open set, and suppose that there is a weakly analytic map $\lambda \mapsto \widetilde{u}_\lambda$, $D \to \mathcal{D}'(\mathbb{R}^n)$ such that for each $\lambda \in D_0$ the distribution $\widetilde{u}_\lambda$ extends the distribution u_λ from Ω to $\mathbb{R}^n$. Then $\widetilde{u}_\lambda$ extends u_λ for each $\lambda \in D$. Moreover, if $\widetilde{u}_\lambda$ is a complex Radon measure on $\mathbb{R}^n$, then f_λ belongs to $L^1_{loc}(\overline{\Omega})$. This means that f_λ is integrable over a sufficiently small neighborhood in $\mathbb{R}^n$ of any point $x \in \overline{\Omega}$.*

To apply this lemma to our situation, fix $k = (k_1, k_2)$ with $\operatorname{Re} k_1 \geq 0$ and $k_2 > 0$ and put

$$D := \{h \in \mathbb{C} : \operatorname{Re} h > -\operatorname{Re} k_1\}, \quad D_0 := \{h \in \mathbb{C} : \operatorname{Re} h > k_2(n-1)\}.$$

Theorem 3.5.   (1) *The mapping $h \mapsto S_{k,h}$ is weakly analytic on D.*

  (2) *Let $h \in D$ and suppose that $S_{k,h}$ is a complex Radon measure on $\mathbb{R}^n$. Then either $h \in D_0$, in which case $S_{k,h} = \rho_{k,h}$, or h is contained in the discrete set $\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{Z}_+$.*

  (3) *Suppose that k_1 is real and $S_{k,h}$ is a positive Radon measure, then in addition to the condition in (2), h must be real.*

Proof. (1) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. By Lemma 2.7, the mapping $k' \mapsto V_{k',k}\,\varphi(\mathbf{1}) = u_{\mathbf{1}}^{k',k}(\varphi)$ is analytic on $\{\operatorname{Re} k' > 0\}$, and therefore $\langle S_{k,h}, \varphi \rangle$ depends analytically on $h \in D$.

(2) Consider formula (3.3), which is valid for $h \in D_0$. The function

$$h \mapsto I_n(k_2, \mu(k), h) = \prod_{j=1}^{n} \frac{\Gamma(1 + jk_2)}{\Gamma(1 + k_2)} \cdot \prod_{j=0}^{n-1} \frac{\Gamma\big(k_1 + jk_2 + \tfrac{1}{2}\big)\Gamma(h - jk_2)}{\Gamma\big(k_1 + jk_2 + h + \tfrac{1}{2}\big)}$$

extends to a meromorphic function on D without zeroes and with pole set

$$D \cap \big(\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{Z}_+\big).$$

Thus the function $h \mapsto f_{k,h}(x)$ extends analytically to D for each $x \in \,]0, 1[^n$. If $h \in D$ is such that $S_{k,h}$ is a complex Radon measure on $\mathbb{R}^n$, then it follows from Lemma 3.4 that $f_{k,h} \in L^1_{loc}([0,1]^n)$. This in turn implies that either $\operatorname{Re} h > k_2(n-1)$ or h is a pole of the function $I_n(k_2, \mu(k), \,.\,)$.

(3) is immediate from part (2).

$\square$

As an important consequence of the previous results, we obtain that in the B_n-case and for arbitrary multiplicity k with $k_1 \geq 0$ and $k_2 > 0$ there exist multiplicities $k' = (k_1 + h, k_2) \geq k$ such that the Bessel function $J_{k'}^B$ has no Sonine integral representation with respect to J_k^B, and that the intertwiner $V_{k',k}$ is not positive. More precisely, the following holds.

Corollary 3.6. *Let $k = (k_1, k_2) \in \mathbb{C}^2$ with $k_2 > 0$, $\operatorname{Re} k_1 \geq 0$ and consider $k' = (k_1 + h, k_2)$ with $\operatorname{Re} h > -\operatorname{Re} k_1$.*

  (1) *Suppose that there exists a bounded complex Radon measure $m \in M_b(\mathbb{R}^n)$ such that the Sonine formula*

$$J_{k'}^B(i\xi, \mathbf{1}) = \int_{\mathbb{R}^n} J_k^B(i\xi, x)\,dm(x) \tag{3.6}$$

holds for all $\xi \in \mathbb{R}^n$. Then either $\operatorname{Re} h > k_2(n-1)$, in which case (3.6) holds with $m = \rho_{k,h}$, or h is contained in $\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{Z}_+$. If $k_1 \geq 0$ and m is positive, then in addition h must be real.

  (2) *Suppose that $k_1 \geq 0$ and that $V_{k',k}$ is positive. Then h is contained in the set*

$$\Sigma(k_2) := \,]k_2(n-1), \infty[\, \cup \big(\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{Z}_+\big).$$

Remarks 3.7. (1) The set $\Sigma(k_2)$ is closely related with the so-called Wallach set

$$]d(n-1), \infty[\, \cup \{0, d, \ldots, d(n-1)\},$$

where $d \in \mathbb{N}$ is the Peirce constant of a symmetric cone. The Wallach set plays an important role in the analysis on symmetric cones, see [FK] for some background. It describes the set of parameters for which Riesz distributions on a symmetric cone are actually positive measures, a result which is due to Gindikin [G].

(2) Corollary 3.6 should be compared with the results of [RV2, Section 4] for Bessel functions on symmetric cones, which are closely related to the Bessel functions J_k^B. In [RV2, Theorem 4] also a sufficient condition for the existence of Sonine formulas between Bessel functions on a symmetric cone is given. It is based on the knowledge of the parameters for which Riesz distributions are actually measures. Corresponding results are not yet available in the Dunkl setting.

4. Consequences for hypergeometric functions and Heckman-Opdam polynomials of type BC

We start with some basic facts from Heckman-Opdam theory, see [HS, O2] for more details. Let again $(\mathfrak{a}, \langle \,.\,,\,.\,\rangle)$ be a finite dimensional Euclidean space, which we identify with its dual $\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ via the given inner product. Let R be a crystallographic, not necessarily reduced root system in $\mathfrak{a}$ with associated reflection group W and fix a positive subsystem R_+ of R as well as a W-invariant multiplicity function k on R, where we assume for simplicity that k is real-valued with $k \geq 0$. The Cherednik operators associated with R_+ and k are defined by

$$D_\xi(k) = \partial_\xi + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{1}{1 - e^{-\alpha}} (1 - \sigma_\alpha) - \langle \rho(k), \xi \rangle, \quad \xi \in \mathbb{R}^n$$

where $e^\lambda(z) := e^{\langle \lambda, z \rangle}$ for $\lambda, z \in \mathfrak{a}$ and $\rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k(\alpha) \alpha$.

The $D_\xi(k), \xi \in \mathfrak{a}$ commute, and for each $\lambda \in \mathfrak{a}_\mathbb{C}$ there exists a unique analytic function $G(\lambda, k; .)$ on a common W-invariant tubular neighborhood of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$, the Opdam-Cherednik kernel, satisfying

$$D_\xi(k) G(\lambda, k; .) = \langle \lambda, \xi \rangle \, G(\lambda, k; .) \ \forall \xi \in \mathfrak{a}; \ G(\lambda, k; 0) = 1.$$

The hypergeometric function associated with R is defined by

$$F(\lambda, k; z) = \frac{1}{|W|} \sum_{w \in W} G(\lambda, k; w^{-1} z).$$

Closely related with the hypegeometric function are the Heckman-Opdam polynomials. To introduce these, write $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ for $\alpha \in R$ and consider the weight lattice and the set of dominant weights associated with R and R_+,

$$P = \{\lambda \in \mathfrak{a} : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in R\}; \quad P_+ = \{\lambda \in P : \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in R_+\}.$$

Note that $R_+ \subset P_+$. We equip P_+ with the usual dominance order, that is, $\mu < \lambda$ iff $\lambda - \mu$ is a sum of positive roots. Denote further by $\mathcal{T} := \mathrm{span}_\mathbb{C}\{e^{i\lambda}, \lambda \in P\}$ the space of trigonometric polynomials associated with R. Notice that the members of $\mathcal{T}$ are $2\pi Q^\vee$-periodic, where $Q^\vee = \mathrm{span}_\mathbb{Z}\{\alpha^\vee, \alpha \in R\}$, and that the orbit sums

$$M_\lambda = \sum_{\mu \in W\lambda} e^{i\mu}, \quad \lambda \in P_+$$

form a basis of the subspace $\mathcal{T}^W$ of W-invariant elements from $\mathcal{T}$. Consider the compact torus $\mathbb{T} = \mathfrak{a}/2\pi Q^\vee$ with the weight function

$$\delta_k(t) := \prod_{\alpha \in R_+} |\sin \frac{\langle \alpha, t \rangle}{2}|^{2k_\alpha}.$$

The Heckman-Opdam polynomials associated with R_+ and k are defined by

$$P_\lambda(k; z) := M_\lambda(z) + \sum_{\nu < \lambda} c_{\lambda\nu}(k) M_\nu(z); \quad \lambda \in P_+\, , z \in \mathfrak{a}_{\mathbb{C}}$$

where the coefficients $c_{\lambda\nu}(k) \in \mathbb{R}$ are uniquely determined by the condition that $P_\lambda(k; .\,)$ is orthogonal to M_μ in $L^2(\mathbb{T}, \delta_k)$ for all $\nu \in P_+$ with $\nu < \lambda$. It is known that the coefficients actually satisfy $c_{\lambda\nu}(k) \geq 0$ for all indices λ, ν ([M, Par.11]), and that the family $\{P_\lambda(k, .\,), \lambda \in P_+\}$ forms an orthonormal basis of $L^2(\mathbb{T}, \delta_k)^W$, the subspace of W-invariant functions from $L^2(\mathbb{T}, \delta_k)$. The renormalized polynomials

$$R_\lambda(k, z) := \frac{P_\lambda(k; z)}{P_\lambda(k; 0)}$$

are related with the hypergeometric function via (see [HS])

$$R_\lambda(k, z) = F(\lambda + \rho(k), k; iz).$$

As $R_0(k, .\,) = 1$, it follows that

$$F(\rho(k), k; .\,) = 1. \tag{4.1}$$

We now consider $\mathfrak{a} = \mathbb{R}^n$ with the nonreduced root system

$$R = BC_n = \{\pm e_i, \pm 2e_i, 1 \leq i \leq n\} \cup \{\pm(e_i \pm e_j), 1 \leq i < j \leq n\} \subset \mathbb{R}^n.$$

Its weight lattice is $P = \mathbb{Z}^n$ and the torus $\mathbb{T}$ is given by $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^n$. We write mulitplicities on R as $k = (k_1, k_2, k_3)$ with k_1, k_2, k_3 the values on the roots $e_i, 2e_i, e_i \pm e_j$. We fix some positive subsystem R_+ and denote the associated Opdam-Cherednik kernel and hypergeometric function by G_{BC} and F_{BC}.

Dunkl operators are scaling limits of Cherednik operators, which implies that Dunkl kernels and Bessel functions can be obtained by a contraction limit from Opdam-Cherednik kernels and hypergeometric functions. We shall need the following variant of Theorem 4.12 in [dJ2] (see also [RV1]) which was originally formulated for reduced root systems. The proof extends to $R = BC_n$ in the obvious way.

Lemma 4.1. *Consider the root systems BC_n with multiplicity $k = (k_1, k_2, k_3)$ and B_n with multiplicity $k_0 := (k_1 + k_2, k_3)$. Let further $K, L \subset \mathbb{C}^n$ be compact, $\delta > 0$ some constant and let $h : (0, \delta) \times L \to \mathbb{C}^n$ a continuous function such that $\lim_{\epsilon \to 0} \epsilon h(\epsilon, \lambda) = \lambda$ uniformly on L. Then*

$$\lim_{\epsilon \to 0} G_{BC}(h(\epsilon, \lambda), k; \epsilon z) = E_{k_0}^B(\lambda, z), \quad \lim_{\epsilon \to 0} F_{BC}(h(\epsilon, \lambda), k; \epsilon z) = J_{k_0}^B(\lambda, z) \tag{4.2}$$

uniformly for $(\lambda, z) \in L \times K$.

Hypergeometric functions associated with root systems generalize the spherical functions of Riemannian symmetric spaces G/K of noncompact type. More precisely, suppose that Σ is the restricted root system of G/K with Weyl group W and geometric multiplicities $m_\alpha, \alpha \in \Sigma$. Let F be the hypergeometric function associated with $R = 2\Sigma$ and define the multiplicity k on R by $k(2\alpha) := \frac{1}{2}m(\alpha)$. Consider the decomposition $G = KAK$ and let $\mathfrak{a} := Lie(A)$, which is a Euclidean space with the Killing form $\langle ., . \rangle$. Then the spherical functions of G/K, considered

as W-invariant functions on $\mathfrak{a}$, are given by $\varphi_\lambda(x) = F(\lambda, k; x)$, $\lambda \in \mathfrak{a}_\mathbb{C}$. From the Harish-Chandra formula [Hel, Theorem IV.4.3] and the Kostant convexity theorem it follows that for R and k as above,

$$F(\lambda + \rho(k), k; x) = \int_{C(x)} e^{\langle \lambda, \xi \rangle} dm_x^k(\xi) \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}$$

where $C(x) \subset \mathfrak{a}$ again denotes the convex hull of the W-orbit of x and m_x^k is a certain W-invariant probability measure. For root system A_n and certain BC_n-cases, this integral representation was recently extended in [Sa1, Sa2] to arbitrary non-negative multiplicities, including a detailed analysis of the representing measures m_x^k. See also [Su] for an alternative approach in the A_n-case. A natural generalization would be an integral representation of Sonine type between hypergeometric functions with different multiplicities. We obtain the following necessary condition on the multiplicities in the BC_n case as a consequence of Corollary 3.6.

Theorem 4.2. *Fix* $k = (k_1, k_2, k_3) \in \mathbb{R}^3$ *with* $k_1, k_2 \geq 0$, $k_3 > 0$ *and consider* $k' = k'(h) := (k_1 + h_1, k_2 + h_2, k_3)$ *with* $h_1 > -k_1, h_2 > -k_2$. *Suppose that for each* $c > 0$ *there exists a positive Radon measure* m_c *on* $\mathbb{R}^n$ *with* $\operatorname{supp} m_c \subseteq \overline{B_c(0)}$ *such that the following Sonine formula holds:*

$$F_{BC}(\lambda + \rho(k'), k'; \mathbf{c}) = \int_{\mathbb{R}^n} F_{BC}(\lambda + \rho(k), k; \xi) \, dm_c(\xi) \quad \forall \lambda \in \mathbb{C}^n, \qquad (4.3)$$

with $\mathbf{c} = (c, \ldots, c)$. *Then* $h_1 + h_2$ *must be contained in the set*

$$\Sigma(k_3) =]k_3(n-1), \infty[\cup \left(\{0, k_3, \ldots, k_3(n-1)\} - \mathbb{Z}_+ \right).$$

In particular, there are multiplicities $k, k' \geq 0$ *on* BC_n *with* $k' \geq k$ *such that a Sonine formula (4.3) does not exist.*

Note that the measures m_c in (4.3) are actually probability measures, which is immediate from the normalization (4.1).

Proof. For $\epsilon > 0$ denote by $m_\epsilon^\epsilon \in M^1(\mathbb{R}^n)$ the image measure of m_ϵ under the mapping $x \mapsto x/\epsilon$. Then by (4.3),

$$F_{BC}\left(\frac{\lambda}{\epsilon} + \rho(k'), k'; \boldsymbol{\epsilon}\right) = \int_{\mathbb{R}^n} F_{BC}\left(\frac{\lambda}{\epsilon} + \rho(k), k; \epsilon \xi\right) dm_\epsilon^\epsilon(\xi) \qquad (4.4)$$

for all $\lambda \in \mathbb{C}^n$. By our assumption, the measures m_ϵ^ϵ are compactly supported in the ball $\overline{B_1(0)}$. Thus by Prohorov's theorem (see e.g. [Bi]), the set $\{m_\epsilon^\epsilon, \epsilon > 0\}$ is relatively (sequentially) compact. Hence there exists a sequence $\epsilon_n \to 0$ and a measure $m \in M^1(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq \overline{B_1(0)}$ such that $m_{\epsilon_n}^{\epsilon_n} \to m$ weakly as $n \to \infty$. Put $k_0 := (k_1 + k_2, k_3)$ and $k_0' := (k_1 + k_2 + h_1 + h_2, k_3)$ on B_n. Using Lemma 4.1 and taking the limit $\epsilon_n \to 0$ in formula (4.4), we obtain that

$$J_{k_0'}^B(\lambda, \mathbf{1}) = \int_{\mathbb{R}^n} J_{k_0}^B(\lambda, \xi) dm(\xi)$$

for all $\lambda \in \mathbb{C}^n$. Corollary 3.6 now implies the assertion. $\qquad\square$

We now turn to the Heckman-Opdam polynomials of type BC_n. These have been extensively studied in the literature, see for instance [BO, La, RR]. We consider

the ($W(B_n)$-invariant) normalized polynomials $R_\lambda = R_\lambda^{BC_n}$, $\lambda \in \mathbb{Z}_+^n$. A short calculation shows that the rescaled polynomials $\widetilde{R}_\lambda$ on $[0,1]^n$ defined by

$$\widetilde{R}_\lambda\big(\frac{1}{2}(1-\cos t)\big) := R_\lambda(k;t), \quad \lambda \in \mathbb{Z}_+^n$$

form an orthogonal basis of $L^2([0,1]^n, \rho_k)$ with the weight function

$$\rho_k(x) = \prod_{i=1}^n x_i^{k_1+k_2-1/2}(1-x_i)^{k_2-1/2}\prod_{i<j}|x_i - x_j|^{2k_3}.$$

In the rank one case, the Heckman-Opdam polynomials can be written in terms of the classical one-variable Jacobi polynomials

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1\big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)\big) \quad (\alpha,\beta > -1, \; n \in \mathbb{Z}_+)$$

as follows, see [HS, Ex.1.3.2]:

$$R_n^{BC_1}(k;t) = R_n^{(\alpha,\beta)}(\cos t) \;\; \text{with} \;\; \alpha = k_1+k_2-\frac{1}{2}, \; \beta = k_2 - \frac{1}{2}.$$

Classical Jacobi polynomials have various interesting integral representations. Of relevance in our context is the following one, which is proven in [A1, eq. (4.19)]: Let $\alpha, \beta > -1$ and $\nu > 0$. Then for each $x \in [-1,1]$ there exists a (necessarily unique) probability measure $\mu_x \in M^1([-1,1])$ such that

$$R_n^{(\alpha+\nu,\beta)}(x) = \int_{-1}^1 R_n^{(\alpha,\beta)}(y)d\mu_x(y) \quad \forall n \in \mathbb{Z}_+. \tag{4.5}$$

In the higher rank case, one may ask for integral representations between Heckman-Opdam polynomials with different multiplicities. Note that for the normalized Heckman-Opdam polynomials R_λ of type BC_n, Lemma (4.1) implies that

$$\lim_{m\to\infty} R_{m\lambda}\big(k;\frac{t}{m}\big) = J_{k_0}^B(\lambda, it). \tag{4.6}$$

The following necessary condition concerns the analogue of formula (4.5) in higher rank; it is a counterpart of Theorem 4.2 with essentially the same proof.

Proposition 4.3. *Consider root system BC_n with multiplicities $k = (k_1, k_2, k_3)$ and $k' = (k_1 + h_1, k_2 + h_2, k_3)$ as in Proposition 4.2. Suppose that for each $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ there exists a positive Radon measure m_τ on $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^n$ such that*

$$R_\lambda(k';\tau) = \int_{\mathbb{T}} R_\lambda(k;s)dm_\tau(s) \quad \forall \lambda \in \mathbb{Z}_+^n.$$

Then $h_1 + h_2$ is contained in $\Sigma(k_3)$.

We finally turn to branching rules for Heckman-Opdam polynomials of type BC_n. For multiplicities k, k' on BC_n and $\lambda \in \mathbb{Z}_+^n$ we have an expansion

$$R_\lambda(k',t) = \sum_{\nu \leq \lambda} c_{\lambda,\nu}(k',k)R_\nu(k;t)$$

with unique connection coefficients $c_{\lambda,\nu}(k',k) \in \mathbb{R}$. In rank one, the following positivity result for the connection coefficients between Jacobi polynomial systems is well-known, see e.g. [A2, (7.33)]: For $\alpha, \beta > -1$ and $\nu > 0$,

$$R_n^{(\alpha+\nu,\beta)} = \sum_{j=0}^n c_{n,j} R_j^{(\alpha,\beta)} \;\; \text{with} \;\; c_{n,j} \geq 0.$$

Heckman-Opdam polynomials of type BC_n generalize the spherical functions of compact Grassmannians. See [RR] for a detailed treatment, where however the notation (scaling of root systems and multiplicities) is slightly different from ours. To become specific, consider for fixed $n \in \mathbb{N}$ and integers $m > n$ the compact Grassmann manifolds U_m/K_m with $U_m = SU(m+n, \mathbb{F}), K_m = S(U(m, \mathbb{F}) \times U(n, \mathbb{F}))$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Via polar decomposition of U_m, the double coset space $U_m//K_m$ may be topologically identified with the fundamental alcove

$$A_0 = \{t \in \mathbb{R}^n : \frac{\pi}{2} \geq t_1 \geq \ldots \geq t_n \geq 0\}$$

with $t \in A_0$ being identified with the matrix

$$a_t = \begin{pmatrix} \cos \underline{t} & 0 & -\sin \underline{t} \\ 0 & I_{m-n} & 0 \\ \sin \underline{t} & 0 & \cos \underline{t} \end{pmatrix} \in U_m \,, \quad \underline{t} = \mathrm{diag}(t_1, \ldots, t_n).$$

The spherical functions of U_m/K_m are given by

$$\varphi_\lambda^m(a_t) = R_\lambda(k_m; 2t), \quad \lambda \in \mathbb{Z}_+^n$$

with

$$k_m = \big(d(m-n)/2, (d-1)/2, d/2\big), \quad d = \dim_\mathbb{R} \mathbb{F}.$$

Here the R_λ are again the normalized Heckman-Opdam polynomials associated with root system BC_n. For integers $l > m$ we consider U_m as a closed subgroup of U_l. Then $K_m = U_m \cap K_l$ and U_m/K_m is a submanifold of U_l/K_l. As a function on U_l, the spherical function φ_λ^l is K_l-biinvariant and positive definite, and its restriction to U_m is K_m-biinvariant and positive definite on U_m. This implies that

$$\varphi_\lambda^l|_{U_m} = \sum_{\nu \in \mathbb{Z}_+^n} c_{\lambda,\nu} \varphi_\nu^m$$

with unique branching coefficients $c_{\lambda,\nu} = c_{\lambda,\nu}(l, m) \geq 0$, only finitely many of them being different from zero. For the Heckman-Opdam polyomials this implies that

$$R_\lambda(k_l; t) = \sum_{\nu \leq \lambda} c_{\lambda,\nu} R_\nu(k_m; t),$$

so the connection coefficients between the two systems are non-negative.

The next result however shows that for general multiplicities $k = (k_1, k_2, k_3) \geq 0$ and $k' = (k_1 + h, k_2, k_3)$ with $h > 0$ there may also occur negative connection coefficients between the associated systems of Heckman-Opdam polynomials.

Theorem 4.4. *Consider root system BC_n with multiplicities $k = (k_1, k_2, k_3)$ and $k' = (k_1 + h_1, k_2 + h_2, k_3)$ as in Theorem 4.2. Suppose that $h_1 + h_2 \notin \Sigma(k_3)$. Then the connection coefficients in the expansion*

$$R_{\mathbf{m}}(k'; t) = \sum_{\nu \leq \mathbf{m}} c_{\mathbf{m},\nu} R_\nu(k; t), \quad \mathbf{m} = (m, \ldots, m) \tag{4.7}$$

satisfy

$$\sup_{m \in \mathbb{N}} \sum_{\nu \leq \mathbf{m}} |c_{\mathbf{m},\nu}| = \infty.$$

In particular, there exist infinitely many $m \in \mathbb{N}$ such that $c_{\mathbf{m},\nu} < 0$ for some ν.

Proof. Assume in the contrary that $S := \sup_{m \in \mathbb{N}} \sum_{\nu \leq \mathbf{m}} |c_{\mathbf{m},\nu}| < \infty$. We proceed similar as in [RV1] and introduce the bounded, discrete signed measures

$$\mu_m := \sum_{\nu \leq \mathbf{m}} c_{\mathbf{m},\nu}\, \delta_{\nu/m} \in M_b(\mathbb{R}^n), \quad m \in \mathbb{N},$$

where δ_x denotes the point measure in $x \in \mathbb{R}^n$. By definition of the dominance order, the support of μ_m is contained in the compact cube $[0,1]^n$. With these measures, expansion (4.7) can be written as

$$R_{\mathbf{m}}\big(k'; \frac{t}{m}\big) = \sum_{\nu \leq \mathbf{m}} c_{\mathbf{m},\nu}\, F_{BC}\big(\nu + \rho(k), k; \frac{it}{m}\big) = \int_{[0,1]^n} F_{BC}\big(mx + \rho(k), k; \frac{it}{m}\big) d\mu_m(x).$$

We consider the Jordan decomposition $\mu_m = \mu_m^1 - \mu_m^2$ where μ_m^i are positive measures whose total variation norm satisfies $\|\mu_m^i\| \leq \|\mu_m\| \leq S$ and which are supported in $[0,1]^n$. Using again Prohorov's theorem we obtain, after passing to subsequences if necessary, that there exist positive bounded Radon measures μ^i on $\mathbb{R}^n$ with $\mathrm{supp}(\mu^i) \subseteq [0,1]^n$ and such that $\mu_m^i \to \mu^i$ as $m \to \infty$. Therefore $\mu_m \to \mu := \mu^1 - \mu^2$ weakly. Taking the limit $m \to \infty$ and employing Lemma 4.1 as well as formula (4.6), we obtain

$$J_{k_0'}^B(\mathbf{1}, it) = \int_{[0,1]^n} J_{k_0}^B(\xi, it) d\mu(\xi) \quad \forall t \in \mathbb{R}^n,$$

with $k_0 = (k_1 + k_2, k_3), k_0' = (k_1 + k_2 + h_1 + h_2, k_3)$. Again, Corollary 3.6 now implies that $h_1 + h_2 \in \Sigma(k_3)$, a contradition. $\qquad\square$

Remark 4.5. Apart from the well-studied rank one case (see [A2] for an overview) and the geometric cases described above, further nontrivial pairs of Heckman-Opdam polynomial families with nonnegative connection coefficients seem to be unknown.

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