THE LAPLACE TRANSFORM IN DUNKL THEORY

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Abstract. In this note, we give an overview of the Laplace transform in Dunkl theory associated with root systems of type A and some of its applications. The results generalize well-known facts in the spherical analysis on symmetric cones.

1. Introduction

In his unpublished manuscript [M13] from the 1980ies, I.G. Macdonald presented a concept generalizing many known properties of the radial analysis on symmetric cones, c.f. [FK94]. His idea was to replace the spherical polynomials of the cone, which are given by Jack polynomials with a certain half-integer index, by Jack polynomials with an arbitrary index. However, many of the statements in [M13] remained conjectural. This was due to the fact that the associated Laplace transform, now involving multivariate Bessel functions instead of the usual exponential function, was not well-understood at that time. Macdonald's ideas were taken up in [BF98] within the study of quantum integrable models of Calogero-Moser type, where also their connection to Dunkl theory was recognized, and later for example in [SZ07]. A rigorous treatment of the relevant Laplace transform in the framework of Dunkl theory was given only much later in [R20] and continued in [BR23], where a new proof for the fundamental Laplace transform identity of Jack polynomials from [BF98] is given and also various statements from [M13, Kan93] are improved or made precise. In the present article, we give an overview of results from [R20, BR23], which constitute natural generalizations of radial analysis on symmetric cones in the framework of Dunkl theory associated with root systems of type A. In particular, we describe inversion theorems for the Laplace transform as well as applications to Riesz distributions and Jack-hypergeometric series.

2. Motivation: Analysis on Hermitian matrices

Consider the space of $n \times n$-Hermitian matrices over one of the (skew-) fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$,

$$H_n(\mathbb{F}) = \{x \in M_n(\mathbb{F}) : x = \overline{x}^t\}.$$

This is a real Euclidean vector space with scalar product $\langle x, y\rangle = \operatorname{Re}\operatorname{tr}(xy)$. The cone of positive definite matrices

$$\Omega_n(\mathbb{F}) = \{x \in H_n(\mathbb{F}) : x \text{ positive definite}\}$$

naturally identifies with the Riemannian symmetric space $GL_n(\mathbb{F})/U_n(\mathbb{F})$. Actually, $H_n(\mathbb{F})$ carries the structure of a Euclidean Jordan algebra and $\Omega = \Omega_n(\mathbb{F})$ is a

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symmetric cone, see [FK94] for some background on these and the subsequent facts. The fundamental objects in the harmonic analysis on Ω are its spherical functions

$$\varphi_\lambda(x) = \int_K \Delta_\lambda(kxk^{-1})dk, \quad x \in \Omega,\, \lambda \in \mathbb{C}^n; \tag{2.1}$$

here the functions $\Delta_\lambda(x)$ are power functions on Ω generalizing the usual powers x^λ for $x \in]0,\infty[$ and $\lambda \in \mathbb{C}$. In particular, if $x = \operatorname{diag}(\xi_1, \dots, \xi_n)$, then $x^\lambda = \xi_1^{\lambda_1} \cdots \xi_n^{\lambda_n}$. The spherical function φ_λ is K-invariant (K acts on Ω by conjugation), and hence depends only on the spectrum of its argument. Of particular importance in the analysis on Ω is their Laplace transform ([FK94, Chapt. VII]): Let $\operatorname{Re}\lambda_j > \frac{d}{2}(j-1)$. Then

$$\int_\Omega e^{-\langle x,y\rangle}\varphi_\lambda(x)\Delta(x)^{-\frac{d}{2}(n-1)-1}dx = \Gamma_\Omega(\lambda)\varphi_\lambda(y^{-1}), \tag{2.2}$$

with Γ_Ω the gamma function associated with Ω, Δ the (Jordan) determinant and $d = \dim_\mathbb{R}\mathbb{F} \in \{1,2,4\}$. Let

$$\Lambda_n^+ := \{\lambda \in \mathbb{N}_0^n : \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}$$

denote the set of partitions of length at most n. Then the spherical functions φ_λ with $\lambda \in \Lambda_n^+$ are polynomials. More precisely, let $C_\lambda^\alpha = C_\lambda^{(\alpha)}$, $\lambda \in \Lambda_n^+$ denote the Jack polynomials in n variables of index $\alpha \in [0,\infty]$, normalized such that

$$(z_1 + \cdots + z_n)^m = \sum_{|\lambda|=m} C_\lambda^\alpha(z) \quad (z \in \mathbb{C}^n,\, m \in \mathbb{N}_0).$$

Then, as observed by Macdonald in [M87],

$$\varphi_\lambda(x) = \frac{C_\lambda^\alpha(\operatorname{spec}(x))}{C_\lambda^\alpha(\underline{1})} \quad \text{with } \alpha = \tfrac{2}{d}, \quad \underline{1} = (1, \dots, 1).$$

The Jack polynomials C_λ^α are homogeneous of degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and symmetric. They are, among others, important in algebraic combinatorics, multivariate statistics, and random matrix theory; see [St89] for their basic properties. For $\alpha = 1$, they coincide with the Schur polynomials. If $n = 1$, then $C_\lambda^\alpha(z) = z^\lambda$.

Let us now consider the Laplace transform of a K-invariant function $f : \Omega \to \mathbb{C}$. Put $\mathbb{R}_+ :=]0,\infty[$. Writing $f(x) = \widetilde{f}(\sigma(x))$ with a symmetric function $\widetilde{f} : \mathbb{R}_+^n \to \mathbb{C}$, calculation in polar coordinates gives

$$\mathcal{L}f(y) = \int_\Omega e^{-\langle x,y\rangle} f(x)dx = \int_{\mathbb{R}_+^n} {}_0F_0^{2/d}(-\xi, \operatorname{spec}(y))\, \widetilde{f}(\xi) \prod_{1\leq i<j\leq n} |\xi_i - \xi_j|^d \, d\xi$$

with the Jack-hypergeometric series

$${}_0F_0^\alpha(z,w) = \sum_{\lambda \in \Lambda_n^+} \frac{1}{|\lambda|!} \frac{C_\lambda^\alpha(z) C_\lambda^\alpha(w)}{C_\lambda^\alpha(\underline{1})}.$$

In [M13], Macdonald presented a formularium involving Jack polynomials of arbitrary index instead of the spherical polynomials on a cone, where he formally replaced the index $\alpha = 2/d$ in the Laplace transform by an arbitrary index $\alpha > 0$. This led to his conjectural formula (C) for the Laplace transform of Jack polynomials substituting (2.2), see Theorem 6.1 below. In [BF98] a first proof of this formula was sketched, still leaving convergence issues open, and it was also observed that ${}_0F_0^\alpha$ coincides with a Bessel function of type A_{n-1} in Dunkl theory.

3. The Dunkl setting and Laplace transform in type A

Dunkl operators are differential-reflection operators associated with root systems which generalize the usual directional derivatives. For a general background, we refer to [DX14, dJ93, R03]. In this note we consider the root system $R = A_{n-1} = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$ (with its standard inner product). The associated reflection group is $\mathcal{S}_n$, the symmetric group on n elements. The rational Dunkl operators associated with R and some fixed multiplicity parameter $k \in [0, \infty[$ are given by

$$T_j = \partial_j + k \cdot \sum_{i \neq j} \frac{1 - s_{ij}}{x_j - x_i} \quad (1 \leq j \leq n),$$

where s_{ij} denotes the orthogonal reflection in the hyperplane $(e_i - e_j)^\perp$, which acts by exchanging the coordinates x_i and x_j. The operators T_j commute and have nice mapping properties similar to usual directional derivatives. In particular, they act continuously on the classical Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and thus by duality also on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. For a polynomial $p \in \mathbb{C}[\mathbb{R}^n]$, we shall write $p(T)$ for the differential-reflection operator obtained from $p(x)$ by replacing x_j by T_j. There is a unique holomorphic function $E = E_k \in \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n)$, the Dunkl kernel of type A_{n-1} associated with k, satisfying

$$T_j E(z, \,.\,) = z_j E(z, \,.\,) \text{ for } j = 1, \ldots, n, \quad E(z, 0) = 1.$$

The Dunkl kernel E is symmetric in its arguments and satisfies $E(sz, w) = E(z, sw)$ and $E(\sigma z, \sigma w) = E(z, w)$ for all $s \in \mathbb{C}, \sigma \in \mathcal{S}_n$. Moreover, $E(x, y) > 0$ and $|E(ix, y)| \leq 1$ for all $x, y \in \mathbb{R}^n$. If $k = 0$, then $E(z, w) = e^{\langle z, w\rangle}$, where $\langle\,.\,,\,.\,\rangle$ is extended to $\mathbb{C}^n \times \mathbb{C}^n$ in a bilinear way. Note that

$$span_{\mathbb{R}}(R) = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\} =: \mathbb{R}^n_0.$$

This easily implies that

$$E(z, w + \underline{s}) = e^{\langle z, \underline{s}\rangle} \cdot E(z, w) \text{ for } \underline{s} := (s, \cdots, s) \in \mathbb{C}^n \text{ with } s \in \mathbb{C}. \tag{3.1}$$

The associated (type A) Bessel function is given by

$$J(z, w) := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} E(\sigma z, w).$$

It is symmetric in both arguments. As observed in [BF98], it can be written as a Jack-hypergeometric series:

$$J(z, w) = {}_0F_0^\alpha(z, w) \quad \text{with } \alpha = 1/k. \tag{3.2}$$

For $x \in \mathbb{R}^n_+$, $a \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq a$ (which is understood componentwise), we have the exponential bound (see [R20])

$$|E(-z, x)| \leq \exp\big(-\|x\|_1 \cdot \min_{1 \leq i \leq n} a_i\big). \tag{3.3}$$

Following [BF98], we define the type A Laplace transform of functions $f \in L^1_{loc}(\mathbb{R}^n_+)$ by

$$\mathcal{L}f(z) = \int_{\mathbb{R}^n_+} f(x) E(-z, x) \omega(x) dx \quad (z \in \mathbb{C}^n),$$

with the Dunkl weight

$$\omega(z) = \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2k} \quad \text{on } \mathbb{C}^n.$$

Identity (3.1) and estimate (3.3), which are very specific for root systems of type A, imply nice properties for the Laplace transform $\mathcal{L}$. For example, if f is exponentially bounded with $|f(x)| \leq Ce^{\langle x, \underline{s}\rangle}$ for some $s \in \mathbb{R}$, then $\mathcal{L}f(z)$ exists and is holomorphic on $\{z \in \mathbb{C}^n : \operatorname{Re} z > \underline{s}\}$.

Theorem 3.1 ([R20]). (1) *Suppose that $\mathcal{L}f(a)$ exists for some $a \in \mathbb{R}^n$. Then $\mathcal{L}f(z)$ exists and is holomorphic on $\{z \in \mathbb{C}^n : Re\, z > a\}$. Moreover, for each polynomial $p \in \mathbb{C}[\mathbb{R}^n]$, $p(-T)(\mathcal{L}f) = \mathcal{L}(pf)$ on $\{Re\, z > a\}$.*

(2) *(Cauchy inversion theorem). Suppose that $\mathcal{L}f(\underline{s})$ exists for some $s \in \mathbb{R}$ and that $y \mapsto \mathcal{L}f(\underline{s} + iy) \in L^1(\mathbb{R}^n, \omega)$. Then*

$$\frac{(-i)^n}{c^2}\int_{Re\, z=\underline{s}} \mathcal{L}f(z)E(x,z)\omega(z)dz = \begin{cases} f(x) & a.e. \text{ on } \mathbb{R}^n_+ \\ 0 & \text{on } \mathbb{R}^n \setminus \mathbb{R}^n_+, \end{cases}$$

with the Mehta-constant $c = \int_{\mathbb{R}^n} e^{-|x|^2/2}\omega(x)dx$.

(3) *(Injectivity) Suppose that $\mathcal{L}f = 0$ on some subspace $\{z \in \mathbb{C}^n : Re\, z = \underline{s}\}$. Then $f = 0$.*

The Laplace transform $\mathcal{L}$ extends naturally to distributions, as follows. Let

$$\mathcal{S}'_+(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\mathbb{R}^n_+}\,\}.$$

Then the Laplace transform of $u \in \mathcal{S}'_+(\mathbb{R}^n)$ is defined, for $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$, by

$$\mathcal{L}u(z) := \langle u, \chi E(\,.\,, -z)\rangle,$$

where $\chi \in C^\infty(\mathbb{R}^n)$ is an arbitrary cutoff function for $\mathbb{R}^n_+$, i.e. $\operatorname{supp}(\chi) \subseteq]-\epsilon, \infty[^n$ for some $\epsilon > 0$ and $\chi = 1$ in a neighborhood of $\overline{\mathbb{R}^n_+}$. Indeed, $\chi E(\,.\,, -z)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and the above definition is independent of the choice of χ. The Laplace transform on $\mathcal{S}'_+(\mathbb{R}^n)$ is also injective.

4. Riesz distributions in the Dunkl setting

We maintain the previous notations and put

$$\mu_0 := k(n-1), \quad \Delta(x) := x_1 \cdots x_n \quad \text{for } x \in \mathbb{R}^n.$$

Moreover, we introduce the multivariate gamma function

$$\Gamma_n(\lambda) := \prod_{j=1}^n \frac{\Gamma(1+jk)}{\Gamma(1+k)} \cdot \prod_{j=1}^n \Gamma(\lambda_j - k(j-1)) \quad (\lambda \in \mathbb{C}^n).$$

and also write $\Gamma_n(\lambda) = \Gamma_n(\underline{\lambda})$ for $\lambda \in \mathbb{C}$. For indices $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > \mu_0$ we define the Riesz measures

$$\langle R_\mu, \varphi\rangle := \frac{1}{\Gamma_n(\mu)} \int_{\mathbb{R}^n_+} \varphi(x)\Delta(x)^{\mu-\mu_0-1}\omega(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

which we consider as tempered distributions on $\mathbb{R}^n$. The following results of [R20] generalize well-known properties of Riesz distributions on a symmetric cone, c.f. [FK94].

Theorem 4.1. (1) $\Delta(T)R_\mu = R_{\mu-1}$ *in* $\mathcal{S}'(\mathbb{R}^n)$.
Via this identity, the mapping $\mu \mapsto R_\mu$ extends to a holomorphic function on $\mathbb{C}$ with values in $\mathcal{S}'(\mathbb{R}^n)$.

(2) *The Riesz distribution $R_\mu \in \mathcal{S}'(\mathbb{R}^n)$ is supported in $\overline{\mathbb{R}^n_+}$.*

(3) *Dunkl-Laplace transform: $\mathcal{L}R_\mu(y) = \Delta(y)^{-\mu}$ for all $y \in \mathbb{R}^n_+$.*

(4) $R_0 = \delta_0$.
(5) R_μ *is a (positive) measure iff* μ *belongs to the generalized Wallach set*

$$\big\{0, k, \dots, k(n-1) = \mu_0\big\} \cup \{\mu \in \mathbb{R} : \mu > \mu_0\}.$$

In fact, the measures R_{kj} with $0 \le j \le n-1$ can be written down recursively. They have shrinking supports in the facets of $\partial(\mathbb{R}^n_+)$. See [R20] for details.

5. The Cherednik kernel and non-symmetric Jack polynomials

Our generalization of the Laplace transform formula (2.2) for the spherical functions of the cone $\Omega = \Omega_n(\mathbb{F})$ shall involve non-symmetric Jack polynomials and the Opdam-Cherednik kernel of type A_{n-1}. In this section, we give the necessary background from [KS97, F10, O95], c.f. also [BR23]. First, we recall the usual dominance order on the set of partitions Λ_n^+, which is given by

$$\mu \le_D \lambda \;\text{ iff }\; |\lambda| = |\mu| \;\text{ and }\; \sum_{j=1}^r \mu_j \le \sum_{j=1}^r \lambda_j \text{ for all } r = 1, \dots, n.$$

This partial order extends from Λ_n^+ to $\mathbb{N}_0^n$ as follows: For each composition $\eta \in \mathbb{N}_0^n$ denote by $\eta_+ \in \Lambda_n^+$ the unique partition in the $\mathcal{S}_n$-orbit of η. The dominance order on $\mathbb{N}_0^n$ is then defined by

$$\kappa \preceq \eta \;\text{ iff }\; \begin{cases} \kappa_+ \le_D \eta_+\,, & \kappa_+ \ne \eta_+ \\ w_\eta \le w_\kappa\,, & \kappa_+ = \eta_+ \end{cases},$$

where $w_\eta \in \mathcal{S}_n$ is the shortest element with $w_\eta \eta_+ = \eta$ and $\le$ is the Bruhat order on $\mathcal{S}_n$. Now consider the (rational) Cherednik operators associated with the positive subsystem $R_+ = \{e_j - e_i : 1 \le i < j \le n\}$ of $R = A_{n-1}$ and multiplicity $k \ge 0$,

$$\mathcal{D}_j := x_j T_j + k(1-n) + k\sum_{i>i} s_{ij} \quad (j = 1, \dots, n),$$

where the T_j are the type A Dunkl operators with multiplicity k as above. The operators $\mathcal{D}_j$ are related by a change of variables to the Cherednik operators D_{e_j} of trigonometric Dunkl theory as introduced in [O95]; we refer to [BR23] for the precise connection. Note that $\mathcal{D}_j$ leaves the space $\mathbb{C}[\mathbb{R}^n]$ invariant and preserves the degree of homogeneity. Indeed, it acts on $\mathbb{C}[\mathbb{R}^n]$ in an upper triangular way:

$$\mathcal{D}_j x^\eta \;=\; \overline{\eta}_j x^\eta + \sum_{\kappa \prec \eta} d_{\kappa\eta} x^\kappa$$

with coefficients $d_{\kappa\eta} \in \mathbb{R}$ and

$$\overline{\eta}_j = \eta_j - k\#\{i < j \mid \eta_i \ge \eta_j\} - k\#\{i > j \mid \eta_i > \eta_j\}.$$

The non-symmetric Jack polynomials of index $\alpha = 1/k$ are defined as the unique basis $\big(E_\eta\big)_{\eta \in \mathbb{N}_0^n}$ of $\mathbb{C}[\mathbb{R}^n]$ satisfying

(1) $E_\eta(x) = x^\eta + \sum_{\kappa \prec \eta} c_{\eta\kappa} x^\kappa$ with $c_{\kappa\eta} \in \mathbb{C}$,
(2) $\mathcal{D}_j E_\eta = \overline{\eta}_j E_\eta$ for all $j = 1, \dots, n$.

By definition, E_η is homogeneous of degree $|\eta| = \eta_1 + \ldots + \eta_n$, and for $k = 0$ we have $E_\eta(x) = x^\eta$.

Property (2) generalizes: For each spectral parameter $\lambda \in \mathbb{C}^n$, there is a unique analytic function $f = \mathcal{G}(\lambda, \,.\,)$ in an open neighborhood of $\mathbb{R}^n$, called the Opdam-Cherednik kernel, satisfying

$$\mathcal{D}_j f = \big(\lambda_j - \frac{k}{2}(n-1)\big) f \quad \text{for } j = 1, \dots, n;\; f(0) = 1. \tag{5.1}$$

Actually, it follows from results of [KO08] that the kernel $\mathcal{G}$ is holomorphic on $\mathbb{C}^n \times \{z \in \mathbb{C}^n : \operatorname{Re} z > 0\}$. Symmetrization of $\mathcal{G}$ gives the Heckman-Opdam hypergeometric function

$$\mathcal{F}(\lambda, z) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \mathcal{G}(\lambda, \sigma z).$$

Both $\mathcal{F}$ and $\mathcal{G}$ differ by a change of variables from the notions used in [O95, HO21]. The uniqueness of $\mathcal{G}$ shows that for $\eta \in \mathbb{N}_0^n$,

$$\frac{E_\eta(x)}{E_\eta(\underline{1})} = \mathcal{G}\big(\overline{\eta} + \frac{k}{2}(n-1)\underline{1}\,, x\big), \quad \overline{\eta} = (\overline{\eta}_1, \dots, \overline{\eta}_n). \tag{5.2}$$

Moreover, the symmetric Jack polynomials can be obtained via symmetrization from the non-symmetric ones: For partitions $\lambda \in \Lambda_n^+$,

$$\frac{C_\lambda(x)}{C_\lambda(\underline{1})} = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{E_\lambda(\sigma x)}{E_\lambda(\underline{1})} = \mathcal{F}(\lambda - \rho, x)$$

with the Weyl vector $\rho = -\frac{k}{2}(n-1, n-3, \dots, -n+1)$. Recall the matrix cone $\Omega = \Omega_n(\mathbb{F})$ with $d = \dim_{\mathbb{R}}(\mathbb{F})$. For $k = \frac{d}{2}$, the functions $\mathcal{F}(\lambda, \,.\,)$ can be identified with the spherical functions of Ω.

6. Laplace transform identities

In this section, we present the main results from [BR23], which generalize the Laplace transform formula (2.2) for the spherical functions of a matrix cone.

Theorem 6.1. *(Master theorem for the type A Laplace transform). Let $\mu \in \mathbb{C}$ with $Re\,\mu > \mu_0$ and $z \in \mathbb{C}^n$ with $Re\, z > 0$. Then for all $\eta \in \mathbb{N}_0^n$ and $\lambda \in \Lambda_n^+$,*

(1) $\displaystyle\int_{\mathbb{R}_+^n} E(-x, z)\, E_\eta(x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \Gamma_n(\eta_+ + \underline{\mu})\, E_\eta\big(\tfrac{1}{z}\big) \Delta(z)^{-\mu}.$

(2) $\displaystyle\int_{\mathbb{R}_+^n} J(-x, z)\, C_\lambda(x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \Gamma_n(\lambda + \underline{\mu})\, C_\lambda\big(\tfrac{1}{z}\big) \Delta(z)^{-\mu}.$

In view of identity (3.2), formula (2) is just Macdonald's [M13] Conjecture (C). It follows immediately from part (1) by symmetrization. Part (1) was first stated (at a formal level) by Baker and Forrester in [BF98], and justified via Laguerre expansions. In [BR23] we give a completely different, rigorous proof by induction on η, using the raising operator of Knop and Sahi [KS97] for the non-symmetric Jack polynomials. By analytic continuation, Theorem 6.1 extends to the Cherednik kernel and the Heckman-Opdam hypergeometric function, as follows.

Theorem 6.2. *Let $\mu \in \mathbb{C}$ with $Re\,\mu > \mu_0$. Then for $\lambda \in \mathbb{C}^n$ with $Re\,\lambda \geq 0$ and $z \in \mathbb{C}^n$ with $Re\, z > 0$, we have*

(1) $\displaystyle\int_{\mathbb{R}_+^n} E(-z, x)\, \mathcal{G}(\lambda, x)\, \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \Gamma_n(\lambda + \rho + \underline{\mu})\, \mathcal{G}(\lambda, \tfrac{1}{z})\, \Delta(z)^{-\mu}.$

$$(2) \quad \int_{\mathbb{R}^n_+} J(-z,x)\,\mathcal{F}(\lambda,x)\,\Delta(x)^{\mu-\mu_0-1}\omega(x)dx \;=\; \Gamma_n(\lambda+\rho+\underline{\mu})\,\mathcal{F}(\lambda,\tfrac{1}{z})\,\Delta(z)^{-\mu}.$$

Formula (2) generalizes the Laplace transform identity (2.2) for the spherical functions of a cone $\Omega_n(\mathbb{F})$.

7. Some applications of the master theorem.

We conclude this overview with two results from [BR23] which are based on Master Theorem 6.1. The first one is a Post-Widder inversion theorem for the type A Laplace transform $\mathcal{L}$, which is the counterpart of an inversion formula of Faraut and Gindikin [FG90] on symmetric cones.

Theorem 7.1 (Post-Widder inversion formula for $\mathcal{L}$)**.** *Let $f:\mathbb{R}^n_+\to\mathbb{C}$ be measurable and bounded, and suppose that f is continuous at $\xi\in\mathbb{R}^n_+$. Then*

$$f(\xi)=\lim_{\nu\to\infty}\frac{(-1)^{n\nu}}{\Gamma_n(\nu+\mu_0+1)}\Delta\Big(\frac{\nu}{\xi}\Big)^{\nu+\mu_0+1}\big(\Delta(T)^\nu(\mathcal{L}f)\big)\Big(\frac{\nu}{\xi}\Big),$$

As a second application, we present some Laplace transform identities for Jack-hypergeometric series. First, one observes that the non-symmetric Jack polynomials E_η have a renormalization $L_\eta=c_\eta E_\eta$ such that

$$\sum_{|\eta|=m} L_\eta(z)=(z_1+\ldots+z_n)^m=\sum_{|\lambda|=m} C_\lambda(z)\quad (m\in\mathbb{N}_0).$$

For parameters $\mu\in\mathbb{C}^p$ and $\nu\in\mathbb{C}^q$ with $p,q\in\mathbb{N}_0$ we define the symmetric and non-symmetric Jack hypergeometric series

$$ {}_pF_q(\mu,\nu;z,w):=\sum_{\lambda\in\Lambda_n^+}\frac{[\mu_1]_\lambda\cdots[\mu_p]_\lambda}{[\nu_1]_\lambda\cdots[\nu_q]_\lambda}\,\frac{C_\lambda(z)C_\lambda(w)}{|\lambda|!\,C_\lambda(\underline{1})}$$

$$ {}_pK_q(\mu,\nu;z,w):=\sum_{\eta\in\mathbb{N}_0^n}\frac{[\mu_1]_{\eta_+}\cdots[\mu_p]_{\eta_+}}{[\nu_1]_{\eta_+}\cdots[\nu_q]_{\eta_+}}\,\frac{L_\lambda(z)L_\lambda(w)}{|\lambda|!\,L_\lambda(\underline{1})}\,,$$

with the generalized Pochhammer symbol

$$[a]_\lambda=\frac{\Gamma_n(\underline{a}+\lambda)}{\Gamma_n(\underline{a})}\quad (a\in\mathbb{C},\,\lambda\in\Lambda_n^+).$$

The convergence properties of these series are made precise in [BR23], improving results for ${}_pF_q$ from [Kan93]. In particular, for $p\le q$ both series are entire functions. For $w=\underline{1}$ and multiplicity $k=\frac{d}{2}$ related to a matrix cone $\Omega_n(\mathbb{F})$, the ${}_pF_q$-series coincide with classical hypergeometric series on Ω, c.f. [FK94, GR89]. They are for instance useful in multivariate statistics. There are interesting special cases leading to special functions from Dunkl theory, such as the type A Dunkl kernel and Bessel function:

$${}_0K_0(z,w)=E(z,w),\quad {}_0F_0(z,w)=J(z,w).$$

Theorem 7.2. (1) *Let $p<q$ and consider $\mu'\in\mathbb{C}$ with $Re\,\mu'>\mu_0$. Then for all $z,w\in\mathbb{C}^n$ with $Re\,z>0$,*

$$\int_{\mathbb{R}^n_+} E(-z,x)\,{}_pK_q(\mu;\nu;w,x)\Delta(x)^{\mu'-\mu_0-1}\omega(x)dx$$

$$=\Gamma_n(\mu')\Delta(z)^{-\mu'}\;{}_{p+1}K_q((\mu',\mu);\nu;w,\tfrac{1}{z}).$$

(2) *If* $p = q$, *then part (1) is valid under the condition* $\|w\|_\infty \cdot \|\frac{1}{Re\, z}\|_\infty < \frac{1}{n}$. *The same formulas hold for* ${}_pF_q$.

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References

[BF98] T.H. Baker, P.J. Forrester, Non-symmetric Jack polynomials and integral kernels. *Duke Math. J.* 95 (1998), 1–50.

[BR23] D. Brennecken, M. Rösler, The Dunkl Laplace transform and Macdonald's hypergeoemetric series, *Trans. Amer. Math. Soc.* 376 (2023), 2419–2447.

[DX14] C.F. Dunkl, Y. Xu, Orthogonal polynomials of Several Variables. Cambridge Univ. Press, 2nd edition, 2014.

[FG90] J. Faraut, S. Gindikin, Deux formules d'inversion pour la transformation de Laplace sur un cône symétrique. *C. R. Acad. Sci. Paris Sér. I Math.* 310 (1990), 5–8.

[FK94] J. Faraut, A. Korányi, Analysis on Symmetric Cones. Oxford Science Publications, Clarendon press, Oxford 1994.

[F10] P.J. Forrester, Log-Gases and Random Matrices. London Mathematical Society Monographs Series, 34. Princeton University Press, Princeton, NJ, 2010.

[GR89] K. Gross, D. Richards, Special functions of matrix argument. I: Algebraic induction, zonal polynomials, and hypergeometric functions. *Trans. Amer Math. Soc.* 301 (1987), 781–811.

[HO21] G. Heckman, E. Opdam, Jacobi polynomials and hypergeometric functions associated with root systems. In: Encyclopedia of Special Functions, Part II: Multivariable Special Functions, eds. T.H. Koornwinder, J.V. Stokman, Cambridge Univ. Press, Cambridge, 2021.

[dJ93] M. de Jeu, The Dunkl transform. *Invent. Math.* 113 (1993), 147–162.

[Kan93] J. Kaneko, Selberg integrals and hypergeometric functions associated with Jack polynomials. *SIAM J. Math. Anal.* 24 (1993), 1086–1100.

[KO08] B. Krötz, E. Opdam, Analysis on the crown domain. *Geom. Funct. Anal. 18* (2008), 1326–1421.

[KS97] F. Knop, S. Sahi, A recursion and a combinatorial formula for Jack polynomials. *Invent. Math.* 128 (1997), 9–22.

[M87] I.G. Macdonald, Commuting differential operators and zonal spherical functions. In: Algebraic groups (Utrecht 1986), eds. A.M. Cohen et al, *Lecture Notes in Mathematics* 1271, Springer-Verlag, Berlin, 1987.

[M13] I.G. Macdonald, Hypergeometric functions I. arXiv: 1309.4568v1 (math.CA).

[O95] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* 175, (1995), 75–112.

[R03] M. Rösler, Dunkl operators: Theory and applications. In: E. Koelink, W. van Assche (Eds.), Lecture Notes in Math. 1817, Springer-Verlag, 2003, pp. 93–136.

[R20] M. Rösler, Riesz distributions and the Laplace transform in the Dunkl setting of type A. *J. Funct. Anal.* 278 (2020), no 12, 108506, 29 pp.

[SZ07] S. Sahi, G. Zhang, Biorthogonal expansion of non-symmetric Jack functions. SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007), Paper 106, 9 pp.

[St89] R.P. Stanley, Some combinatorial properties of Jack symmetric functions. *Adv. Math.* 77 (1989), 76–115.

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