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# Contributions to the Theory of Dunkl Operators

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# Zusammenfassung

Im Gegensatz zur traditionsreichen Theorie der speziellen Funktionen einer Veränderlichen besteht erst seit jüngerer Zeit zunehmendes Interesse auch an speziellen Funktionen in mehreren Variablen. Eine besonders intensive Entwicklung hat dabei in den letzten Jahren im Gebiet der speziellen Funktionen zu Wurzelsystemen stattgefunden, mit bedeutenden Beiträgen durch Heckmann und Opdam, Dunkl, Macdonald und Cherednik. Wesentliche Motivation bezieht dieses Gebiet aus der harmonischen Analyse Riemannscher symmetrischer Räume, deren sphärische Funktionen sich als spezielle Funktionen mehrerer Variabler mit gewissen diskreten Parametern schreiben lassen. Für einen Überblick über den aktuellen Stand der Forschung und weitere Literaturangaben verweisen wir auf [He3], [M2], [H-Sc] und [Ki]. Ein zentrales Hilfsmittel in der Untersuchung spezieller Funktionen zu Wurzelsystemen sind Dunkl-Operatoren und ihre Varianten, die trigonometrischen Dunkl-Operatoren von Heckman und Opdam sowie die eng verwandten Cherednik-Operatoren ([D2], [H-Sc], [Che]). Dunkl-Operatoren sind mit endlichen Spiegelungsgruppen assoziierte, durch Spiegelungsterme modifizierte Differentialoperatoren. Ihre Bedeutung für das Studium spezieller Funktionen mehrerer Variabler ist eng gekoppelt mit ihrer Interpretation im Kontext gewisser Darstellungen (degenerierter) affiner Hecke-Algebren (siehe [Che], [O2] und [Ki]). Dunkl-Operatoren – in ihrer ursprünglichen Version – gehen zurück auf C. F. Dunkl, der sie gegen Ende der 80iger Jahre im Rahmen seiner Untersuchungen zu verallgemeinerten sphärischen Harmonischen einführte ([D1-D5]); zur Abgrenzung von den weiteren Varianten werden diese Operatoren heute auch “rationale” Dunkl-Operatoren genannt.

Eine ganz wesentliche Motivation für die Beschäftigung mit Dunkl-Operatoren liegt in ihrer Bedeutung für die Analyse quantenmechanischer Mehrteilchenmodelle vom Calogero-Moser-Sutherland-Typ. Dies sind exakt integrierbare eindimensionale Systeme, die erstmals von Calogero und Sutherland ([Ca], [Su]) studiert wurden und in den letzten Jahren innerhalb der Mathematischen Physik zunehmend an Bedeutung gewonnen haben. Calogero-Moser-Sutherland-Modelle sind unter anderem in der konformen Feldtheorie von Interesse und werden eingesetzt, um Modelle der fraktionalen Statistik zu testen ([Ha], [Hal]). Der Dunkl-Operator-Formalismus liefert explizite Operatorlösungen für eine Vielzahl von Systemen dieses Typs ([L-V], [K1], [BHKV], [B-F3], [U-W]).

Die vorliegende Arbeit beschäftigt sich ausschließlich mit den klassischen, rationalen Dunkl-

Operatoren; wir werden sie der Einfachheit halber stets "Dunkl-Operatoren" nennen.

Das erste Kapitel liefert eine Einführung in die Theorie der Dunkl-Operatoren und der Dunkl-Tranformation. Dabei ist keine inhaltliche Vollständigkeit angestrebt; vielmehr ist Wert darauf gelegt, einen Überblick über die für das Weitere grundlegenden Konzepte zu vermitteln. Verschaffen wir uns nun einen Einstieg in den Gegenstand der Arbeit: Gegeben sei eine endliche Spiegelungsgruppe $G$ auf dem $\mathbb{R}^N$ mit zugehörigem Wurzelsystem $R$. Die assoziierten Dunkl-Operatoren sind dann definiert durch

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in R_+} k(\alpha)\,\langle\alpha,\xi\rangle\,\frac{f(x)-f(\sigma_\alpha x)}{\langle\alpha,x\rangle}\,,\quad \xi\in\mathbb{R}^N.$$

Dabei ist $R_+$ ein positives Teilsystem von $R$ und $\langle\,.\,,\,.\,\rangle$ ist das euklidische Standard-Skalarprodukt auf $\mathbb{R}^N$; ferner bezeichnet $\sigma_\alpha$ die Spiegelung an der zu $\alpha$ orthogonalen Hyperebene und $k: R\to\mathbb{C}$ eine sogenannte Multiplizitätsfunktion auf $R$, d.h. eine unter der natürlichen Operation von $G$ invariante Funktion $k: R\to\mathbb{C}$. Der Dunkl-Operator $T_\xi(k)$ kann insbesondere als eine mittels $k$ parameterisierte Störung des gewöhnlichen partiellen Ableitungsoperators in Richtung $\xi$ aufgefaßt werden. Ist die Multiplizitätsfunktion $k$ identisch Null, so stimmt $T_\xi(k)$ mit der gewöhnlichen Ableitung in Richtung $\xi$ überein. Die Aktion der Dunkl-Operatoren $T_\xi(k)$ auf dem Vektrorraum der Polynomfunktionen auf $\mathbb{R}^N$ ist $G$-äquivariant und homogen vom Grad $-1$. Darüberhinaus haben die Dunkl-Operatoren (bei festem $k$) die bemerkenswerte Eigenschaft daß sie kommutieren:

$$T_\xi(k)\,T_\eta(k) \;=\; T_\eta(k)\,T_\xi(k) \quad \text{für alle } \xi,\eta\in\mathbb{R}^N.$$

In [D4] wurde zunächst für nichtnegative Parameter $k$ die Existenz eines homogenen linearen Operators $V_k$ auf dem Raum der Polynome bewiesen, welcher die Algebra der Dunkl-Operatoren mit der Algebra der gewöhnlichen Differentialoperatoren vertauscht, d.h.

$$T_\xi(k)V_k \;=\; V_k\,\partial_\xi \quad \text{für alle } \xi\in\mathbb{R}^N.$$

Unter der Forderung $V_k(1)=1$ ist dieser auch eindeutig. Eine gründliche Analyse in [D-J-O] ergab später, daß ein solcher Vertauschungsoperator tatsächlich für genau diejenigen Multiplizitätsfunktionen existiert, für welche der gemeinsame Kern der $T_\xi(k)$, aufgefaßt als lineare Operatoren auf dem Vektorraum der Polynome in $N$ Variablen, keine "singulären" Polynome außer den Konstanten enthält.

Ein zentraler Teil dieser Arbeit ist dem weiteren Studium des Vertauschungsoperators $V_k$ für nichtnegative Multiplizitätsfunktionen $k$ gewidmet. Obwohl eine explizite Darstellung nur in sehr wenigen speziellen Fällen bekannt ist, äußerte Dunkl in [D4] die Vermutung, daß der Operator $V_k$ im Fall $k\geq 0$ stets positivitätserhaltend auf Polynomen ist und eine Integraldarstellung vom Bochner-Typ auf gewissen Algebren analytischer Funktionen besitzt. Diese Vermutung wird in Kapitel 2 bewiesen. Der Beweis erfordert mehrere Schritte; der zentrale Teil ist dabei der Nachweis der Positivität von $V_k$ auf Polynomen. Entscheidend hierfür ist eine

Reduktion des $N$-dimensionalen auf ein eindimensionales Problem, welches dann explizit zu lösen ist. Die erforderliche Reduktion wird mittels einer Charakterisierung vom Hille-Yosida-Typ für die Erzeuger positiver Halbgruppen auf Räumen von Polynomen bewerkstelligt. Das Hauptresultat von Kapitel 2 basiert dann auf einer Fortsetzung des Vertauschungsoperators $V_k$ auf gewisse Algebren analytischer Funktionen. Es besagt, daß für jedes $x \in \mathbb{R}^N$ ein eindeutiges Wahrscheinlichkeitsmaß $\mu_x^k$ auf der Borel-$\sigma$-Algebra des $\mathbb{R}^N$ existiert, so daß $V_k$ für analytische Funktionen $f$ einer geeignet großen Klasse, und insbesondere für alle Polynome, die Darstellung

$$V_k f(x) \,=\, \int_{\mathbb{R}^N} f(\xi)\, d\mu_x^k(\xi)$$

besitzt. Dieses Resultat erlaubt allerdings nur sehr begrenzte Aussagen bezüglich spezifischer Eigenschaften der darstellenden Maße, nämlich ihre Träger und gewisse Invarianz-Eigenschaften betreffend. Es ist eine interessante und noch offene Frage, unter welchen Bedingungen an die Multiplizitätsfunktion $k$ die Maße $\mu_x^k$ absolut stetig bezüglich des Lebesgue-Maßes sind oder aber diskrete Anteile haben.

Eine wichtige Konsequenz unseres Hauptresultats betrifft den verallgemeinerten Exponentialkern $E_k(z,w)$ zu $G$ und $k$, welcher im Rahmen der Dunkl-Theorie die übliche Exponentialfunktion auf $\mathbb{C}^N \times \mathbb{C}^N$ ersetzt. Nach einem Resultat aus [O2] ist dieser Exponentialkern als eindeutige Lösung eines simultanen Eigenwert-Problems für die Dunkl-Operatoren $\{T_\xi(k),\ \xi \in \mathbb{R}^N\}$ charakterisiert; er läßt sich auch schreiben als

$$E_k(z,w) \,=\, V_k\big(e^{\langle\,\cdot\,,\,w\rangle}\big)(z).$$

Zu dem Kern $E_k$ gibt es eine Integraltransformation im $\mathbb{R}^N$, die Dunkl-Transformation, welche in [D5] eingeführt und in [dJ1] detailliert untersucht wurde. Sie besitzt viele Eigenschaften der klassischen Fourier-Transformation im $\mathbb{R}^N$; das gewöhnliche Lebesgue-Maß ist dabei allerdings durch die $G$-invariante Gewichtsfunktion

$$w_k(x) \,=\, \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}$$

modifiziert. Obiges Positivitätsresultat impliziert für reelle $x$ eine Bochner-Darstellung des Dunkl-Kerns mittels der Maße $\mu_x^k$ – was insbesondere zeigt, daß die Funktion $y \mapsto E_k(x, iy)$ für festes $x \in \mathbb{R}^N$ positiv-definit auf $\mathbb{R}^N$ ist. Hieraus ergeben sich auch (weitgehend) schärfere Schranken für den Dunkl-Kern als die in [dJ1] angegebenen.

An dieser Stelle scheint eine Bemerkung zum eindimensionalen Fall angebracht, der mit der Spiegelungsgruppe $\mathbb{Z}_2$ auf $\mathbb{R}$ und einem Parameter $k > 0$ assoziiert ist. Hier ist der Dunkl-Kern explizit bekannt; er läßt sich mittels einer konfluenten hypergeometrischen Funktion vom Typ ${}_1F_1$ ausdrücken (oder auch als Kombination zweier normalisierter sphärischer Bessel-Funktionen vom Index $k-1/2$ und $k+1/2$), und besitzt die positive Integraldarstellung

$$E_k^{\mathbb{Z}_2}(z,w) \,=\, C_k \int_{-1}^{1} e^{tzw}(1-t)^{k-1}(1+t)^k\, dt \quad (z, w \in \mathbb{C}),$$

mit einer Normierungskonstanten $C_k > 0$. Die Dunkl-Transformation $\mathbb{Z}_2$-invarianter Funktionen stimmt überein mit der Hankel-Transformation zum Index $k - 1/2$.

Das dritte Kapitel ist Verallgemeinerungen der klassischen Hermite-Polynome im Kontext endlicher Spiegelungsgruppen gewidmet. Es handelt sich dabei um orthogonale und, allgemeiner, biorthogonale Polynomsysteme bezüglich einer Gewichtsfunktion der Form $w_k(x)e^{-\omega|x|^2}$ auf dem $\mathbb{R}^N$. Wir beziehen die Motivation für das Studium solcher Polynome aus ihrer Bedeutung im Zusammenhang mit exakt lösbaren quantenmechanischen Mehrteilchen-Systemen vom Calogero-Moser-Sutherland (CMS)-Typ. CMS-Modelle beschreiben Systeme von $N$ Teilchen auf einem Kreis oder einer Geraden, die durch invers-quadratische Potentiale gekoppelt sind. Während die Integrabilität solcher Modelle schon von Calogero und Sutherland ([Ca], [Su]) festgestellt worden war, führten erst jüngere Beobachtungen von [Po] und [He2] zu einem wesentlich tieferen Verständnis ihrer algebraischen Struktur. Polychronakos betrachtete den rationalen Calogero-Operator mit externem quadratischen Potential in $L^2(\mathbb{R}^N)$,

$$\mathcal{H}_C \,=\, -\Delta + \omega^2|x|^2 \,+\, 2k(k-1)\sum_{1\leq i<j\leq N}\frac{1}{(x_i - x_j)^2}.$$

Hierbei ist $\omega > 0$ ein Frequenz-Parameter und $k \geq 0$ eine Kopplungskonstante. Er erkannte, daß sich dieser Operator, nach Modifikation durch zusätzliche Austausch-Terme, in entkoppelter Form mittels Dunkl-Operatoren vom Typ $A_{N-1}$ schreiben läßt, und so als eine Variante des Schrödinger-Operators für den klassischen harmonischen Oszillator betrachtet werden kann. Durch die Einführung entsprechender Erzeugungs- und Vernichtungsoperatoren erhielt er dann eine explizite Operator-Lösung für das Ausgangsmodell. Diese beobachtungen waren Ausgangspunkt eines intensiven und anhaltenden Studiums expliziter Lösungen von CMS-Modellen mittels geeigneter Differential-Spiegelungs-Formalismen ([L-V], [K1], [BHKV], [B-F3], [U-W]). Um unsere weiteren Konstruktionen zu motivieren, beginnen wir das 3. Kapitel mit einer kurzen Diskussion klassischer CMS-Operatoren. Wir wenden uns dann, auf der Grundlage beliebiger Wurzelsysteme, abstrakten CMS-Operatoren mit quadratischem Potential zu. Diese sind von der Form

$$\mathcal{H}_k \,=\, -\Delta_k \,+ \omega^2|x|^2\,,$$

mit dem Dunkl'schen Laplace-Operator

$$\Delta_k \,=\, \sum_{i=1}^N T_{e_i}(k)^2\,.$$

Für ein Wurzelsystem vom Typ $A_{N-1}$ stimmt der Operator $\mathcal{H}_k$ bis auf eine Eichtransformation mit der durch Austausch-Operatoren modifizierten Version des Operators $H_C$ überein. Die Spektraleigenschaften abstrakter CMS-Operatoren sind mittels Dunkl-Operator-Formalismus leicht vollständig zu klären; sie sind denen des isotropen harmonischen Oszillators sehr ähnlich. Insbesondere sind die Spektren dieser Operatoren diskret und hochgradig entartet, und es gibt eine Vielzahl möglicher Eigenfunktionsbasen. In Abschnitt 3.2 konstruieren wir natürliche Klassen von Eigenfunktionen im Rahmen eines einheitlichen Konzepts verallgemeinerter

Hermite-Polynome und biorthogonaler Polynomsysteme. Die Hermite-Polynome sind dabei gekennzeichnet durch Orthogonalität im entsprechend gewichteten $L^2$-Raum, während natürliche biorthogonale Systeme sich mittels Erzeugungs-Operatoren gewinnen lassen. Viele der bekannten Eigenschaften klassischer Hermite-Polynome und Hermite-Funktionen erlauben eine Ausdehnung auf diese allgemeineren Systeme, darunter die Erzeugenden-Funktion, Rodrigues-Formeln, und eine Mehler-Formel. Im $A_{N-1}$-Fall schließen unsere Hermite-Systeme die verallgemeinerten nicht-symmetrischen Hermite-Polynome von Baker und Forrester [B-F3] als spezielle Klasse ein. Im eindimensionalen Fall erhalten wir die verallgemeinerten Hermite-Polynome auf $\mathbb{R}$, die bereits in [Ros] untersucht wurden. Die Ergebnisse von [B-F3] und [Ros] haben viele unserer vorliegenden allgemeinen Konstruktionen angeregt. Das Kapitel schließt mit einer Version des klassischen Weyl-Heisenberg'schen Unschärfeprinzips für die Dunkl-Transformation; sein Beweis basiert auf Entwicklungen bezüglich verallgemeinerter Hermite-Funktionen.

In Kapitel 4 werden mit Spiegelungsgruppen assoziierte Wärmeleitungskerne sowie verschiedene Familien von Operatorhalbgruppen untersucht, die alle in Verbindung mit dem Dunkl-schen Laplace-Operator stehen. Für eine gegebene Spiegelungsgruppe $G$ und eine Multiplizitätsfunktion $k \geq 0$ wird der zugehörige Wärmeleitungskern durch eine verallgemeinerte Translation gewonnen, und zwar aus der Gauß'schen Fundamentallösung des "Wärmeleitungs-Operators" $\Delta_k - \partial_t$. Der entsprechende Wärmeleitungskern ist gegeben durch

$$\Gamma_k(t,x,y) := \frac{M_k}{t^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x,y \in \mathbb{R}^N,\ t>0,$$

mit einer Normierungskonstanten $M_k > 0$ und dem Parameter $\gamma = \sum_{\alpha\in R_+} k(\alpha)$. Die Positivität des Dunkl-Kerns $E_k$ für reelle Argumente (sie ergibt sich aus unserem Hauptresultat in Kapitel 2) gewährleistet, daß auch $\Gamma_k$ positiv ist. Die Situation ist in der Tat weitgehend analog zur klassischen: Der Dunkl'sche Laplace-Operator $\Delta_k$ erzeugt positive Kontraktions-Halbgruppen auf einer ganzen Reihe von Funktionenräumen einschließlich $L^2(\mathbb{R}^N, w_k)$ und $(C_0(\mathbb{R}^N), \|.\|_\infty)$, und $\Gamma_k$ tritt als Integralkern dieser Wärmeleitungshalbgruppen auf. Weitere verwandte Halbgruppen ergeben sich in der üblichen Weise; wir erhalten insbesondere verallgemeinerte Cauchy-Kerne und Cauchy-Halbgruppen durch Subordination aus den Wärmeleitungshalbgruppen, und greifen die abstrakten Oszillator-Operatoren aus dem vorangehenden Kapitel nochmals auf. Unsere Resultate für die Wärmeleitungs- und Cauchy-Halbgruppen vom Dunkl-Typ schließen auch die explizite Lösung entsprechend (klassisch) gestellter Anfangswert-Probleme mit Ortsgebiet $\mathbb{R}^N$ und gewissen Wachstumsbedingungen im Unendlichen ein, so etwa dasjenige für den Wärmeleitungsoperator $\Delta_k - \partial_t$. Die Eindeutigkeit der Lösungen läßt sich hier wie im klassischen Fall durch ein Maximumprinzip für den Wärmleitungsoperator in unbeschränkten Gebieten sicherstellen. Den Abschluß dieses Kapitels bilden Untersuchungen zur Kurzzeit-Asymptotik des verallgemeinerten Wärmeleitungskerns $\Gamma_k$. Die Struktur des Dunkl'schen Laplace-Operators läßt erwarten, daß der Kern $\Gamma_k$, nach Übergang von $L^2(\mathbb{R}^N, w_k)$ in den ungewichteten $L^2(\mathbb{R}^N)$, für sehr kurze Zeiten die Spiegelungshyperebenen "nicht spürt" und sich wie der freie Gaußkern $\Gamma_0$ verhält; wir vermuten, genauer ausgedrückt, daß für alle $x, y$

innerhalb einer festen (offenen) Weyl-Kammer die folgende asymptotische Relation besteht:

$$\lim_{t\downarrow 0} \frac{\sqrt{w_k(x)\,w_k(y)}\,\Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} \,=\, 1\,.$$

Wir geben zwei Teilresultate in Richtung dieser Vermutung an; beide basieren auf dem Maximumprinzip für den klassischen Wärmeleitungsoperator. Das erste Resultat liefert die behauptete Asymptotik unter einer gewissen Einschränkung an die Argumente $x$, $y$ (die auf den Einfluß der Spiegelungshyperebenen zurückzuführen ist), das zweite beinhaltet eine scharfe untere Schranke.

# Contents

# Introduction

While the theory of special functions in one variable has a long and rich history, the growing interest in special functions of several variables is comparably recent. In particular, there has been a rapid development in the area of special functions related to root systems, with important contributions during the last ten years by Heckman and Opdam, Dunkl, Macdonald, and Cherednik. The motivation for this subject comes to some extent from the harmonic analysis on Riemannian symmetric spaces, whose spherical functions can be written as multivariable special functions depending on certain discrete sets of parameters. For an overview and further references we refer to the Bourbaki lecture of Heckman [He3], to [M2], [H-Sc], and the survey of Kirillov [Ki]. A key tool in the analysis of special functions related with root systems are Dunkl operators and their variants, the trigonometric Dunkl operators of Heckman and Opdam as well as the closely related Cherednik operators ([D2], [H-Sc], [Che]). Dunkl operators are differential-reflection operators, associated to a finite reflection group on some finite-dimensional Euclidean space. Their relevance for the analysis of multivariable special functions is closely connected with their interpretation in the context of (degenerate) affine Hecke algebras (see [Che], [O2] and [Ki]). In their original version, Dunkl operators were introduced and first studied by Dunkl in a series of papers ([D1-D5]) in the context of a theory of generalized spherical harmonics. These operators are now sometimes called "rational" Dunkl operators.

An equally important motivation to study Dunkl operators originates in their relevance for the analysis of quantum many body systems of Calogero-Moser-Sutherland type. These are exactly solvable models in one dimension, which were first studied by Calogero and Sutherland ([Ca], [Su]). During the last years, such models have gained considerable interest in mathematical physics. They are, for example, of interest in conformal field theory, and are being used to test the ideas of fractional statistics ([Ha], [Hal]). The Dunkl operator formalism provides explicit operator solutions for a variety of such systems ([L-V], [K1], [BHKV], [B-F3], [U-W]).

In this thesis, we present various contributions to the theory of the classical rational Dunkl operators, which we always call "Dunkl operators" for short. The first chapter is intended to provide an essentially self-contained introduction to the theory of Dunkl operators and the Dunkl transform; it does not aim at completeness, but concentrates on those aspects which will be important in our context. Let us briefly describe our setting: Given a finite reflection group

$G$ on $\mathbb{R}^N$ with root system $R$, the associated Dunkl operators are defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in R_+} k(\alpha)\,\langle\alpha,\xi\rangle\,\frac{f(x)-f(\sigma_\alpha x)}{\langle\alpha,x\rangle}\,, \quad \xi\in\mathbb{R}^N.$$

Here $R_+$ is a positive subsystem of $R$, $\langle\,.\,,\,.\,\rangle$ is the standard Euclidean scalar product in $\mathbb{R}^N$, $\sigma_\alpha$ denotes the reflection in the hyperplane orthogonal to $\alpha$, and $k:\, R\to\mathbb{C}$ is a function which is invariant under the natural action of $G$ on the root system, called a multiplicity function on $R$. The Dunkl operator $T_\xi(k)$ can therefore be considered as a perturbation in the parameter $k$ of the usual partial derivative in direction $\xi\in\mathbb{R}^N$. If the multiplicity function $k$ is identically zero, then $T_\xi(k)$ coincides with the partial derivative $\partial_\xi$ (independently of the group $G$). The action of the Dunkl operators $T_\xi(k)$ on the vector space of polynomial functions on $\mathbb{R}^N$ is $G$-equivariant and homogeneous of degree $-1$. Moreover, they have the remarkable property that they commute:

$$T_\xi(k)\,T_\eta(k) \,=\, T_\eta(k)\,T_\xi(k) \quad \text{for all } \xi,\eta\in\mathbb{R}^N.$$

It was first shown in [D4] that for non-negative multiplicity functions, there exists a unique linear and homogeneous isomorphism $V_k$ on polynomials such that $V_k(1)=1$ and

$$T_\xi(k)V_k \,=\, V_k\,\partial_\xi \quad \text{for all } \xi\in\mathbb{R}^N,$$

i.e. $V_k$ intertwines the commutative algebra of Dunkl operators with the algebra of partial differential operators. A thorough analysis in [D-J-O] subsequently revealed that such an intertwining operator exists if and only if the common kernel of the $T_\xi(k)$, considered as linear operators on the vector space of polynomials, contains no "singular" polynomials besides the constants.

A central part of this thesis is devoted to a further study of the intertwining operator $V_k$ for non-negative multiplicity functions. Although an explicit form for this operator is known only in very special cases, it was conjectured by Dunkl in [D4] that $V_k$ should always be positivity-preserving on polynomials, and allow a Bochner-type integral representation on certain algebras of analytic functions. This conjecture will be confirmed in Chapter 2. Its proof affords several steps, the crucial part being a reduction from the $N$-dimensional to a one-dimensional problem, which is then solved explicitly. This reduction is achieved by a Hille-Yosida type characterization for the generators of positivity-preserving semigroups of linear operators on spaces of polynomials. The main result of Chapter 2 is then obtained by an extension of $V_k$ to larger function algebras – essentially the same as introduced by Dunkl [D4]. More precisely, we shall prove that for each $x\in\mathbb{R}^N$, there exists a unique probability measure $\mu_x^k$ on the Borel-$\sigma$-algebra of $\mathbb{R}^N$ such that

$$V_kf(x) \,=\, \int_{\mathbb{R}^N} f(\xi)\,d\mu_x^k(\xi)$$

for all polynomials, and in fact for all functions from a certain algebra of analytic functions in a Euclidean ball of sufficiently large radius. As to the nature of the representing measures, this result allows only limited conclusions, refering to invariance properties and the support. It is an interesting and still open problem to find criteria on the multiplicities $k$ under which the measures $\mu_x^k$ are absolutely continuous with respect to Lebesgue measure or do have discrete parts.

An important consequence of our main theorem concerns the generalized exponential kernel $E_k(z,w)$ on $\mathbb{C}^N \times \mathbb{C}^N$ associated to $G$ and $k$, which generalizes the usual exponential function $e^{\langle z,w\rangle}$. This kernel is also called the Dunkl kernel; by a result of [O2], it is characterized as the unique solution of a joint eigenfunction problem for the Dunkl operators $\{T_\xi(k),\, \xi \in \mathbb{R}^N\}$, and it can be written as

$$E_k(z,w) \,=\, V_k\big(e^{\langle\,\cdot\,,\,w\rangle}\big)(z).$$

The Dunkl kernel gives rise to an integral transform on $\mathbb{R}^N$, called the Dunkl-transform, which was introduced in [D5] and studied in detail in [dJ3]. This transform shares many properties of the classical Fourier transform on $\mathbb{R}^N$, with the Lebesgue measure being modified by the $G$-invariant weight function

$$w_k(x) := \prod_{\alpha\in R_+} |\langle\alpha,x\rangle|^{2k(\alpha)}.$$

The above positivity result implies, for real $x$, a Bochner-type representation of the Dunkl kernel by means of the representing measures $\mu_x^k$ above. This shows in particular that the function $y \mapsto E_k(x,iy)$ is positive definite on $\mathbb{R}^N$ for each fixed $x \in \mathbb{R}^N$. We also obtain essentially sharper bounds on the kernel $E_k$ than those in [dJ3].

We mention that in the one-dimensional case, associated to the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$ and a single multiplicity parameter $k > 0$, the corresponding Dunkl kernel is known in terms of a confluent hypergeometric function of type ${}_1F_1$ (or equivalently, as a certain combination of two normalized spherical Bessel functions of index $k-1/2$ and $k+1/2$ respectively); it allows the explicit integral representation

$$E_k^{\mathbb{Z}_2}(z,w) \,=\, C_k \int_{-1}^{1} e^{tzw}(1-t)^{k-1}(1+t)^k\, dt \quad (z,w\in\mathbb{C}),$$

with some normalization constant $C_k > 0$. In this case, the Dunkl transform of group-invariant functions coincides with the Hankel transform of order $k-1/2$.

Chapter 3 is concerned with generalizations of the classical multivariable Hermite polynomials to the Dunkl setting. For a finite reflection group $G$ on $\mathbb{R}^N$, we consider polynomial systems which are orthogonal or, more generally, biorthogonal with respect to weight functions of the form $w_k(x)e^{-\omega|x|^2}$ on $\mathbb{R}^N$. The motivation to study such polynomials originates in their connection with exactly solvable quantum many particle systems of Calogero-Moser-Sutherland (CMS) type. CMS models describe a system of $N$ particles on a circle or line which interact pairwise through potentials of inverse square type. While the quantum integrability of such models was already observed by Calogero and Sutherland ([Ca], [Su]), a new aspect in the understanding of their algebraic structure was only recently initiated by [Po] and [He2]. Polychronakos considered the so-called rational Calogero Hamiltonian with harmonic confinement

$$\mathcal{H}_C \,=\, -\Delta + \omega^2|x|^2 \,+\, 2k(k-1)\sum_{1\leq i<j\leq N}\frac{1}{(x_i-x_j)^2}$$

in $L^2(\mathbb{R}^N)$; here $\omega > 0$ is a frequency parameter and $k \geq 0$ a coupling constant. He observed that after a modification of this operator by additional "exchange operators", the resulting abstract Hamiltonian can be written in a decoupled form involving Dunkl operators of type $A_{N-1}$. It can in fact be considered as a Dunkl-type variant of the classical, $N$-dimensional isotropic oscillator Hamiltonian. Introducing analogues of the classical lowering and raising operators, Polychronakos obtained an explicit operator solution for his original model in an elegant way. Since then, there has been an extensive and ongoing study of CMS models and explicit operator solutions for them via differential-reflection operator formalisms ([L-V], [K1], [BHV], [B-F3], [U-W]).

To motivate our further constructions, we start Chapter 3 with a discussion of classical CMS Hamiltonians. We then turn to abstract CMS operators with harmonic confinement, based on arbitrary root systems. They are of the form

$$\mathcal{H}_k \,=\, -\Delta_k \,+ \omega^2|x|^2\,,$$

where

$$\Delta_k \,=\, \sum_{i=1}^N T_{e_i}(k)^2$$

is the Dunkl Laplacian. If the root system is of type $A_{N-1}$, then the operator $\mathcal{H}_k$ coincides – up to some gauge transform – with the exchange operator modification of $\mathcal{H}_C$ considered by Polychronakos. The spectral properties of abstract CMS operators are easily determined within the Dunkl operator formalism. They are very similar to those of the isotropic oscillator Hamiltonian. In particular, the spectra of these operators are discrete and highly degenerate, and there are various possible choices of eigenfunction bases. In Section 3.2, we present natural choices of eigenfunction bases within a unified concept of generalized Hermite polynomials and biorthogonal polynomials. In particular, the generalized Hermite polynomials are characterized by orthogonality in the underlying weighted $L^2$-space, while certain biorthogonal systems can be obtained by a ladder formalism. Many of the well-known properties of classical Hermite polynomials and Hermite functions allow extensions to generalized Hermite- and biorthogonal systems, such as the Rodrigues formula, generating function, and a Mehler formula. In the $A_{N-1}$-case, our Hermite systems include the generalized Hermite polynomials of Baker and Forrester [B-F2], [B-F3] as a particular class. In the one-dimensional case, we obtain the generalized Hermite polynomials on $\mathbb{R}$ which were studied in [Ros] in connection with a Bose-type oscillator calculus. The results of [B-F3] and [Ros] have inspired many of our present, more general considerations. Chapter 3 is concluded by an analogue of the classical Weyl-Heisenberg uncertainty principle for the Dunkl transform; its proof is based on expansions with respect to generalized Hermite functions.

In Chapter 4, we study heat kernels for finite reflection groups as well as several families of operator semigroups which are related to Dunkl's Laplacian $\Delta_k$. For a given reflection group $G$ and a non-negative multiplicity function $k$, the associated heat kernel on $\mathbb{R}^N$ is obtained, by a generalized translation, from the Gaussian "fundamental solution" of the Dunkl-type heat operator $\Delta_k - \partial_t$. This heat kernel is given by

$$\Gamma_k(t,x,y) := \frac{M_k}{t^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x,y \in \mathbb{R}^N,\ t>0,$$

with a suitable normalization constant $M_k > 0$ and $\gamma = \sum_{\alpha\in R_+} k(\alpha)$. The positivity of the Dunkl kernel $E_k$ for real arguments, due to our main theorem of Chapter 2, ensures that the kernel $\Gamma_k$ is positive as well. In fact, there is a complete analogy to the classical situation: the Dunkl Laplacian $\Delta_k$ generates positivity-preserving contraction semigroups on a variety of Banach spaces including $L^2(\mathbb{R}^N, w_k)$ and $(C_0(\mathbb{R}^N), \|.\|_\infty)$, and $\Gamma_k$ serves as the integral kernel of these generalized heat semigroups. Interesting related semigroups can be obtained by the usual methods; in particular, we consider generalized Cauchy kernels and Cauchy semigroups, and take up the Dunkl-type oscillator Hamiltonians from the previous Chapter. Their spectral properties being completely known, we are directly led to a generalization of the classical oscillator semigroup. Our results for heat- and Cauchy semigroups lead also to explicit solutions of the associated (classical) initial value problems with spatial domain $\mathbb{R}^N$ and certain growth conditions at infinity, such as for the generalized heat operator $\Delta_k - \partial_t$. Uniqueness of the solutions is guaranteed, just as in the classical case, by a maximum principle for the generalized heat operator on unbounded domains. The last section of this Chapter is concerned with the asymptotic behaviour of the generalized heat kernel $\Gamma_k$ for short times. The structure of the Dunkl Laplacian suggests that after being transfered from $L^2(\mathbb{R}^N, w_k)$ to the unweighted $L^2(\mathbb{R}^N)$, the Dunkl heat kernel should not feel the reflecting hyperplanes and behave like the free Gaussian heat kernel $\Gamma_0$ for short times; more precisely, it is conjectured that for all $x, y$ within a fixed open Weyl chamber,

$$\lim_{t\downarrow 0} \frac{\sqrt{w_k(x)\, w_k(y)}\, \Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} = 1\,.$$

We present two partial results towards this conjecture; both are based on the maximum principle for the classical heat operator. The first result gives the stated asymptotics under a certain restriction on the arguments (due to the influence of the reflecting hyperplanes), the second one provides a sharp lower bound.

Basic notations, in particular those for spaces of functions and measures, but also frequently used specific notations, are collected in the appendix.

Major parts of this thesis are already published or accepted for publication. In their present form, however, they differ from the original versions by revisions and extensions at many places.

In particular, [R5] contains the essential parts of Chapter 2, except of the detailed discussion of the algebras of homogeneous series in Section 2.3.. Proposition 1.4.8 as well as partial results of Section 4.3 are published in [R-V2]. Chapter 3.2 is a seriously revised and unified treatment of the material on generalized Hermite polynomials and biorthogonal systems which has been published in [R4] and [R-V1], respectively. The result of Section 3.3 is contained in [R6]. Finally, parts of Section 4.1 as well as Section 4.2 are published in [R4].

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# Chapter 1

# Basic concepts

The aim of this chapter is to provide an introductory overview of the theory of (rational) Dunkl operators and the Dunkl transform. General references are [D2-5], [D-J-O], [dJ1] and [O1]; for a background on reflection groups and root systems we refer to [Hu] and [G-B]. The material of this chapter is essentially well-known; there are, however, also a few new aspects included which are at least not explicitly contained in the literature, such as the behaviour of Dunkl operators and the Dunkl kernel under orthogonal transformations, the Dunkl transform of radial functions in Section 1.4, and the discussion of generalized translations in the last section. We do not intend to give a complete survey, but rather focus on those aspects which will be important in our context. In particular, we do not talk about generalized spherical harmonics (see [D4] and [X2-X3] for this interesting subject), and we restrict major parts of our discussion to non-negative multiplicity functions, because this case will be the only relevant one in the following chapters. The proofs are omitted in as much as they can be found in the quoted literature.

## 1.1 Dunkl operators

Dunkl operators are differential-reflection operators associated with a finite reflection group, acting on some Euclidean space $(E, \langle\,.\,,\,.\,\rangle)$ of finite dimension $N$. We shall always assume that $E = \mathbb{R}^N$ with the standard Euclidean scalar product $\langle x, y\rangle = \sum_{j=1}^N x_j y_j$. For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote by $\sigma_\alpha$ the reflection in the hyperplane $H_\alpha$ orthogonal to $\alpha$, i.e.

$$\sigma_\alpha(x) = x - 2\,\frac{\langle \alpha, x\rangle}{|\alpha|^2}\,\alpha\,,$$

where $|x| := \sqrt{\langle x, x\rangle}$. We use the notation $\langle\,.\,,\,.\,\rangle$ also for the bilinear extension of the Euclidean scalar product to $\mathbb{C}^N \times \mathbb{C}^N$, whereas $z \mapsto |z|$ stands for the standard Hermitean norm on $\mathbb{C}^N$,

$$|z| = \big(|z_1|^2 + \ldots + |z_N|^2\big)^{1/2} \quad \text{for } z = (z_1, \ldots, z_N) \in \mathbb{C}^N.$$

A finite, non-empty set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a (reduced) root system if

$$R \cap \mathbb{R}\alpha = \{\pm\alpha\} \quad \text{and} \quad \sigma_\alpha R = R \quad \text{for all } \alpha \in R.$$

There are no cristallographic conditions imposed on the roots, and we do not require that $R$ spans $\mathbb{R}^N$. For a given root system $R$ the reflections $\sigma_\alpha$ $(\alpha \in R)$ generate a finite group $G \subset O(N, \mathbb{R})$; it is called the reflection group associated with $R$. All reflections in $G$ correspond to suitable pairs of roots, and the orbits in $R$ under the natural action of $G$ correspond to the conjugacy classes of reflections in $G$. The connected components of $\mathbb{R}^N \setminus H$, where $H = \bigcup_{\alpha \in R} H_\alpha$, are called the Weyl chambers of $G$. If $W$ is an arbitrary fixed Weyl chamber of $G$, then its closure $\overline{W}$ in $\mathbb{R}^N$ is a fundamental domain of $G$, i.e. $\overline{W}$ is naturally homeomorphic to the space $(\mathbb{R}^N)^G$ of all $G$-orbits in $\mathbb{R}^N$, endowed with the quotient topology. For details, we refer to Chapter 1 of [Hu]. We further fix a positive subsystem $R_+ = \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$, where $\beta \in \mathbb{R}^N \setminus H$. Then for each $\alpha \in R$, either $\alpha \in R_+$ or $-\alpha \in R_+$. From now on we assume that the root system $R$ is normalized in the sense that $|\alpha| = \sqrt{2}$ for all $\alpha \in R$; this simplifies formulas, but is no loss of generality for our purposes.

Let $M$ be the vector space of $\mathbb{C}$-valued functions on $R$ which are invariant under the action of the associated reflection group $G$. Its dimension is equal to the number of $G$-orbits in $R$. An element $k \in M$ is called a multiplicity function on $R$. We write $\operatorname{Re} k \geq 0$ if $\operatorname{Re} k(\alpha) \geq 0$ for all $\alpha \in R$ and $k \geq 0$ if $k(\alpha) \geq 0$ for all $\alpha \in R$. For abbreviation, we introduce the index

$$\gamma := \gamma(k) := \sum_{\alpha \in R_+} k(\alpha). \tag{1.1}$$

Since $k$ is $G$-invariant, and therefore in particular $k(-\alpha) = k(\alpha)$ for all $\alpha \in R$, this definition is independent of the special choice of $R_+$. For $k \geq 0$, we denote by $w_k$ the weight function

$$w_k(x) \;=\; \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \tag{1.2}$$

which is $G$-invariant and homogeneous of degree $2\gamma$, with $\gamma = \gamma(k)$ as defined in (1.1). We further fix (again for $k \geq 0$) the Mehta-type constants

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x) dx\,. \tag{1.3}$$

Macdonald [M1] conjectured a closed expression for these constants, which was confirmed for arbitrary root systems in Corollary 9.10 of [O1].
Now fix some root system $R$ on $\mathbb{R}^N$, corresponding to a reflection group $G$ on $\mathbb{R}^N$. The Dunkl operators attached to $G$ are first-order differential-reflection operators on $\mathbb{R}^N$ which are parametrized by some multiplicity function $k \in M$.

**1.1.1 Definition.** For $\xi \in \mathbb{R}^N$, the Dunkl operator $T_\xi(k)$ is defined by

$$T_\xi(k) f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \, \langle \alpha, \xi \rangle \, \frac{f(x) - \sigma_\alpha f(x)}{\langle \alpha, x \rangle}\,, \qquad f \in C^1(\mathbb{R}^N);$$

here $\partial_\xi$ denotes the directional derivative corresponding to $\xi$.

For the $i$-th standard basis vector $\xi = e_i \in \mathbb{R}^N$, we use the abbreviation $T_i(k) = T_{e_i}(k)$. Again, the above definition is independent of the choice of $R_+$. In case $k=0$, the $T_\xi(k)$ reduce to the corresponding directional derivatives. The operators $T_\xi(k)$ were introduced and first studied by Dunkl in a series of papers ([D2-5]) in connection with a generalization of the classical theory of spherical harmonics; here the uniform surface measure on the $(N-1)$-dimensional unit sphere is modified by a weight function $w_k$ as defined above. The most important features of Dunkl operators are visible already by their action on polynomials. Before going on, we have to provide some additional notation.
Let $\Pi^N := \mathbb{C}[\mathbb{R}^N]$ denote the $\mathbb{C}$ -algebra of polynomial functions on $\mathbb{R}^N$. $\Pi^N$ has a natural grading $\Pi^N = \bigoplus_{n\geq 0} \mathcal{P}_n^N$, where $\mathcal{P}_n^N$ $(n \in \mathbb{Z}_+ = \{0,1,\ldots\})$ is the subspace of homogeneous polynomials of (total) degree $n$. Let further $\Pi_n^N := \bigoplus_{k=0}^n \mathcal{P}_k^N$, the space of polynomials of degree at most $n$.

We use the standard multi-index notation; in particular, for $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^N$ we write $\nu! := \nu_1! \cdot \ldots \cdot \nu_N!$ and $|\nu| := \nu_1 + \ldots + \nu_N$, as well as

$$z^\nu := z_1^{\nu_1} \cdot \ldots \cdot z_N^{\nu_N} \quad \text{ and } A^\nu := A_1^{\nu_1} \cdots A_N^{\nu_N}$$

for $z \in \mathbb{C}^N$ and any family $A = (A_1, \ldots, A_N)$ of commuting operators on $\Pi^N$. The natural action of $O(N,\mathbb{R})$ on $C^1(\mathbb{R}^N)$ is given by

$$h \cdot f(x) := f(h^{-1}x), \quad h \in O(N,\mathbb{R}).$$

We continue with our discussion of the Dunkl operators $T_\xi(k)$. First of all, they have the following regularity properties:

**1.1.2 Lemma.** *([D2], [dJ1]) Let $k \in M$ and $\xi \in \mathbb{R}^N$. Then the following assertions hold:*

(1) *If $f \in C^n(\mathbb{R}^N)$ with $n \geq 1$, then $T_\xi(k)f \in C^{n-1}(\mathbb{R}^N)$.*

(2) *If $f$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$, then also $T_\xi(k)f \in \mathscr{S}(\mathbb{R}^N)$.*

(3) *$T_\xi(k)$ is homogeneous of degree $-1$ on $\Pi^N$, that is, $T_\xi(k)\,p \in \mathcal{P}_{n-1}^N$ for $p \in \mathcal{P}_n^N$.*

*Proof.* All statements follow from the representation

$$\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle} = \int_0^1 \partial_\alpha f\big(x - t\langle \alpha, x\rangle \alpha\big)dt \quad \text{ for } f \in C^1(\mathbb{R}^N),\ \alpha \in R.$$

(1) and (3) are immediate; for details concerning (2) we refer to [dJ1]. □

We note further that by the $G$-invariance of $k$,

$$g \circ T_\xi(k) \circ g^{-1} = T_{g\xi}(k) \quad (g \in G). \tag{1.4}$$

Moreover, the $T_\xi(k)$ satisfy a product rule, which is easily verified by a short calculation: If $f, g \in C^1(\mathbb{R}^N)$ and at least one of them is $G$-invariant, then

$$T_\xi(k)(fg) = T_\xi(k)(f) \cdot g + f \cdot T_\xi(k)(g). \tag{1.5}$$

The most striking property of the Dunkl operators, which was verified in [D2] by direct computation, is the following

**1.1.3 Proposition.** *For each $k \in M$, the family $\{T_\xi(k),\, \xi \in \mathbb{R}^N\}$ generates a commutative algebra of linear operators on $\Pi^N$.*

Together with Lemma 1.1.2 (3), this fact implies that for any real-analytic function $f : \mathbb{R}^N \to \mathbb{C}$ with power series $f(x) = \sum_{\nu \in \mathbb{Z}_+^N} a_\nu x^\nu$ there is a unique linear operator $f(T(k))$ on $\Pi^N$ defined by the terminating series

$$f(T(k))(p) := \sum_{\nu \in \mathbb{Z}_+^N} a_\nu\, T(k)^\nu(p) := \sum_{\nu \in \mathbb{Z}_+^N} a_\nu\, T_1(k)^{\nu_1} \cdot \ldots \cdot T_N(k)^{\nu_N}(p).$$

The classical case $k = 0$ will be distinguished by the notation $f(\partial)$. Of particular importance is the generalized Laplacian associated with $G$ and $k$, which is defined by $\Delta_k := p(T(k))$ with $p(x) = |x|^2$. It is homogeneous of degree $-2$ and satisfies

$$\Delta_k = \sum_{i=1}^N T_{\xi_i}(k)^2 \tag{1.6}$$

for any orthonormal basis $\{\xi_1, \ldots, \xi_N\}$ of $\mathbb{R}^N$, see [D2]. By our convention $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R_+$, the generalized Laplacian is given explicitly by

$$\Delta_k \;=\; \Delta + 2 \sum_{\alpha \in R_+} k(\alpha)\delta_\alpha \quad \text{ with } \quad \delta_\alpha f(x) \;=\; \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - \sigma_\alpha f(x)}{\langle \alpha, x \rangle^2}; \tag{1.7}$$

here $\Delta$ and $\nabla$ denote the usual Laplacian and gradient respectively.

*1.1.4 Remark.* 1. It follows from (1.4) together with (1.6) that $\Delta_k$ is equivariant under $G$, i.e.

$$g \circ \Delta_k = \Delta_k \circ g \quad \text{for all } g \in G.$$

However, $\Delta_k$ is not fully rotationally equivariant: For $h \in O(N, \mathbb{R})$, define the multiplicity function $k_h$ on the transformed root system $h(R)$ (which belongs to the conjugate reflection group $hGh^{-1}$) by

$$k_h(h\alpha) := k(\alpha),\ \alpha \in R. \tag{1.8}$$

Then a short calculation yields

$$h \circ T_\xi^R(k) = T_{h\xi}^{h(R)}(k_h) \circ h \quad \text{ for } h \in O(N, \mathbb{R}), \tag{1.9}$$

where the additional superscript stands for the underlying root system. It follows, again in view of (1.6), that

$$h \circ \Delta_k^R \;=\; \Delta_{k_h}^{h(R)} \circ h \quad \text{for all } h \in O(N, \mathbb{R}). \tag{1.10}$$

Let us consider some important special cases:

*1.1.5 Examples.* **(1) The one-dimensional case.** In case $N = 1$, the only choice of $R$ is $R = \{\pm\sqrt{2}\}$, corresponding to the reflection group $G = \{id, \sigma\} \cong \mathbb{Z}_2$ on $\mathbb{R}$, where $\sigma(x) = -x$. The Dunkl operator $T(k) := T_{\sqrt{2}}(k)$ associated with the multiplicity parameter $k \in \mathbb{C}$ is given by

$$T(k)f(x) \;=\; f'(x) + k\,\frac{f(x) - f(-x)}{x}\,,\;\; f \in C^1(\mathbb{R}).$$

Its square $T(k)^2$, when restricted to the even subspace $C^1(\mathbb{R})^e := \{f \in C^1(\mathbb{R}) : f(x) = f(-x)\}$, coincides with the Bessel differential operator of index $k - 1/2$ on $\mathbb{R}$:

$$T(k)^2|_{C^1(\mathbb{R})^e} f(x) \;=\; f''(x) + \frac{2k}{x} \cdot f'(x)\,.$$

**(2) Dunkl operators of type $A_{N-1}$.** These belong to the symmetric group $G = S_N$, which acts in a canonical way on $\mathbb{R}^N$ by permuting the standard basis vectors $e_1, \ldots, e_N$. Each transposition $(ij)$ acts as a reflection $\sigma_{ij}$, sending $e_i - e_j$ to its negative. On $C^1(\mathbb{R}^N)$, $\sigma_{ij}$ acts by transposing the coordinates $x_i$ and $x_j$ with respect to the standard basis. The attached root system, of type $A_{N-1}$, is given by $R = \{e_i - e_j\,,\; 1 \le i, j \le N,\; i \neq j\}$. Since all transpositions are conjugate in $S_N$, the vector space of multiplicity functions on $R$ is one-dimensional. The Dunkl operators associated with the multiplicity parameter $k \in \mathbb{C}$ are given by

$$T_i(k)^S \;=\; \partial_i \,+\, k \cdot \sum_{j \neq i} \frac{1 - \sigma_{ij}}{x_i - x_j} \quad (i = 1, \ldots, N),$$

and the generalized Laplacian is

$$\Delta_k^S \;=\; \Delta + 2k \sum_{1 \le i < j \le N} \frac{1}{x_i - x_j}\Big[(\partial_i - \partial_j) - \frac{1 - \sigma_{ij}}{x_i - x_j}\Big]\,.$$

**(3) Dunkl operators of type $B_N$.** Let $G$ be the Weyl group of type $B_N$ $(N \ge 2)$, i.e. the reflection group on $\mathbb{R}^N$ which is generated by the transpositions $\sigma_{ij}$ as above, as well as the sign changes $\sigma_i : e_i \to -e_i\,,\; i = 1, \ldots, N$. The group of sign changes is isomorphic to $\mathbb{Z}_2^N$, intersects $S_N$ trivially and is normalized by $S_N$, so $G \cong S_N \ltimes \mathbb{Z}_2^N$. There are two conjugacy classes of reflections in $G$, leading to multiplicity functions of the form $k = (k_0, k_1)$ with $k_i \in \mathbb{C}$. The associated Dunkl operators are given by

$$T_i(k)^B \;=\; \partial_i \,+\, k_1 \frac{1 - \sigma_i}{x_i} \,+\, k_0 \cdot \sum_{j \neq i} \Big[\frac{1 - \sigma_{ij}}{x_i - x_j} + \frac{1 - \tau_{ij}}{x_i + x_j}\Big] \quad (i = 1, \ldots, N),$$

where $\tau_{ij} := \sigma_{ij}\sigma_i\sigma_j$.

## 1.2 Dunkl's intertwining operator

It was first shown in [D4] that for non-negative multiplicity functions, the associated commutative algebra of Dunkl operators is intertwined with the algebra of usual partial differential operators by a unique linear and homogeneous isomorphism on polynomials. A thorough analysis in [D-J-O] subsequently revealed that such an intertwining operator exists if and only if the common kernel of the $T_\xi(k)$, considered as linear operators on $\Pi^N$, contains no "singular" polynomials besides the constants. More precisely, the following characterization holds:

**1.2.1 Theorem.** *([D-J-O]) Let* $M^{reg} := \big\{k \in M : \bigcap_{\xi\in\mathbb{R}^N} Ker\big(T_\xi(k)\big) = \mathbb{C}\cdot 1\big\}$. *Then the following assertions are equivalent*

(1) $k \in M^{reg}$;

(2) *There exists a unique linear isomorphism ("intertwining operator")* $V_k$ *of* $\Pi^N$ *such that*

$$V_k(\mathcal{P}_n^N) = \mathcal{P}_n^N\,,\ \ V_k\,|_{\mathcal{P}_0^N} = id \quad \textit{and} \quad T_\xi(k)V_k = V_k\,\partial_\xi \quad \textit{for all } \xi \in \mathbb{R}^N.$$

The singular parameter set $M \setminus M^{reg}$ is explicitly determined in [D-J-O]. From [D-J-O] together with the results of [O1] it follows in particular that $M^{reg}$ is an open subset of $M$ which is invariant under complex conjugation, and that

$$\{k \in M : \operatorname{Re} k \geq 0\} \subseteq M^{reg}.$$

In most parts of this thesis, we will in fact restrict our attention to non-negative multiplicity functions. The operator $V_k$ plays an important role in Dunkl's theory and its applications. In [D4] it was used, for $k \geq 0$, to define a generalized exponential kernel (the Dunkl kernel) and an associated integral transform (the Dunkl transform); these subjects will be introduced in the subsequent sections. An explicit form of $V_k$, however, is known so far only in very special cases:

1. *The one-dimensional case 1.1.5 (1).* It follows from the results of [D-J-O] that

   $$M^{reg} = \mathbb{C} \setminus \big\{-\frac{1}{2} - n,\ n \in \mathbb{Z}_+\big\} =: M_*.$$

   The associated intertwining operator is given explicitly by

   $$V_k\big(x^{2n}\big) = \frac{\big(\frac{1}{2}\big)_n}{\big(k+\frac{1}{2}\big)_n}\,x^{2n}; \quad V_k\big(x^{2n+1}\big) = \frac{\big(\frac{1}{2}\big)_{n+1}}{\big(k+\frac{1}{2}\big)_{n+1}}\,x^{2n+1},$$

   where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. For $\operatorname{Re} k > 0$, this amounts to the following integral representation (see [D4], Th. 5.1):

   $$V_k\,p(x) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)}\int_{-1}^{1} p(xt)\,(1-t)^{k-1}(1+t)^k\,dt. \tag{1.11}$$

2. *The direct product case* $G = \mathbb{Z}_2^N$ *on* $\mathbb{R}^N$. Here $M^{reg} = (M_*)^N$ and for $k = (k_1,\ldots,k_N) \in M^{reg}$, the operator $V_k$ is defined by $V_k(x^\nu) = \prod_{j=1}^N V_{k_j}\big(x_j^{\nu_j}\big)$. For $\operatorname{Re} k > 0$, an integral representation is easily obtained by an $N$-fold iteration of (1.11), c.f. [X1].

3. *The case* $G = S_3$, which was studied in [D6]. Here

   $$M^{reg} = \mathbb{C} \setminus \{-\frac{1}{2} - n, -\frac{1}{3} - n, -\frac{2}{3} - n,\ n \in \mathbb{Z}_+\}.$$

   For $k \geq 0$, an integral formula of Harish-Chandra on the unitary group $U(3)$ (whose Weyl group is $S_3$) was used in [D6] to construct an integral representation for $V_k$ over a compact subset of $\mathbb{R}^{2\times 2}$.

The proof of Theorem 1.2.1 is based on a general inductive construction of operators that intertwine the action of two sets of operators in a graded vector space (Theorem 2.1 of [D-J-O]). When applied to the operators $\{\partial_i\,, i = 1,\ldots,N\}$ and $\{T_i(k)\,, i = 1,\ldots,N\}$ on $\Pi^N$, the quoted theorem implies that for every $k \in M$ there exists a unique linear map $W_k : \Pi^N \to \Pi^N$ such that

$$W_k|_{\mathcal{P}_0^N} = id,\ \ W_k(\mathcal{P}_n^N) \subseteq \mathcal{P}_n^N \quad \text{and} \ \ \partial_\xi\, W_k \,=\, W_k T_\xi(k) \quad \text{for all} \ \ \xi \in \mathbb{R}^N.$$

In addition, $W_k$ is bijective for $k \in M^{reg}$.

It is easily verified that the operator $W_k$, which of course coincides with $V_k^{-1}$ for every $k \in M^{reg}$, is given explicitly by

$$(W_k\, p)(x) \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \frac{x^\nu}{\nu!}\, T(k)^\nu p\,(0) \,=\, \big(e^{\langle x, T(k)\rangle} p\big)(0). \tag{1.12}$$

An immediate consequence of (1.12) is the following Taylor-type formula:

**1.2.2 Corollary.** *Let $k \in M^{reg}$, and suppose that $f : B \to \mathbb{C}$ is holomorphic in an open ball $B \subseteq \mathbb{C}^N$ around $0$. Then*

$$f(z) \,=\, \sum_{n=0}^{\infty} \sum_{|\nu|=n} \frac{V_k(z^\nu)}{\nu!}\, T(k)^\nu f(0),$$

*where the series $\sum_{n=0}^{\infty}$ converges normally in $B$.*

We finally mention the behaviour of $V_k$ under orthogonal transformations: Let $h \in O(N,\mathbb{R})$, and use the notations from Remark 1.1.4. The transformation property (1.9) for the Dunkl operators, together with the intertwining property of $V_{k_h}$, implies that

$$T_\xi^R(k)\, h^{-1} V_{k_h} h \,=\, h^{-1}\, T_{h\xi}^{h(R)}(k_h) V_{k_h} h \,=\, h^{-1} V_{k_h} \partial_{h\xi}\, h \,=\, h^{-1} V_{k_h} h\, \partial_\xi.$$

In view of the characterizing properties of $V_k$ it follows that

$$h^{-1} V_{k_h} h \,=\, V_k \quad \text{for all} \ \ h \in O(N,\mathbb{R}). \tag{1.13}$$

In particular,

$$g^{-1} V_k\, g \,=\, V_k \quad \text{for all} \ \ g \in G. \tag{1.14}$$

In [D4], the intertwining operator $V_k$ is, for $k \geq 0$, extended to a bounded linear operator on a suitably normed algebra of homogeneous series on the unit ball of $\mathbb{R}^N$, as follows: Let $K := \{x \in \mathbb{R}^N : |x| \leq 1\}$ denote the unit ball in $\mathbb{R}^N$, and define

$$A := \Big\{ f : K \to \mathbb{C},\ f = \sum_{n=0}^{\infty} f_n \quad \text{with} \ \ f_n \in \mathcal{P}_n^N \ \ \text{and} \ \ \|f\|_A := \sum_{n=0}^{\infty} \|f_n\|_{\infty,K} < \infty \Big\}. \tag{1.15}$$

It is easily checked that for $f \in A$ the homogeneous expansion $f = \sum_{n=0}^{\infty} f_n$ is unique, and that $A$ is a commutative Banach-*-algebra (with complex conjugation as involution); moreover, each $f \in A$ is real-analytic in the open ball $K^\circ$ and continuous on $K$. For more information we refer to Section 2.3, where this algebra will be studied in some detail. We have to point out, however, that $A$ as defined above is a complex algebra, whereas in [D4] only series of real-valued polynomials are considered. In [D4] it is shown that $\|V_k\, p\,\|_{\infty,K} \leq \|p\|_{\infty,K}$ for every homogeneous polynomial $p$ with real coefficients; as a consequence, $V_k$ extends to a continuous linear operator on $A$ by

$$V_k f := \sum_{n=0}^{\infty} V_k f_n \quad \text{for } f = \sum_{n=0}^{\infty} f_n \in A\,. \tag{1.16}$$

Up to now, it has been an open question whether for $k \geq 0$ the intertwining operator $V_k$ is always positive, i.e. $V_k\, p \geq 0$ on $\mathbb{R}^N$ for each nonnegative polynomial $p \in \Pi^N$. More generally, Dunkl stated the following conjecture:

**1.2.3 Conjecture.** *([D4]) Let $k \geq 0$. Then for every $x \in \mathbb{R}^N$ with $|x| \leq 1$, the functional $f \mapsto V_k f(x)$ is positive on $A$.*

This statement can be derived from the explicit representation of $V_k$ in the above listed special cases 1 and 2; in the $S_3$-case however, the integral representations derived in [D6] failed to infer this result - at least for a large range of $k$. The above conjecture will be proved in Section 2.4 for general reflection groups and nonnegative multiplicity functions. One of the tools in our proof is the following bilinear form on $\Pi^N$, associated with $G$ and $k \geq 0$, which was introduced in [D4] in the context of generalized spherical harmonics:

$$[p,q]_k \,:=\, \big(p(T(k))\, q\big)(0) \quad \text{ for } p,q \in \Pi^N.$$

We collect some fundamental properties of this bilinear form.

**1.2.4 Lemma.** (1) *If $p \in \mathcal{P}_n^N$ and $q \in \mathcal{P}_m^N$ with $n \neq m$, then* $[p,q]_k = 0$.

(2) $[x_i\, p\,, q]_k \,=\, [p, T_i(k) q]_k \quad (p,q \in \Pi^N,\ i = 1, \ldots, N)$.

(3) $[\,.,\,.]_k$ *is symmetric and non-degenerate.*

(4) $[g \cdot p\,, g \cdot q]_k \,=\, [p,q]_k \quad (p,q \in \Pi^N,\ g \in G)$.

(5) $[V_k\, p, q]_k \,=\, [p,q]_0 \quad (p,q \in \Pi^N)$.

*Proof.* (1) follows from the homogenity of the Dunkl operators, (2) is clear from the definition, (3) is shown in [D4] and [D-J-O] respectively, while (4) follows from (1.4). It remains to prove (5). In view of (1), it is enough to consider $p,q \in \mathcal{P}_n^N$ with $n \in \mathbb{Z}_+$. Then

$$[V_k\, p, q]_k \,=\, [q, V_k\, p]_k \,=\, q(T(k))(V_k\, p) \,=\, V_k(q(\partial)p) \,=\, q(\partial)(p) \,=\, [p,q]_0\,;$$

here the characterizing properties of $V_k$ and the fact that $q(\partial)p$ is a constant have been used. □

The pairing $[\,.\,,.\,]_k$ is closely related to the scalar product in $L^2(\mathbb{R}^N, e^{-|x|^2/2} w_k)$; in fact, we have the following identity due to Dunkl [D4], which generalizes a result of Macdonald [M1] for the classical case:

**1.2.5 Proposition.** *For all* $p, q \in \Pi^N$,

$$[p,q]_k \;=\; c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}p(x)\, e^{-\Delta_k/2}q(x)\, e^{-|x|^2/2}\, w_k(x)dx. \tag{1.17}$$

This identity implies in particular that $[\,.\,,\,.\,]_k$ is in fact a scalar product on the vector space $\Pi_{\mathbb{R}}^N = \mathbb{R}[\mathbb{R}^N]$ of real valued polynomials on $\mathbb{R}^N$. Later on, it will be of some importance in several contexts; we therefore include an elegant and instructive proof which is taken from an unpublished part of de Jeu's thesis ([dJ3], Chap. 3.3). It involves the following commutator results in $\mathrm{End}_{\mathbb{C}}(\Pi^N)$, where as usual, $[A,B] = AB - BA$ for $A, B \in \mathrm{End}_{\mathbb{C}}(\Pi^N)$.

**1.2.6 Lemma.** *For* $i = 1, \ldots, N$,

(1) $\big[x_i\,,\,\Delta_k/2\big] \;=\; -T_i(k);$

(2) $\big[x_i\,,\,e^{-\Delta_k/2}\big] \;=\; T_i(k)\,e^{-\Delta_k/2}.$

*Proof.* (1) follows by a short calculation (c.f. [D2], Prop. 2.2). Induction then yields that

$$\big[x_i\,,(\Delta_k/2)^n\big] \;=\; -n\,T_i(k)\big(\Delta_k/2\big)^{n-1} \quad \text{for } n \geq 1,$$

and this implies (2). □

*Proof of Proposition 1.2.5.* Let $i \in \{1, \ldots, N\}$, and denote the right-hand side of (1.17) by $\big(p,q\big)_k$. Then by the anti-symmetry of $T_i(k)$ in $L^2(\mathbb{R}^N, w_k)$ and the above Lemma,

$$\begin{aligned}
\big(p, T_i(k)\,q\big)_k &= c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}p\cdot\big(T_i(k)\,e^{-\Delta_k/2}q\big)\,e^{-|x|^2/2}\,w_k\,dx\\
&= -\,c_k^{-1}\int_{\mathbb{R}^N} T_i(k)\big(e^{-|x|^2/2}e^{-\Delta_k/2}p\big)\cdot\big(e^{-\Delta_k/2}q\big)\,w_k\,dx\\
&= c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}(x_i\,p)\cdot\big(e^{-\Delta_k/2}q\big)\,e^{-|x|^2/2}\,w_k\,dx \;=\; \big(x_i\,p,q\big)_k.
\end{aligned}$$

But the form $[\,.\,,\,.\,]_k$ has the same property by Lemma 1.2.4(3). Since the $T_i(k)$ are homogeneous of degree $-1$, an easy induction argument with respect to $\max(\deg p, \deg q)$ now finishes the proof. □

## 1.3 The generalized exponential kernel

For regular multiplicity parameters, there exists a generalization of the usual exponential kernel $e^{\langle x,y\rangle}$, which can be characterized as a solution of the joint eigenfunction problem for the Dunkl operators $\{T_\xi(k),\ \xi \in \mathbb{R}^N\}$. The following theorem is a weakened version of [O1], Prop. 6.7. (c.f. Theorem 2.6 of [dJ1]).

**1.3.1 Theorem.** *For each* $k \in M^{reg}$ *and* $w \in \mathbb{C}^N$, *the system*

$$T_\xi(k)f \;=\; \langle \xi, w\rangle f \quad (\xi \in \mathbb{R}^N)$$

*has a unique solution* $x \mapsto E_k(x,w)$ *which is real-analytic on* $\mathbb{R}^N$ *and satisfies* $f(0) = 1$. *Moreover, the mapping* $(x,k,w) \mapsto E_k(x,w)$ *extends to a meromorphic function on* $\mathbb{C}^N \times M \times \mathbb{C}^N$ *with pole set* $\mathbb{C}^N \times (M \setminus M^{reg}) \times \mathbb{R}^N$.

The function $E_k$ is called the Dunkl kernel, or generalized exponential kernel, attached to the reflection group $G$ and the multiplicity function $k$. The group-invariant counterpart of this kernel, the "generalized Bessel function"

$$J_k(z,w) := \frac{1}{|G|}\sum_{g\in G} E_k(gz,w) \qquad (z,w\in\mathbb{C}^N), \tag{1.18}$$

is also of some importance.

*1.3.2 Remarks.* (1) If $k=0$, then $E_k(z,w)=e^{\langle z,w\rangle}$ for all $z,w\in\mathbb{C}^N$. (Recall that $\langle\,.\,,.\rangle$ was defined to be bilinear on $\mathbb{C}^N\times\mathbb{C}^N$.)

(2) The kernel $J_k$ was introduced in [O1], where in particular the pole set of the mapping $(z,k,w)\mapsto J_k(z,w)$ was determined explicitly (Prop. 9.6 of [O1]); it coincides with the pole set of $E_k$ (Cor. 6.10 of [O1]). The intimate connection of this pole set with the singular parameter set of the intertwining operator and its identification with $\mathbb{C}^N\times(M\setminus M^{reg})\times\mathbb{R}^N$ were observed in [D-J-O] (c.f. the remark after Theorem 4.8).

(3) For nonnegative multiplicity functions, the kernel $E_k$ was originally constructed in [D4] by means of the intertwining operator $V_k$. In fact, the exponential function $x\mapsto e^{\langle x,w\rangle}$, with $w\in\mathbb{C}^N$ fixed, obviously belongs to the algebra $A$. Hence one can define $\widetilde{E}_k(\,.\,,w)\in A$ by

$$\widetilde{E}_k(x,w) := V_k\big(e^{\langle\,.\,,w\rangle}\big)(x), \quad |x|\le 1.$$

The homogeneity of $V_k$ implies that for $\lambda\in\mathbb{R}$ with $|\lambda x|\le 1$,

$$\widetilde{E}_k(\lambda x,w) \;=\; \sum_{n=0}^{\infty} V_k\Big(\frac{\langle\,.,w\rangle^n}{n!}\Big)(\lambda x) \;=\; \sum_{n=0}^{\infty} V_k\Big(\frac{\langle\,.,\lambda w\rangle^n}{n!}\Big)(x) \;=\; \widetilde{E}_k(x,\lambda w).$$

We therefore have a unique extension of $\widetilde{E}_k(.,w)$ to a real-analytic function on $\mathbb{R}^N$ satisfying $\widetilde{E}_k(\lambda x,w)=\widetilde{E}_k(x,\lambda w)$ for all $\lambda\in\mathbb{R}$. Moreover, using the intertwining property of $V_k$ as well as its normalization $V_k(1)=1$, it is easily checked that $\widetilde{E}_k$ fulfills the characterization of $E_k$ according to the above proposition; hence

$$E_k(x,w) \;=\; V_k\big(e^{\langle\,.\,,w\rangle}\big)(x) \quad \text{for } |x|\le 1. \tag{1.19}$$

(4) It is usually not required in the theory of Dunkl operators that the reflection group $G$ is essential relative to $\mathbb{R}^N$, i.e. acts on $\mathbb{R}^N$ with no nonzero fixed points. (Notice that this condition is equivalent to $\langle R\rangle=\mathbb{R}^N$, i.e. the associated root system spans $\mathbb{R}^N$.) Such an additional requirement would however impose no serious restrictions. In fact, if $G$ is not essential relative to $\mathbb{R}^N$, then $V:=\{x\in\mathbb{R}^N: gx=x \text{ for all } g\in G\}$ is a nontrivial subspace of $\mathbb{R}^N$, and $G$ is essential on the orthogonal complement $V^\perp$ of $V$ in $\mathbb{R}^N$. Moreover, we have $R\subset V^\perp$, and therefore $T_\xi(k)=\partial_\xi$ for all $\xi\in V$ and arbitrary multiplicity functions $k$. Thus the relevant action of the Dunkl operators takes place in $V^\perp$, which is also reflected by the form of the associated Dunkl kernel. For $z\in\mathbb{C}^N$ write

$z = z' + z''$, with $z' \in V + iV$ and $z'' \in V^\perp + iV^\perp$. Then it is immediately seen from the characterization of $E_k$ according to Theorem 1.3.1 that

$$E_k(z,w) = e^{\langle z', w'\rangle} \cdot E_k(z'', w'') \quad \text{for all } z, w \in \mathbb{C}^N. \tag{1.20}$$

Just as for the intertwining operator, the kernels $E_k$ and $J_k$ are known explicitly for some particular cases only. An important example is the one-dimensional situation:

*1.3.3 Example.* For the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$ and multiplicity parameter $k$ with $\operatorname{Re} k > 0$, the integral representation (1.11) for $V_k$ implies that for all $z, w \in \mathbb{C}$,

$$E_k^{\mathbb{Z}_2}(z,w) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)} \int_{-1}^1 e^{tzw}(1-t)^{k-1}(1+t)^k\,dt = e^{zw} \cdot {}_1F_1(k, 2k+1, -2zw).$$

We therefore have

$$E_k^{\mathbb{Z}_2}(z,w) = j_{k-1/2}(izw) + \frac{zw}{2k+1}\, j_{k+1/2}(izw), \quad \text{and} \quad J_k^{\mathbb{Z}_2}(z,w) = j_{k-1/2}(izw),$$

where for $\alpha \geq -1/2$, $j_\alpha$ is the normalized spherical Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \cdot \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \cdot \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}. \tag{1.21}$$

We list some general and basic properties of the Dunkl kernel.

**1.3.4 Proposition.** *Let $k \in M^{reg}$, $g \in G$, $h \in O(N,\mathbb{R})$, $z, w \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$. Then*

(1) $E_k(z,0) = 1$.

(2) $E_k(z,w) = E_k(w,z)$.

(3) $E_k(gz, gw) = E_k(z,w)$ *and* $E_k(\lambda z, w) = E_k(z, \lambda w)$.

(4) $E_{k_h}(hz, hw) = E_k(z,w)$.

(5) $\overline{E_k(z,w)} = E_{\overline{k}}(\overline{z}, \overline{w})$.

*The same properties hold for the generalized Bessel function $J_k$; moreover, $J_k(gz,w) = J_k(z,w)$ for all $g \in G$, $z, w \in \mathbb{C}^N$.*

*Proof.* (1) is clear from the definition and (2) was shown in [D5]. The remaining properties follow from corresponding homogeneity properties of the Dunkl kernel; for (3) and (5) we refer to [dJ1], while (4) can be seen as follows: The transformation property (1.9) implies that for fixed $w \in \mathbb{C}^N$, the function $F(x) := h^{-1} \cdot E_{k_h}(x, hw)$ on $\mathbb{R}^N$ satisfies $T_\xi(k)F = \langle \xi, w\rangle F$ as well as $F(0) = 1$. The assertion now follows from Theorem 1.3.1 and analytic continuation. □

*1.3.5 Remark.* For fixed $z \in \mathbb{C}^N$, the generalized Bessel function $f_z(x) := J_k(x,z)$ solves the eigenvalue problem

$$L_k f = \langle z, z\rangle f \quad \text{on } \mathbb{R}^N,$$

with the differential operator

$$L_k f(x) := \Delta f(x) + 2 \sum_{\alpha\in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle}\,.$$

In fact, the $G$-equivariance of the Dunkl Laplacian $\Delta_k$ implies that

$$\Delta_k f_z \,=\, \langle z, z\rangle f_z\,,$$

and $\Delta_k$ coincides with $L_k$ for $G$-invariant functions from $C^2(\mathbb{R}^N)$. The operator $L_k$ can also be written in divergence form,

$$L_k f(x) \,=\, \frac{1}{w_k(x)} \sum_{i=1}^N \partial_i\big(w_k(x)\partial_i f(x)\big).$$

This is a canonical multivariable generalization of the Sturm-Liouville operator for the classical spherical Bessel function $j_{k-1/2}$, which is obtained in the one-dimensional case, c.f. Example 1.1.5 (1).

We conclude this section by a closer look at the homogeneous expansion of the Dunkl kernel. Since $E_k$ is holomorphic on $\mathbb{C}^N\times\mathbb{C}^N$, it admits a unique normally convergent expansion into a series of homogeneous polynomials. In fact, Corollary 1.2.2 with $f = E_k(\,.\,,w)$, $w\in\mathbb{C}^N$ fixed, shows that

$$E_k(z,w) \,=\, \sum_{n=0}^\infty E_{k,n}(z,w) \quad (z,w\in\mathbb{C}^N) \tag{1.22}$$

with

$$E_{k,n}(z,w) \,=\, \sum_{|\nu|=n} \frac{V_k(z^\nu)}{\nu!}\, w^\nu \,=\, V_k\Big(\frac{\langle\,.\,,w\rangle^n}{n!}\Big)(z). \tag{1.23}$$

Here $E_{k,n}$ is a homogeneous polynomial of degree $n$ in each of its arguments; so the expansion (1.22) coincides with the homogeneous expansion of $E_k$ and converges normally on $\mathbb{C}^N\times\mathbb{C}^N$. Comparison of the homogeneous parts shows that the properties listed in Proposition 1.3.4 hold for each of the $E_{k,n}$ instead of $E_k$ as well. Moreover,

$$T_\xi(k)E_{k,n}(\,.\,,w) \,=\, \langle \xi, w\rangle\, E_{k,\,n-1}(\,.\,,w) \quad \text{for all } n\geq 1,\ \xi\in\mathbb{R}^N. \tag{1.24}$$

We shall also need the following estimates, valid for $k\geq 0$ ([D4], [D5]):

$$|E_{k,n}(z,w)| \,\leq\, \frac{|z|^n|w|^n}{n!} \quad (z,w\in\mathbb{C}^N,\ n\in\mathbb{Z}_+). \tag{1.25}$$

## 1.4 The Dunkl transform

The generalized exponential function $E_k$ gives rise to an integral transform on $\mathbb{R}^N$, called the Dunkl transform. It was introduced in [D5] for non-negative multiplicity functions and further studied in [dJ1] for the more general case $\mathrm{Re}\, k\geq 0$. Since several results are known for non-negative multiplicity functions only, we shall throughout this section restrict to the case $k\geq 0$.

On suitable function spaces, the Dunkl transform establishes a natural correspondence between the action of multiplication operators on the one hand and the associated Dunkl operators on the other. Definition and essential properties of the Dunkl transform rely on suitable growth estimates for the kernel $E_k$. Such estimates were first derived in [D4]; the following sharper ones were proven in [dJ1] (in fact, for the larger range $\operatorname{Re} k \geq 0$).

**1.4.1 Proposition.** *For all $z, w \in \mathbb{C}^N$,*

$$|E_k(z,w)| \leq \sqrt{|G|} \cdot e^{\max_{g\in G} Re\langle gz,w\rangle}.$$

*In particular,* $|E_k(-ix,y)| \leq \sqrt{|G|}$ *for all $x, y \in \mathbb{R}^N$.*

*1.4.2 Remark.* We shall prove in Section 2.4 that for fixed $w \in \mathbb{C}^N$, the function $x \mapsto E_k(x,w)$ has a positive, Bochner-type integral representation (Corollary 2.4.3). This in particular implies that the factor $\sqrt{|G|}$ can be omitted, see Corollary 2.4.5.

At this point, we include two important reproducing properties of the Dunkl kernel; notice that the above estimates on $E_k$ assure the convergence of the integrals involved.

**1.4.3 Proposition.** *([D5]) Let $k \geq 0$. Then*

(1) $\displaystyle\int_{\mathbb{R}^N} e^{-\Delta_k/2}p\,(x)\, E_k(x,z)\, e^{-|x|^2/2}w_k(x)dx \;=\; c_k\, e^{\langle z,z\rangle/2}p(z) \quad (p \in \Pi^N,\ z \in \mathbb{C}^N).$

(2) $\displaystyle\int_{\mathbb{R}^N} E_k(x,z)\, E_k(x,w)\, e^{-|x|^2/2}w_k(x)dx \;=\; c_k\, e^{(\langle z,z\rangle + \langle w,w\rangle)/2}E_k(z,w) \quad (z,w \in \mathbb{C}^N).$

The Dunkl operators $T_\xi(k)$ can be considered as linear operators on $L^2(\mathbb{R}^N, w_k)$ with domain $\mathscr{S}(\mathbb{R}^N)$ or $C_c^\infty(\mathbb{R}^N)$; both spaces are dense in $L^2(\mathbb{R}^N, w_k)$ according to Lemma 4.5 of [dJ1]. It is of basic importance in context with the Dunkl transform that the $T_\xi(k)$ are anti-symmetric in $L^2(\mathbb{R}^N, w_k)$:

**1.4.4 Proposition.** *([D5]) Let $k \geq 0$. Then for every $f \in \mathscr{S}(\mathbb{R}^N)$ and $g \in C_b(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} T_\xi(k)f(x)\, g(x)\, w_k(x)dx \;=\; -\int_{\mathbb{R}^N} f(x)\, T_\xi(k)g(x)\, w_k(x)dx\,.$$

**1.4.5 Definition.** The Dunkl transform associated with $G$ and $k \geq 0$ is given by

$$\widehat{.}^{\,k} : L^1(\mathbb{R}^N, w_k) \to C_b(\mathbb{R}^N); \quad \widehat{f}^{\,k}(\xi) := c_k^{-1}\int_{\mathbb{R}^N} f(x)E_k(-i\xi,x)\, w_k(x)dx \;\; (\xi \in \mathbb{R}^N).$$

The inverse transform is defined by $f^{\vee k}(y) = \widehat{f}^{\,k}(-y)$.

The Dunkl transform has many properties analogous to the classical Fourier transform. The results listed in the following proposition are proven in [D5], [dJ1].

**1.4.6 Proposition.** (1) *The Dunkl transform $f \mapsto \widehat{f}^{\,k}$ is a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$.*

(2) $(T_j(k)f)^{\wedge k}(\xi) = i\xi_j \widehat{f}^{\,k}(\xi)$ *for all $f \in \mathscr{S}(\mathbb{R}^N)$ and $j = 1, \ldots, N$.*

(3) *(Lemma of Riemann-Lebesgue) $(L^1(\mathbb{R}^N, w_k))^{\wedge k}$ is a $\|.\|_\infty$-dense subspace of $C_0(\mathbb{R}^N)$.*

(4) *($L^1$-inversion) For all* $f \in L^1(\mathbb{R}^N, w_k)$ *with* $\widehat{f}^k \in L^1(\mathbb{R}^N, w_k)$,

$$f = (\widehat{f}^k)^{\vee k} \quad a.e..$$

(5) *(Plancherel theorem) The Dunkl transform has a unique extension to an isometric isomorphism of* $L^2(\mathbb{R}^N, w_k)$, *which is again denoted by* $f \mapsto \widehat{f}^k$.

The following fact is also completely analogous to the classical setting.

**1.4.7 Proposition.** *The linear operator* $\Delta_k$ *in* $L^2(\mathbb{R}^N, w_k)$, *with domain* $\mathcal{D}(\Delta_k) = \mathscr{S}(\mathbb{R}^N)$, *is essentially self-adjoint. Its closure is given by*

$$\overline{\Delta}_k f := -\big(|\xi|^2 \widehat{f}^k\big)^{\vee k},$$

*with the Sobolev-type domain*

$$\mathcal{D}(\overline{\Delta}_k) = \{f \in L^2(\mathbb{R}^N, w_k) : |\xi|^2 \widehat{f}^k(\xi) \in L^2(\mathbb{R}^N, w_k)\}.$$

*Proof.* By Proposition 1.4.4, $\Delta_k$ is symmetric in $L^2(\mathbb{R}^N, w_k)$. The rest is straightforward as in the classical case. □

We conclude this section by a special feature concerning the Dunkl transform of radial (i.e. $O(N,\mathbb{R})$-invariant) functions: if $f \in L^1(\mathbb{R}^N, w_k)$ is radial, then its Dunkl transform $\widehat{f}^k$ is again radial and given by a classical Hankel transform. This result is not obvious, as the weight $w_k$ is usually invariant under the reflection group $G$ only. Our proof is based on the explicit integration of the operator $V_k$ over spheres in [X2]. We first have to provide some notation and facts concerning Hankel transforms: For $\alpha \geq -1/2$, define the measure $\omega_\alpha$ on $[0,\infty)$ by

$$d\omega_\alpha(r) = (2^\alpha \Gamma(\alpha+1))^{-1} r^{2\alpha+1}\, dr.$$

The Hankel transform $\mathcal{H}^\alpha$ of order $\alpha$ on $L^1([0,\infty), \omega_\alpha)$ is then defined by

$$(\mathcal{H}^\alpha f)(\lambda) = \int_0^\infty f(r)\, j_\alpha(\lambda r)\, d\omega_\alpha(r);$$

Here $j_\alpha$ is the normalized spherical Bessel function as defined in (1.21). The transform $\mathcal{H}^\alpha$ can be uniquely extended to an isometric isomorphism on $L^2([0,\infty), \omega_\alpha)$.

**1.4.8 Proposition.** *There is a one-to-one correspondence between the space of all radial functions* $f$ *in* $L^1(\mathbb{R}^N, w_k)$ *and the space of all functions* $F \in L^1([0,\infty), \omega_{\gamma+N/2-1})$, *via*

$$f(x) = F(|x|) \qquad \text{for } x \in \mathbb{R}^N.$$

*Moreover, the Dunkl transform of* $f$ *is related to the Hankel transform* $\mathcal{H}^{\gamma+N/2-1}F$ *of* $F$ *by*

$$\widehat{f}^k(y) = (\mathcal{H}^{\gamma+N/2-1}F)(|y|) \qquad \text{for} \quad y \in \mathbb{R}^N.$$

*Proof.* The result is obvious in case $N=1, \gamma=0$; we may therefore assume that $\gamma+N/2-1 > -1/2$. Let $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere in $\mathbb{R}^N$ with normalized surface measure $d\sigma$. Put

$$d_k := \int_{S^{N-1}} w_k(x)\, d\sigma(x) = \frac{c_k}{2^{\gamma+N/2-1}\Gamma(\gamma+N/2)}.$$

Let $f$ and $F$ be related as in the proposition. Then the homogenity of $w_k$ leads to

$$\int_{\mathbb{R}^N} |f(x)|\, w_k(x)\, dx = \int_0^\infty \Big(\int_{S^{N-1}} w_k(ry)\, d\sigma(y)\Big)|F(r)|\, r^{N-1}\, dr = d_k \int_0^\infty |F(r)|\, r^{2\gamma+N-1}\, dr.$$

This yields the first statement. We now turn to the second assertion. Corollary 2.2 of [X2] states that for each polynomial $p$ in one variable and $x \in \mathbb{R}^N$,

$$\int_{S^{N-1}} V_k p\,(\langle x, .\,\rangle)(y)\, w_k(y)\, d\sigma(y) = d_k' \int_{-1}^1 p(t|x|)(1-t^2)^{\gamma+(N-3)/2}\, dt$$

with some constant $d_k' > 0$ depending on $k$ only. The homogeneous expansion (1.22) of $E_k$ and Mehler's integral representation for Bessel functions ([Sz], (1.71.6)) lead to

$$\int_{S^{N-1}} E_k(ix,y)\, w_k(y)\, d\sigma(y) = d_k' \int_{-1}^1 e^{it|x|}(1-t^2)^{\gamma+(N-3)/2}\, dt = d_k \cdot j_{\gamma+N/2-1}(|x|). \tag{1.26}$$

Moreover, by Prop. 1.11 and the homogeneity of $w_k$,

$$\begin{aligned}\widehat{f}^k(y) &= c_k^{-1} \cdot \int_{\mathbb{R}^N} F(|x|) E_k(-ix,y)\, w_k(x)\, dx \\ &= c_k^{-1} \cdot \int_0^\infty \Big(\int_{S^{N-1}} E_k(-iry,z) w_k(z)\, d\sigma(z)\Big) F(r)\, r^{2\gamma+N-1}\, dr.\end{aligned}$$

It follows from (1.26) that

$$\widehat{f}^k(y) = \frac{d_k}{c_k} \cdot \int_0^\infty j_{\gamma+N/2-1}(r|y|) F(r) r^{2\gamma+N-1}\, dr = (\mathcal{H}^{\gamma+N/2-1}F)(|y|),$$

which completes the proof. □

## 1.5 Generalized translations

If the multiplicity parameter is zero, the Dunkl transform coincides with the usual Fourier transform on the group $(\mathbb{R}^N, +)$, and the functions

$$\{x \mapsto E_0(-i\xi, x) = e^{\langle -i\xi, x\rangle};\ \xi \in \mathbb{R}^N\}$$

constitute the dual space of this group. It is an interesting question whether the Dunkl transform and the Dunkl kernels admit a similar interpretation for arbitrary parameters $k \geq 0$. At least in the one-dimensional case, there exists an algebraic structure on $\mathbb{R}$ which replaces the usual group addition. To describe this structure, we recall from Example 1.3.3 that for the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$ and a multiplicity parameter $k \geq 0$, the Dunkl kernel is given by

$$E_k^{\mathbb{Z}_2}(z,w) = e^{zw} \cdot {}_1F_1(k, 2k+1, -2zw) = j_{k-1/2}(izw) + \frac{zw}{2k+1}\, j_{k+1/2}(izw) \quad (z,w \in \mathbb{C}).$$

Moreover, the generalized Bessel function is

$$J_k^{\mathbb{Z}_2}(z,w) \,=\, j_{k-1/2}(izw).$$

It is well-known that the normalized spherical Bessel functions $j_{k-1/2}$ with $k>0$ satisfy the product formula

$$j_{k-1/2}(x)j_{k-1/2}(y) \,=\, \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)}\int_0^\pi j_{k-1/2}(\sqrt{x^2+y^2-2xy\cos\theta})\sin^{2k-1}\theta\,d\theta$$

for all $x,y\geq 0$, see e.g. Section 11.4 of [W]. This induces a commutative hypergroup structure (in the sense of Dunkl, Spector and Jewett) on $[0,\infty)$, with the convolution of point measures being defined by

$$\delta_x * \delta_y := \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)}\int_0^\pi \delta_{\sqrt{x^2+y^2-2xy\cos\theta}}\,\sin^{2k-1}\theta\,d\theta\,.$$

For an introduction to hypergroups, we refer to [B-H] and [Je]. The functions

$$\{x\mapsto\, j_{k-1/2}(\lambda x),\;\; \lambda\geq 0\}$$

form the dual space of this hypergroup, i.e. the space of bounded, multiplicative and symmetric functions on it.

It can be shown that the Dunkl kernels $E_k^{\mathbb{Z}_2}$ satisfy a similar product linearization which leads to a convolution structure on the whole real line, providing a natural extension of the usual group structure. This convolution was found and studied independently in [R3] and [Ros]. In contrast to a hypergroup convolution, it is not positivity-preserving for $k>0$, but endows $\mathbb{R}$ with the structure of a so-called signed hypergroup as introduced in [R1] (see also [R2]). More precisely, the following was proven in [R3]:

**1.5.1 Theorem.** *Let $k\geq 0$. Then there is a unique bilinear and separately $\sigma(M_b(\mathbb{R}),C_0(\mathbb{R}))$-continuous convolution $*_k$ on $M_b(\mathbb{R})$ such that the product of point measures satisfies*

$$E_k^{\mathbb{Z}_2}(\lambda,x)\,E_k^{\mathbb{Z}_2}(\lambda,y) \,=\, \int_{\mathbb{R}} E_k^{\mathbb{Z}_2}(\lambda,z)\,d(\delta_x *_k \delta_y)(z) \quad \textit{for } x,y\in\mathbb{R},\ \lambda\in\mathbb{C}.$$

*This convolution has the following properties:*

(1) *It is associative, commutative, and norm-continuous with*

$$(\mu *_k \nu)(\mathbb{R}) = \mu(\mathbb{R})\cdot\nu(\mathbb{R}) \quad \textit{and} \quad \|\mu *_k \nu\| \leq\, 4\cdot\|\mu\|\cdot\|\nu\| \quad \textit{for }\; \mu,\nu\in M_b(\mathbb{R}).$$

*Moreover, if $k>0$, then*

$$\textit{supp}(\delta_x *_k \delta_y) \,=\, \big[-|x|-|y|,-\big||x|-|y|\big|\big]\cup\big[\big||x|-|y|\big|,|x|+|y|\big] \quad \textit{for }\; x,y\neq 0.$$

(2) *$(M_b(\mathbb{R}),*_k)$ is a commutative Banach-$*$-algebra with unit $\delta_0$, involution $\mu\mapsto\mu^*$ (where $\mu^*(A) := \overline{\mu(-A)}$ for Borel sets $A\subseteq\mathbb{R}$), and with the norm $\|\mu\|' := \|L_\mu\|$, the operator $L_\mu$ on $M_b(\mathbb{R})$ being defined by $L_\mu(\nu) := \mu *_k \nu$.*

The convolution $*_k$ is given explicitly in [R3]. In case $k=0$, it coincides with the usual group convolution of $(\mathbb{R},+)$. The measure $w_k(x)dx = |x|^{2k}dx \in M^+(\mathbb{R})$ is $*_k$-invariant, i.e. for each $f \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} L_k^y f(x)\, w_k(x)dx = \int_{\mathbb{R}} f(x)\, w_k(x)dx\,, \quad \text{where } L_k^y f(x) := \int_{\mathbb{R}} f\, d\,(\delta_y *_k \delta_x).$$

The dual space of this signed hypergroup is given by

$$\{x \mapsto E_k^{\mathbb{Z}_2}(-i\xi, x);\ \xi \in \mathbb{R}\},$$

and the Fourier transform on it coincides with the Dunkl transform associated with $\mathbb{Z}_2$ on $\mathbb{R}$ and the parameter $k$.
In the higher dimensional Dunkl setting it is an open question whether there exists an analogous convolution structure on $\mathbb{R}^N$ which substitutes the standard group convolution and matches the action of the corresponding Dunkl transform as above. In some particular cases, namely for Weyl groups $G$ and certain discrete sets of non-negative multiplicity functions, the generalized Bessel functions $J_k(.,y)$ allow an interpretation as the spherical functions of a Cartan motion group; for details we refer to [O1] and [dJ3]. In these cases, they satisfy a positive product formula, which leads to a commutative hypergroup structure on $(\mathbb{R}^N)^G \cong \overline{W}$, where $W$ is one of the Weyl chambers of $G$. We conjecture that for all reflection groups and arbitrary non-negative multiplicity functions, the associated generalized Bessel functions satisfy a product formula which leads to a commutative hypergroup structure on $(\mathbb{R}^N)^G$, and that the dual space of this hypergroup consists of the functions

$$\{\, \varphi_{k,\lambda} : \overline{W} \to \mathbb{C},\ x \mapsto J_k(-i\lambda, x),\ \lambda \in \overline{W}\,\}.$$

We further conjecture that for all reflection groups and all multiplicities $k \geq 0$ there exists an associated signed hypergroup structure on $\mathbb{R}^N$ such that the functions

$$\{\, \psi_{k,\xi} : \mathbb{R}^N \to \mathbb{C},\ x \mapsto E_k(-i\xi, x),\ \xi \in \mathbb{R}^N\,\}$$

constitute its dual space. Although not having a signed hypergroup structure at our disposal, we may introduce the notion of a generalized translation in the $N$-dimensional Dunkl case at least for certain function spaces as the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N, w_k)$ . The definition is natural:

$$L_k^y f(x) := c_k^{-1} \int_{\mathbb{R}^N} \widehat{f}^{\,k}(\xi)\, E_k(ix,\xi)E_k(iy,\xi)\, w_k(\xi)d\xi \quad (x,\, y \in \mathbb{R}^N). \tag{1.27}$$

Notice that that for $k=0$, we just have $L_0^y f(x) = f(x+y)$. In the one-dimensional case, (1.27) coincides with the translation in the signed hypergroup $(\mathbb{R}, *_k)$, i.e. $L_k^y f(x) = (\delta_y *_k \delta_x)(f)$. We collect some properties of the generalized translation (1.27):

**1.5.2 Lemma.** *For all $f \in \mathscr{S}(\mathbb{R}^N)$ and $\xi,\, y \in \mathbb{R}^N$,*

(1) $L_k^y f(x) = L_k^x f(y)\,$;

(2) $L_k^0 f \,=\, f\,$;

(3) $L_k^y f \,\in \mathscr{S}(\mathbb{R}^N)$, *and* $(L_k^y f)^{\wedge k}(\xi) \,=\, E_k(iy,\xi)\,\widehat{f}^{\,k}(\xi)$;

(4) $\displaystyle\int_{\mathbb{R}^N} L_k^y f(x)\, w_k(x)dx \,=\, \int_{\mathbb{R}^N} f(x)\, w_k(x)dx\,$;

(5) $T_\xi(k) L_k^y f \,=\, L_k^y T_\xi(k) f\,$.

*Moreover, if* $f \in L^2(\mathbb{R}^N, w_k)$, *then*

(6) $L_k^0 f \,=\, f\,$;

(7) $L_k^y f \in L^2(\mathbb{R}^N, w_k)$ *for all* $y \in \mathbb{R}^N$, *and* $(L_k^y f)^{\wedge k} \,=\, E_k(iy,\,.\,)\,\widehat{f}^{\,k}\,$;

(8) $L_k^y f(x) = L_k^x f(y)$ *for almost all* $x, y \in \mathbb{R}^N$.

*Proof.* (1) is obvious, while (2) follows from the inversion theorem for the Dunkl transform. The first part of (3) results from the invariance of $\mathscr{S}(\mathbb{R}^N)$ under the Dunkl transform, together with the bounds of Proposition 1.4.1, the second one follows from the inversion theorem for the Dunkl transform. (4) is an immediate consequence of (3). Property (5) is obtained by applying the Dunkl transform and using (3) as well as Proposition 1.4.6(2):

$$\begin{aligned}
(T_\xi(k) L_k^y f)^{\wedge k}(\eta) &= i\,\langle \xi,\eta\rangle\,(L_k^y f)^{\wedge k}(\eta) \,=\, i\,\langle \xi,\eta\rangle\, E_k(iy,\eta)\widehat{f}^{\,k}(\eta) \\
&= E_k(iy,\eta)\,(T_\xi(k) f)^{\wedge k}(\eta) \,=\, (L_k^y T_\xi(k) f)^{\wedge k}(\eta).
\end{aligned}$$

Finally, properties (6) and (7) are clear from the Plancherel theorem for the Dunkl transform, and (8) follows from the definition. □

# Chapter 2

# Positivity of Dunkl's intertwining operator

This chapter is devoted to a further study of the intertwining operator $V_k$, where $k$ is a non-negative multiplicity function attached to a finite reflection group on $\mathbb{R}^N$. We prove that $V_k$ is positivity-preserving on polynomials and allows a positive, Bochner-type integral representation on certain algebras of analytic functions. This confirms Conjecture 1.2.3 and implies in particular that the generalized exponential kernel of the Dunkl transform is positive definite. The proof of our central result, positivity of $V_k$ on polynomials, affords several steps, the crucial one being a reduction from the $N$-dimensional to a one-dimensional problem. For this, we invoke semigroup techniques for linear operators on spaces of polynomials. The generators of the semigroups under consideration are certain differential-reflection operators whose common decisive property is that they are "degree-lowering". This setting is introduced in Section 2.1, together with a Hille-Yosida type theorem which characterizes positivity of such semigroups by means of their generator. In Section 2.2 we then prove positivity of $V_k$ on polynomials. Section 2.3 contains a discussion of certain algebras of homogeneous series and their spectral properties; parts of these results are the basis for the main theorem in Section 2.4, which establishes the announced positive integral representation of $V_k$.

## 2.1 Semigroups generated by degree-lowering operators on polynomials

In the following, $\Pi_+^N := \{p \in \Pi^N : p(x) \geq 0 \text{ for all } x \in \mathbb{R}^N\}$ denotes the cone of non-negative polynomials on $\mathbb{R}^N$.

**2.1.1 Definition.** A linear operator $A$ on $\Pi^N$ is called

(i) positive, if $Ap \in \Pi_+^N$ for each $p \in \Pi_+^N$.

(ii) degree-lowering, if $A\big(\Pi_n^N\big) \subseteq \Pi_{n-1}^N$ for all $n \in \mathbb{Z}_+$; here $\Pi_{-1}^N := \{0\}$.

Important examples of degree-lowering operators are linear operators on $\Pi^N$ which are homogeneous of some degree $-n$ with $n \geq 1$. This includes in particular usual partial derivatives and Dunkl operators, as well as products and linear combinations of those. If $A$ is degree-lowering on $\Pi^N$, then for every analytic function $f: \mathbb{R} \to \mathbb{C}$ with power series $f(x) = \sum_{k=0}^{\infty} c_k x^k$, there is a linear operator $f(A)$ on $\Pi^N$ defined by the terminating series

$$f(A)p(x) := \sum_{k=0}^{\infty} c_k A^k p(x).$$

Notice that $f(A)(\Pi_n^N) \subseteq \Pi_n^N$ for each $n \in \mathbb{Z}_+$. This yields a natural restriction of $f(A)$ to a linear operator on the finite-dimensional vector space $\Pi_n^N$. In particular, the well-known product and exponential formulas for linear operators on finite-dimensional vector spaces (see, e.g. §4.7 of [Ka2]) imply corresponding exponential formulas for degree-lowering operators on $\Pi^N$, where the topology may be chosen to be the one of pointwise convergence. We note two results of this type, which will be used later on:

**2.1.2 Lemma.** *Suppose that $A$ and $B$ are degree-lowering linear operators on $\Pi^N$. Then for all $p \in \Pi^N$ and $x \in \mathbb{R}^N$,*

(i) $e^A p(x) = \lim_{n\to\infty} \Big(I - \dfrac{A}{n}\Big)^{-n} p(x).$

(ii) $e^{A+B} p(x) = \lim_{n\to\infty} \big(e^{A/n} e^{B/n}\big)^n p(x)$ *(Trotter product formula).*

Each degree-lowering operator $A$ on $\Pi^N$ generates a semigroup $(e^{tA})_{t\geq 0}$ of linear operators on $\Pi^N$ and, in fact, on each of the $\Pi_n^N$. Conversely, it follows from general semigroup theory that any semigroup $(T(t))_{t\geq 0}$ of linear operators on $\Pi^N$ with $T(t)\big(\Pi_n^N\big) \subseteq \Pi_n^N$ for all $t \geq 0$, is of the form $T(t) = e^{tA}$, the generator $A \in \mathrm{End}_{\mathbb{C}}(\Pi^N)$ being uniquely determined by

$$Ap(x) = \lim_{t\downarrow 0} \frac{1}{t}\,(T(t) - 1)p(x) \quad \text{for all } p \in \Pi^N.$$

The following key-result characterizes positive semigroups generated by degree-lowering operators; it is an adaption of a well-known Hille-Yosida type characterization theorem for Feller-Markov semigroups on $C(K)$, $K$ a compact Hausdorff space (see, e.g. §II.4 of [G-S]):

**2.1.3 Theorem.** *Let $A$ be a degree-lowering linear operator on $\Pi^N$. Then the following assertions are equivalent:*

(1) *$e^{tA}$ is positive on $\Pi^N$ for all $t \geq 0$.*

(2) *$A$ satisfies the "positive minimum principle"*

(M) *For every $p \in \Pi_+^N$ and $x_0 \in \mathbb{R}^N$, $p(x_0) = 0$ implies $Ap(x_0) \geq 0$.*

*Proof.* (1) $\Rightarrow$ (2): Let $p \in \Pi_+^N$ with $p(x_0) = 0$. Then

$$Ap(x_0) = \lim_{t\downarrow 0} \frac{e^{tA}p(x_0) - p(x_0)}{t} = \lim_{t\downarrow 0} \frac{1}{t}\, e^{tA}p(x_0) \geq 0.$$

(2) $\Rightarrow$ (1): Notice first that for each $\lambda \neq 0$, the operator $\lambda I - A$ is bijective on $\Pi^N$. In fact, $\lambda I - A$ is injective on $\Pi^N$, because otherwise there would exist some $p \in \Pi^N$, $p \neq 0$, with $Ap = \lambda p$, in contradiction to the degree-lowering character of $A$. Since $(\lambda I - A)(\Pi_n^N) \subseteq \Pi_n^N$, this already proves bijectivity of $\lambda I - A$ on each $\Pi_n^N$, hence on $\Pi^N$ as well. We next claim that for every $\lambda > 0$ the resolvent operator $R(\lambda; A) := (\lambda I - A)^{-1}$ is positive on $\Pi^N$. For this, let $p \in \Pi_+^N$ and $q := R(\lambda; A)\,p$. If $p$ is constant, then $q = \frac{1}{\lambda}p \geq 0$. We may therefore assume that the total degree $n$ of $p$ (which must be even) is greater than $0$. Suppose first that $p(x) \geq c|x|^n$ for all $x \in \mathbb{R}^N$, with some constant $c > 0$. Since $A$ lowers the degree, we may write $q = \frac{1}{\lambda}p + r$ with a polynomial $r$ of total degree less than $n$. Hence $\lim_{|x|\to\infty} q(x) = \infty$, which shows that $q$ attains an absolute minimum, let us say at $x_0 \in \mathbb{R}^N$. Put $\widetilde{q}(x) := q(x) - q(x_0)$. Then $\widetilde{q} \in \Pi_+^N$ with $\widetilde{q}(x_0) = 0$, and property (M) assure that $Aq(x_0) = A\widetilde{q}(x_0) \geq 0$. For $\lambda > 0$ and $x \in \mathbb{R}^N$ we therefore obtain

$$\lambda q(x) \;\geq\; \lambda q(x_0) \;=\; (\lambda I - A)q(x_0) + Aq(x_0) \;\geq\; p(x_0) \;\geq 0.$$

If $p \in \Pi_+^N$ is arbitrary, then consider the polynomials $p_\epsilon(x) := p(x) + \epsilon\,|x|^n$ for $\epsilon > 0$, where $n$ is the degree of $p$. As $A$ is degree-lowering, and by the above result, we obtain

$$R(\lambda; A)\,p\,(x) \;=\; \lim_{\epsilon\to 0} R(\lambda; A)\,p_\epsilon(x) \;\geq 0 \quad \text{for all } x \in \mathbb{R}^N.$$

This proves the stated positivity of $R(\lambda; A)$ for $\lambda > 0$. Now let $p \in \Pi_+^N$ and $t > 0$. Then according to Lemma 2.2.(i),

$$e^{tA}p\,(x) \;=\; \lim_{n\to\infty}\Big(I - \frac{tA}{n}\Big)^{-n} p\,(x) \;=\; \lim_{n\to\infty}\Big(\frac{n}{t}\,R\big(\frac{n}{t}; A\big)\Big)^{n} p\,(x) \;\geq 0$$

for all $x \in \mathbb{R}^N$. This finishes the proof. □

## 2.2 Positivity of $V_k$ on polynomials

Throughout this section, $G$ is a finite reflection group on $\mathbb{R}^N$ and $k \geq 0$ is a non-negative multiplicity function on its root system $R$. We have

**2.2.1 Theorem.** *The intertwining operator $V_k$ is positive on $\Pi^N$.*

As the proof of this result affords several reductions, we start with a general outline: In a first step, the statement will be reduced to an equivalent one, which involves exponentials of Laplacians:

**2.2.2 Proposition.** *The following statements are equivalent:*

(i) *$V_k$ is positive on $\Pi^N$.*

(ii) *The operator $e^{-\Delta/2}e^{\Delta_k/2}$ is positive on $\Pi^N$.*

We are thus led to prove positivity of $e^{-\Delta/2}e^{\Delta_k/2}$ on polynomials. For this, we first consider the (one-dimensional) operators

$$\Lambda_s := e^{-sD^2}\delta\, e^{sD^2}, \quad s \geq 0$$

on $\Pi^1$. Here $D$ denotes the usual first derivative, i.e. $Dp(x) = p'(x)$ for $x \in \mathbb{R}$, and $\delta$ is the linear operator on $\Pi^1$ given by

$$\delta p(x) := \frac{p'(x)}{x} - \frac{p(x)-p(-x)}{2x^2} = \frac{1}{2}\int_{-1}^{1}\big(D^2p\big)(tx)(1+t)\,dt. \tag{2.1}$$

This operator is related to the Dunkl operator $T(k)$ attached to the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ and the multiplicity parameter $k \geq 0$ by

$$T(k)^2 = D^2 + 2k\delta.$$

Since both $D^2$ and $\delta$ are homogeneous of degree $-2$ on $\Pi^1$, the operators $\Lambda_s$ are well-defined and degree-lowering on $\Pi^1$. We shall prove that they have the following decisive property:

**2.2.3 Proposition.** *The operators $\Lambda_s$, $s \geq 0$, satisfy the positive minimum principle* (M) *on* $\Pi^1$.

We next turn to the general $N$-dimensional setting: Here $G$ is an arbitrary finite reflection group on $\mathbb{R}^N$ with multiplicity function $k \geq 0$. We consider the generalized Laplacian $\Delta_k$ associated with $G$ and $k$, which is homogeneous of degree $-2$ on $\Pi^N$. With the notation introduced in (1.7), it can be written as

$$\Delta_k = \Delta + L_k, \quad \text{where } L_k = 2\sum_{\alpha\in R_+} k(\alpha)\delta_\alpha\,. \tag{2.2}$$

We shall derive the following multivariable extension of the previous result:

**2.2.4 Proposition.** *The operators $e^{-s\Delta}L_k e^{s\Delta}$ $(s \geq 0)$, satisfy the positive minimum principle* (M) *on* $\Pi^N$.

The characterization of Theorem 2.1.3 is now the key to the following corollary, whose second part finally implies the assertion of Theorem 2.2.1:

**2.2.5 Corollary.** (i) *The operators $e^{-s\Delta}e^{t\,L_k}e^{s\Delta}$ $(s,t \geq 0)$ are positive on* $\Pi^N$.

(ii) *The operator $e^{-\Delta/2}e^{\Delta_k/2}$ is positive on* $\Pi^N$.

*Proof.* For fixed $s \geq 0$, the operators $\big(e^{-s\Delta}e^{t\,L_k}e^{s\Delta}\big)_{t\geq 0}$ form a semigroup on $\Pi^N$ with generator $e^{-s\Delta}L_k\, e^{s\Delta}$. Hence (i) follows from the above proposition, together with Theorem 2.1.3. In order to prove (ii), we apply Trotter's product formula of Lemma 2.1.2. We can write

$$\begin{aligned} e^{-\Delta/2}e^{\Delta_k/2}\,p\,(x) &= e^{-\Delta/2}e^{\Delta/2+L_k/2}\,p\,(x) = \lim_{n\to\infty} e^{-\Delta/2}\Big(e^{\Delta/2n}\,e^{L_k/2n}\Big)^n p\,(x) \\ &= \lim_{n\to\infty}\prod_{j=1}^{n}\Big(e^{-(1-j/n)\cdot\Delta/2}\,e^{L_k/2n}\,e^{(1-j/n)\cdot\Delta/2}\Big)p\,(x) \qquad (p \in \Pi^N,\ x \in \mathbb{R}^N). \end{aligned}$$

By Part (i), each of the $n$ factors in the above product is a positive operator on $\Pi^N$. Hence $e^{-\Delta/2}e^{\Delta_k/2}$ is also positive on $\Pi^N$. □

We now turn to the proof of Proposition 2.2.2. We need the following positivity criterion for polynomials:

**2.2.6 Lemma.** *Let $\alpha > 0$ and suppose that $h \in C_b(\mathbb{R}^N)$ satisfies*

$$\int_{\mathbb{R}^N} h(x)\, p(x)\, e^{-\alpha|x|^2}\, w_k(x)dx \,\geq 0 \qquad \text{for all } p \in \Pi_+^N. \tag{2.3}$$

*Then $h(x) \geq 0$ for all $x \in \mathbb{R}^N$.*

*Proof.* For abbreviation, put

$$dm_k(x) := e^{-\alpha|x|^2}\, w_k(x)dx \,\in M_b^+(\mathbb{R}^N).$$

*Step 1.* We shall use the fact that $\Pi^N$ is dense in $L^2(\mathbb{R}^N, dm_k)$. This is proved (for $\alpha = 1/2$) in Theorem 2.5 of [D5] by refering to a well-known theorem of Hamburger for one-dimensional distributions, but it can also be seen directly as follows: Suppose on the contrary that $\Pi^N$ is not dense in $L^2(\mathbb{R}^N, dm_k)$. Then there exists some $f \in L^2(\mathbb{R}^N, dm_k)$, $f \neq 0$, with $\int_{\mathbb{R}^N} fp\, dm_k = 0$ for all $p \in \Pi^N$. Now consider the measure $\nu := fm_k \in M_b(\mathbb{R}^N)$ and its (classical) Fourier-Stieltjes transform

$$\widehat{\nu}(\lambda) \,=\, \int_{\mathbb{R}^N} e^{-i\,\langle\lambda,x\rangle}\, d\nu(x) \,=\, \int_{\mathbb{R}^N} f(x)\, e^{-i\,\langle\lambda,x\rangle}\, dm_k(x).$$

Since $x \mapsto e^{|\lambda||x|}$ belongs to $L^2(\mathbb{R}^N, dm_k)$ for all $\lambda \in \mathbb{R}^N$, the dominated convergence theorem yields

$$\widehat{\nu}(\lambda) \,=\, \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^N} f(x)\, \langle\lambda, x\rangle^n\, dm_k(x) \,=\, 0.$$

By injectivity of the Fourier-Stieltjes transform on $M_b(\mathbb{R}^N)$, it follows that $\nu = 0$ and hence $f = 0$ a.e., a contradiction.
*Step 2.* Now assume that $h \in C_b(\mathbb{R}^N)$ satisfies (2.3). In order to prove $h \geq 0$, it suffices to check that

$$\int_{\mathbb{R}^N} fh\, dm_k \,\geq 0 \qquad \text{for all } f \in C_c^+(\mathbb{R}^N). \tag{2.4}$$

For this, let $f \in C_c^+(\mathbb{R}^N)$ and $\epsilon > 0$. By density of $\Pi^N$ in $L^2(\mathbb{R}^N, dm_k)$ there exists some $p = p_\epsilon \in \Pi^N$ with $\|\sqrt{f} - p\,\|_{2,m_k} < \epsilon$. With $M := \|h\|_{\infty,\mathbb{R}^N}$ it follows that

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} fh\, dm_k - \int_{\mathbb{R}^N} p^2 h\, dm_k\Big| \,&\leq\, M \int_{\mathbb{R}^N} |f - p^2|\, dm_k \\ &\leq M \cdot \|\sqrt{f} - p\,\|_{2,m_k}\, \|\sqrt{f} + p\,\|_{2,m_k} \,\leq\, M\epsilon \cdot \big(2\,\|\sqrt{f}\,\|_{2,m_k} + \epsilon\big), \end{aligned}$$

which tends to $0$ as $\epsilon \to 0$. This proves (2.4) and yields the assertion. □

The proof of Proposition 2.2.2 is now easily accomplished:

**Proof of Proposition 2.2.2.** Combining the Macdonald-type identity (1.17) with part (5) of Lemma 1.2.4, we obtain for all $p, q \in \Pi^N$ the identity

$$c_k^{-1} \int_{\mathbb{R}^N} e^{-\Delta_k/2}(V_k\, p)(x)\, e^{-\Delta_k/2} q(x)\, e^{-|x|^2/2}\, w_k(x) dx \,=\, c_0^{-1} \int_{\mathbb{R}^N} e^{-\Delta/2} p(x)\, e^{-\Delta/2} q(x)\, e^{-|x|^2/2} dx.$$

Since $e^{-\Delta_k/2}(V_k\, p) \,=\, V_k\big(e^{-\Delta/2} p\big)$ , and since we may also replace $p$ by $e^{\Delta/2} p$ and $q$ by $e^{\Delta_k/2} q$ in the above identity, it follows that for all $p, q \in \Pi^N$

$$c_k^{-1} \int_{\mathbb{R}^N} V_k p\,(x)\, q(x)\, e^{-|x|^2/2}\, w_k(x) dx \,=\, c_0^{-1} \int_{\mathbb{R}^N} p(x)\, e^{-\Delta/2} e^{\Delta_k/2} q\,(x)\, e^{-|x|^2/2} dx. \tag{2.5}$$

Now suppose that (i) is satisfied, i.e. $V_k$ is positive on $\Pi^N$. Then the left-hand side of (2.5) is non-negative for all $p, q \in \Pi_+^N$. Now fix $p \in \Pi_+^N$ and apply Lemma 2.2.6 with, let us say, $\alpha = 1/4$, to the function $h(x) := e^{-|x|^2/4}\, e^{-\Delta/2} e^{\Delta_k/2} q(x) \in C_b(\mathbb{R}^N)$. This yields (ii). The converse direction is obtained by the same argument. □

We next turn to the proof of Proposition 2.2.3. We start with two elementary auxiliary results:

**2.2.7 Lemma.** *For each $p \in \Pi^1$ and $c \in \mathbb{R}$,*

$$e^{cD^2}(xp(x)) \,=\, x\, e^{cD^2} p\,(x) + 2c\, e^{cD^2} p'(x).$$

*Proof.* Power series expansion of $e^{cD^2}$ yields

$$\begin{aligned} e^{cD^2}(xp(x)) \,&=\, \sum_{n=0}^{\infty} \frac{c^n}{n!}\, D^{2n}(xp(x)) \,=\, xp(x) + \sum_{n=1}^{\infty} \frac{c^n}{n!}\big(x\, D^{2n} p\,(x) + 2n D^{2n-1} p(x)\big) \\ &= x\, e^{cD^2} p\,(x) + \, 2c \sum_{n=1}^{\infty} \frac{c^{n-1}}{(n-1)!}\, D^{2n-1} p\,(x) \,=\, x\, e^{cD^2} p\,(x) + \, 2c\, e^{cD^2} p'(x). \end{aligned}$$

□

**2.2.8 Lemma.** *Let $p \in \Pi^1_{2n+1}$, $n \in \mathbb{Z}_+$, be an odd polynomial. Then the differential equation*

$$c\, y' - xy \,=\, p \qquad (c > 0) \tag{2.6}$$

*has exactly one polynomial solution (which belongs to $\Pi^1_{2n}$), namely*

$$y_p(x) \,=\, \frac{1}{c}\, e^{x^2/2c} \int_{-\infty}^{x} e^{-t^2/2c}\, p(t)\, dt.$$

*Proof.* The general solution of (2.6) is

$$y(x) \,=\, a\, e^{x^2/2c} + \frac{1}{c}\, e^{x^2/2c} \int_{-\infty}^{x} e^{-t^2/2c}\, p(t)\, dt, \quad a \in \mathbb{R}.$$

It therefore remains to prove that

$$x \mapsto e^{x^2/2c} \int_{-\infty}^{x} e^{-t^2/2c}\, p(t)\, dt$$

is a polynomial. We use induction by $n$: For $n=0$, the statement is obvious. For $n\geq 1$, write $p(x) = -c^{-1}xr(x)$ with $r\in \Pi^1_{2n}$. Partial integration then yields

$$\int_{-\infty}^x e^{-t^2/2c}\,p(t)\,dt \,=\, -\frac{1}{c}\int_{-\infty}^x te^{-t^2/2c}\,r(t)\,dt \,=\, e^{-x^2/2c}\,r(x) \,-\, \int_{-\infty}^x e^{-t^2/2c}\,r'(t)\,dt.$$

By our induction hypothesis, this equals $e^{-x^2/2c}(r(x)-\widetilde{r}(x))$ with some polynomial $\widetilde{r}\in \Pi^N_{2n-2}$. This finishes the proof. $\square$

**Proof of Proposition 2.2.3.** The proof will be divided into several steps.
*Step 1.* The case $s=0$ is easy and may be treated separately: Let $p\in \Pi^1_+$ with $p(x_0)=0$. Then $p'(x_0)=0$ and $p''(x_0)\geq 0$. Thus if $x_0\neq 0$, then $\delta p\,(x_0) = p(-x_0)/(2x_0^2)\geq 0$. In case $x_0 = 0$, it is seen from the integral representation (2.1) that $\delta p\,(0) = p''(0)\geq 0$.
From now on, we may therefore assume that $s>0$.
*Step 2.* We first derive an explicit representation of the operator $\Lambda_s$ $(s>0)$, which allows to check property (M) easily: We claim that

$$\begin{aligned}\Lambda_s p\,(x) \,&=\, -\frac{1}{2s}p(x) \,-\, \frac{1}{8s^2}\,e^{x^2/4s}\Big(\int_{-\infty}^x g_{p,\,x}(t)\,dt \,-\, \int_{-x}^\infty g_{p,\,x}(t)\,dt\Big) \quad \text{for } p\in \Pi^1,\\ &\text{with}\quad g_{p,\,x}(t) \,=\, e^{-t^2/4s}\,(t+x)\,p(t).\end{aligned} \tag{2.7}$$

This may of course be verified by a (tedious) direct computation of $\Lambda_s(x^k)$, $k\in\mathbb{Z}_+$, and an explicit evaluation of the corresponding integrals on the right side by series expansions of the involved exponentials. We prefer, however, to give a more instructive proof:

Note first that the operators $D^2$ and $\delta$ map even polynomials to even ones and odd polynomials to odd ones, and that

$$\delta p\,(x) \,=\, \begin{cases}\dfrac{1}{x}p'(x) & \text{if } p \text{ is even,}\\ \Big(\dfrac{1}{x}\,p(x)\Big)' & \text{if } p \text{ is odd.}\end{cases}$$

Now fix $s>0$ and suppose that $p\in\Pi^1$ is even. Then the polynomials $e^{sD^2}p$ and $q:=\Lambda_s p$ are also even, and we obtain the following equivalences:

$$q = \Lambda_s p \iff \delta\big(e^{sD^2}p\big) \,=\, e^{sD^2}q \iff p'(x) \,=\, e^{-sD^2}\big(x\,e^{sD^2}q\big)(x).$$

By use of Lemma 2.2.7, this becomes

$$p'(x) \,=\, xq(x) \,-\, 2sq'(x),$$

which is a differential equation of type (2.6) for $q$. Lemma 2.2.8, together with a further partial integration, now implies that

$$\begin{aligned}\Lambda_s p\,(x) \,&=\, -\,\frac{1}{2s}\,e^{x^2/4s}\int_{-\infty}^x e^{-t^2/4s}\,p'(t)\,dt\\ &=\, -\,\frac{1}{2s}\,p(x) \,-\, \frac{1}{4s^2}\,e^{x^2/4s}\int_{-\infty}^x e^{-t^2/4s}\,t\,p(t)\,dt \qquad (p \text{ even}).\end{aligned} \tag{2.8}$$

In a similar way, we calculate $q = \Lambda_s p$ for odd $p \in \Pi^1$: In this case, $e^{sD^2}p$ and $q = \Lambda_s p$ are odd as well, and we have the equivalence

$$q = \Lambda_s p \iff \frac{d}{dx}\Big(\frac{1}{x}\, e^{sD^2}p\,(x)\Big) \;=\; e^{sD^2}q\,(x).$$

Hence there exists a constant $c_1 \in \mathbb{R}$ such that

$$e^{sD^2}p\,(x) \;=\; x(c_1 + h(x)), \qquad \text{with} \;\; h(x) \;=\; \int_0^x e^{sD^2}q\,(t)\,dt.$$

Applying Lemma 3.4 again, we obtain

$$p(x) \;=\; c_1 e^{-sD^2}(x) + xe^{-sD^2}h\,(x) - 2s\,e^{-sD^2}h'(x) \;=\; c_1 x + x\,e^{-sD^2}h\,(x) - 2sq(x). \tag{2.9}$$

In order to determine $e^{-sD^2}h$, note that

$$\frac{d}{dx}\big(e^{-sD^2}h\,(x)\big) \;=\; e^{-sD^2}h'(x) \;=\; q(x).$$

Consequently, there exists a constant $c_2 \in \mathbb{R}$ such that

$$e^{-sD^2}h\,(x) \;=\; c_2 + \int_0^x q(t)\,dt. \tag{2.10}$$

Now write $p(x) = x\,P(x)$ and $q(x) = x\,Q(x)$ with even $P,\,Q \in \Pi^1$. Then by (2.9) and (2.10),

$$P(x) \;=\; c_1 + c_2 + \int_0^x t\,Q(t)\,dt \;-\; 2s\,Q(x),$$

and therefore

$$P'(x) \;=\; x\,Q(x) - 2s\,Q'(x).$$

This is exactly the same differential equation as we had in the even case before, and the transfer of (2.8) gives

$$\Lambda_s p\,(x) \;=\; -\frac{1}{2s}\,p(x) \;-\; \frac{1}{4s^2}\,e^{x^2/4s}\,x\int_{-\infty}^x e^{-t^2/4s}p(t)\,dt \qquad (p \text{ odd}). \tag{2.11}$$

If finally $p \in \Pi^1$ is arbitrary, then write $p = p_e + p_o$ with even part $p_e(x) = (p(x) + p(-x))/2$ and odd part $p_o(x) = (p(x) - p(-x))/2$. The combination of (2.8) for $p_e$ with (2.11) for $p_o$ then leads to

$$\Lambda_s p\,(x) \;=\; -\frac{1}{2s}\,p(x) \;-\; \frac{1}{4s^2}\,e^{x^2/4s}\int_{-\infty}^x e^{-t^2/4s}\Big(\frac{t+x}{2}\,p(t) \;+\; \frac{t-x}{2}\,p(-t)\Big)dt,$$

and an easy reformulation yields the stated representation (2.7).

*Step 3.* In order to prove that $\Lambda_s$ satisfies the positive minimum principle (M), define

$$F_p(x) := \int_{-\infty}^x g_{p,\,x}(t)\,dt \;-\; \int_{-x}^\infty g_{p,\,x}(t)\,dt, \quad \text{for } p \in \Pi^1 \text{ and } x \in \mathbb{R}\,.$$

Now let $p \in \Pi^1_+$ with $p(x_0) = 0$. Then in view of (2.7),

$$\Lambda_s p\,(x_0) \;=\; -\frac{1}{8s^2}\,e^{x_0^2/4s}\,F_p(x_0),$$

and it remains to check that $F_p(x_0) \leq 0$. For this, we rewrite $F_p$ as

$$F_p(x) \,=\, \int_{-\infty}^{-|x|} g_{p,\,x}(t)\,dt \,-\, \int_{|x|}^{\infty} g_{p,\,x}(t)\,dt.$$

Since $p$ is non-negative, the sign of $g_{p,\,x}(t)$ coincides with the sign of $x+t$ for all $x,t \in \mathbb{R}$. This shows that in fact, $F_p(x) \leq 0$ for all $x \in \mathbb{R}$, which completes the proof. $\square$

We come to the final step in the proof of Theorem 2.2.1.

**Proof of Proposition 2.2.4.** Since $L_k \,=\, 2\sum_{\alpha\in R_+} k(\alpha)\delta_\alpha$ with $k(\alpha) \geq 0$ for all $\alpha$, it is enough to make sure that each of the operators

$$\rho_\alpha^s := \, e^{-s\Delta}\,\delta_\alpha\, e^{s\Delta} \quad (\alpha \in R_+)$$

satisfies the positive minimum principle (M). (Here the assumption $k \geq 0$ is crucial!) Now fix $\alpha \in R_+$. A short calculation shows that $\delta_\alpha$ and hence also $\rho_\alpha^s$ is equivariant under orthogonal transformations, i.e.

$$g \circ \rho_\alpha^s \circ g^{-1} \,=\, \rho_{g\alpha}^s \quad \text{for } g \in O(N,\mathbb{R}).$$

Moreover, it is easily checked that $\rho_\alpha^s$ satisfies (M) if and only if $g \circ \rho_\alpha^s \circ g^{-1}$ does so for some (and hence all) $g \in O(N,\mathbb{R})$. We may therefore assume that $\alpha \,=\, \sqrt{2}e_1 \,=\, (\sqrt{2}, 0, \ldots, 0)$. As $\delta_{\sqrt{2}e_1}$ obviously commutes with each of the partial derivatives $\partial_2\,, \ldots, \partial_N$ on $\mathbb{R}^N$, we obtain

$$\rho_{\sqrt{2}e_1}^s \,=\, e^{-s\partial_1^2}\,\delta_{\sqrt{2}e_1}\, e^{s\partial_1^2}\,.$$

But this operator acts in the first variable only, namely via $\Lambda_s$:

$$\rho_{\sqrt{2}e_1}^s\, p\,(x_1\,, \ldots, x_N) \,=\, \Lambda_s\, p_{x_2\,\ldots,\,x_N}\,(x_1), \;\text{ where }\; p_{x_2\,,\ldots,\,x_N}\,(x_1) := p(x_1, x_2, \ldots, x_N), \quad p \in \Pi^N.$$

Proposition 2.2.3 now yields the assertion. $\square$

We conclude this section by an immediate application of Theorem 2.2.1.

**2.2.9 Summability of orthogonal series in generalized harmonics.** The study of generalized spherical harmonics associated with a finite reflection group and a multiplicity function $k \geq 0$ was one of the starting points of Dunkl's theory in [D4] and has been extended in [X2] and [X3]. Many results for classical spherical harmonics carry over to these spherical $k$-harmonics, where harmonicity is now meant with respect to $\Delta_k$. In particular, there is a natural decomposition of $\mathcal{P}_n^N|_{S^{N-1}}$ into subspaces of $k$-spherical harmonics, which are orthogonal in $L^2(S^{N-1}, w_k)$. In [X2], Cesàro summability of generalized Fourier expansions with respect to an orthonormal basis of spherical $k$-harmonics is studied. Recall that a sequence $\{s_n\}_{n\in\mathbb{Z}_+}$ is called Cesàro summable of order $\delta$ to $s$, for short, $C_\delta$-summable to $s$, if

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k \,\longrightarrow\, s \quad \text{with } n \to \infty.$$

The following result is proven in [X2] under the requirement that the intertwining operator $V_k$ is positive on $\Pi^N$; Theorem 2.2.1 now assures its validity for all $k \geq 0$:

**2.2.10 Theorem.** *Let $f : S^{N-1} \to \mathbb{C}$ be continuous, and let $\{s_n\}$ denote the sequence of partial sums in the expansion of $f$ as a Fourier series with respect to a fixed orthonormal basis of spherical $k$-harmonics. Then $\{s_n\}$ is uniformly $C_\delta$-summable over $S^{N-1}$ to $f$, provided that $\delta > \gamma + N/2 - 1$ with $\gamma = \sum_{\alpha \in R_+} k(\alpha)$.*

## 2.3 An algebra of homogeneous series

More information about $V_k$ will be obtained by its extension to Dunkl's algebra

$$A = \Big\{f : K \to \mathbb{C},\ f = \sum_{n=0}^{\infty} f_n \quad \text{with } f_n \in \mathcal{P}_n^N \text{ and } \|f\|_A := \sum_{n=0}^{\infty} \|f_n\|_{\infty,K} < \infty\Big\},$$

as introduced in Section 1.2. In the present section, we will discuss this algebra in some detail. First of all, we note that for $f \in A$, the homogeneous expansion $f = \sum_{n=0}^{\infty} f_n$ is unique. For this, suppose that $f \in A$ vanishes identically on the ball $K = \{x \in \mathbb{R}^N : |x| \leq 1\}$. For any fixed $x \in K$ and $-1 < \lambda < 1$ we then have $0 = \sum_{n=0}^{\infty} f_n(x)\lambda^n$, which is a power series in $\lambda$. Hence $f_n(x) = 0$ for all $n$, which implies the stated uniqueness. Notice further that the topology of $A$ is stronger than the topology induced by the uniform norm on $K$, and that $A$ is not closed with respect to $\|.\|_{\infty,K}$, in view of the Stone-Weierstraß theorem. In particular, $A$ is not a uniform algebra on $K$. The following result is straightforward:

**2.3.1 Lemma.** $(A, \|.\|_A)$ *is a commutative Banach-$*$-algebra with the pointwise multiplication of functions, complex conjugation as involution, and with unit* $1$.

*Proof.* To show completeness, let $(f^m)_{m\in\mathbb{Z}_+}$ be a Cauchy sequence in $A$. Then for $\epsilon > 0$ there exists an index $m(\epsilon) \in \mathbb{Z}_+$ such that

$$\sum_{n=0}^{\infty} \|f_n^m - f_n^{m'}\|_{\infty,K} < \epsilon \quad \text{for } m, m' > m(\epsilon). \tag{2.12}$$

In particular, for each degree $n$ the homogeneous parts $(f_n^m)_{m\in\mathbb{Z}_+}$ converge uniformly on $K$, and hence within $\mathcal{P}_n^N$ to some $g_n \in \mathcal{P}_n^N$. It further follows from (2.12) that

$$\sum_{n=0}^{\infty} \|g_n - f_n^m\|_{\infty,K} < \epsilon \quad \text{for } m > m(\epsilon).$$

Therefore $g := \sum_{n=0}^{\infty} g_n$ belongs to $A$ with $\|g - f^m\|_A \to 0$ for $m \to \infty$. It is also easily checked by a Cauchy-product argument that $A$ is an algebra with $\|fg\|_A \leq \|f\|_A \cdot \|g\|_A$ for all $f, g \in A$. The rest is obvious. □

It is now in particular clear that the extension of $V_k$ according to (1.16) is a well-defined, continuous linear operator on $(A, \|.\|_A)$. Our main theorem in the following section will be based on a Bochner-theorem for positive functionals on commutative Banach-*-algebras; for this, we need the symmetric spectrum of $A$, i.e. the subspace of the spectrum $\Delta(A)$ given by

$$\Delta_S(A) := \{\varphi \in \Delta(A) : \varphi(\overline{f}) = \overline{\varphi(f)} \quad \text{for all } f \in A\}.$$

As usual, $\Delta(A)$ and $\Delta_S(A)$ are equipped with the Gelfand-topology. While the determination of the complete spectrum requires some further knowledge about $A$ and will be carried out later, its symmetric part is obtained quite easily: For $x \in K$, define the evaluation homomorphism at $x$ by $\varphi_x : A \to \mathbb{C}$, $\varphi_x(f) := f(x)$.

**2.3.2 Lemma.** *$\Delta_S(A) = \{\varphi_x : x \in K\}$, and the mapping $x \mapsto \varphi_x$ is a homeomorphism from $K$ onto $\Delta_S(A)$.*

*Proof.* It is obvious that $\varphi_x$ belongs to $\Delta_S(A)$ for each $x \in K$, and that the mapping $x \mapsto \varphi_x$ is injective and continuous on $K$. As $K$ and $\Delta_S(A)$ are compact Hausdorff spaces, it remains to show that $x \mapsto \varphi_x$ is surjective, i.e. each $\varphi \in \Delta_S(A)$ is of the form $\varphi_x$ with some $x \in K$. To this end, put $\lambda_i := \varphi(x_i)$ for $i = 1, \ldots, N$. By symmetry of $\varphi$ we have $\lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$. Moreover,

$$|\lambda|^2 = \varphi(|x|^2) \leq \| \, |x|^2 \|_A = 1 .$$

This shows that $\lambda \in K$. By definition of $\lambda$, the identity $p(\lambda) = \varphi(p)$ holds for all polynomials $p \in \Pi^N$. The assertion now follows from the density of $\Pi^N$ in $(A, \|.\|_A)$. □

To achieve a more comprehensive knowledge of $A$ and its spectrum, we have to extend functions from $A$ to complex arguments. Besides the real unit ball $K \subset \mathbb{R}^N$, we introduce the complex unit ball

$$U := \{z \in \mathbb{C}^N : |z| \leq 1\}.$$

We shall see that each function $f \in A$ has a unique continuation to the well-known ball algebra

$$\mathcal{B} := \{g : U \to \mathbb{C}, g \text{ is continuous on } U \text{ and holomorphic in its interior } U^\circ\}.$$

The following observation will be crucial:

**2.3.3 Lemma.** *For each homogeneous polynomial $p \in \mathbb{C}[\mathbb{C}^N]$,*

$$\|p\|_{\infty,K} = \|p\|_{\infty,U} .$$

*Proof.* As $p$ is homogeneous, there exists some $z_0 \in U$ with $|z_0| = 1$ such that $\|p\|_{\infty,U} = |p(z_0)|$. Write $z_0 = x_0 + iy_0$ with $x_0, y_0 \in \mathbb{R}^N$, and choose matrices $M_1, M_2 \in O(N, \mathbb{R})$ such that

$$M_1 x_0 = \frac{|x_0|}{\sqrt{N}} \cdot (1, \ldots, 1), \quad M_2 y_0 = \frac{|y_0|}{\sqrt{N}} \cdot (1, \ldots, 1).$$

Then $L(x + iy) := M_1 x + iM_2 y$ is bijective and $\mathbb{R}$-linear on $\mathbb{C}^N$ with $|L(z)| = |z|$ for all $z \in \mathbb{C}^N$. Moreover, $L$ maps $K$ onto itself and satisfies $L(z_0) = \frac{\lambda}{\sqrt{N}}(1, \ldots, 1)$ with $\lambda = |x_0| + i|y_0|$. Notice that $|\lambda| = 1$, because $|z_0| = 1$. Now define $p_L \in \mathbb{C}[\mathbb{C}^N]$ by $p_L(z) := p(L^{-1}z)$. Then $p_L$ is again homogeneous, and

$$\|p\|_{\infty,U} = \|p_L\|_{\infty,U} = \left|p_L\left(\frac{\lambda}{\sqrt{N}}, \ldots, \frac{\lambda}{\sqrt{N}}\right)\right| = \left|p_L\left(\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}\right)\right| \leq \|p_L\|_{\infty,K} = \|p\|_{\infty,K} .$$

This yields the assertion. □

**2.3.4 Proposition.** *Each $f \in A$ has a unique extension $\widetilde{f} \in \mathcal{B}$. For $f = \sum_{n=0}^{\infty} f_n \in A$, this extension is given by $\widetilde{f}(z) := \sum_{n=0}^{\infty} f_n(z)$.*

*2.3.5 Remark.* The inclusion $A \hookrightarrow \mathcal{B}, f \mapsto \widetilde{f}$, is not surjective. To see this, consider first the case $N=1$. Let $F$ be a continuous function on the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, whose Fourier series does not converge absolutely. Consider the Poisson modification of $F$, i.e. the unique function $f \in \mathcal{B}$ with $f|_{\mathbb{T}} = F$. For $|z| < 1$, it can be written as

$$f(z) = \sum_{n=0}^{\infty} \widehat{F}(n) z^n \quad \text{with } \widehat{F}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it}) e^{-int} dt .$$

It follows that $f|_K$ does not belong to $A$. This counterexample for $N=1$ gives also counterexamples in higher dimensions: We distinguish dimension 1 from dimensions $N > 1$ by an additional subscript. For $F \in C(\mathbb{T})$ and $f \in \mathcal{B}_{(1)}$ as above, the function $g(z) := f(z_1)$, with $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, obviously belongs to $\mathcal{B}_{(N)}$. The expansion of $g$ as a power series in the complex unit ball $U^\circ$ is given by $g(z) = \sum_{n=0}^{\infty} \widehat{F}(n) z_1^n$, which shows that the restriction of $g$ to $K_{(N)}$ does not belong to $A_{(N)}$.

*Proof of Proposition 2.3.4.* As a consequence of the above lemma, the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on the complex ball $U$. This shows that $\widetilde{f}$ is well defined with $\widetilde{f} \in \mathcal{B}$. An easy induction argument, based on the identity theorem for holomorphic functions in one dimension, shows that the extension $f \mapsto \widetilde{f}$ is unique: For this, put $M_i := \{z \in U^\circ : z_j \in \mathbb{R} \text{ for } j \geq i\}, \quad i = 1, \ldots, N$. We have to show that for functions $f, g$ holomorphic in $U^\circ$, the equality $f = g$ on $K^\circ = M_1$ implies that $f = g$ on $U^\circ = M_{N+1}$. But if $f = g$ on $M_i$ $(1 \leq i \leq N)$, then the identity theorem in one dimension implies that $f = g$ on $M_{i+1}$ as well. $\square$

We are now able to determine the complete spectrum of $A$. Notice first that for each $z \in U$, a well-defined evaluation homomorphism on the algebra $A$ is given by

$$\varphi_z : A \to \mathbb{C}, \quad \varphi_z(f) := \widetilde{f}(z).$$

**2.3.6 Lemma.** *The spectrum of $A$ is given by $\Delta(A) = \{\varphi_z : z \in U\}$. Moreover, the mapping $z \mapsto \varphi_z$ is a homeomorphism from $U$ onto $\Delta(A)$.*

*Proof.* It is again obvious that each $\varphi_z$, $z \in U$, belongs to $\Delta(A)$, that $z \mapsto \varphi_z$ is injective and that the mapping $z \mapsto \varphi_z$ is continuous on $U$. Again, it remains to show that each $\varphi \in \Delta(A)$ is of the stated form. For this, put $\lambda_i := \varphi(x_i)$ for $i = 1, \ldots, N$ and define $p_\lambda \in \mathcal{P}_2^N$ by $p_\lambda(z) := \sum_{i=1}^{N} \frac{\overline{\lambda}_i}{\lambda_i} z_i^2$. Then $p(\lambda) = \varphi(p)$ for all polynomials $p \in \Pi^N$, and thus in particular,

$$|\lambda|^2 = |p_\lambda(\lambda)| = |\varphi(p_\lambda)| \leq \|p_\lambda\|_A = \|p_\lambda\|_{\infty,K} \leq 1 .$$

This proves $\lambda \in U$, and the same density argument as in Lemma 2.3.2 implies that $\varphi = \varphi_\lambda$. $\square$

## 2.4 The main theorem

As before, $G$ is a finite reflection group on $\mathbb{R}^N$ and $k \geq 0$ is a non-negative multiplicity function on its root system $R$. It will be convenient to have a slightly extended notion of Dunkl's algebra of homogeneous series: For $r > 0$ let $K_r := \{x \in \mathbb{R}^N : |x| \leq r\}$ denote the ball of radius $r$ and define

$$A_r := \{f : K_r \to \mathbb{C},\ x \mapsto f_r(x) := f(rx) \in A\}, \tag{2.13}$$

as well as $\|f\|_{A_r} := \|f_r\|_A$. Notice that $A_r \subset A_s$ with $\|.\|_{A_r} \geq \|.\|_{A_s}$ for $s \geq r$. The dilated algebra $(A_r\,, \|.\|_{A_r})$ is again a commutative Banach-*-algebra, and $V_k$ extends uniquely to a continuous linear operator on $A_r$, by $V_k f := (V_k f_r)_{1/r}$. Moreover, the results of Section 2.3 for $A$ transfer to $A_r$ in the obvious way; in particular, $\Delta_S(A_r)$ can be naturally identified with $K_r$. Of particular interest in the following is the exponential function $x \mapsto e^{\langle x,z\rangle}$ $(z \in \mathbb{C}^N)$, which belongs to $A_r$ for each $r > 0$. Notice that by (1.19),

$$E_k(x,z) \,=\, V_k(e^{\langle\,.\,,z\rangle})(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{2.14}$$

Here is the main result of this chapter.

**2.4.1 Theorem.** *For each $x \in \mathbb{R}^N$ there exists a unique probability measure $\mu_x^k \in M^1(\mathbb{R}^N)$ such that*

$$V_k f(x) \,=\, \int_{\mathbb{R}^N} f(\xi)\, d\mu_x^k(\xi) \quad \textit{for all } f \in A_{|x|}. \tag{2.15}$$

*The representing measures $\mu_x^k$ are compactly supported with $\operatorname{supp}\mu_x^k \subseteq \{\xi \in \mathbb{R}^N : |\xi| \leq |x|\}$, and the mapping $\mathbb{R}^N \to M^1(\mathbb{R}^N)$, $x \mapsto \mu_x^k$ is weakly continuous. Moreover, the measures $\mu_x^k$ satisfy*

$$\mu_{rx}^k(B) \,=\, \mu_x^k(r^{-1}B), \quad \mu_{gx}^k(B) \,=\, \mu_x^k(g^{-1}B), \quad \textit{and } \mu_{hx}^{k_h}(B) \,=\, \mu_x^k(h^{-1}B) \tag{2.16}$$

*for each $r > 0$, $g \in G$, $h \in O(N,\mathbb{R})$ and each Borel set $B \in \mathcal{B}(\mathbb{R}^N)$.*

*Proof.* Fix $x \in \mathbb{R}^N$ and put $r = |x|$. Then the mapping

$$\Phi_x : f \mapsto V_k f(x)$$

is a bounded linear functional on $A_r$, and Theorem 2.2.1 implies that it is positive on the dense subalgebra $\Pi^N$ of $A_r$, i.e. $\Phi_x(|p|^2) \geq 0$ for all $p \in \Pi^N$. Consequently, $\Phi_x$ is a positive functional on the whole Banach-$*$-algebra $A_r$. Now, by a well-known Bochner-type representation theorem for positive functionals on commutative Banach-$*$-algebras (see e.g. Theorem 21.2 of [F-D]), there exists a unique measure $\nu_x^k \in M_b^+(\Delta_S(A_r))$ such that

$$\Phi_x(f) \,=\, \int_{\Delta_S(A_r)} \widehat{f}(\varphi)\, d\nu_x^k(\varphi) \quad \text{for all } f \in A_r, \tag{2.17}$$

with $\widehat{f}$ the Gelfand transform of $f$. Denote by $\mu_x^k$ the image measure of $\nu_x^k$ under the homeomorphism $\Delta_S(A_r) \to K_r$, $\varphi_\xi \to \xi$. Equation (2.17) then becomes

$$V_k f(x) \,=\, \int_{\{|\xi|\leq |x|\}} f(\xi)\, d\mu_x^k(\xi) \quad \text{ for all } f \in A_{|x|}.$$

The normalization $V_k 1 = 1$ implies that $\mu_x^k$ is a probability measure on $\{\xi \in \mathbb{R}^N : |\xi| \leq |x|\}$. To see the uniqueness of the representing measures $\mu_x^k$, we use representation (2.15) with $f(x) = e^{\langle x, -iy\rangle}$, $y \in \mathbb{R}^N$. Together with (2.14), this implies that the classical Fourier-Stieltjes transform of $\mu_x^k$, $x \in \mathbb{R}^N$, is just given by

$$(\mu_x^k)^\wedge(y) \,=\, \int_{\mathbb{R}^N} e^{-i\langle \xi, y\rangle}\, d\mu_x^k(\xi) \,=\, E_k(x, -iy). \tag{2.18}$$

Thus the uniqueness of $\mu_x^k$ follows from the injectivity of the Fourier-Stieltjes transform on $M^1(\mathbb{R}^N)$. In order to check that $x \mapsto \mu_x^k$ is weakly continuous, take a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ with $\lim_{n\to\infty} x_n = x_0 \in \mathbb{R}^N$. Then (2.18) and the continuity of the kernel $E_k$ yield that $(\mu_{x_n}^k)^\wedge \to (\mu_{x_0}^k)^\wedge$ pointwise on $\mathbb{R}^N$. By Lévy's continuity theorem (Satz 23.8 of [Ba]), this implies that the $\mu_{x_n}^k$ converge weakly to $\mu_{x_0}^k$. Finally, the transformation properties (2.16) follow immediately from the homogenity-preserving character of $V_k$ on $\Pi^N$ and the invariance property (1.13). □

*2.4.2 Remark.* In the one-dimensional case, associated with the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$ and multiplicity parameter $k > 0$, the representing measures $\mu_x^k \in M^1(\mathbb{R})$, $x > 0$, are given by

$$d\mu_x^k(t) \,=\, \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)} \cdot \frac{1_{[-x,x]}(t)}{x^{2k}}\,(x-t)^{k-1}(x+t)^k\, dt.$$

This is immediate from the explicit representation (1.11) for $V_k$.

As an important consequence of Theorem 2.4.1 we obtain that for fixed $y \in \mathbb{R}^N$, the function $x \mapsto E_k(x, iy)$ is positive definite on $\mathbb{R}^N$, and the same holds for the generalized Bessel function $x \mapsto J_k(x, iy)$.

**2.4.3 Corollary.** *For each $z \in \mathbb{C}^N$, the function $x \mapsto E_k(x,z)$ has the Bochner-type representation*

$$E_k(x,z) \,=\, \int_{\mathbb{R}^N} e^{\langle \xi, z\rangle} d\mu_x^k(\xi), \tag{2.19}$$

*where the $\mu_x^k$ are the representing measures from Theorem 2.4.1. In particular, $E_k(x,y) > 0$ for all $x, y \in \mathbb{R}^N$, and for each $x \in \mathbb{R}^N$ the function $y \mapsto E_k(x, iy)$ is positive definite on $\mathbb{R}^N$. Moreover, for each fixed $x \in \mathbb{R}^N$ the generalized Bessel function $y \mapsto J_k(x, iy)$ is positive definite on $\mathbb{R}^N$.*

*Proof.* This is immediate from Theorem 2.4.1 and representation (2.14), together with Bochner's theorem. □

In those cases where the generalized Bessel functions $J_k(.,y)$ allow an interpretation as the spherical functions of a Cartan motion group, the positive definiteness of these functions is an immediate consequence. Moreover, for the group $G = S_3$ and non-negative multiplicity functions, it results from the integral representations in [D6]. There are, however, no group-theoretical interpretations known for the kernel $E_k$. Nevertheless, the conjecture that it should be positive definite has been confirmed by several of its properties (see [dJ1]); moreover, it can be shown by classical methods characterizing positive semigroups, namely a variant of the Lumer-Phillips theorem, that $E_k(x,y) > 0$ for all $x,y \in \mathbb{R}^N$. This was carried out in [R4] in connection with the study of generalized heat semigroups for Dunkl operators. These semigroups will be studied in Chapter 4 of this thesis – now however on the basis of the already known positivity of the Dunkl kernel.

From (2.19) we obtain further knowledge about the support of the representing measures $\mu_x^k$:

**2.4.4 Corollary.** *The measures* $\mu_x^k$, $x \in \mathbb{R}^N$, *satisfy*

(i) $\operatorname{supp}\mu_x^k$ *is contained in* $\operatorname{co}\{gx,\, g\in G\}$, *the convex hull of the orbit of* $x$ *under* $G$.

(ii) $\operatorname{supp}\mu_x^k \cap \{gx,\, g\in G\} \neq \emptyset$.

*Proof.* (i) follows from Corollary 3.3 of [dJ1]. For the proof of (ii) it is therefore enough to show that

$$\operatorname{supp}\mu_x^k \cap \{\xi\in\mathbb{R}^N : |\xi| = |x|\} \neq \emptyset.$$

Suppose on the contrary that $\operatorname{supp}\mu_x^k \cap \{\xi\in\mathbb{R}^N : |\xi| = |x|\} = \emptyset$ for some $x\in\mathbb{R}^N$. Then there exists a constant $\sigma \in ]0,1[$ such that $\operatorname{supp}\mu_x^k \subseteq \{\xi\in\mathbb{R}^N : |\xi|\leq \sigma|x|\}$. This leads to the estimate

$$E_k(x,y) = \int_{|\xi|\leq\sigma|x|} e^{\langle \xi, y\rangle}\, d\mu_x^k(\xi) \leq e^{\sigma|x||y|}$$

for all $y\in\mathbb{R}^N$. On the other hand, Proposition 1.4.3 with $w=0$ says that

$$c_k^{-1}\int_{\mathbb{R}^N} E_k(x,y)\, e^{-(|x|^2+|y|^2)/2}\, w_k(y)dy = 1.$$

Now let $r>0$. As both formulas above remain valid if $x$ is replaced by $rx$, it follows that

$$1 \leq c_k^{-1}\int_{\mathbb{R}^N} e^{-(|rx|^2+|y|^2)/2}\, e^{\sigma|rx||y|}\, w_k(y)dy \leq c_k^{-1}\int_{\mathbb{R}^N} e^{(\sigma-1)(r^2|x|^2+|y|^2)/2}\, w_k(y)dy,$$

which tends to $0$ as $r\to\infty$, a contradiction. □

This result implies useful estimates for $E_k$ and its derivatives, which partially sharpen those of [dJ1], Lemma 3.5.

**2.4.5 Corollary.** *For all* $x,y\in\mathbb{R}^N$ *and* $z\in\mathbb{C}^N$, *the kernel* $E_k$ *satisfies*

(1) $|\,\partial_z^\nu E_k(x,z)| \leq |x|^{|\nu|}\cdot \max_{g\in G} e^{\langle gx,\, \mathrm{Re}\, z\rangle} \quad (\nu\in\mathbb{Z}_+^N,\ |\nu| = \nu_1+\ldots+\nu_N).$

(2) $|E_k(x, iy)| \leq 1$.

(3) $E_k(x,y) \geq \min_{g\in G} e^{\langle gx,\, y\rangle}$.

*Proof.* Each $\xi \in \mathrm{co}\{gx, g\in G\}$ is of the form $\xi = \sum_{g\in G}\lambda_g\cdot gx$ with $\lambda_g > 0$ and $\sum_{g\in G}\lambda_g = 1$. This leads for all $z\in\mathbb{C}^N$ to the estimates

$$\min_{g\in G}\langle gx, \mathrm{Re}\, z\rangle \;\leq\; \mathrm{Re}\,\langle \xi, z\rangle \;\leq\; \max_{g\in G}\,\langle gx, \mathrm{Re}\, z\rangle\,. \tag{2.20}$$

Part (1) is now obtained by differentiating the expression under the integral in (2.19), and using then (2.20). Part (2) is a special case of part (1), and part (3) is also immediate from (2.20). □

**2.4.6 Corollary.** *The Dunkl transform on* $L^1(\mathbb{R}^N, w_k)$ *satisfies*

$$\|\widehat{f}^k\|_\infty \;\leq\; \|f\|_{1,w_k}\,.$$

A further consequence of Theorem 2.4.1 is the possibility to extend $V_k$ to larger function spaces:

*2.4.7 Remark.* The integral representation (2.15) for $V_k$ allows an extension to $\mathscr{L}^\infty_{loc}(\mathbb{R}^N)$, the space of locally bounded, Borel measurable functions on $\mathbb{R}^N$, via

$$\widetilde{V}_k f(x) \;=\; \int_{\mathbb{R}^N} f(y)\, d\mu_x^k(y).$$

On the subspace $W := \{f\in C(\mathbb{R}^N)\cap L^1(\mathbb{R}^N) : \widehat{f}\in L^1(\mathbb{R}^N)\}\subset \mathscr{L}^\infty_{loc}(\mathbb{R}^N)$ an equivalent representation of $\widetilde{V}_k$ is obtained by the inversion theorem for the classical Fourier transform: Let $f\in W$. Then

$$\widetilde{V}_k f(x) \;=\; c_0^{-1}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\widehat{f}(\xi)\, e^{i\langle\xi,y\rangle}d\xi\Big)d\mu_x^k(y) \;=\; c_0^{-1}\int_{\mathbb{R}^N}\widehat{f}(\xi)\, E_k(i\xi, x)\, d\xi\,. \tag{2.21}$$

In particular, $\widetilde{V}_k$ also satisfies the intertwining property $T_\xi(k)\widetilde{V}_k \;=\; \widetilde{V}_k\,\partial_\xi \quad (\xi\in\mathbb{R}^N)$. We mention that on $W\cap C^\infty(\mathbb{R}^N)$, the operator $\widetilde{V}_k$ coincides with the extension of $V_k$ to $C^\infty(\mathbb{R}^N)$ which was already obtained in [dJ3], Chap. 3.4 as a consequence of a Paley-Wiener theorem for the Dunkl transform.

At the end of this chapter we would like to indicate an application of Theorem 2.4.1 to the study of probabilistic aspects of Dunkl's theory.

**2.4.8 Generalized moment functions.** The classical moments of probability measures on $\mathbb{R}^N$ have many applications to sums of independent random variables. The definition of these moments is based on the monomial "moment functions" $x^\nu = x_1^{\nu_1}x_2^{\nu_2}\ldots x_N^{\nu_N}$, $x\in\mathbb{R}^N$, $\nu\in\mathbb{Z}_+^N$. Recently, in [R-V2] a concept of Markov kernels and Markov processes which are homogeneous with respect to a given Dunkl transform has been developed. In this context, generalized moment functions on $\mathbb{R}^N$ provide a useful tool. They generalize the classical moment functions and are defined as the unique coefficients $m_{k,\nu}$ in the expansion

$$E_k(x,y) \;=\; \sum_{\nu\in\mathbb{Z}_+^N}\frac{m_{k,\nu}(x)}{\nu!}\, y^\nu \qquad (x\in\mathbb{R}^N,\; y\in\mathbb{C}^N).$$

Comparison with the homogeneous expansion of $E_k$, (1.22) and (1.23), shows that

$$m_{k,\nu}(x) = V_k(x^\nu) \in \mathcal{P}^N_{|\nu|} \qquad \text{for } \nu \in \mathbb{Z}_+^N .$$

Theorem 2.4.1 in particular implies the following useful relations for the generalized moment functions, which are obvious only in the classical case (again, we assume $k \geq 0$):

$$|m_{k,\nu}(x)| \leq |x|^{|\nu|} \quad \text{and} \quad 0 \leq m_{k,\nu}(x)^2 \leq m_{k,2\nu}(x) \quad \text{for all } x \in \mathbb{R}^N,\ \nu \in \mathbb{Z}_+^N .$$

The first inequality is clear from the support properties of the measures $\mu_x$ while the second one follows from Jensen's inequality. Among the applications of these moments, we mention the construction of martingales from Dunkl-type Markov processes; for details, we refer to [R-V2].

# Chapter 3

# Generalized Hermite polynomials and biorthogonal systems

This chapter presents a general concept of multivariable biorthogonal polynomials with respect to weight functions of the form $w_k(x)e^{-\omega|x|^2}$ on $\mathbb{R}^N$, where $k$ is a non-negative multiplicity function on the root system of a given finite reflection group and $\omega > 0$ a frequency parameter. This concept includes, as a particular case, orthogonal polynomial systems with respect to such weight functions, which are called generalized Hermite polynomials. The definition and properties of these polynomials extend naturally those of the classical multivariable Hermite polynomials; partial derivatives and the usual exponential kernel are replaced by Dunkl operators and the Dunkl kernel here. In particular, for root systems of type $A$ and $B$ there is some physical relevance of the associated biorthogonal systems: They arise as the eigenfunctions of Hamiltonians which describe certain exactly solvable quantum many body systems of Calogero-Moser-Sutherland type.

In Section 3.1 we give a short explanation of linear Calogero-Moser-Sutherland models and the relevance of Dunkl operators in their algebraic description; we also solve the spectral problem for abstract Calogero-Moser-Sutherland operators with harmonic confinement. The spectra of these operators are highly degenerate, and there are several favourable choices of eigenfunction bases, having in common that they fit into the above mentioned concept of biorthogonality. This concept is developed in Section 3.3, and several classes of examples are studied. Some emphasis is put on the particular class of generalized Hermite polynomials. In the one-dimensional case, associated with the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$, these generalized Hermite polynomials coincide with those introduced in [Chi] and studied in [Ros]. Our setting also includes, for the symmetric group $G = S_N$, the so-called non-symmetric generalized Hermite polynomials which were recently introduced by Baker and Forrester in [B-F2], [B-F3]. These are non-symmetric analogues of the symmetric (i.e. permutation-invariant) generalized Hermite polynomials associated with the group $S_N$, which were first introduced by Lassalle [La2]. Moreover, the "generalized Laguerre polynomials" of [B-F2], [B-F3], which are non-symmetric analogues of those in [La1], can be considered as a subsystem of Hermite polynomials associated with a reflection group of type $B_N$. For a thorough study of the symmetric multivariable Hermite- and Laguerre systems we refer to [B-F1], [vD] and [K2]. Besides of the second order differential equations, many of the well-known properties of classical Hermite polynomials and Hermite functions allow extensions to our biorthogonal systems, such as the Rodrigues formula, generating function and the Mehler formula. We conclude this chapter by an application of generalized Hermite expansions: They are used to derive an analogue of the classical Heisenberg-Weyl uncertainty principle for the Dunkl transform.

In order to make notations more transparent, we shall frequently omit the explicit reference to the underlying multiplicity function $k$ in the Dunkl operators, and therefore e.g. write $T_\xi$ instead of $T_\xi(k)$.

## 3.1 Linear Calogero-Moser-Sutherland models

The Calogero-Moser-Sutherland (CMS) models are quantum many-body models in one dimension; they describe a system of $N$ particles on a circle or line which interact pairwise through long range potentials of inverse square type. These models are exactly solvable and therefore of great interest for the understanding of quantum many-body physics. They have in particular attracted some attention in conformal field theory and are being used to test the ideas of fractional statistics ([Hal], [Ha]). The study of CMS models was initiated by Calogero ([Ca]). He considered a translation invariant $N$-particle system with a potential of the form

$$V(x) \,=\, \sum_{1\leq i<j\leq N}\Big[\frac{g_0}{(x_i-x_j)^2} + g_1(x_i-x_j)^2\Big],$$

for which he computed the spectrum and determined the structure of the eigenfunctions and scattering states. Up to a center of mass motion $\big(\sum_{1\leq j<k\leq N}(x_j-x_k)^2 = N|x|^2 - \big(\sum_j x_j\big)^2\big)$ this system is equivalent to the so-called rational Calogero model with harmonic confinement, whose quantum Hamiltonian on $L^2(\mathbb{R}^N)$ is

$$\mathcal{H}_C \,=\, -\Delta + \omega^2|x|^2 \,+\, 2k(k-1)\sum_{1\leq i<j\leq N}\frac{1}{(x_i-x_j)^2}, \tag{3.1}$$

with frequency parameter $\omega > 0$ and a coupling parameter $k \geq 0$. $\mathcal{H}_C$ is symmetric and bounded from below with the ground state

$$f_0(x) \,=\, e^{-\omega|x|^2/2}\sqrt{w_k(x)};$$

here $w_k$ is the $S_N$-type weight function

$$w_k(x) \,=\, \prod_{1\leq i<j\leq N}|x_i-x_j|^{2k}.$$

The standard circle model, whose study was initiated by Sutherland ([Su]), is characterized by the Hamiltonian

$$\mathcal{H}_S \,=\, -\Delta + g\sum_{i<j}\frac{1}{d(x_i,x_j)^2} \tag{3.2}$$

on $L^2([0,1]^N)$; here $g \geq -1/2$ is again a coupling constant and

$$d(x_i,x_j) \,=\, \frac{1}{\pi}\sin\big(\pi(x_i-x_j)\big)$$

is the chord length between the positions of the particles $i$ and $j$ on a circle of circumference 1. It was first observed by Perelomov [Pe] that (3.1) is completely quantum integrable, i.e. there exist $N$ algebraically independent symmetric linear operators in $L^2(\mathbb{R}^N)$ which commute with each other and with $\mathcal{H}_C$. The quantum integrability of (3.2) was proven in [CRM], while the complete integrability of the classical Hamiltonian systems associated with (3.1) and (3.2) goes back to Moser [Mo].

There are natural extensions of these systems in the context of abstract root systems, see e.g. [O-P2], [H-Sc]. In particular, if $R$ is an arbitrary root system on $\mathbb{R}^N$ (not necessarily cristallographic), and $k$ is a nonnegative multiplicity function on it, then the corresponding abstract linear CMS operator with harmonic confinement is given by

$$\widetilde{\mathcal{H}}_k \,=\, -\widetilde{\mathcal{F}}_k + \omega^2|x|^2$$

with the formal expression

$$\widetilde{\mathcal{F}}_k \,=\, \Delta - 2\sum_{\alpha\in R_+} k(\alpha)(k(\alpha)-1)\frac{1}{\langle\alpha,x\rangle^2}.$$

If $R$ is of type $A_{N-1}$, then $\widetilde{\mathcal{H}}_k$ just coincides with $\mathcal{H}_C$. For both the classical and the quantum case, partial results on the integrability of this model, as well as periodic variants, are due to Olshanetsky and Perelomov [O-P1], [O-P2]. A new aspect in the understanding of the algebraic structure and the quantum integrability of CMS systems was later initiated by Polychronakos [Po] and Heckman [He2]. The underlying idea is to construct quantum integrals for (linear) CMS models from differential-reflection operators. Polychronakos introduced them in terms of an "exchange-operator formalism" for the linear CMS model (3.1). He thus obtained a complete set of commuting observables for (3.1) in an elegant way. In [He2] it was observed in general that the complete algebra of quantum integrals for abstract linear CMS models - in case of an arbitrary root system, but without harmonic confinement - is intimately connected with the corresponding algebra of Dunkl operators. Since then, there has been an extensive and ongoing study of CMS models and explicit operator solutions for them via differential-difference operator formalisms; among the broad literature, we refer to [L-V], [K1], [BHV], [BHKV], [B-F3], [U-W]. We briefly describe the connection of (abstract) linear CMS models to Dunkl operators: Consider the following modification of $\widetilde{\mathcal{F}}_k$, involving reflection terms:

$$\mathcal{F}_k \,=\, \Delta - 2\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}\,(k(\alpha)-\sigma_\alpha)\,. \tag{3.3}$$

In order to avoid singularities in the reflecting hyperplanes, it is suitable to carry out a gauge transform by $\sqrt{w_k}$. This leads to the following

**3.1.1 Lemma.** *The operator $\mathcal{F}_k$, with domain*

$$\mathcal{D}(\mathcal{F}_k) := \{w_k^{1/2} f \,:\, f \in \mathscr{S}(\mathbb{R}^N)\} \subset L^2(\mathbb{R}^N),$$

*is essentially self-adjoint in $L^2(\mathbb{R}^N)$. $\mathcal{D}(\mathcal{F}_k)$ is invariant under $\mathcal{F}_k$, and*

$$\mathcal{F}_k \,=\, w_k^{1/2}\,\Delta_k\, w_k^{-1/2}.$$

*Proof.* A short calculation yields that for $f \in C^2(\mathbb{R}^N)$ and $x \notin \bigcup_{\alpha\in R_+} H_\alpha$,

$$\begin{aligned} w_k^{-1/2}(x)\Delta(w_k^{1/2}f)(x) \,&= \Delta f(x) \,+\, 2\sum_{\alpha\in R_+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x)}{\langle \alpha, x\rangle^2}\Big) \\ &\quad + f(x) \sum_{\alpha,\beta\in R_+} k(\alpha)k(\beta)\,\frac{\langle \alpha, \beta\rangle}{\langle \alpha, x\rangle\langle \beta, x\rangle}\,. \end{aligned} \tag{3.4}$$

Applying Proposition 1.7.(i) of [D2] to the bilinear form $B(x,y) := \langle x, y\rangle$ on $\mathbb{R}^N$, we obtain that for every plane rotation $g \in G$ with $g \neq e$,

$$\sum_{\alpha,\beta\in R_+:\,\sigma_\alpha\sigma_\beta = g} k(\alpha)k(\beta)\frac{\langle \alpha, \beta\rangle}{\langle \alpha, x\rangle\langle \beta, x\rangle} \,=\, 0.$$

Therefore the last sum in (3.4) reduces to

$$2f(x)\cdot \sum_{\alpha\in R_+} \frac{k(\alpha)^2}{\langle \alpha, x\rangle^2}$$

(recall our normalization $|\alpha|^2 \,=\, 2$ for all $\alpha \in R_+$). As $w_k$ is $G$-invariant, this implies

$$\begin{aligned} w_k^{-1/2}\mathcal{F}_k\, w_k^{1/2} f(x) \,&= w_k^{-1/2}(x)\Delta(w_k^{1/2}f)(x) \,-2\sum_{\alpha\in R_+} k(\alpha)\,\frac{k(\alpha)f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2} \\ &= \Delta f(x) \,+\, 2\sum_{\alpha\in R_+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} \,-\, \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}\Big) \\ &= \,\Delta_k f(x), \end{aligned} \tag{3.5}$$

where the last identity follows from the explicit representation (1.7) for $\Delta_k$. The assertion now follows from the facts that $\Delta_k$, with domain $\mathscr{S}(\mathbb{R}^N)$, is essentially self-adjoint in $L^2(\mathbb{R}^N, w_k)$ (c.f. Chapter 1), and that the mapping

$$\Phi : L^2(\mathbb{R}^N) \,\to\, L^2(\mathbb{R}^N, w_k),\;\; f \mapsto w_k^{-1/2} f$$

is an isometric isomorphism. □

Consider now the algebra of $G$-invariant polynomials on $\mathbb{R}^N$,

$$(\Pi^N)^G = \{p \in \Pi^N : g \cdot p = p \text{ for all } g \in G\}.$$

If $p \in (\Pi^N)^G$, then the Dunkl operator $p(T)$ leaves $(\Pi^N)^G$ invariant (by (1.4)). For such $p$, denote the restriction of $p(T)$ to $(\Pi^N)^G$ by $\mathrm{Res}\,(p(T))$. Then, as observed in [He2], the family

$$\big\{\mathrm{Res}\,(p(T)) : p \in (\Pi^N)^G\big\}$$

is a commutative algebra of differential operators, containing the operator

$$\mathrm{Res}\,(\Delta_k) = w_k^{-1/2}\widetilde{\mathcal{F}}_k w_k^{1/2}.$$

This implies the integrability of the CMS operator $\widetilde{\mathcal{F}}_k$. For the special case of the symmetric group, this is essentially also contained in [Po] (where, however, the above gauge transform is omitted). As already mentioned, Polychronakos also succeeded to determine a complete set of quantum integrals for the $S_N$-type Hamiltonian $\mathcal{H}_C$ with harmonic confinement - at least in the physically relevant bosonic and fermionic subspaces of $L^2(\mathbb{R}^N)$. He constructed the integrals by a Lax formalism involving the lowering and raising operators

$$a_j = w_k^{1/2}(\omega x_j + T_j^S)w_k^{-1/2} \quad \text{and} \quad a_j^+ = w_k^{1/2}(\omega x_j - T_j^S)w_k^{-1/2}.$$

Then $a_j^+$ is the adjoint of $a_j$ in $L^2(\mathbb{R}^N)$, and the families $\{a_j,\ j = 1,\dots,N\}$ as well as $\{a_j^+,\ j = 1,\dots,N\}$ commute (c.f. the more general treatment in Lemma 3.1.6.) It was further shown in [Po] that the operators

$$I_m = \sum_{j=1}^N (a_j^+ a_j)^m, \quad m \in \mathbb{N}$$

commute in $L^2(\mathbb{R}^N)$. Moreover, the restriction of $I_1$ to the bosonic subspace

$$L^2(\mathbb{R}^N)^{S_N} := \{f \in L^2(\mathbb{R}^N) : g \cdot f = f \text{ for all } g \in S_N\}$$

coincides, up to an additive constant, with the Hamiltonian $\mathcal{H}_C$. In a similar way, the restriction of $I_1$ to the fermionic subspace of completely anti-symmetric functions coincides up to a constant with $\mathcal{H}_C$, the coupling constant $2k(k-1)$ being replaced by $2k(k+1)$. Concerning more general root systems, there are only particular results of this kind for systems of type $B_N$ and $G_2$, see [B-F2] and [Hi-K]. For the abstract CMS operator $\widetilde{\mathcal{H}}_k$ with harmonic confinement, the general question of how to obtain an algebra of quantum integrals is, to the author's knowledge, still open. We shall however see from the results below that the spectral properties of $\widetilde{\mathcal{H}}_k$ in the "bosonic" subspace

$$L^2(\mathbb{R}^N)^G = \{f \in L^2(\mathbb{R}^N) : g \cdot f = f \text{ for all } g \in G\}$$

are rather easy to describe, and that it is possible to obtain complete bases of eigenfunctions by a suitable ladder formalism. We again work with the gauge-transformed version with reflection terms,

$$\mathcal{H}_k := w_k^{-1/2}(-\mathcal{F}_k + \omega^2|x|^2)\, w_k^{1/2} \;=\; -\Delta_k + \omega^2|x|^2.$$

This operator is symmetric and densely defined in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathcal{D}(\mathcal{H}_k) := \mathscr{S}(\mathbb{R}^N)$. Notice that in case $k = 0$, $\mathcal{H}_k$ is just the Hamiltonian of the $N$-dimensional isotropic harmonic oscillator. We further consider the Hilbert space $L^2(\mathbb{R}^N, dm_k^\omega)$, where $m_k^\omega$ is the probability measure

$$dm_k^\omega(x) \;:=\; c_k^{-1}(2\omega)^{\gamma+N/2}\, e^{-\omega|x|^2} w_k(x)dx \;\in M^1(\mathbb{R}^N) \quad (\omega > 0), \tag{3.6}$$

and the operator

$$\mathcal{J}_k :=\; -\Delta_k + 2\omega \sum_{j=1}^N x_j\partial_j$$

in $L^2(\mathbb{R}^N, dm_k^\omega)$, with the dense domain $\mathcal{D}(\mathcal{J}_k) \;:=\; \Pi^N$. The subsequent theorem gives a complete description of the spectral properties of $\mathcal{H}_k$ and $\mathcal{J}_k$ and generalizes well-known facts for the corresponding classical operators. Its proof relies on the $sl(2)$-commutation relations of the operators

$$E := \frac{1}{2}|x|^2,\;\; F := -\frac{1}{2}\Delta_k \quad \text{and} \;\; H := \sum_{i=1}^N x_i\partial_i + (\gamma + N/2)$$

on $\Pi^N$ (with the index $\gamma = \gamma(k)$ as defined in (1.1)), which can be found in [He2]. They are

$$\big[H, E\big] = 2E, \;\; \big[H, F\big] = -2F, \;\; \big[E, F\big] = H. \tag{3.7}$$

Notice that the first two relations are immediate consequences of the fact that the Euler operator

$$\rho := \sum_{i=1}^N x_i\partial_i \tag{3.8}$$

satisfies $\rho(p) = np$ for each homogeneous $p \in \mathcal{P}_n$. We start with the following

**3.1.2 Lemma.** *On* $\mathcal{D}(\mathcal{J}_k) = \Pi^N$,

$$\mathcal{J}_k \;=\; e^{\omega|x|^2/2}\big(\mathcal{H}_k - (2\gamma + N)\omega\big)\, e^{-\omega|x|^2/2}.$$

*In particular,* $\mathcal{J}_k$ *is symmetric in* $L^2(\mathbb{R}^N, dm_k^\omega)$.

*Proof.* From (3.7) it is easily verified by induction that

$$\big[\Delta_k, E^n\big] = 2nE^{n-1}H + 2n(n-1)E^{n-1} \quad \text{for all} \quad n \in \mathbb{N},$$

and therefore $\big[\Delta_k, e^{-\omega E}\big] = -2\omega e^{-\omega E}H + 2\omega^2 E e^{-\omega E}$. Thus on $\Pi^N$,

$$\mathcal{H}_k\, e^{-\omega E} \;=\; -\Delta_k e^{-\omega E} \;+ 2\omega^2 E e^{-\omega E} \;=\; -e^{-\omega E}\Delta_k \;+ 2\omega e^{-\omega E} H \;=\; e^{-\omega E}\big(\mathcal{J}_k + (2\gamma + N)\omega\big).$$

□

**3.1.3 Theorem.** *For $\omega > 0$ and $n \in \mathbb{Z}_+$ define*

$$V_n^\omega := \{e^{-\Delta_k/4\omega}\, p : p \in \mathcal{P}_n^N\} \subset \Pi_n^N \quad \text{and} \quad W_n^\omega := \{e^{-\omega|x|^2/2}\, q(x),\ q \in V_n^\omega\} \subset \mathscr{S}(\mathbb{R}^N).$$

*Then the following assertions hold:*

(1) *The spaces $L^2(\mathbb{R}^N, dm_k^\omega)$ and $L^2(\mathbb{R}^N, w_k)$ admit the orthogonal Hilbert space decompositions*

$$L^2(\mathbb{R}^N, dm_k^\omega) = \bigoplus_{n\in\mathbb{Z}_+} V_n^\omega \quad \text{and} \quad L^2(\mathbb{R}^N, w_k) = \bigoplus_{n\in\mathbb{Z}_+} W_n^\omega\,;$$

*here $V_n^\omega$ is the eigenspace of $\mathcal{J}_k$ corresponding to the eigenvalue $2n\omega$, and $W_n^\omega$ is the eigenspace of $\mathcal{H}_k$ corresponding to the eigenvalue $(2n + 2\gamma + N)\omega$.*

(2) *The operators $\mathcal{H}_k$ and $\mathcal{J}_k$ are essentially self-adjoint; the spectra of their closures are discrete and given by $\sigma(\overline{\mathcal{H}_k}) = \{(2n + 2\gamma + N)\omega,\ n \in \mathbb{Z}_+\}$ and $\sigma(\overline{\mathcal{J}_k}) = \{2n\omega,\ n \in \mathbb{Z}_+\}$ respectively.*

*Proof.* (1) Equation (3.7) and an induction argument yield the commuting relations $\big[\rho, \Delta_k^n\big] = -2n\Delta_k^n$ for all $n \in \mathbb{Z}_+$, and hence

$$\big[2\omega\rho, e^{-\Delta_k/4\omega}\big] = \Delta_k\, e^{-\Delta_k/4\omega}.$$

If $q \in \Pi^N$ is arbitrary and $p := e^{\Delta_k/4\omega}q$, it follows that

$$\rho(q) = (\rho e^{-\Delta_k/4\omega})(p) = e^{-\Delta_k/4\omega}\rho(p) + \frac{1}{2\omega}\Delta_k e^{-\Delta_k/4\omega} p = e^{-\Delta_k/4\omega}\rho(p) + \frac{1}{2\omega}\Delta_k q.$$

Hence for $a \in \mathbb{C}$ the following relations are equivalent:

$$(-\Delta_k + 2\omega\rho)(q) = 2a\omega q \iff \rho(p) = ap \iff a = n \in \mathbb{Z}_+ \text{ and } p \in \mathcal{P}_n^N.$$

Thus each function from $V_n^\omega$ is an eigenfunction of $\mathcal{J}_k$ corresponding to the eigenvalue $2n\omega$, and $V_n^\omega \perp V_m^\omega$ for $n \neq m$ by the symmetry of $\mathcal{J}_k$. This proves the statements for $\mathcal{J}_k$, because $\Pi^N = \bigoplus V_n^\omega$ is dense in $L^2(\mathbb{R}^N, dm_k^\omega)$. The statements for $\mathcal{H}_k$ are then immediate by the previous Lemma.
(2) follows from (1) by a well-known criterion for self-adjointness of symmetric operators on a Hilbert space which have a complete set of orthogonal eigenfunctions within their domain (Lemma 1.2.2 of [Da3]). □

*3.1.4 Remark.* Part (1) of the above theorem implies in particular that the operator $\mathcal{J}_k$ has for each given $p \in \mathcal{P}_n^N$ a unique polynomial eigenfunction $q$ of the form $q = p + r$, where the degree of $r$ is strictly less than $n$; it is given by $q = e^{-\Delta_k/4\omega}p$.

By the $G$-equivariance of $\Delta_k$, the spectral resolution of the CMS operator $\widetilde{\mathcal{H}}_k$ in the bosonic subspace $L^2(\mathbb{R}^N)^G$ is now an easy consequence of Theorem 3.1.3.

**3.1.5 Corollary.** *For $n \in \mathbb{Z}_+$, put* $W_n^{\omega,G} = \{e^{-\omega|x|^2/2}e^{-\Delta_k/4\omega}\, p : p \in \mathcal{P}_n^N \cap (\Pi^N)^G\}$. *Then*

$$L^2(\mathbb{R}^N)^G = \bigoplus_{n\in\mathbb{Z}_+} W_n^{\omega,G},$$

*and* $W_n^{\omega,G}$ *is the eigenspace of* $\widetilde{\mathcal{H}}_k$ *in* $L^2(\mathbb{R}^N)^G$ *corresponding to the eigenvalue* $(2n+2\gamma+N)\omega$.

We next prove Rodrigues formulas for the eigenfunctions of $\mathcal{J}_k$ and $\mathcal{H}_k$; they generalize results from [BHV] and [BHKV] for the $S_N$- case, where explicit bases of bosonic and fermionic eigenfunctions for the Calogero Hamiltonian $\mathcal{H}_C$ were constructed from its vacuum state. We first introduce an obvious analogue of the $S_N$-type ladder operators: We define "lowering" and "raising" operators in $L^2(\mathbb{R}^N, w_k)$ by

$$A_j := \frac{1}{\sqrt{2}}(\omega x_j + T_j) \quad \text{and} \quad A_j^+ := \frac{1}{\sqrt{2}}(\omega x_j - T_j) \quad (j = 1, \ldots, N).$$

Then, by the anti-symmetry of the $T_j$ in $L^2(\mathbb{R}^N, w_k)$ (Prop. 1.4.4), $A_j^+$ is the adjoint of $A_j$ in this Hilbert space, and

$$\mathcal{H}_k = \sum_{j=1}^N \big(A_j A_j^+ + A_j^+ A_j\big).$$

Moreover, the following commutation relations are verified by short calculations:

**3.1.6 Lemma.** *For $i, j = 1, \ldots, N$,*

(1) $[A_i, A_j^+] = \omega \cdot [T_i, x_j] = \omega\Big(\delta_{i,j} \cdot id + \sum_{\alpha\in R_+} k(\alpha)\alpha_i\alpha_j\, \sigma_\alpha\Big)$;
(2) $[A_i, A_j] = [A_i^+, A_j^+] = 0.$

In particular, each $p \in \Pi^N$ defines unique linear operators $p(A)$, $p(A^+)$ in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathscr{S}(\mathbb{R}^N)$. We shall further use the following rescaling formula:

**3.1.7 Lemma.** *Let $p \in \mathcal{P}_n$. Then for $c \in \mathbb{C}$ and $a \in \mathbb{C}\setminus\{0\}$,*

$$\big(e^{c\Delta_k}p\big)(ax) = a^n\big(e^{a^{-2}c\,\Delta_k}\big)p(x) \quad (x \in \mathbb{R}^N).$$

*Proof.* For $m \in \mathbb{Z}_+$ with $2m \le n$, the polynomial $\Delta_k^m p$ is homogeneous of degree $n - 2m$. Hence

$$\big(e^{c\Delta_k}p\big)(ax) = \sum_{m=0}^{\lfloor n/2\rfloor} \frac{c^m}{m!}(\Delta_k^m p)(ax) = \sum_{m=0}^{\lfloor n/2\rfloor} \frac{c^m}{m!}a^{n-2m}(\Delta_k^m p)(x) = a^n\big(e^{a^{-2}c\,\Delta_k}p\big)(x).$$

□

**3.1.8 Theorem.** *(Rodrigues formulas for the eigenfunctions of $\mathcal{J}_k$ and $\mathcal{H}_k$.)*

(1) *Let $f = e^{-\Delta_k/4\omega}p \in V_n^\omega$, with $p \in \mathcal{P}_n^N$. Then*

$$f(x) = \Big(\frac{-1}{2\omega}\Big)^n e^{\omega|x|^2}\, p(T)\, e^{-\omega|x|^2}. \tag{3.9}$$

(2) *Let* $f(x) = e^{-\omega|x|^2/2}\, e^{-\Delta_k/4\omega}p(x) \in W_n^\omega$, *with* $p \in \mathcal{P}_n^N$. *Then*

$$f(x) \,=\, \Big(\frac{1}{\sqrt{2\omega}}\Big)^n p(A^+)\, e^{-\omega|x|^2/2}\,. \tag{3.10}$$

*Proof.* For the proof of (1), it suffices to consider the case $\omega = 1/2$; in fact, Lemma 3.1.7 with $c = -1/2$ and $a = \sqrt{2\omega}$ then implies that

$$\begin{aligned} e^{-\Delta_k/4\omega}p(x) &= (2\omega)^{-n/2}\, e^{-\Delta_k/2}\, p(\sqrt{2\omega}\, x) \,=\, (2\omega)^{-n/2}(-1)^n\, e^{\omega|x|^2}\big(p(T)\, e^{-|\xi|^2/2}\big)(\sqrt{2\omega}\, x) \\ &= (-2\omega)^{-n}\, e^{\omega|x|^2}\, p(T)\, e^{-\omega|x|^2}. \end{aligned}$$

Hence let $\omega = 1/2$, and consider the Dunkl operator $T_j,\ j = 1, \dots, N,$ as a linear operator on $L^2(\mathbb{R}^N, e^{-|x|^2/2}w_k(x)dx)$, with domain $\Pi^N$. Its adjoint is given by

$$T_j^* \,=\, -e^{|x|^2/2} \circ T_j \circ e^{-|x|^2/2} \,=\, x_j - T_j\,; \tag{3.11}$$

here the first identity follows again from the anti-symmetry of $T_j$ in $L^2(\mathbb{R}^N, w_k)$, and the second one is obtained by the product rule (1.5). Our assertion is therefore equivalent to

$$e^{-\Delta_k/2}p \,=\, p(T^*)(1) \tag{3.12}$$

This identity is now easily checked by induction with respect to the degree $n$ of $p$: The case $n = 0$ is clear. Moreover, if (3.12) holds for $p \in \mathcal{P}_n^N$, then by (3.11) and Lemma 1.2.6(2),

$$T_j^* p(T^*)(1) \,=\, T_j^* e^{-\Delta_k/2}p \,=\, x_j\, e^{-\Delta_k/2}p \,-\, T_j\, e^{-\Delta_k/2}\, p \,=\, e^{-\Delta_k/2}(x_j p),$$

which finishes the proof of (1). To obtain (2) from (1), write

$$f(x) \,=\, (2\omega)^{-n}\big(e^{\omega|x|^2/2} \circ p(-T) \circ e^{-\omega|x|^2/2}\big)\big(e^{-\omega|x|^2/2}\big).$$

Again by the product rule (1.5), we have

$$e^{\omega|x|^2/2} \circ p(-T) \circ e^{-\omega|x|^2/2} \,=\, p\,(\omega x_j - T_j) \,=\, \sqrt{2}^{\,n} p(A^+). \tag{3.13}$$

This yields the assertion. □

We finish this section by the spectral resolution for the Dunkl transform, which is also in analogy to the classical case $k = 0$. Recall that by the Plancherel theorem 1.4.6, the Dunkl transform $\mathcal{D}_k : f \mapsto \widehat{f}^{\,k}$ is unitary on $L^2(\mathbb{R}^N, w_k)$.

**3.1.9 Proposition.** *The Dunkl transform* $\mathcal{D}_k$ *on* $L^2(\mathbb{R}^N, w_k)$ *has the spectrum* $\{(-i)^j,\ j = 0, \dots, 3\}$. *The eigenspace belonging to the eigenvalue* $(-i)^j$ *is*

$$E_j := \bigcup_{n \equiv j \, mod\, 4} W_n^1\,.$$

*Proof.* Take $f(x) = e^{-|x|^2/2}q(x) \in W_n^1$ with $q = e^{-\Delta_k/4}p,\ p \in \mathcal{P}_n^N$. Let further $\widetilde{p} := e^{\Delta_k/2}q$. Employing Proposition 1.4.3(1), we obtain

$$\begin{aligned}\widehat{f}^k(\xi) &= \frac{1}{c_k}\int_{\mathbb{R}^N} E_k(-i\xi, x)\, e^{-\Delta_k/2}\widetilde{p}(x)\, e^{-|x|^2/2} w_k(x)dx \\ &= e^{-|\xi|^2/2}\widetilde{p}(-i\xi) \;=\; e^{-|\xi|^2/2}\left(e^{\Delta_k/4}p\right)(-i\xi).\end{aligned}$$

Application of Lemma 3.1.7 with $a = i$ and $c = -1/4$ now shows that the last expression equals

$$e^{-|\xi|^2/2}(-i)^n e^{-\Delta_k/4}p(\xi) \;=\; (-i)^n f(\xi).$$

Since $L^2(\mathbb{R}^N, w_k) = \bigoplus_{n\in\mathbb{Z}_+} W_n^1$, this finishes the proof. $\square$

## 3.2 Biorthogonal polynomials associated with reflection groups

Up to now, there are ambiguities in the possible choices for the bases of eigenfunctions of the Hamiltonians $\mathcal{J}_k$ and $\mathcal{H}_k$, caused by the high degeneracy of their spectra. One natural requirement is orthogonality in the underlying Hilbert space, which leads to the concept of generalized Hermite polynomials and Hermite functions. Another criterion is the possibility to generate the basis within a suitable operator formalism; this is not conveniently met by generalized Hermite systems, but can be achieved by weakening the orthogonality requirement and considering, more generally, biorthogonal systems. Such systems also admit interpretations and applications in probability theory (see [R-V1] and [R-V2]).

The definition and essential properties of biorthogonal systems will be based on the Macdonald relation (1.17) for Dunkl's scalar product $[\,.\,,.\,]_k$. We start from a rescaled version, which is easily obtained from Lemma 3.1.7: For all $p \in \mathcal{P}_n^N$, $q \in \mathcal{P}_m^N$ and all $\omega > 0$,

$$[p,q]_k \;=\; \sqrt{2\omega}^{\,n+m}\int_{\mathbb{R}^N} e^{-\Delta_k/4\omega}p(x)\, e^{-\Delta_k/4\omega}q(x)\, dm_k^\omega(x) \tag{3.14}$$

with the measure $m_k^\omega$ as defined in (3.6). Notice that both sides of (3.14) are zero for $n \neq m$. Identity (3.14) suggests to construct orthogonal polynomial systems in $L^2(\mathbb{R}^N, dm_k^\omega)$ from orthogonal homogeneous polynomial systems with respect to $[\,.\,,.\,]_k$ via $p \mapsto e^{-\Delta_k/4\omega}p$, and, more generally, biorthogonal systems in $L^2(\mathbb{R}^N, dm_k^\omega)$ from bidual homogeneous polynomial systems with respect to $[\,.\,,.\,]_k$. In the following, we denote by $\mathcal{P}_n^N(\mathbb{R}) := \Pi_{\mathbb{R}}^N \cap \mathcal{P}_n^N$ the vector space of real-valued homogeneous polynomials of degree $n \in \mathbb{Z}_+$ on $\mathbb{R}^N$.

**3.2.1 Definition.** A family $\{\varphi_\nu,\ \psi_\nu\,,\ \nu \in \mathbb{Z}_+^N\} \subset \Pi_{\mathbb{R}}^N$ is called a homogeneous dual system with respect to $[\,.\,,.\,]_k$, if for every $n \in \mathbb{Z}_+$, the sets $\{\varphi_\nu\,,\ \nu \in \mathbb{Z}_+^N,\ |\nu| = n\}$ and $\{\psi_\nu\,,\ \nu \in \mathbb{Z}_+^N,\ |\nu| = n\}$ are dual $\mathbb{R}$-bases of $\mathcal{P}_n^N(\mathbb{R})$ with respect to the scalar product $[\,.\,,.\,]_k$ on $\Pi_{\mathbb{R}}^N$.

Notice that a homogeneous dual system $\{\varphi_\nu\, \psi_\nu\,,\ \nu \in \mathbb{Z}_+^N\}$ always satisfies

$$\varphi_0 \;=\; \psi_0 \;=\; 1\,.$$

Moreover, as $\mathcal{P}_n^N \perp \mathcal{P}_m^N$ for $n \neq m$ by Lemma 1.2.4, the complete sets $\{\varphi_\nu\,,\, \nu \in \mathbb{Z}_+^N\}$ and $\{\psi_\nu\,,\, \nu \in \mathbb{Z}_+^N\}$ constitute dual bases of $\Pi_{\mathbb{R}}^N$ with respect to $[\,.\,,.\,]_k$, i.e.

$$[\varphi_\nu, \psi_\mu]_k \,=\, \delta_{\nu,\mu} \quad \text{for all } \nu,\, \mu \in \mathbb{Z}_+^N,$$

where $\delta$ is the Kronecker-symbol. Of particular importance is the special case that the sets $\{\varphi_\nu\}$ and $\{\psi_\nu\}$ coincide; $\{\varphi_\nu\}$ is then called a homogeneous orthonormal system with respect to $[\,.\,,.\,]_k$. Such a system can for example be constructed by Gram-Schmidt orthogonalization within each $\mathcal{P}_n^N(\mathbb{R})$ from an arbitrary ordered $\mathbb{R}$-basis.

*3.2.2 Examples.* (1) If $k = 0$, then the natural choice of a homogeneous orthonormal system with respect to $[\,.\,,.\,]_k$ is $\varphi_\nu(x) \,=\, (\nu!)^{-1/2}x^\nu$.

(2) A simple case is the one-dimensional situation ($G = \mathbb{Z}_2$ acting on $\mathbb{R}$, with multiplicity $k \geq 0$). Here every homogeneous dual system with respect to $[\,.\,,.\,]_k$ is of the form $\{c_n x^n,\, d_n x^n\}$ with suitable normalization constants $c_n,\, d_n \in \mathbb{R}\setminus\{0\}$. In particular there exists, up to sign changes, only one orthonormal system $\{\varphi_n\}$ with respect to $[\,.\,,.\,]_k$.

(3) Suppose that $\{\rho_\nu,\, \psi_\nu\}$ is a homogeneous dual system with respect to $[\,.\,,\,.\,]_0$ (i.e. $k = 0$). Since for $k \geq 0$ the intertwining operator $V_k$ is an isomorphism of $\mathcal{P}_n^N(\mathbb{R})$, the family $\{\varphi_\nu := V_k(\rho_\nu),\, |\nu| = n\}$ is a basis of $\mathcal{P}_n^N(\mathbb{R})$, and Lemma 1.2.4(5) implies that $\{\varphi_\nu,\, \psi_\nu\}$ constitutes a homogeneous dual system with respect to $[\,.\,,.\,]_k$. The most important special case of this construction is obtained for the natural choice $\rho_\nu = \psi_\nu = (\nu!)^{-1/2}x^\nu$.

(4) *The $A_{N-1}$-case: Non-symmetric Jack polynomials.* For the definition of these polynomials, we need a partial order $<_P$ on $\mathbb{Z}_+^N$, which was introduced in [O2] in a more general context; see also [K-S]. For this we need the usual dominance order on the set of partitions with at most $N$ nonzero parts, $\Lambda_N \,=\, \{\lambda = (\lambda_1, \ldots, \lambda_N) \subset \mathbb{Z}_+^N \,:\, \lambda_1 \geq \ldots \geq \lambda_N\}$; it is given by

$$\lambda \leq_D \mu \;:\Longleftrightarrow\; |\lambda| = |\mu| \quad \text{and} \quad \sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i \quad \text{for } 1 \leq j \leq N.$$

This partial order is extended to all compositions as follows: For each $\nu \in \mathbb{Z}_+^N$ there exists a unique permutation $w_\nu \in S_N$ of minimal length and a unique partition $\nu^+ \in \Lambda_N$ such that $\nu \,=\, w_\nu\, \nu^+$. For $\mu,\, \nu \in \mathbb{Z}_+^N$ one then defines $\mu \leq_P \nu$, if either $\mu^+ <_D \nu^+$ or $\mu^+ \,=\, \nu^+$ and $w_\mu \leq w_\nu$ in the Bruhat order of $S_N$ (see Chapter 5 of [Hu] for its definition). Moreover, one defines $\mu <_P \nu$ if and only if $\mu \leq_P \nu$ and $\mu \neq \nu$.

Now consider $G \,=\, S_N$ and fix a multiplicity parameter $k > 0$. Then the associated non-symmetric Jack polynomials $E_\nu\,,\, \nu \in \mathbb{Z}_+^N$, as introduced in [O2] (see also [K-S]), are uniquely defined by the following conditions:

(i) $E_\nu(x) \,=\, x^\nu \,+\, \sum_{\mu <_P \nu} c_{\nu,\mu} x^\mu$ with $c_{\nu,\mu} \in \mathbb{R}$;

(ii) For all $\mu <_P \nu$, $\big(E_\nu(x), x^\mu\big)_k \,=\, 0$;

here the inner product $(.,.)_k$ on $\Pi^N$ is given by

$$(f,g)_k := \int_{\mathbb{T}^N} f(z)\overline{g(z)} \prod_{i<j} |z_i - z_j|^{2k} dz,$$

with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $dz$ being the Haar measure on $\mathbb{T}^N$. Notice that by definition, the set $\{E_\nu, |\nu| = n\}$ forms a vector space basis of $\mathcal{P}_n^N(\mathbb{R})$. We claim that the $E_\nu$, which are orthogonal with respect to $(.,.)_k$ by construction, also form a homogeneous orthogonal basis with respect to the Dunkl inner product $[.,.]_k$ (for $S_N$). This is easily seen as follows: According to [O2], Prop. 2.10, the $E_\nu$ are simultaneous eigenfunctions of the Cherednik operators $\xi_i$ for $S_N$, which were introduced in [Che] and can be written as

$$\xi_i = \frac{1}{k} x_i T_i^S + 1 - N + \sum_{j>i} \sigma_{ij} \qquad (i = 1, \dots, N), \tag{3.15}$$

where the $T_i^S$ are the Dunkl operators of type $A_{N-1}$, c.f. Example 1.1.5 (2). In fact, the $E_\nu$ satisfy $\xi_i E_\nu = \overline{\nu}_i E_\nu$, where the eigenvalues $\overline{\nu} = (\overline{\nu}_1, \dots, \overline{\nu}_N)$ are given explicitly in [O2]. They are distinct, i.e. if $\nu \neq \mu$, then $\overline{\nu} \neq \overline{\mu}$. On the other hand, it follows from (3.15), together with the properties of $[.,.]_k$ (Lemma 1.2.4), that the Cherednik operators $\xi_i$ are symmetric with respect to $[.,.]_k$. This proves that the $E_\nu$ are orthogonal with respect to $[.,.]_k$.

**3.2.3 Definition.** Let $\{\varphi_\nu, \psi_\nu\}$ be a homogeneous dual system with respect to $[.,.]_k$. Then for fixed $\omega > 0$, the associated biorthogonal polynomial systems $\{R_\nu(\omega;.), \nu \in \mathbb{Z}_+^N\}$ and $\{S_\nu(\omega;.), \nu \in \mathbb{Z}_+^N\}$ in $L^2(\mathbb{R}^N, dm_k^\omega)$ are given by

$$R_\nu(\omega;x) := \sqrt{2\omega}^{|\nu|} e^{-\Delta_k/4\omega} \varphi_\nu(x), \quad S_\nu(\omega;x) := \sqrt{2\omega}^{|\nu|} e^{-\Delta_k/4\omega} \psi_\nu(x).$$

Moreover, we define the associated biorthonormal functions

$$r_\nu(\omega;x) := d_k^\omega\, e^{-\omega|x|^2/2} R_\nu(\omega;x), \quad s_\nu(\omega;x) := d_k^\omega\, e^{-\omega|x|^2/2} S_\nu(\omega;x), \quad (\nu \in \mathbb{Z}_+^N)$$

in $L^2(\mathbb{R}^N, w_k)$, with the normalization constant

$$d_k^\omega := \frac{\sqrt{2\omega}^{\gamma+N/2}}{\sqrt{c_k}}.$$

We list elementary properties of these systems, which also justify our terminology.

**3.2.4 Lemma.** *For all $\omega > 0$, the following assertions hold:*

(1) *The $R_\nu(\omega;x)$ and $S_\nu(\omega;x)$ are real-valued polynomials of degree $|\nu|$. They satisfy*

$$R_\nu(\omega;-x) = (-1)^{|\nu|} R_\nu(\omega;x); \qquad R_\nu(\omega;x) = R_\nu(1;\sqrt{\omega}x),$$

*and the same relations hold for the $S_\nu(\omega;.)$.*

(2) *The systems* $\{R_\nu(\omega;.)\}$ *and* $\{S_\nu(\omega;.)\}$ *constitute biorthogonal bases of* $L^2(\mathbb{R}^N, dm_k^\omega)$ *with*

$$\int_{\mathbb{R}^N} R_\nu(\omega;.)\, S_\mu(\omega;.)\, dm_k^\omega \;=\; \delta_{\nu,\mu}.$$

*Moreover, the systems* $\{r_\nu(\omega;.)\}$ *and* $\{s_\nu(\omega;.)\}$ *form biorthogonal bases of* $L^2(\mathbb{R}^N, w_k)$ *with*

$$\int_{\mathbb{R}^N} r_\nu(\omega;.)\, s_\mu(\omega;.)\, w_k(x)dx \;=\; \delta_{\nu,\mu}.$$

(3) *For each* $n \in \mathbb{Z}_+$, *the family* $\{R_\nu(\omega;.),\, |\nu| = n\}$ *constitutes a basis of* $V_n^\omega$, *while the family* $\{r_\nu(\omega;.),\, |\nu| = n\}$ *is a basis of* $W_n^\omega$. *The same statements hold for the systems* $\{S_\nu\}$ *and* $\{s_\nu\}$.

*Proof.* Part (1) follows from Lemma 3.1.7 with $c = -1/4$ and $a = \sqrt{\omega}$. Part (2) is an immediate consequence of formula (3.14), together with the density of $\Pi^N$ in $L^2(\mathbb{R}^N, dm_k^\omega)$. Part (3) is obvious. $\square$

The orthonormal case gives rise to a separate notation:

**3.2.5 Definition.** If $\{\varphi_\nu\}$ is a homogeneous orthonormal system with respect to $[\,.\,,.\,]_k$, then the associated generalized Hermite polynomials and Hermite functions are defined by

$$H_\nu(\omega;x) := \sqrt{2\omega}^{\,|\nu|} e^{-\Delta_k/4\omega}\varphi_\nu(x) \qquad \text{and} \qquad h_\nu(\omega;x) := d_k^\omega\, e^{-\omega|x|^2/2} H_\nu(\omega;x), \quad \nu \in \mathbb{Z}_+^N.$$

By construction, the Hermite polynomials $\{H_\nu(\omega;.),\ \nu \in \mathbb{Z}_+^N\}$ and the Hermite functions $\{h_\nu(\omega;.),\ \nu \in \mathbb{Z}_+^N\}$ form orthonormal bases of $L^2(\mathbb{R}^N, dm_k^\omega)$ and of $L^2(\mathbb{R}^N, w_k)$ respectively.

As a consequence of Lemma 3.2.4 (3), biorthogonal and generalized Hermite systems satisfy second order differential-difference equations according to Theorem 3.1.3 as well as Rodrigues formulas according to Theorem 3.1.8. Moreover, they constitute bases of eigenfunctions for the Dunkl transform. We recapitulate these properties here only for the special case of generalized Hermite systems; for biorthogonal systems they are completely analogous.

**3.2.6 Proposition.** *The generalized Hermite polynomials* $\{H_\nu(\omega;.)\}$ *and Hermite functions* $\{h_\nu(\omega;.)\}$ *associated with the basis* $\{\varphi_\nu\}$ *have the following properties:*

(1) *The* $H_\nu(\omega;.),\ \nu \in \mathbb{Z}_+^N$ *are a basis of eigenfunctions of* $-\Delta_k + 2\omega\rho$ *in* $L^2(\mathbb{R}^N, dm_k^\omega)$, *with*

$$(-\Delta_k + 2\omega\rho)\, H_\nu(\omega;.) \;=\; 2|\nu|\omega \cdot H_\nu(\omega;.).$$

(2) *The* $h_\nu(\omega;.),\ \nu \in \mathbb{Z}_+^N$ *are a basis of eigenfunctions of* $-\Delta_k + \omega^2|x|^2$ *in* $L^2(\mathbb{R}^N, w_k)$, *with*

$$(-\Delta_k + \omega^2|x|^2)\, h_\nu(\omega;.) \;=\; (2|\nu| + 2\gamma + N)\omega \cdot h_\nu(\omega;.).$$

*Moreover, the functions* $h_\nu(1;.)$ *are a basis of eigenfunctions for the Dunkl transform in* $L^2(\mathbb{R}^N, w_k)$, *satisfying* $h_\nu(1;.)^{\wedge k} \;=\; (-i)^{|\nu|} h_\nu(1;.)$.

(3) *(Rodrigues formulas)*

$$H_\nu(\omega;x) \,=\, \Big(\frac{-1}{\sqrt{2\omega}}\Big)^{|\nu|} e^{\omega|x|^2}\varphi_\nu(T)\, e^{-\omega|x|^2}; \quad h_\nu(\omega;x) \,=\, \frac{d_k^\omega}{\sqrt{\omega}^{\,|\nu|}}\,\varphi_\nu(A^+)\, e^{-\omega|x|^2/2}.$$

Notice that in view of (3.13), the Rodrigues formula for the generalized Hermite polynomials can also be written as

$$H_\nu(\omega;x) \,=\, (2\omega)^{-|\nu|/2}\,\varphi_\nu(2\omega x - T)(1). \tag{3.16}$$

We continue with some examples:

*3.2.7 Examples.* (1) *Classical multivariable Hermite polynomials.* Let $k=0$, and choose the standard orthonormal system $\varphi_\nu(x) = (\nu!)^{-1/2}x^\nu$, with respect to $[\,.\,,.\,]_0$. The associated Hermite polynomials are given by

$$H_\nu(\omega;x) \,=\, \frac{\sqrt{2\omega}^{\,|\nu|}}{\sqrt{\nu!}}\prod_{i=1}^N e^{-\partial_i^2/4\omega}(x_i^{\nu_i}) \,=\, \frac{2^{-|\nu|/2}}{\sqrt{\nu!}}\prod_{i=1}^N \widehat{H}_{\nu_i}(\sqrt{\omega}\,x_i), \tag{3.17}$$

where the $\widehat{H}_n$, $n\in\mathbb{Z}_+$ are the classical Hermite polynomials on $\mathbb{R}$ defined by

$$\widehat{H}_n(x) \,=\, (-1)^n\, e^{x^2}\,\frac{d^n}{dx^n}e^{-x^2}\,.$$

(2) *The natural biorthogonal systems associated with $G$ and $k$.* These are the systems $\{R_\nu^0(\omega;.)\}$ and $\{S_\nu^0(\omega;.)\}$ which are obtained for the special choice

$$\varphi_\nu^0(x) := (\nu!)^{-1/2}V_k(x^\nu), \quad \psi_\nu^0(x) := (\nu!)^{-1/2}x^\nu\,.$$

The systems $\{R_\nu^0(\omega;.)\}$ and $\{S_\nu^0(\omega;.)\}$ have been studied in [R-V1], [R-V2], where they have been called generalized Appell systems. Notice that

$$R_\nu^0(\omega;.) \,=\, V_k(H_\nu(\omega;.))$$

with the classical Hermite polynomials $H_\nu(\omega;.)$ from (3.17). The intertwining property of $V_k$ now implies that

$$T_jR_{\nu+e_j}^0(\omega;x) \,=\, \sqrt{2\omega(\nu_j+1)}\cdot R_\nu^0(\omega;x); \quad (j=1,\ldots,N).$$

Finally, the $A_j^+$ are creation operators in the literal sense for the natural eigenstates $s_\nu^0$ of $\mathcal{H}_k$:

$$s_\nu^0(\omega;x) \,=\, d_k^\omega\,\omega^{-|\nu|/2}\cdot\frac{(A^+)^\nu}{\sqrt{\nu!}}\,e^{-\omega|x|^2/2} \,=\, \omega^{-|\nu|/2}\cdot\frac{(A^+)^\nu}{\sqrt{\nu!}}\,s_0^0(\omega;x).$$

In case $G=S_N$, complete symmetrization and anti-symmetrization of these states gives the bosonic and fermionic eigenstates for the $N$-body Calogero Hamiltonian which were constructed in [BHV].

(3) *The one-dimensional case* ($G = \mathbb{Z}_2$ on $\mathbb{R}$, multiplicity parameter $k \geq 0$). Recall that up to sign changes, there exists only one homogeneous orthonormal system with respect to $[\,.,.\,]_k$. The associated Hermite polynomials are given, up to multiplicative constants, by the generalized Hermite polynomials $H_n^k(\sqrt{\omega}x)$ on $\mathbb{R}$. These polynomials can be found e.g. in [Chi] and were further studied in [Ros] in connection with a Bose-like oscillator calculus. The $H_n^k$ are orthogonal with respect to $|x|^{2k}e^{-|x|^2}$ and can be written as

$$\begin{cases} H_{2n}^k(x) = (-1)^n 2^{2n} n!\, L_n^{k-1/2}(x^2), \\ H_{2n+1}^k(x) = (-1)^n 2^{2n+1} n!\, x L_n^{k+1/2}(x^2); \end{cases}$$

here the $L_n^\alpha$ are the Laguerre polynomials of index $\alpha \geq -1/2$, given by

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}\Big(x^{n+\alpha}e^{-x}\Big).$$

All biorthogonal systems coincide, up to multiplicative constants, with the Hermite systems $H_n^k(\sqrt{\omega}x)$.

The following further examples concern generalized Hermite polynomials of type $A_{N-1}$ and $B_N$. They show that our concept of generalized Hermite polynomials includes in particular the generalized Hermite and Laguerre systems of Baker and Forrester [B-F2], [B-F3].

*3.2.8 Examples.* (1) *The $A_{N-1}$-case.* We again assume that the multiplicity parameter $k$ is positive. To stress the dependence on the symmetric group, we use the notation $T_i^S$ and $\Delta_S$ (for the generalized Laplacian), as well as $[\,.\,,.\,]_S$ for Dunkl's bilinear form. In [B-F2], Baker and Forrester study "non-symmetric generalized Hermite polynomials" $E_\nu^{(H)}$, which they define as the unique eigenfunctions of $\Delta_S - 2\rho$ of the form

$$E_\nu^{(H)} \;=\; E_\nu + \sum_{|\mu|<|\nu|} c_{\mu,\nu} E_\mu \qquad (c_{\mu,\nu} \in \mathbb{R}).$$

Here the $E_\nu$, $\nu \in \mathbb{Z}_+^N$ are the non-symmetric Jack polynomials (associated with $S_N$ and $k$) as defined in Example 3.2.2 (3). Remark 3.1.4 now implies that $E_\nu^{(H)} = e^{-\Delta_S/4}E_\nu$. Therefore the $E_\nu^{(H)}$, up to some normalization factors, make up a system of generalized Hermite polynomials for $S_N$ in our sense, with parameter $\omega = 1$.

(2) *A remark on the $B_N$-case.* Suppose that $R$ is a root system of type $B_N$, and $k = (k_0, k_1)$ a nonnegative multiplicity function on it. The associated Dunkl operators and Dunkl Laplacian are denoted by $T_i^B$ and $\Delta_B$, cf. Examples 1.1.5 (3). We consider the space

$$W := \{f \in C^1(\mathbb{R}^N) :\; f(x) = F(x^2) \quad \text{for some } F \in C^1(\mathbb{R}^N)\}$$

of "completely even" $C^1$-functions; here $x^2 = (x_1^2, \ldots, x_N^2)$. The restriction of $\Delta_B$ to $W$

is given by

$$\Delta_B|_W = \Delta + 2k_1\sum_{i=1}^N \frac{1}{x_i}\,\partial_i \,+\, 2k_0\sum_{i<j}\Big(\frac{1}{x_i-x_j}\,(\partial_i-\partial_j) + \frac{1}{x_i+x_j}\,(\partial_i+\partial_j)\Big)$$
$$- 2k_0\sum_{i<j}\Big(\frac{1}{(x_i-x_j)^2} + \frac{1}{(x_i+x_j)^2}\Big)\big(1-s\sigma_{ij}\big).$$

It is easily checked that for completely even $f$, $\Delta_B f$ is also completely even. The operator $(\Delta_B - 2\rho)|_W$ is also of CMS-type. Its completely even polynomial eigenfunctions are discussed in [B-F2] and [B-F3] separately from the Hermite-case; they are called "non-symmetric Laguerre polynomials" and denoted by $E_\nu^{(L)}(x^2)$. It is easy to see that they make up the completely even subsystem of a suitably chosen generalized Hermite system $\{H_\nu(1;.)\}$ for $B_N$ associated with $(k_0,\,k_1)$, where we assume $k_0>0$.

To this end, let again $E_\nu$ denote the $S_N$-type non-symmetric Jack polynomials, corresponding to the multiplicity parameter $k_0$. For $\nu \in \mathbb{Z}_+^N$ set $\widehat{E}_\nu(x) := E_\nu(x^2)$. These modified Jack polynomials form a homogeneous basis of $\Pi^N\cap W$. The non-symmetric Laguerre polynomials of Baker and Forrester can be written as

$$E_\nu^{(L)}(x^2) \,=\, e^{-\Delta_B/4}\widehat{E}_\nu(x)\,.$$

(Notice that the polynomials on the right-hand side are in fact completely even and eigenfunctions of $\Delta_B - 2\rho$, according to Theorem 3.1.3). Involving again the $S_N$-type Cherednik operators (3.15), it is easily checked that the $\widehat{E}_\nu$ are orthogonal with respect to Dunkl's pairing $[\,.\,,.]_B$. In fact, the $\xi_i$ induce operators $\widehat{\xi}_i$ $(i=1,\ldots,N)$ on $W$ by

$$\widehat{\xi}_i\, f(x) := (\xi_i F)(x^2) \quad \text{if } f(x) = F(x^2),$$

c.f. [B-F3]. Thus $\widehat{\xi}_i\,\widehat{E}_\nu = \overline{\nu}_i\,\widehat{E}_\nu$, with the same eigenvalues $\overline{\nu}_i$ as in Example 3.2.2 (3). A short calculation gives

$$\widehat{\xi}_i\, f(x) = \frac{1}{k_0}x_i^2(T_i^S F)(x^2) + \big(1-N+\sum_{j>i}\sigma_{ij}\big)F(x^2)$$
$$= \Big(\frac{1}{2k_0}x_iT_i^B + 1 - N + \sum_{j>i}\sigma_{ij}\Big)f(x).$$

Employing Lemma 1.2.4 again, we obtain that the $\widehat{\xi}_i$ are symmetric with respect to $[\,.\,,.]_B$ on $\Pi^N\cap W$. This yields our assertion by the same argument as in the previous example. We therefore obtain a homogeneous orthonormal system $\{\varphi_\nu\}$ with respect to $[\,.\,,.]_B$ by setting $\varphi_\nu := d_\nu\widehat{E}_\eta$ for $\nu = (2\eta_1,\ldots,2\eta_N)$ (with suitable normalization constants $d_\nu>0$), and completing the set $\{\varphi_\nu\,,\,\nu\in(2\mathbb{Z}_+)^N\}$ by a Gram-Schmidt procedure.

We continue by a simple but useful representation of the Dunkl kernel $E_k$ and its homogeneous parts $E_{k,n}$ in terms of a given dual system $\{\varphi_\nu,\,\psi_\nu\}$. It is the foundation for several further properties of our (bi-)orthogonal systems, which generalize results from [B-F2], [B-F3] and [Ros] obtained for special cases.

**3.2.9 Lemma.** (i) *If $\{\varphi_\nu,\, \psi_\nu\}$ is a homogeneous dual system with respect to $[\,.\,,.\,]_k$, then*

$$E_{k,n}(x,y) \,=\, \sum_{|\nu|=n} \varphi_\nu(x)\,\psi_\nu(y) \quad \textit{for all } n\in\mathbb{Z}_+\,,\; x,y\in\mathbb{C}^N.$$

(ii) *If $\{\varphi_\nu\}$ is orthonormal with respect to $[\,.\,,.\,]_k$, then*

$$E_k(x,y) \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \varphi_\nu(x)\,\varphi_\nu(y) \quad \textit{for all } x,y\in\mathbb{C}^N,$$

*where the convergence is normal on $\mathbb{C}^N\times\mathbb{C}^N$.*

*Proof.* For (i), it suffices to consider the case $x,y\in\mathbb{R}^N$. As a function of $x$, the polynomial $E_{k,n}(x,y)$ is homogeneous of degree $n$. Hence we have

$$E_{k,n}(x,y) = \sum_{|\nu|=n} c_{\nu,\,y}\,\varphi_\nu(x) \quad \text{with} \quad c_{\nu,y} = [E_{k,n}(\,.\,,y),\psi_\nu]_k\,.$$

Repeated application of formula (1.24) for $E_{k,n}$ gives

$$c_{\nu,y} = \psi_\nu(T)E_{k,n}(\,.,y) \,=\, \psi_\nu(y)\,E_{k,0}(\,.,y) \,=\, \psi_\nu(y).$$

This shows (i). To prove (ii), note first that $\overline{\varphi_\nu(x)} = \varphi_\nu(\overline{x})$ for all $\nu$. We can therefore estimate

$$\begin{aligned}\sum_{|\nu|=n} |\varphi_\nu(x)\varphi_\nu(y)| \,&\leq\, \Big(\sum_{|\nu|=n}|\varphi_\nu(x)|^2\Big)^{1/2}\cdot\Big(\sum_{|\nu|=n}|\varphi_\nu(y)|^2\Big)^{1/2}\\ &= E_{k,n}(x,\overline{x})^{1/2}\cdot E_{k,n}(y,\overline{y})^{1/2} \,\leq\, \frac{|x|^n|y|^n}{n!}\,,\end{aligned}$$

where for the last step, inequality (1.25) was used. This implies the assertion. □

We mention the following interesting consequence for the Dunkl kernel:

**3.2.10 Corollary.** *The Dunkl kernel associated with $G$ and $k\geq 0$ satisfies*

$$E_k(x,y) \,\leq\, \sqrt{E_k(x,x)E_k(y,y)} \quad \textit{for all } x,y\in\mathbb{R}^N.$$

*In particular, $E_k(x,x)\geq 1$ for all $x\in\mathbb{R}^N$.*

*Proof.* Let $\{\varphi_\nu\}$ be an arbitrary homogeneous orthonormal system with respect to $[\,.\,,\,.]_k$, and recall that the polynomials $\varphi_\nu$ are real-valued on $\mathbb{R}^N$. The first statement is therefore obtained by applying the Cauchy-Schwarz inequality in representation (ii) of Lemma 3.2.9; the second one then follows by setting $y=0$. □

As a further consequence of Lemma 3.2.9, biorthogonal systems on $\mathbb{R}^N$ can be obtained from a common generating function, which generalizes the well-known generating function of the classical multivariate Hermite polynomials in a canonical way.

**3.2.11 Proposition.** *Suppose that* $\{R_\nu(\omega;.)\}$ *and* $\{S_\nu(\omega;.)\}$ *are biorthogonal systems on* $\mathbb{R}^N$ *corresponding to the homogeneous dual system* $\{\varphi_\nu,\, \psi_\nu\}$ *with respect to* $[.\,,.]_k$. *Then for all* $x, y \in \mathbb{C}^N$,

$$e^{-\langle y,y\rangle/2} E_k(\sqrt{2\omega}\, x, y) \;=\; \sum_{n=0}^{\infty}\sum_{|\nu|=n} R_\nu(\omega;x)\,\psi_\nu(y) \;=\; \sum_{n=0}^{\infty}\sum_{|\nu|=n} S_\nu(\omega;x)\,\varphi_\nu(y).$$

*Both series* $\sum_{n=0}^\infty$ *converge normally on* $\mathbb{C}^N \times \mathbb{C}^N$.

*Proof.* It suffices to prove the first identity; moreover, by Lemma 3.2.4 (1), we may restrict ourselves to the case $\omega = 1/2$. For this, put $L_n(x,y) := \sum_{|\nu|=n} R_\nu(1/2;x)\,\psi_\nu(y)$, and suppose first that $x, y \in \mathbb{R}^N$. By definition of the $R_\nu$ and in view of formula (1.24) for $E_{k,n}$ we may write

$$L_n(x,y) \;=\; e^{-\Delta_k^x/2} E_{k,n}(x,y) \;=\; \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(-1)^j}{2^j j!} \langle y,y\rangle^j\, E_{k,n-2j}(x,y).$$

By analytic continuation, this holds for all $x, y \in \mathbb{C}^N$ as well. Using (1.25), one further obtains

$$S_n(x,y) := \sum_{j=0}^{\lfloor n/2\rfloor} \frac{1}{2^j\, j!} |y|^{2j}\, |E_{k,\,n-2j}(x,y)| \;\leq\; \sum_{j=0}^{\lfloor n/2\rfloor} \frac{|y|^{2j}}{2^j\, j!} \cdot \frac{|x|^{n-2j}|y|^{n-2j}}{(n-2j)!}\,.$$

If $n$ is even, set $m := n/2$ and estimate further as follows:

$$S_n(x,y) \;\leq\; \frac{|y|^{2m}}{2^m\, m!} \sum_{j=0}^{m} \binom{m}{j} |x|^{2(m-j)} \;=\; \frac{1}{m!}\Big(\frac{|y|^2}{2}(1+|x|^2)\Big)^m.$$

A similar estimate holds if $n$ is odd. This entails the normal convergence of the series $\sum_{n=0}^\infty L_n(x,y)$ on $\mathbb{C}^N \times \mathbb{C}^N$, and also that

$$\begin{aligned}\sum_{n=0}^{\infty} L_n(x,y) \;&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j}{2^j\, j!}\, \langle y,y\rangle^j E_{k,\,n-2j}(x,y) \quad (\text{with } \; E_{k,l} := 0 \; \text{for} \; l<0)\\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j\, j!}\, \langle y,y\rangle^j \sum_{n=0}^{\infty} E_{k,\,n-2j}(x,y) \;=\; e^{-\langle y,y\rangle/2} E_k(x,y)\end{aligned}$$

for all $x, y \in \mathbb{C}^N$. □

We conclude this section by a Mehler formula for biorthogonal systems. For this, we need the following integral representation.

**3.2.12 Lemma.** *Let* $p \in \mathcal{P}_n^N$. *Then for all* $x, y \in \mathbb{R}^N$,

$$e^{-|x|^2/2}\, e^{-\Delta_k/2} p(x) \;=\; \int_{\mathbb{R}^N} E_k(x,-iy)\, p(iy)\, dm_k^{1/2}(y).$$

*Proof.* Put $q := e^{-\Delta_k/2}p$. Lemma 3.1.7 with $a = -i$ and $c = -1/2$ shows that $e^{\Delta_k/2}p(x) = i^n q(-ix)$, hence $p = (-i)^n\, e^{-\Delta_k/2} q^*$ with $q^*(x) = q(ix)$. Employing Proposition 1.4.3(2), we thus obtain

$$\int_{\mathbb{R}^N} E_k(x,-iy)\, p(iy)\, dm_k^{1/2}(y) \;=\; \int_{\mathbb{R}^N} E_k(y,-ix)\big(e^{-\Delta_k/2} q^*\big)(y)\, dm_k^{1/2}(y) \;=\; e^{-|x|^2/2}\, q^*(-ix),$$

which yields the stated identity. □

**3.2.13 Theorem.** (Mehler-formula for biorthogonal systems). *Suppose that* $\{R_\nu(\omega;.)\}$ *and* $\{S_\nu(\omega;.)\}$ *are biorthogonal polynomial systems associated with* $G$ *and* $k$*. Then for* $r\in\mathbb{C}$ *with* $|r|<1$ *and all* $x,y\in\mathbb{R}^N$,

$$\sum_{n=0}^\infty\sum_{|\nu|=n}R_\nu(\omega;x)S_\nu(\omega;y)\,r^{|\nu|} \;=\; \frac{1}{(1-r^2)^{\gamma+N/2}}\exp\Big\{-\frac{\omega r^2(|x|^2+|y|^2)}{1-r^2}\Big\}\,E_k\Big(\frac{2\omega r x}{1-r^2},\,y\Big).$$

*Proof.* We may again assume that $\omega=1/2$. Consider the integral

$$M_k(x,y,r):=\int_{\mathbb{R}^N\times\mathbb{R}^N}E_k(-rz,v)E_k(-iz,x)E_k(-iv,y)\,d(m_k^{1/2}\otimes m_k^{1/2})(z,v).$$

The bounds of Corollary 2.4.5 on $E_k$ assure that it converges for all $r\in\mathbb{C}$ with $|r|<1$ and all $x,y\in\mathbb{R}^N$. Now write $E_k(-rz,v)=\sum_{n=0}^\infty r^nE_{k,n}(iz,iv)$ for the above integral, and remember the representation of the $E_{k,n}$ in terms of the underlying homogeneous system $\{\varphi_\nu,\,\psi_\nu\}$. Since

$$\sum_{n=0}^\infty r^n|E_{k,n}(iz,iv)|\le e^{|r||z||v|}$$

(by (1.25)), the dominated convergence theorem yields that

$$\begin{aligned}M_k(x,y,r)&=\sum_{n=0}^\infty r^n\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}E_{k,n}(iz,iv)\,E_k(-iz,x)\,E_k(-iv,y)\,dm_k^{1/2}(z)\,dm_k^{1/2}(v)\\&=\sum_{n=0}^\infty r^n\sum_{|\nu|=n}\Big(\int_{\mathbb{R}^N}E_k(-iz,x)\,\varphi_\nu(iz)\,dm_k^{1/2}(z)\Big)\Big(\int_{\mathbb{R}^N}E_k(-iv,y)\,\psi_\nu(iv)\,dm_k^{1/2}(v)\Big);\end{aligned}$$

From the above lemma we thus obtain

$$M_k(x,y,r)\;=\;e^{-(|x|^2+|y|^2)/2}\sum_{n=0}^\infty r^n\sum_{|\nu|=n}R_\nu(1/2;x)S_\nu(1/2;y). \tag{3.18}$$

On the other hand, iterated integration and repeated application of Theorem 1.3.4(3) and the reproducing formula Prop. 1.4.3(2) show that for real $r$ with $|r|<1$,

$$\begin{aligned}M_k(x,y,r)&=c_k^{-1}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}E_k(-rz,v)\,E_k(-iy,v)\,dm_k^{1/2}(v)\Big)E_k(-iz,x)e^{-|z|^2/2}w_k(z)dz\\&=c_k^{-1}e^{-|y|^2/2}\int_{\mathbb{R}^N}e^{(r^2-1)|z|^2/2}E_k(iry,z)\,E_k(-ix,z)\,w_k(z)\,dz\\&=c_k^{-1}(1-r^2)^{-(\gamma+N/2)}e^{-|y|^2/2}\int_{\mathbb{R}^N}e^{-|u|^2/2}E_k\Big(u\,,\,\frac{iry}{\sqrt{1-r^2}}\Big)E_k\Big(u\,,\,\frac{-ix}{\sqrt{1-r^2}}\Big)w_k(u)\,du\\&=(1-r^2)^{-(\gamma+N/2)}\exp\Big\{-\frac{|x|^2+|y|^2}{2(1-r^2)}\Big\}\,E_k\Big(\frac{rx}{1-r^2},\,y\Big).\end{aligned}$$

By analytic continuation, this holds for $\{r\in\mathbb{C}:|r|<1\}$ as well. Together with (3.18), this finishes the proof. □

**3.2.14 Corollary.** *Let* $\{H_\nu(\omega;.)\}$ *be a generalized Hermite system associated with* $G$ *and* $k$*. Then*

$$\sum_{\nu\in\mathbb{Z}_+^N}H_\nu(\omega;x)H_\nu(\omega;y)\,r^{|\nu|}\;=\;\frac{1}{(1-r^2)^{\gamma+N/2}}\exp\Big\{-\frac{\omega r^2(|x|^2+|y|^2)}{1-r^2}\Big\}\,E_k\Big(\frac{2\omega rx}{1-r^2},\,y\Big), \tag{3.19}$$

*the sum on the left hand side being absolutely convergent for all* $x,y\in\mathbb{R}^N$ *and* $0<r<1$.

## 3.3 An uncertainty principle for the Dunkl transform

In his paper [dJ2], de Jeu proved a quite general uncertainty principle for integral operators with bounded kernel which applies to the Dunkl transform; this result has the form of an $\epsilon-\delta$-concentration principle as first stated in [D-S] for the Fourier transform. However, analogues of the classical variance-based Weyl-Heisenberg uncertainty principle for the Dunkl transform have up to now only been given in the one-dimensional case ([R-V3] and [Roo]). Here we present an extension to general Dunkl transforms in arbitrary dimensions. Our main result is the following generalization of the classical Heisenberg-Weyl uncertainty principle:

**3.3.1 Theorem.** *Let* $f \in L^2(\mathbb{R}^N, w_k)$. *Then*

$$\| \, |x| f \|_{2,w_k} \cdot \| \, |\xi| \widehat{f}^k \|_{2,w_k} \, \geq \, (\gamma + N/2) \cdot \|f\|_{2,w_k} \, . \tag{3.20}$$

*Moreover, equality holds if and only if* $f(x) = c\, e^{-d|x|^2}$ *for some constants* $c \in \mathbb{C}$ *and* $d > 0$.

If the multiplicity function $k$ is identically $0$, then the above result reduces to the classical Weyl-Heisenberg inequality on $L^2(\mathbb{R}^N)$.

Our proof of Theorem 3.3.1 is based on expansions in terms of generalized Hermite functions. This generalizes a well-known method for the (one-dimensional) classical situation, see e.g. [dBr]. Essentially the same method was used in [Roo], where the result of Theorem 3.3.1 was proven for the one-dimensional case. The additional effort in the general Dunkl setting is only of technical nature, but it requires a zero-centered situation. This is a certain restriction, which cannot easily be removed in the general case. In the one-dimensional case, an uncentered version was proven in [R-V3]. It is based on commutator methods which become difficult to handle in higher dimensions, as a consequence of the involved reflection terms, c.f. Lemma (3.1.6). For comparison with Theorem 3.3.1, we briefly recapitulate the result from [R-V3]:

Let $G = \mathbb{Z}_2$ on $\mathbb{R}$ and $k$ a nonnegative multiplicity parameter on its root system. Let further $Q$ be the multiplication operator on $L^2(\mathbb{R}, w_k)$ defined by $Qf(x) := xf(x)$, with domain

$$\mathcal{D}(Q) \, = \, \{f \in L^2(\mathbb{R}, w_k) : xf \in L^2(\mathbb{R}, w_k)\}.$$

For $f \in \mathcal{D}(Q)$ with $\|f\|_{2,w_k} = 1$ the $k$-variance of $f$ is defined by

$$var_k(f) := \|(x - \langle xf, f\rangle_k) f\|_{2,w_k}^2 \, ,$$

with $\langle \, .,. \rangle_k$ denoting the scalar product in $L^2(\mathbb{R}, w_k)$. Put further

$$f_e(x) \, = \, \frac{1}{2}\big(f(x) + f(-x)\big), \quad f_o(x) \, = \, \frac{1}{2}\big(f(x) - f(-x)\big).$$

Then we have

**3.3.2 Theorem.** *([R-V3]) Let* $f \in \mathcal{D}(Q)$ *with* $\widehat{f}^k \in \mathcal{D}(Q)$ *and* $\|f\|_{2,w_k} = 1$. *Then*

$$var_k(f) \cdot var_k(\widehat{f}^k) \, \geq \, \Big(k\big(\|f_e\|_{2,w_k}^2 - \|f_o\|_{2,w_k}^2\big) + \frac{1}{2}\Big)^2. \tag{3.21}$$

*Moreover, equality holds if and only if* $f$ *has the form* $f(x) = de^{-cx^2} \cdot E_k^{\mathbb{Z}_2}(ib, x)$, *where* $c > 0$, $b \in \mathbb{C}$ *and* $d > 0$ *is a suitable normalization constant.*

Notice that this result coincides with the one of Theorem 3.3.1 only for even functions, and that the lower bound in (3.21) is not uniform.

For the proof of Theorem 3.3.1, we fix an arbitrary system $\{H_\nu\,,\ \nu\in\mathbb{Z}_+^N\}$ of generalized Hermite polynomials associated with $G$ and $k\geq 0$ according to Definition 3.2.5, with frequency parameter $\omega=1$. Let further $\{h_\nu\,,\ \nu\in\mathbb{Z}_+^N\}$ be the associated system of generalized Hermite functions. Recall from Proposition 3.2.6 that the $h_\nu$, $\nu\in\mathbb{Z}_+^N$, form an orthonormal basis of eigenfunctions for the Dunkl transform in $L^2(\mathbb{R}^N,w_k)$, with

$$\widehat{h}_\nu^{\,k} \,=\, (-i)^{|\nu|}h_\nu\,. \tag{3.22}$$

Moreover, they satisfy the differential-reflection equation

$$\big(-\Delta_k+|x|^2\big)\,h_\nu \,=\, (2|\nu|+2\gamma+N)\,h_\nu. \tag{3.23}$$

Since $H_\nu$ is a polynomial of degree $|\nu|$ with real coefficients, we have (3N-term) recurrencies of the following form: For $\nu\in\mathbb{Z}_+^N$, let $I_\nu=\{\mu\in\mathbb{Z}_+^N\,:\,||\mu|-|\nu||\leq 1\}$. Then

$$x_jH_\nu \,=\, \sum_{\mu\in I_\nu} c_{\nu,\,\mu}^{j}H_\mu \quad\text{ and }\quad x_jh_\nu \,=\, \sum_{\mu\in I_\nu} c_{\nu,\,\mu}^{j}h_\mu \quad\text{ for } j=1,\ldots,N, \tag{3.24}$$

with coefficients $c_{\nu,\,\mu}^{j}\in\mathbb{R}$. We shall also need the dual counterparts of these recurrences:

**3.3.3 Lemma.**

$$T_jh_\nu \,=\, \sum_{\mu\in I_\nu} i^{1-|\nu|+|\mu|}\,c_{\nu,\,\mu}^{j}\,h_\mu \quad (j=1,\ldots,N,\ \ \nu\in\mathbb{Z}_+^N.) \tag{3.25}$$

*Proof.* By (3.22) and Prop. 1.4.6 (2), we have

$$x_jh_\nu \,=\, i^{|\nu|}x_j\widehat{h}_\nu^{\,k} \,=\, i^{|\nu|-1}(T_jh_\nu)^{\wedge k}\,.$$

On the other hand, it follows from (3.24) that

$$x_jh_\nu \,=\, \sum_{\mu\in I_\nu} c_{\nu,\,\mu}^{j}h_\mu \,=\, \sum_{\mu\in I_\nu} c_{\nu,\,\mu}^{j}i^{|\mu|}\,\widehat{h}_\mu^{\,k}.$$

The assertion now follows form the injectivity of the Dunkl transform. □

We write $\langle\,.,.\rangle_k$ for the scalar product in $L^2(\mathbb{R}^N,w_k)$. The main part in the proof of Theorem 3.3.1 is the following Parseval-type identity.

**3.3.4 Lemma.** *Let $f\in L^2(\mathbb{R}^N,w_k)$. Then*

$$\int_{\mathbb{R}^N}|x|^2\big(|f(x)|^2+|\widehat{f}^{\,k}(x)|^2\big)w_k(x)\,dx \,=\, \sum_{\nu\in\mathbb{Z}_+^N}(2|\nu|+2\gamma+N)\cdot\big|\langle f,h_\nu\rangle_k\big|^2.$$

*Proof.* Fix $j \in \{1,\dots,N\}$. Since the $h_\nu$ form an orthonormal basis of $L^2(\mathbb{R}^N, w_k)$, we can write

$$\int_{\mathbb{R}^N} |x_j|^2 |f(x)|^2 w_k(x)dx \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \left|\langle x_j f, h_\nu\rangle_k\right|^2 \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \left|\langle f, x_j h_\nu\rangle_k\right|^2.$$

By use of (3.24), this becomes

$$\sum_{\nu\in\mathbb{Z}_+^N}\sum_{\mu,\,\rho\in I_\nu} c^j_{\nu,\,\mu} c^j_{\nu,\,\rho}\cdot \langle f, h_\mu\rangle_k \overline{\langle f, h_\rho\rangle_k} \,=\, \sum_{\mu,\,\rho\in\mathbb{Z}_+^N}\Big(\sum_{\nu\in I_\mu\cap I_\rho} c^j_{\nu,\,\mu} c^j_{\nu,\,\rho}\Big)\langle f, h_\mu\rangle_k \overline{\langle f, h_\rho\rangle_k}.$$

Here the last equality is justified by the facts that the involved index sets $I_\nu$ are finite, and that $\mu\in I_\nu \iff \nu\in I_\mu$ holds for all $\nu,\,\mu\in\mathbb{Z}_+^N$. Exploiting (3.24), Prop. 1.4.6(2) and the Parseval identity for the Dunkl transform, one further obtains

$$\int_{\mathbb{R}^N} |x_j|^2 |\widehat{f}^k(x)|^2 w_k(x)dx \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \left|\langle x_j \widehat{f}^k, h_\nu\rangle_k\right|^2 \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \left|\langle \widehat{f}^k, x_j\, \widehat{h}_\nu^k\rangle_k\right|^2 \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \left|\langle f, T_j h_\nu\rangle_k\right|^2.$$

With the recurrency (3.25), this becomes

$$\begin{aligned}\sum_{\nu\in\mathbb{Z}_+^N}\sum_{\mu,\,\rho\in I_\nu} & i^{|\nu|-|\mu|-1} c^j_{\nu,\,\mu}\cdot i^{1-|\nu|+|\rho|}\, c^j_{\nu,\,\rho}\cdot \langle f, h_\mu\rangle_k \overline{\langle f, h_\rho\rangle_k} \\ &= \sum_{\mu,\,\rho\in\mathbb{Z}_+^N}\Big(\sum_{\nu\in I_\mu\cap I_\rho} c^j_{\nu,\,\mu} c^j_{\nu,\,\rho}\Big) i^{|\rho|-|\mu|}\cdot \langle f, h_\mu\rangle_k \overline{\langle f, h_\rho\rangle_k}.\end{aligned}$$

Thus we arrive at

$$\int_{\mathbb{R}^N} |x|^2\big(\left|f(x)\right|^2 + \left|\widehat{f}^k(x)\right|^2\big) w_k(x)\,dx \,=\, \sum_{\mu,\,\rho\in\mathbb{Z}_+^N} A_{\mu,\,\rho}\langle f, h_\mu\rangle\overline{\langle f, h_\rho\rangle}, \tag{3.26}$$

where

$$A_{\mu,\,\rho} \,=\, \big(1 + i^{|\rho|-|\mu|}\big)\cdot \sum_{j=1}^N \sum_{\nu\in I_\mu\cap I_\rho} c^j_{\nu,\,\mu} c^j_{\nu,\,\rho}.$$

On the other hand, a short calculation, using (3.24) and (3.25), shows that

$$\big(|x|^2 - \Delta_k\big)h_\nu \,=\, \sum_{j=1}^N \sum_{\mu\in I_\nu}\sum_{\rho\in I_\mu} c^j_{\nu,\,\mu} c^j_{\mu,\,\rho}\big(1 + i^{|\rho|-|\nu|}\big)h_\rho \,=\, \sum_{\rho\in\mathbb{Z}_+^N} A_{\nu,\,\rho}\, h_\rho, \tag{3.27}$$

where for the last identity, the fact was used that the coefficients $c^j_{\nu,\,\mu}$ are symmetric in their subscripts:

$$c^j_{\nu,\,\mu} \,=\, \int_{\mathbb{R}^N} x_j h_\nu(x) h_\mu(x) w_k(x)dx \,=\, c^j_{\mu,\nu}.$$

But by equation (3.23), the left-hand side of (3.27) is equal to $(2|\nu| + 2\gamma + N)\, h_\nu$ . The linear independence of the $h_\nu$ now implies that

$$A_{\nu,\,\rho} \,=\, \begin{cases} 0 & \text{if } \rho\neq\nu, \\ 2|\nu| + 2\gamma + N & \text{if } \rho = \nu. \end{cases}$$

Together with (3.26), this yields the assertion. □

Since $h_0$ is a constant multiple of $e^{-|x|^2/2}$, we obtain as an immediate consequence the following

**3.3.5 Corollary.** *For* $f \in L^2(\mathbb{R}^N, w_k)$,

$$\int_{\mathbb{R}^N} |x|^2 \big(|f(x)|^2 + |\widehat{f}^k(x)|^2\big) w_k(x)\, dx \;\geq\; (2\gamma + N) \cdot \|f\|_{2,w_k}^2 .$$

*Moreover, equality holds if and only if* $f(x) \,=\, c\, e^{-|x|^2/2}$ *with some constant* $c \in \mathbb{C}$.

*Proof of Theorem 3.3.1.* We may assume that $\|f\|_{2,w_k} = 1$. For $s > 0$ define $f_s(x) := s^{-\gamma - N/2} f\big(\frac{x}{s}\big)$. Short calculations, having in mind that $w_k$ is homogeneous of degree $2\gamma$, show that

$$\|f_s\|_{2,w_k} = 1 \quad \text{and} \quad \widehat{f_s}^k(\xi) \,=\, s^{\gamma + N/2} \cdot \widehat{f}^k(s\xi) \quad \text{for all } s > 0 \text{ and } \xi \in \mathbb{R}^N.$$

The above corollary implies that

$$\Phi_f(s) := \int_{\mathbb{R}^N} |x|^2 \big(|f_s(x)|^2 + |\widehat{f_s}^k(x)|^2\big) w_k(x)\, dx \;\geq\; 2\gamma + N .$$

On the other hand, we can write

$$\Phi_f(s) \;=\; s^2 \cdot \|\, |x| f\|_{2,w_k}^2 + \frac{1}{s^2} \cdot \|\, |x| \widehat{f}^k \|_{2,w_k}^2 .$$

It is easily checked that $s \mapsto \Phi_f(s)$ attains a minimum on $]0, \infty[$, namely

$$2 \cdot \|\, |x| f\|_{2,w_k} \cdot \|\, |x| \widehat{f}^k\|_{2,w_k} .$$

This implies (3.20). Further, equality in (3.20) holds if and only if $\min_{s \in (0,\infty)} \Phi_f(s) \;=\; 2\gamma + N$. By the second part of the corollary, this condition is satisfied if and only if $f(x) = c\, e^{-s^2|x|^2/2}$ with some constants $c \in \mathbb{C}$ and $s > 0$. This finishes the proof. $\square$

# Chapter 4

# Heat kernels for finite reflection groups

The positivity of the Dunkl kernel $E_k(x,y)$ for real arguments, due to our main theorem in Chapter 2, is the cornerstone for the investigations of this final Chapter. In its first section, we introduce generalized heat kernels for Dunkl operators and construct various classes of semigroups by them, following well-known classical concepts. The most important semigroup in this context will be the generalized heat semigroup, which in particular leads to a solution of the Cauchy problem for the Dunkl-type heat operator $\Delta_k - \partial_t$ on $(0,\infty)\times\mathbb{R}^N$, with initial data in $C_b(\mathbb{R}^N)$. To obtain uniqueness results, Section 4.2 provides analogues of the maximum principles for the classical heat operator in bounded and unbounded domains; of course, if unbounded, the underlying domain has to be group-invariant in our setting. In Section 4.3, several related semigroups are constructed, such as a variant of the classical Cauchy semigroup, which can be constructed from the heat semigroup via subordination, generalized oscillator semigroups, and the unitary semigroup of the time-dependent, Dunkl-type Schrödinger equation in $L^2(\mathbb{R}^N, w_k)$. The last section is devoted to a study of the short-time asymptotic behaviour of the generalized heat kernel.

As before, $G$ is a finite reflection group on $\mathbb{R}^N$, with root system $R$ and a non-negative multiplicity function $k$ on $R$. Moreover, $\gamma := \sum_{\alpha\in R_+} k(\alpha)$.

## 4.1 Heat semigroups

This section deals with strongly continuous one-parameter semigroups related to Dunkl's Laplacian $\Delta_k$ and with associated Cauchy problems. The basic semigroups under consideration are those which are generated by $\Delta_k$ (more precisely, by its closure) on several function spaces, including $L^2(\mathbb{R}^N, w_k)$ and $(C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$. These semigroups are positivity-preserving contraction semigroups; in fact, they are governed by an integral kernel which can be considered as a natural generalization of the classical Gaussian heat kernel; it is nonnegative as a consequence of our positivity result for Dunkl's intertwining operator. We start with an explicit construction of this generalized heat kernel.

**4.1.1 Lemma.** *For parameters $a \geq 0$ and $b \in \mathbb{R} \setminus \{0\}$, the function*

$$u(t,x) = \frac{1}{(a-bt)^{\gamma+N/2}} \exp\Big\{\frac{b|x|^2}{4(a-bt)}\Big\}$$

*solves the generalized heat equation $\Delta_k u = \partial_t u$ on $(-\infty, a/b) \times \mathbb{R}^N$ in case $b > 0$, and on $(a/b, \infty) \times \mathbb{R}^N$ in case $b < 0$.*

*Proof.* It is easily checked that $\sum_{i=1}^N T_i x_i = N + 2\gamma$. Together with the product rule (1.5), this shows that for each $\lambda \in \mathbb{C}$,

$$\Delta_k\big(e^{\lambda|x|^2}\big) = \sum_{i=1}^N T_i\big(2\lambda x_i\, e^{\lambda|x|^2}\big) = 2\lambda\,(N + 2\gamma + 2\lambda|x|^2)\, e^{\lambda|x|^2}. \tag{4.1}$$

From this the assertion is obtained by a short calculation. □

In particular, the function

$$F_k(t,x) = \frac{M_k}{t^{\gamma+N/2}} e^{-|x|^2/4t}, \quad \text{with} \quad M_k := \frac{1}{2^{\gamma+N/2}\, c_k},$$

is a solution of the heat equation $\Delta_k u - \partial_t u = 0$ on $(0,\infty)\times\mathbb{R}^N$. It generalizes the fundamental solution for the classical heat equation $\Delta u - \partial_t u = 0$, which is given by

$$F_0(t,x) = (4\pi t)^{-N/2} e^{-|x|^2/4t}.$$

The normalization constant $M_k$ is chosen such that

$$\int_{\mathbb{R}^N} F_k(t,x)\, w_k(x) dx \;=\; 1 \quad \text{for all } t > 0. \tag{4.2}$$

In the classical case, the free heat kernel on $\mathbb{R}^N$ is obtained from $F_0$ simply by translations. This corresponds to the fact that the solution of “Cauchy problems” for the classical heat operator, with suitable decay of the given initial data, is obtained from the initial data by convolution with the fundamental solution. In the Dunkl setting, we may replace the classical convolution on $\mathbb{R}^N$ by the weak generalized translation on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ as introduced in (1.27). For this, we use the reproducing formula of Proposition 1.4.3(2) and conclude that

$$\widehat{F}_k^k(t,\xi) \;=\; \frac{1}{c_k} e^{-t|\xi|^2}. \tag{4.3}$$

Applying the quoted reproducing formula again, we obtain from the generalized translation (1.27) the representation

$$L_k^{-y} F_k(t,x) \;=\; \frac{M_k}{t^{\gamma+N/2}} e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big). \tag{4.4}$$

This motivates the following

**4.1.2 Definition.** The generalized heat kernel $\Gamma_k$ is given by

$$\Gamma_k(t,x,y) := \frac{M_k}{t^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x, y \in \mathbb{R}^N,\ t > 0.$$

Notice in particular that $y \mapsto \Gamma_k(t,x,y)$ belongs to $\mathscr{S}(\mathbb{R}^N)$ for fixed $t>0,\, x\in\mathbb{R}^N$. We collect a series of further fundamental properties of this kernel.

**4.1.3 Lemma.** *The heat kernel $\Gamma_k$ has the following properties on $(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$:*

(1) $\Gamma_k(t,x,y) \,=\, c_k^{-2}\displaystyle\int_{\mathbb{R}^N} e^{-t|\xi|^2}\,E_k(ix,\xi)\,E_k(-iy,\xi)\,w_k(\xi)d\xi\,.$

(2) $\displaystyle\int_{\mathbb{R}^N}\Gamma_k(t,x,y)\,w_k(y)dy \,=\, 1.$

(3) $\dfrac{M_k}{t^{\gamma+N/2}}\,\min_{g\in G} e^{-|gx-y|^2/4t} \,\le\, \Gamma_k(t,x,y) \,\le\, \dfrac{M_k}{t^{\gamma+N/2}}\,\max_{g\in G} e^{-|gx-y|^2/4t}\,.$

(4) $\Gamma_k(t+s,x,y) \,=\, \displaystyle\int_{\mathbb{R}^N}\Gamma_k(t,x,z)\,\Gamma_k(s,y,z)\,w_k(z)dz.$

(5) *For fixed $y\in\mathbb{R}^N$, the function $u(t,x):=\Gamma_k(t,\,x,y)$ solves the generalized heat equation $\Delta_k u=\partial_t u$ on $(0,\infty)\times\mathbb{R}^N$.*

*Proof.* (1) is clear from (4.3) and the definition of $\Gamma_k$, and (2) follows from (4.2) together with Proposition 1.5.2 (4). The estimates (3) are an immediate consequence of Corollary 2.4.5. For the proof of (4), use (1) and Fubini's theorem to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Gamma_k(t,x,z)\,\Gamma_k(s,y,z)\,w_k(z)dz\\
&\qquad = c_k^{-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-t|\xi|^2}E_k(ix,\xi)E_k(-iz,\xi)\,\Gamma_k(s,y,z)\,w_k(z)w_k(\xi)dz\,d\xi\\
&\qquad = c_k^{-1}\int_{\mathbb{R}^N} e^{-t|\xi|^2}E_k(ix,\xi)\Gamma_k(s,y,.)^{\wedge k}(\xi)w_k(\xi)d\xi\\
&\qquad = c_k^{-2}\int_{\mathbb{R}^N} e^{-(s+t)|\xi|^2}E_k(ix,\xi)\,E_k(-iy,\xi)w_k(\xi)d\xi \,=\, \Gamma_k(t+s,x,y).
\end{aligned}$$

Finally, for (5) remember that $\Delta_k^x\,E_k(ix,\xi) \,=\, -|\xi|^2E_k(ix,\xi)$. Hence the assertion follows at once from representation (1) by interchanging differentiation and integration. This is justified by the decay properties of the integrand and its derivatives in question (see Corollary 2.4.5). □

*4.1.4 Remark.* In the one-dimensional case, the generalized heat kernel was already introduced in [Ros]. In the general case, but only for integer-valued multiplicities, Berest and Molchanov [B-M] constructed the heat kernel for the $G$-invariant part of $\Delta_k$ (in a conjugated version) by shift-operator techniques.

**4.1.5 Definition.** For $f\in L^p(\mathbb{R}^N,w_k)$ $(1\le p\le\infty)$ and $t\ge 0$ set

$$H_k(t)f(x) := \begin{cases}\displaystyle\int_{\mathbb{R}^N}\Gamma_k(t,x,y)f(y)\,w_k(y)dy & \text{if } t>0,\\ f(x) & \text{if } t=0\end{cases} \tag{4.5}$$

Notice that the decay properties of $\Gamma_k$ assure that the integral defining $H_k(t)f(x)$ converges for all $t>0,\, x\in\mathbb{R}^N$. The properties of the operators $H_k(t)$ are most easily described on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$:

**4.1.6 Theorem.** *Let* $f \in \mathscr{S}(\mathbb{R}^N)$. *Then* $u(t,x) := H_k(t)f(x)$ *belongs to* $C_b([0,\infty) \times \mathbb{R}^N) \cap C^2((0,\infty) \times \mathbb{R}^N)$ *and solves the Cauchy problem*

$$\begin{cases} (\Delta_k - \partial_t)\, u = 0 & \text{on } (0,\infty) \times \mathbb{R}^N, \\ u(0,.) = f. \end{cases} \tag{4.6}$$

*Moreover,* $H_k(t)f$ *has the following properties:*

(1) $H_k(t)f \in \mathscr{S}(\mathbb{R}^N)$ *for all* $t > 0$.

(2) $H_k(t+s)\, f \;=\; H_k(t)H_k(s)f$ *for all* $s, t \geq 0$.

(3) $\|H_k(t)f - f\|_{\infty,\mathbb{R}^N} \to 0$ *with* $t \to 0$.

*Proof.* By use of Lemma 4.1.3 (1) and Fubini's theorem, we write

$$\begin{aligned} u(t,x) &= H_k(t)f(x) \\ &= c_k^{-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} E_k(ix,\xi)\, E_k(-iy,\xi)\, e^{-t|\xi|^2} f(y) w_k(\xi) w_k(y)\, d\xi dy \\ &= c_k^{-1} \int_{\mathbb{R}^N} e^{-t|\xi|^2} \widehat{f}^{\,k}(\xi)\, E_k(ix,\xi)\, w_k(\xi) d\xi \quad (t > 0). \end{aligned} \tag{4.7}$$

The invariance of $\mathscr{S}(\mathbb{R}^N)$ under the Dunkl transform makes clear that (1) is satisfied. Moreover, part (2) is an immediate consequence of the reproducing formula for $\Gamma_k$ (Lemma 4.1.3 (4)). As before, it is seen that differentiation may be interchanged with integration in (4.7), and that $\Delta_k u = \partial_t u$ on $(0,\infty) \times \mathbb{R}^N$. In view of the inversion theorem for the Dunkl transform, (4.7) holds for $t = 0$ as well. Using the bound of Corollary 2.4.5 on $E_k$, we thus obtain the inequality

$$\|H_k(t)f - f\|_{\infty,\mathbb{R}^N} \;\leq\; c_k^{-1} \int_{\mathbb{R}^N} |\widehat{f}^{\,k}(\xi)| \big(1 - e^{-t|\xi|^2}\big) w_k(\xi) d\xi,$$

and this integral tends to 0 as $t \to 0$. This yields (3). In particular, it follows that $u$ is continuous and bounded on $\mathbb{R}^N \times [0,\infty)$. $\square$

*4.1.7 Remark.* The integral operators $H_k(t)$, $t \geq 0$, are well-defined also on $\Pi^N$. In fact,

$$H_k(t)p(x) \;=\; e^{t\Delta_k} p(x) \quad \text{for all } t \geq 0,\; p \in \Pi^N. \tag{4.8}$$

This follows from Proposition 1.4.3(1) in case $t = 1/2$, and by use of the rescaling formula 3.1.7 in the general case. Thus for each $p \in \Pi^N$, the function $u(t,x) = H_k(t)p(x)$ is a polynomial solution of the heat equation $\Delta_k u \;=\; \partial_t u$ on $(0,\infty) \times \mathbb{R}^N$ with $u(0,.) = p$.

**4.1.8 Lemma.** *For every* $t > 0$, $H_k(t)$ *defines a continuous linear operator on each of the Banach spaces* $L^p(\mathbb{R}^N, w_k)$ $(1 \leq p \leq \infty)$, $(C_b(\mathbb{R}^N), \|.\|_\infty)$ *and* $(C_0(\mathbb{R}^N), \|.\|_\infty)$, *with norm* $\|H_k(t)\| \leq 1$.

*Proof.* The estimates for the kernel $\Gamma_k$ in Lemma 4.1.3(3) and its normalization ensure that for every $f \in L^\infty(\mathbb{R}^N, w_k)$, we have $H(t)f \in C_b(\mathbb{R}^N)$ with $\|H_k(t)f\|_\infty \leq \|f\|_\infty$. Moreover, if $f \in L^p(\mathbb{R}^N, w_k)$, then Jensen's inequality implies that

$$|H_k(t)f(x)|^p \leq \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\,|f(y)|^p\,w_k(y)dy,$$

and therefore $\|H_k(t)f\|_{p,w_k} \leq \|f\|_{p,w_k}$. Finally, the invariance of $C_0(\mathbb{R}^N)$ under $H_k(t)$ follows from part (1) of the previous theorem, together with the density of $\mathscr{S}(\mathbb{R}^N)$ in $C_0(\mathbb{R}^N)$. □

In the following, $X$ is one of the Banach spaces $L^p(\mathbb{R}^N, w_k)(1 \leq p < \infty)$ or $(C_0(\mathbb{R}^N), \|.\|_\infty)$. We consider the Dunkl Laplacian $\Delta_k$ as a linear operator in $X$ with dense domain $\mathcal{D}(\Delta_k) := \mathscr{S}(\mathbb{R}^N)$.

**4.1.9 Theorem.** (1) *$(H_k(t))_{t\geq 0}$ is a strongly continuous, positivity-preserving contraction semigroup on $X$.*

(2) *$\Delta_k$ is closable, and its closure $\overline{\Delta}_k$ is the generator of the semigroup $(H_k(t))_{t\geq 0}$ on $X$.*

In view of this result, we call $(H_k(t))_{t\geq 0}$ the generalized Gaussian or heat semigroup on $X$.

*Proof.* (1) Theorem 4.1.6(2), together with Lemma 4.1.8 and the density of $\mathscr{S}(\mathbb{R}^N)$ in $X$, ensures that $(H_k(t))_{t\geq 0}$ forms a semigroup of continuous linear operators on $X$. Its positivity is clear by the positivity of $\Gamma_k$. Moreover, in case of $X = (C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$, its strong continuity follows from part (3) of Theorem 4.1.6. It remains to check strong continuity in the case $X = L^p(\mathbb{R}^N, w_k)$, $1 \leq p < \infty$. In view of Lemma 4.1.8, and as $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N, w_k)$, it suffices to show that $\lim_{t\downarrow 0} \|H(t)f - f\|_{p,w_k} = 0$ for all $f \in C_c(\mathbb{R}^N)$; here we may further assume that $f \geq 0$. We then obtain

$$\|H_k(t)f\|_{1,w_k} = \int_{\mathbb{R}^N} H_k(t)f(x)\,w_k(x)dx = \int_{\mathbb{R}^N} f(x)w_k(x)dx = \|f\|_{1,w_k} \quad \text{for } t > 0.$$

Since $\lim_{t\downarrow 0} \|H_k(t)f - f\|_\infty = 0$, a well-known convergence criterion (see for instance [H-St]) implies that $\lim_{t\downarrow 0} \|H_k(t)f - f\|_{1,w_k} = 0$. The estimate

$$\|H_k(t)f - f\|_{p,w_k}^p \leq \|H_k(t)f - f\|_{1,w_k} \cdot \|H_k(t)f - f\|_{\infty,w_k}^{p-1}$$

then entails that $\lim_{t\downarrow 0} \|H_k(t)f - f\|_{p,w_k} = 0$ as well.

(2) The proof is similar as in the classical case. Let $A$ be the generator of the semigroup $(H_k(t))_{t\geq 0}$ on $X$, and let $f \in \mathscr{S}(\mathbb{R}^N)$. Then by Theorem 4.1.6(1), $H_k(t)f \in \mathscr{S}(\mathbb{R}^N)$ for all $t > 0$, and application of the Dunkl transform yields

$$\Big[\frac{1}{t}\big(H_k(t) - id\big)\,f\Big]^{\wedge k}(\xi) = \frac{1}{t}\big(e^{-t|\xi|^2} - 1\big)\widehat{f}^k(\xi).$$

It is easily checked that as $t \downarrow 0$, this tends to $-|\xi|^2\widehat{f}^k(\xi)$ in the topology of $\mathscr{S}(\mathbb{R}^N)$. The Dunkl transform being a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$, we therefore obtain

$$\lim_{t\downarrow 0} \frac{1}{t}\big(H_k(t) - id\,\big)f = (-|\xi|^2\widehat{f}^k)^{\vee k} = \Delta_k f$$

in the topology of $\mathscr{S}(\mathbb{R}^N)$, and therefore in $\|\,.\,\|_{p,w_k}$ as well. It follows that $f$ belongs to the domain $\mathcal{D}(A)$ of $A$. Thus $\mathscr{S}(\mathbb{R}^N) \subset \mathcal{D}(A)$, and $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k$. Moreover, $\mathscr{S}(\mathbb{R}^N)$ is dense in $X$ and invariant under $(H_k(t))_{t\geq 0}$. A well-known characterization of cores for the generators of strongly continuous semigroups (see, for instance, Theorem 1.9 of [Da1]) now implies that $\mathscr{S}(\mathbb{R}^N)$ is a core of $A$. This finishes the proof. □

The above theorem says in particular that $(H_k(t))_{t\geq 0}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^N, w_k)$ in the following sense:

**4.1.10 Definition.** ([Da2]) Let $\mu \in M^+(\mathbb{R}^N)$. A strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on $L^2(\mathbb{R}^N, \mu)$ is called a symmetric Markov semigroup, if it satisfies the following conditions:

(1) The generator $A$ of $(T(t))_{t\geq 0}$ is self-adjoint and non-positive, i.e. $\langle Af, f\rangle \leq 0$ for all $f \in \mathcal{D}(A)$;
(2) $(T(t))_{t\geq 0}$ is positivity-preserving for all $t \geq 0$, i.e. $T(t)f \geq 0$ for $f \geq 0$;
(3) If $f \in L^\infty(\mathbb{R}^N, \mu) \cap L^2(\mathbb{R}^N, \mu)$ then $\|T(t)f\|_{\infty,\mu} \leq \|f\|_{\infty,\mu}$ for all $t \geq 0$.

Theorem 1.4.2 of [Da2] implies the following

**4.1.11 Corollary.** *For $1 < p < \infty$, the semigroup $(H_k(t))_{t\geq 0}$ on $L^p(\mathbb{R}^N, w_k)$ is a bounded holomorphic semigroup (in the sense of [Da1]) in the sector*

$$\Big\{z \in \mathbb{C} : |Arg(z)| < \frac{\pi}{2}\cdot\big(1 - |2/p - 1|\big)\Big\}.$$

*4.1.12 Remarks.* 1. The result that $\overline{\Delta}_k$ generates a symmetric Markov semigroup on $L^2(\mathbb{R}^N, w_k)$ can of course also been seen directly via the Dunkl transform. In fact, according to Proposition 1.4.7, we have $\overline{\Delta}_k = \mathcal{D}_k^{-1} M \mathcal{D}_k$, where $\mathcal{D}_k$ denotes the Dunkl-Plancherel transform on $L^2(\mathbb{R}^N, w_k)$ and $M$ is the non-positive self-adjoint multiplication operator in $L^2(\mathbb{R}^N, w_k)$ defined by

$$Mf(x) = -|x|^2 f(x) \quad \text{with domain } \mathcal{D}(M) = \{f \in \mathcal{H} : |x|^2 f(x) \in \mathcal{H}\}.$$

The operator $M$ generates the strongly continuous contraction semigroup $e^{tM}f(x) = e^{-t|x|^2}f(x)$ on $L^2(\mathbb{R}^N, w_k)$. Since $\overline{\Delta}_k$ is unitarily equivalent to $M$, the linear operator $e^{t\overline{\Delta}_k}$, $t \geq 0$, (defined via spectral calculus) is unitarily equivalent to $e^{tM}$, and $(e^{t\overline{\Delta}_k})_{t\geq 0}$ forms a strongly continuous contraction semigroup on $L^2(\mathbb{R}^N, w_k)$ which is unitarily equivalent to $(e^{tM})_{t\geq 0}$; it is given by

$$e^{t\overline{\Delta}_k} f(x) = c_k^{-1} \int_{\mathbb{R}^N} e^{-t|\xi|^2}\, \widehat{f}^k(\xi)\, E_k(ix, \xi)\, w_k(\xi) d\xi.$$

The Parseval identity for the Dunkl transform shows that this semigroup indeed coincides with the semigroup $(H_k(t))_{t\geq 0}$ on $L^2(\mathbb{R}^N, w_k)$.

2. For $X = (C_0(\mathbb{R}^N), \|.\,,.\|_\infty)$, Theorem 4.1.9 just says that the generalized heat semigroup is a Feller-Markov semigroup, i.e. a (strongly continuous) positive contraction semigroup on

$C_0(\mathbb{R}^N)$. This observation was the starting point in [R-V2] for the construction of an associated semigroup of Markov kernels on $\mathbb{R}^N$, which can be considered as a generalization of the semigroup of a Brownian motion. It is defined as follows:
For $x \in \mathbb{R}^N$ and $A \subset \mathscr{B}(\mathbb{R}^N)$ put

$$P_t(x,A) := \int_A \Gamma_k(t,x,y)w_k(y)dy \quad (t>0), \quad P_0(x,A) := \delta_x(A),$$

with $\delta_x$ denoting the point measure in $x \in \mathbb{R}^N$. Then $(P_t)_{t\geq 0}$ is a semigroup of Markov kernels on $\mathbb{R}^N$ in the following sense:

(1) Each $P_t$ is a Markov kernel, and for all $s,t \geq 0$, $x \in \mathbb{R}^N$ and $A \in \mathscr{B}(\mathbb{R}^N)$,

$$P_s \circ P_t(x,A) := \int_{\mathbb{R}^N} P_t(z,A)\,P_s(x,dz) \;=\; P_{s+t}(x,A).$$

(2) The mapping $[0,\infty) \to M^1(\mathbb{R}^N)$, $t \mapsto P_t(0,\,.\,)$, is weakly continuous.

Moreover, the semigroup $(P_t)_{t\geq 0}$ has the following special property:

(3) $P_t(0,\,.\,)^{\wedge k}(\xi) \;=\; e^{-t|\xi|^2}$ and $P_t(x,.\,)^{\wedge k}(\xi) = E_k(-ix,\xi)\,P_t(0,\,.\,)^{\wedge k}(\xi)$ for all $\xi \in \mathbb{R}^N$.

Here the Dunkl transform of the probability measures $P_t(x,\,.\,)$, $t \geq 0$, is defined by

$$P_t(x,\,.\,)^{\wedge k}(\xi) := \int_{bR^N} E_k(-i\xi,x)\,P_t(x,d\xi).$$

Property (3) is a substitute for translation invariance ($P_t(x+y,A+y) \;=\; P_t(x,A)$ for all $y \in \mathbb{R}^N$), which is satisfied only in the classical case $k=0$. The proof of (1) – (3) is straightforward by the properties of $\Gamma_k$ and Theorem 4.1.9(1). For details and a further study of the semigroup $(P_t)_{t\geq 0}$ and the associated Markov process, we refer to [R-V2].

Now we come back to the general case. It is a basic fact in semigroup theory that for given initial data $f \in \mathcal{D}(\overline{\Delta}_k) \subset X$, the function $u(t) := H_k(t)f$ provides the unique classical solution of the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) \;=\; \overline{\Delta}_k u(t) & \text{for } \; t>0, \\ u(0) \;=\; f; & \end{cases}$$

here “classical” means $u \in C^1([0,\infty),X)$ with $u(t) \in \mathcal{D}(\overline{\Delta}_k)$ for all $t \geq 0$. Of course, Theorem 4.1.9 also leads to a solution of the following classical initial-boundary-value problem for the generalized heat operator $\Delta_k - \partial_t$:
Find $u \in C^2((0,\infty)\times\mathbb{R}^N)$ which is continuous on $[0,\infty)\times\mathbb{R}^N$ and satisfies

$$\begin{cases} (\Delta_k - \partial_t)\,u \;=\; 0 & \text{on } (0,\infty)\times\mathbb{R}^N, \\ u(0,.\,) \;=\; f & \in C_b(\mathbb{R}^N). \end{cases} \tag{4.9}$$

We have the following

**4.1.13 Theorem.** *Let* $f \in C_b(\mathbb{R}^N)$. *Then* $u(t,x) := H_k(t)f(x)$ *is bounded on* $[0,\infty)\times\mathbb{R}^N$ *and solves* (4.9).

*Proof.* In order to see that $u$ is twice continuously differentiable on $(0,\infty)\times\mathbb{R}^N$ with $(\Delta_k - \partial_t)u = 0$, we only have to make sure that the necessary differentiations of $u$ may be carried out under the integral sign in (4.5). One has to use again the estimates of Corollary 2.4.5 for the partial derivatives of $E_k$; these provide sufficient decay properties for the derivatives of $\Gamma_k$, allowing the necessary differentiations under the integral by use of the dominated convergence theorem. From the positivity and normalization of $\Gamma_k$ it is clear that $u$ is bounded on $[0,\infty)\times\mathbb{R}^N$, with $\|u\|_\infty = \|f\|_\infty$. It remains to prove the continuity of $u$ as $t\downarrow 0$. For fixed $x\in\mathbb{R}^N$, consider the net $(m_{x,t})_{t\geq 0}$ of probability measures on $\mathbb{R}^N$, which are defined by

$$dm_{x,t}(y) = \Gamma_k(t,x,y)\, w_k(y)dy.$$

The strong continuity of the semigroup $(H_k(t))_{t\geq 0}$ on $(C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$ implies in particular that for $t\downarrow 0$, the measures $m_{x,t}$ converge weakly to the point measure $\delta_x$, i.e. in the $\sigma(M^1(\mathbb{R}^N), C_0(\mathbb{R}^N))$-topology, which coincides with the $\sigma(M^1(\mathbb{R}^N), C_b(\mathbb{R}^N))$-topology on $M^1(\mathbb{R}^N)$. Therefore $\lim_{t\downarrow 0} H_k(t)f(x) = f(x)$. Together with the continuity of $f$ and the already known continuity of $u$ on $(0,\infty)\times\mathbb{R}^N$, this proves that $u$ is continuous on $[0,\infty)\times\mathbb{R}^N$ as well. □

At this point, it is still open whether our solution of the "Cauchy problem" (4.9) is unique within an appropriate class of functions. As in the classical case, this follows from a maximum principle for the generalized heat operator on $\mathbb{R}^N\times(0,\infty)$, which will be derived in the following section.

## 4.2 Maximum principles

Recall the action of Dunkl's Laplacian on $C^2(\mathbb{R}^N)$, which is given by

$$\Delta_k f = \Delta f + 2\sum_{\alpha\in R_+} k(\alpha)\,\delta_\alpha f$$

with

$$\delta_\alpha f(x) = \frac{\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{f(x)-f(\sigma_\alpha x)}{\langle\alpha,x\rangle^2}\,.$$

This operator may as well be considered on $C^2(\Omega)$ for any open set $\Omega\subset\mathbb{R}^N$ which is invariant under the group operation of $G$. In this section, we first prove a weak maximum principle for $\Delta_k$ on bounded, $G$-invariant domains, and then present maximum principles for the generalized heat operator $\Delta_k - \partial_t$ on domains of the form $(0,T)\times\Omega$, where either $\Omega$ is bounded and $G$-invariant or $\Omega = \mathbb{R}^N$. These results are to a large extent straightforward generalizations of there classical counterparts; the crucial ingredient is the following observation:

**4.2.1 Lemma.** *Let $\Omega \subseteq \mathbb{R}^N$ be open and $G$-invariant. If a real-valued function $f \in C^2(\Omega)$ attains an absolute maximum at $x_0 \in \Omega$, i.e. $f(x_0) = \sup_{x\in\Omega} f(x)$, then*

$$\Delta_k f(x_0) \leq 0 .$$

*Proof.* Let $D^2 f(x)$ denote the Hessian of $f$ at $x \in \Omega$. The given situation enforces that $\nabla f(x_0) = 0$ and $D^2 f(x_0)$ is negative semi-definite; in particular, $\Delta f(x_0) \leq 0$. Moreover, $f(x_0) \geq f(\sigma_\alpha x_0)$ for all $\alpha \in R$, so the statement is obvious in the case that $\langle \alpha, x_0\rangle \neq 0$ for all $\alpha \in R$. If $\langle \alpha, x_0\rangle = 0$ for some $\alpha \in R$, we have to argue more carefully: Choose an open ball $B \subseteq \Omega$ with center $x_0$. Then $\sigma_\alpha x \in B$ for $x \in B$, and $\sigma_\alpha x - x = -\langle \alpha, x\rangle\alpha$. Now Taylor's formula yields

$$f(\sigma_\alpha x) - f(x) = -\langle \alpha, x\rangle \, \langle \nabla f(x), \alpha\rangle + \frac{1}{2}\langle \alpha, x\rangle^2 \, \alpha^t D^2 f(\xi)\alpha \, ,$$

with some $\xi$ on the line segment between $x$ and $\sigma_\alpha x$. It follows that for $x \in B$ with $\langle \alpha, x\rangle \neq 0$ we have $\delta_\alpha f(x) = \frac{1}{2}\alpha^t D^2 u(\xi)\alpha$. Passing to the limit $x \to x_0$ now leads to $\delta_\alpha f(x_0) = \frac{1}{2}\alpha^t D^2 f(x_0)\alpha \leq 0$, which finishes the proof. □

We call a function $f \in C^2(\Omega)$ $k$-subharmonic on $\Omega$, if $\Delta_k f \geq 0$ on $\Omega$. Based on the previous lemma, it is now easy to obtain a weak maximum principle for $k$-subharmonic functions on bounded, $G$-invariant subsets of $\mathbb{R}^N$. Its range of validity is quite general, in contrast to the strong maximum principle in [D2], which is restricted to $k$-harmonic polynomials on the unit ball. Our proof follows the classical one for the usual Laplacian, as it can be found e.g. in [Jo].

**4.2.2 Theorem.** *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and $G$-invariant, and let $f \in C^2(\Omega) \cap C(\overline{\Omega})$ be real-valued and $k$-subharmonic on $\Omega$. Then*

$$\max\nolimits_{\overline{\Omega}}(f) \, = \, \max\nolimits_{\partial\Omega}(f) \, .$$

*Proof.* Fix $\epsilon > 0$ and put $g := f + \epsilon|x|^2$. A short calculation gives $\Delta_k(|x|^2) = 2N + 4\gamma > 0$. Hence $\Delta_k g > 0$ on $\Omega$, and Lemma 4.2.1 shows that $g$ cannot achieve its maximum on $\overline{\Omega}$ at any $x_0 \in \Omega$. It follows that

$$\max\nolimits_{\overline{\Omega}}(f + \epsilon|x|^2) \, = \, \max\nolimits_{\partial\Omega}(f + \epsilon|x|^2)$$

for each $\epsilon > 0$. Consequently,

$$\max\nolimits_{\overline{\Omega}}(f) \, + \, \epsilon\min\nolimits_{\overline{\Omega}}|x|^2 \, \leq \, \max\nolimits_{\partial\Omega}(f) \, + \, \epsilon\max\nolimits_{\partial\Omega}|x|^2.$$

The assertion now follows by letting $\epsilon \to 0$. □

A similar method leads to the following maximum principle for the generalized heat operator $\Delta_k - \partial_t$ on bounded domains. By virtue of Lemma 4.2.1, the proof can be adapted literally from the standard one in the classical case (see e.g. [Jo]); it is therefore omitted here.

**4.2.3 Proposition.** *Suppose that* $\Omega \subset \mathbb{R}^N$ *is open, bounded and* $G$*-invariant. For* $T > 0$ *put*

$$\Omega_T := \Omega \times (0,T) \quad \textit{and} \quad \partial_*\Omega_T := \{(x,t) \in \partial\Omega_T : \, t = 0 \textit{ or } x \in \partial\Omega\}.$$

*Assume further that* $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ *satisfies* $(\Delta_k - \partial_t)u \geq 0$ *in* $\Omega_T$. *Then*

$$\max_{\overline{\Omega}_T}(u) \, = \, \max_{\partial_*\Omega_T}(u).$$

Under a suitable growth condition on the solution, this maximum principle may be extended to the case where $\Omega = \mathbb{R}^N$. The proof is adapted from the one in [dB] for the classical case.

**4.2.4 Theorem.** *(Weak maximum principle for* $\Delta_k - \partial_t$ *on* $\mathbb{R}^N$*.)* *Let* $S_T := \mathbb{R}^N \times (0,T)$ *and suppose that* $u \in C^2(S_T) \cap C(\overline{S}_T)$ *satisfies*

$$\begin{cases} (\Delta_k - \partial_t)\, u \, \geq \, 0 \quad \textit{in } S_T, \\ u(\,.\,,0) \, = \, f\,, \end{cases}$$

*where* $f \in C_b(\mathbb{R}^N)$ *is real-valued. Assume further that there exist positive constants* $C, \lambda, r$ *such that*

$$u(x,t) \, \leq \, C \cdot e^{\lambda |x|^2} \quad \textit{for all } (x,t) \in S_T \quad \textit{with } |x| > r.$$

*Then* $\quad \sup_{\overline{S}_T}(u) \, \leq \, \|f\|_\infty$.

*Proof.* Let us first assume that $8\lambda T < 1$. For fixed $\epsilon > 0$ set

$$v(x,t) := u(x,t) - \epsilon \cdot \frac{1}{(2T-t)^{\gamma + N/2}} \exp\left\{\frac{|x|^2}{4(2T-t)}\right\}, \quad (x,t) \in \mathbb{R}^N \times [0, 2T).$$

By Lemma 4.1.1, $v$ satisfies $(\Delta_k - \partial_t)v \, = \, (\Delta_k - \partial_t)u \, \geq \, 0$ in $S_T$. Now fix some constant $\rho > r$ and consider the bounded cylinder $\Omega_T = \Omega \times (0,T)$ with $\Omega = \{x \in \mathbb{R}^N : |x| < \rho\}$. Setting $M := \|f\|_\infty$, we have $v(x,0) < u(x,0) \leq M$ for $x \in \overline{\Omega}$. Moreover, for $|x| = \rho$ and $t \in (0,T]$

$$v(x,t) \, \leq \, Ce^{\lambda\rho^2} - \epsilon \cdot \frac{1}{(2T)^{\gamma + N/2}} \, e^{\rho^2/8T}.$$

Since $\lambda < (8T)^{-1}$, we see that $v(x,t) \leq M$ on $\partial_*\Omega_T$, provided that $\rho$ is large enough. Then by Proposition 4.2.3 we also have $v(x,t) \leq M$ on $\overline{\Omega}_T$. Since $\rho > r$ was arbitrary, it follows that $v(x,t) \leq M$ on $\overline{S}_T$. Since $\epsilon > 0$ was arbitrary as well, this implies that $u(x,t) \leq M$ on $\overline{S}_T$. If $8\lambda T \geq 1$, we may subdivide $S_T$ into finitely many adjacent open strips of width less than $1/8\lambda$ and apply the above conclusion repeatedly. □

**4.2.5 Corollary.** *The solution of the Cauchy problem* (4.9) *according to Theorem 4.1.13 is unique within the class of functions* $u \in C^2(S_T) \cap C(\overline{S}_T)$ *which satisfy the following exponential growth condition: There exist positive constants* $C, \lambda, r$ *such that*

$$|u(x,t)| \, \leq \, C \cdot e^{\lambda |x|^2} \quad \textit{for all } (x,t) \in S_T \textit{ with } |x| > r.$$

## 4.3 Some further semigroups related to Dunkl's Laplacian

**1. Subordination and Cauchy semigroups**

A standard procedure to obtain new one-parameter semigroups from known ones is by subordination. This principle is based on convolution semigroups of probability measures on the group $(\mathbb{R},+)$ which are supported on $[0,\infty)$. For details, we refer to [Da1], [Be-F], where the subordination principle is worked out very clearly in the related setting of translation invariant Markov kernels on locally compact abelian groups. For the connection with fractional powers we refer to [Go]. One of the most prominent examples in this context is the semigroup of the $N$-dimensional Cauchy process, which is obtained by subordination from the $N$-dimensional Gaussian semigroup. In the Dunkl setting, the same construction will lead to a generalization of the classical Cauchy semigroup and to a solution of the following Dirichlet problem in the upper half space: Find $u\in C([0,\infty)\times\mathbb{R}^N)\cap C^2((0,\infty)\times\mathbb{R}^N)$ which satisfies

$$\begin{cases}(\Delta_k\,+\,\partial_t^2)\,u=0 \quad \text{on } (0,\infty)\times\mathbb{R}^N,\\ u(0,.)=f.\end{cases} \tag{4.10}$$

Consider the heat semigroup $(H_k(t))_{t\geq 0}$ on one of the Banach spaces $X$ as in Theorem 4.1.9. Let further $(\mu_t)_{t\geq0}\subset M^1(\mathbb{R})$ be a convolution semigroup of probability measures on the group $(\mathbb{R},+)$ which is supported on $[0,\infty)$, i.e. satisfying

(1) $\mu_0\,=\,\delta_0$;

(2) $\mu_t\,*\,\mu_s\,=\,\mu_{t+s}$ for all $s,t\geq 0$;

(3) the mapping $t\mapsto\mu_t$ is weakly continuous on $[0,\infty)$;

(4) $\operatorname{supp}\mu_t\subseteq[0,\infty)$ for all $t\geq 0$.

Then the $X$-valued integral

$$S_\mu(t)(f):=\int_0^\infty H_k(s)f\,d\mu_t(s),\quad (t\geq 0)$$

defines a strongly continuous, positivity-preserving contraction semigroup on $X$, the semigroup subordinated to $(H_k(t))_{t\geq0}$ by $(\mu_t)_{t\geq 0}$; (see Section 4 of [Da1]). It is explicitly given by

$$S_\mu(t)(f)\;=\;\int_{\mathbb{R}^N}\Lambda_\mu(t,x,y)f(y)w_k(y)dy\quad\text{for } t>0,$$

with the kernel

$$\Lambda_\mu(t,x,y)\;=\;\int_0^\infty\Gamma_k(s,x,y)\,d\mu_t(s)\quad (t>0,\; x,y\in\mathbb{R}^N). \tag{4.11}$$

We collect some properties of the subordinated kernels $\Lambda_\mu$; they follow immediately from the corresponding properties of $\Gamma_k$ (see Lemma 4.1.3) and the fact that $(\mu_t)_{t\geq0}$ is a convolution semigroup of probability measures. Recall that the Laplace transform of a measure $\mu\in M^1([0,\infty))$ is defined by

$$\mathcal{L}\mu(z):=\int_0^\infty e^{-zs}\,d\mu(s)\quad (\operatorname{Re}z\geq 0).$$

**4.3.1 Lemma.** *The kernels $\Lambda_\mu$ have the following properties on $(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$:*

(1) $\Lambda_\mu(t,x,y) \,=\, c_k^{-2}\displaystyle\int_{\mathbb{R}^N} \mathcal{L}\,\mu_t(|\xi|^2)\, E_k(ix,\xi)\, E_k(-iy,\xi)\, w_k(\xi)d\xi\,.$

(2) $\Lambda_\mu(t,x,y) \,>\, 0$ *and* $\displaystyle\int_{\mathbb{R}^N} \Lambda_\mu(t,x,y)\, w_k(y)dy \,=\, 1.$

(3) $\Lambda_\mu(t+s,x,y) \,=\, \displaystyle\int_{\mathbb{R}^N} \Lambda_\mu(t,x,z)\,\Lambda_\mu(s,y,z)\, w_k(z)dz.$

The Laplace transforms $\mathcal{L}\mu_t$ can be written as

$$\mathcal{L}\mu_t(z) = e^{-t\,f(z)} \qquad (t\geq 0,\;\operatorname{Re} z>0), \tag{4.12}$$

with a unique analytic function $f$ on $\{\operatorname{Re} z>0\}$ which is continuous on $\{\operatorname{Re} z\geq 0\}$. It is well known (see Section 9 of [Be-F]) that the functions which are affiliated in this way with some convolution semigroup of probability measures on $(\mathbb{R},+)$, supported on $[0,\infty)$, are exactly those analytic functions on $\{\operatorname{Re} z>0\}$ whose restrictions to $(0,\infty)$ are Bernstein functions, i.e., $f|_{(0,\infty)}\geq 0$ and $(-1)^nD^nf|_{(0,\infty)}\leq 0$ for all $n\in\mathbb{N}$.

*4.3.2 Example.* For $0<\alpha<1$, the function $f(x) \,=\, x^\alpha$ is a Bernstein function. The corresponding convolution semigroup on $(\mathbb{R},+)$ is the so-called one-sided stable semigroup of order $\alpha$ and is denoted by $(\sigma_t^\alpha)_{t\geq 0}$. The generator of the corresponding semigroup $(S_\alpha(t))_{t\geq 0}$ subordinated to $(H_k(t))_{t\geq 0}$ is the fractional power $(\overline{\Delta}_k)^\alpha$.

Of particular interest within this class of examples is the case $\alpha = 1/2$, corresponding to the Bernstein function $f(x) \,=\, \sqrt{x}$. The convolution semigroup $(\sigma_t^{1/2})_{t\geq 0}$ is given explicitly by

$$d\sigma_t^{1/2}(s) \,=\, 1_{[0,\infty)}(s)\cdot\frac{1}{\sqrt{4\pi}}ts^{-3/2}e^{-t^2/4s}\,; \tag{4.13}$$

c.f. Example 9.23 of [Be-F]. The corresponding subordinated kernel is called the generalized Cauchy kernel associated with the reflection group $G$ and the multiplicity function $k$; we denote it by $C_k$. Here are some further properties of this kernel, complementing those contained in Lemma 4.3.1.

**4.3.3 Lemma.** *The generalized Cauchy kernel $C_k$ has the following properties:*

(1) $C_k(t,x,y) \,=\, c_k^{-2}\displaystyle\int_{\mathbb{R}^N} e^{-t|\xi|}E_k(ix,\xi)E_k(-iy,\xi)\, w_k(\xi)d\xi \quad (t>0,\; x,y\in\mathbb{R}^N).$

(2) $C_k(t,x,0) \,=\, \lambda_k\cdot\dfrac{t}{(t^2+|x|^2)^{\gamma+(N+1)/2}}\,,$ *with* $\lambda_k \,=\, \dfrac{M_k}{\sqrt{\pi}}\cdot 4^{\gamma+N/2}\,\Gamma\big(\gamma+(N+1)/2\big).$

(3) *For fixed $y\in\mathbb{R}^N$, the function $u(t,x) := C_k(t,x,y)$ satisfies $\big(\Delta_k+\partial_t^2\big)\,u \,=\, 0$ on $(0,\infty)\times\mathbb{R}^N$.*

*Proof.* (1) This follows from part (1) of Lemma 4.3.1, as $\mathcal{L}\,\sigma_t^{1/2}(|\xi|^2) \,=\, \exp\{-t|\xi|\}$.
(2) By (4.13), (4.11) and the definition of $\Gamma_k$, the Cauchy kernel $C_k$ can be written as

$$C_k(t,x,y) \,=\, \frac{M_k\,t}{\sqrt{4\pi}}\int_0^\infty s^{-(\gamma+(N+3)/2)}\,e^{-(t^2+|x|^2+|y|^2)/4s}\,E_k\Big(\frac{x}{\sqrt{2s}},\frac{y}{\sqrt{2s}}\Big)ds\,.$$

In case $y = 0$, this can be simplified by use of the substitution $r = (t^2 + |x|^2)/4s$ and the integral representation of the Gamma function:

$$\begin{aligned} C_k(t,x,0) &= \frac{M_k\, t}{\sqrt{4\pi}} \int_0^\infty s^{-(\gamma+(N+3)/2))}\, e^{-(t^2+|x|^2)/4s}\, ds \\ &= \frac{M_k\, t}{\sqrt{4\pi}} \Big(\frac{4}{t^2+|x|^2}\Big)^{\gamma+(N+1)/2} \cdot \int_0^\infty e^{-r}\, r^{\gamma+(N-1)/2}\, dr \\ &= \frac{M_k}{\sqrt{\pi}} \cdot 4^{\gamma+N/2}\, \Gamma\big(\gamma+(N+1)/2\big) \cdot \frac{t}{(t^2+|x|^2)^{\gamma+(N+1)/2}}\,. \end{aligned}$$

(3) This follows from the representation (1) by differentiation under the integral sign. $\square$

The strongly continuous, positivity-preserving semigroup on $X$ which is subordinated to the heat semigroup $(H_k(t))_{t\geq 0}$ by $(\sigma_t^{1/2})$ is a natural generalization of the classical Cauchy semigroup; it is given explicitly by

$$S_k(t)f(x) \,=\, \begin{cases} \displaystyle\int_{\mathbb{R}^N} C_k(t,x,y)f(y)w_k(y)dy & \text{if } \ t>0, \\ f(x) & \text{if } \ t=0. \end{cases}$$

We conclude our discussion of the generalized Cauchy semigroup with the solution of the Dirichlet problem (4.10) on the upper half space.

**4.3.4 Theorem.** *Let $f \in C_b(\mathbb{R}^N)$. Then the function*

$$u(t,x) \,=\, \begin{cases} \displaystyle\int_{\mathbb{R}^N} C_k(t,x,y)f(y)\, w_k(y)dy & \text{if } \ t>0, \\ f(x) & \text{if } \ t=0 \end{cases}$$

*belongs to $C_b([0,\infty)\times\mathbb{R}^N) \cap C^2((0,\infty)\times\mathbb{R}^N)$ and solves the Dirichlet problem* (4.10).

*Proof.* The argument is exactly the same as it was used for the generalized heat equation (Theorem 4.1.13). $\square$

## 2. Generalized oscillator semigroups

For a fixed parameter $\omega > 0$, consider the generalized oscillator Hamiltonian

$$\mathcal{J}_k \,=\, -\Delta_k + 2\omega \sum_{j=1}^N x_j \partial_j$$

with domain $\mathcal{D}(\mathcal{J}_k) := \Pi^N$ in the weighted Hilbert space $L^2(\mathbb{R}^N, dm_k^\omega)$; here $dm_k^\omega$ is the probability measure

$$dm_k^\omega(x) \,=\, c_k^{-1}(2\omega)^{\gamma+N/2}\, e^{-\omega|x|^2} w_k(x)dx$$

as defined in (3.6). In Theorem 3.1.3 it was shown that $\mathcal{J}_k$ is essentially self-adjoint with discrete spectrum; moreover, according to Proposition 3.2.6, each system $\{H_\nu(\omega;\,.),\, \nu \in \mathbb{Z}_+^N\}$

of generalized Hermite polynomials with respect to $G$ and $k$ provides an orthonormal basis of $L^2(\mathbb{R}^N, dm_k^\omega)$ consisting of eigenfunctions of $\mathcal{J}_k$; recall that

$$\mathcal{J}_k\, H_\nu(\omega;\, .) \,=\, 2|\nu|\omega \cdot H_\nu(\omega;\, .). \tag{4.14}$$

To abbreviate notations, we put $H_\nu := H_\nu(\omega;\, .)$ and denote the closure of $\mathcal{J}_k$ by $A$. Moreover, we write $\langle\, .\, ,\, .\, \rangle$ for the scalar product in $L^2(\mathbb{R}^N, dm_k^\omega)$. Then $A$ is just given by

$$A(f) \,=\, \sum_{\nu\in\mathbb{Z}_+^N} 2|\nu|\omega\, \langle f, H_\nu\,\rangle\, f\,,$$

with domain

$$\mathcal{D}(A) \,=\, \{f \in L^2(\mathbb{R}^N, dm_k^\omega) : \sum_{\nu\in\mathbb{Z}_+^N} |\nu|^2 |\langle f, H_\nu\,\rangle|^2 \,<\, \infty\}.$$

The spectral resolution of $A$ directly implies that $-A$ is the generator of a strongly continuous contraction semigroup on $L^2(\mathbb{R}^N, dm_k^\omega)$, namely

$$e^{-tA} f \,=\, \sum_{\nu\in\mathbb{Z}_+^N} e^{-2|\nu|\omega t}\, \langle f, H_\nu\,\rangle H_\nu\,, \quad t \geq 0.$$

Remember now the Mehler formula (3.19) for the generalized Hermite polynomials. It states that for all $x, y \in \mathbb{R}^N$ and $t > 0$,

$$\sum_{\nu\in\mathbb{Z}_+^N} e^{-2|\nu|\omega t}\, H_\nu(x) H_\nu(y) \,=\, M(e^{-2t}, x, y), \tag{4.15}$$

with the generalized Mehler kernel

$$M_k(r, x, y) \,=\, \frac{1}{(1-r^2)^{\gamma+N/2}} \exp\Big\{-\frac{\omega r^2(|x|^2+|y|^2)}{1-r^2}\Big\}\, E_k\Big(\frac{2\omega r x}{1-r^2},\, y\Big), \quad (0<r<1).$$

It is easily seen from the absolute convergence of the sum in (4.15), together with the orthogonality of the generalized Hermite polynomials, that the function $y \mapsto M_k(e^{-2t}, x, y)$ belongs to $L^2(\mathbb{R}^N, dm_k^\omega)$ for each fixed $x \in \mathbb{R}^N$. This shows that for all $t > 0$,

$$e^{-tA} f(x) \,=\, \int_{\mathbb{R}^N} M_k(e^{-2t}, x, y) f(y)\, dm_k^\omega(y) \quad a.e..$$

**4.3.5 Proposition.** *$\big(e^{-tA}\big)_{t\geq 0}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^N, dm_k^\omega)$ in the sense of Definition 4.1.10.*

*Proof.* $A$ is self-adjoint and non-negative, and the semigroup $\big(e^{-tA}\big)_{t\geq 0}$ is positivity-preserving on $L^2(\mathbb{R}^N, dm_k^\omega)$, because the kernel $M_k$ is strictly positive. The $\{H_\nu\,, \nu \in \mathbb{Z}_+^N\}$ being orthonormal with $H_0 \,=\, 1$, we further have

$$\int_{\mathbb{R}^N} M_k(e^{-2t}, x, y)\, dm_k^\omega(y) \,=\, 1 \quad \text{for all } t > 0,\, x \in \mathbb{R}^N. \tag{4.16}$$

This implies that the operators $e^{-tA}$, $t \geq 0$ are also contractive with respect to $\|\, .\, \|_\infty$. □

As a consequence, the generalized oscillator semigroup $\big(e^{-tA}\big)_{t\geq 0}$ also allows an extension to a strongly continuous contraction semigroup on each of the Banach spaces $L^p(\mathbb{R}^N, dm_k^\omega)$. We introduce the following notation:

**4.3.6 Definition.** For $f \in L^1(\mathbb{R}^N, dm_k^\omega)$ and $t \geq 0$ set

$$O_k(t)f(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} M_k(e^{-2t}, x, y) f(y)\, dm_k^\omega(y)dy & \text{if } \ t > 0, \\ f(x) & \text{if } \ t = 0 \end{cases} \tag{4.17}$$

**4.3.7 Corollary.** $(O_k(t))_{t\geq 0}$ *is a strongly continuous, positivity-preserving contraction semigroup on each of the Banach spaces* $L^p(\mathbb{R}^N, dm_k^\omega)$, $1 \leq p < \infty$. *It is a bounded holomorphic semigroup in the sector*

$$\Big\{z \in \mathbb{C} : |Arg(z)| < \frac{\pi}{2} \cdot \big(1 - |2/p - 1|\big)\Big\}.$$

*Proof.* This follows from Proposition 4.3.5 together with Theorems 1.4.1 and 1.4.2 of [Da2]. □

Direct inspection shows that the Mehler kernel is related to the Gaussian kernel $\Gamma_k$ via

$$M_k(e^{-2t}, x, y)\, dm_k^\omega(y) = \Gamma_k\Big(\frac{1 - e^{-4\omega t}}{4\omega}, \, e^{-2\omega t}x, y\Big)\, w_k(y)dy \quad (t > 0, \, x \in \mathbb{R}^N). \tag{4.18}$$

The operators $O_k(t)$ can be expressed in terms of the heat operators $H_k(t)$:

$$O_k(t)f(x) = H_k\Big(\frac{1 - e^{-4\omega t}}{4\omega}\Big) f\big(e^{-2\omega t}x\big) \tag{4.19}$$

for all $f \in C_0(\mathbb{R}^N)$ and all $t > 0$. This implies that $(O_k(t))_{t\geq 0}$ leaves both $C_0(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$ invariant.

**4.3.8 Proposition.** $(O_k(t))_{t\geq 0}$ *defines a strongly continuous, positivity-preserving contraction semigroup on* $(C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$. *The Schwartz space* $\mathscr{S}(\mathbb{R}^N)$ *is a core of its generator* $A_0$, *and* $A_0|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k - 2\omega \sum_{j=1}^N x_j \partial_j$.

*Proof.* It is clear from Proposition (4.3.5) that $(O_k(t))_{t\geq 0}$ is a positivity-preserving contraction semigroup on $(C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$. For the remaining parts, we use our knowledge about the heat semigroup on this space, c.f. Theorem 4.1.9. For abbreviation, we put

$$\varphi(t) := (1 - e^{-4\omega t})/4\omega \quad \text{for } \ t \geq 0.$$

If $f \in C_0(\mathbb{R}^N)$, then

$$\|O_k(t)f(x) - f(x)\|_\infty \leq \big\|H_k\big(\varphi(t)\big)f - f\big\|_\infty + \sup_{x\in\mathbb{R}^N} \big|f(e^{-2\omega t}x) - f(x)\big|,$$

and both terms tend to 0 as $t \downarrow 0$. This proves the strong continuity. Now let $f \in \mathscr{S}(\mathbb{R}^N)$, and suppose that $f$ is real-valued. Then

$$\frac{O_k(t)f(x) - f(x)}{t} = \frac{H_k\big(\varphi(t)\big)f\big(e^{-2\omega t}x\big)}{\varphi(t)} \cdot \frac{\varphi(t)}{t} + \frac{f\big(e^{-2\omega t}x\big) - f(x)}{t}. \tag{4.20}$$

Moreover,

$$\lim_{t\downarrow 0}\frac{\varphi(t)}{t} = 1 \quad \text{and} \quad \lim_{t\downarrow 0}\big\|\big(H_k(\varphi(t))(f)-f\big)/\varphi(t) \,-\, \Delta_k f\,\big\|_\infty = 0.$$

This shows that the first summand in (4.20) tends to $\Delta_k f$ uniformly on $\mathbb{R}^N$ as $t\downarrow 0$. Further, the mean value theorem ensures that there exists some $\lambda_t\in(e^{-2\omega t},1)$ with

$$\frac{f(e^{-2\omega t}x)-f(x)}{t} = \langle\nabla f(\lambda_t x),x\rangle\cdot\frac{e^{-2\omega t}-1}{t},$$

which uniformly tends to $-2\omega\langle\nabla f(x),x\rangle$ as $t\downarrow 0$. This shows that $\mathscr{S}(\mathbb{R}^N)\subseteq D(A_0)$ and $A_0|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j$. Since $\mathscr{S}(\mathbb{R}^N)$ is invariant under $(O_k(t))_{t\geq 0}$, it follows again from Theorem 1.9 of [Da1] that it provides a core of $A_0$. □

**4.3.9 Proposition.** *For each $f\in C_b(\mathbb{R}^N)$, the function $u(t,x) := O_k(t)f(x)$ belongs to $C_b([0,\infty)\times\mathbb{R}^N)\cap C^2((0,\infty)\times\mathbb{R}^N)$ and solves the Cauchy problem*

$$\begin{cases}\partial_t u = (\Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j)\,u \quad \text{on } (0,\infty)\times\mathbb{R}^N,\\ u(0,.) = f.\end{cases} \tag{4.21}$$

*Proof.* With the same notations as above, we write $u(t,x) = H_k\big(\varphi(t)\big)f(e^{-2\omega t}x)$ for $t\geq 0$ and $x\in\mathbb{R}^N$. Thus, it follows from Theorem 4.1.13 that $u\in C_b([0,\infty)\times\mathbb{R}^N)\cap C^2((0,\infty)\times\mathbb{R}^N)$. Further, formula (4.15) for the Mehler kernel $M_k$ and the differential equation (4.14) for the generalized Hermite polynomials imply that the function $\widetilde{M}_k(t,x) := M_k(e^{-2\omega t},x,y)$ satisfies

$$(\Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j)\widetilde{M}_k = \partial_t\widetilde{M}_k \quad \text{on } (0,\infty)\times\mathbb{R}^N.$$

The proof is then finished by differentiation under the integral in (4.17). □

**3. The free, time-dependent Schrödinger equation**

Consider again the self-adjoint Dunkl Laplacian $\overline{\Delta}_k$ on $L^2(\mathbb{R}^N,w_k)$. By Stone's Theorem (see, e.g. Theorem 4.7 of [Go]), the skew-adjoint operator $i\overline{\Delta}_k$ generates a strongly continuous unitary semigroup $\big(e^{it\overline{\Delta}_k}\big)_{t\geq 0}$ on $L^2(\mathbb{R}^N,w_k)$. The explicit determination of this semigroup can be achieved by standard arguments, see for instance Chapter IX. 1.8 of [Ka1] for the classical case. First, notice that the heat kernel $\Gamma_k$ extends naturally to complex "time" arguments, by

$$\Gamma_k(z,x,y) = \frac{M_k}{z^{\gamma+N/2}}\,e^{-(|x|^2+|y|^2)/4z}\,E_k\Big(\frac{x}{2z},y\Big)$$

for $x,y\in\mathbb{R}^N$ and $z\in\mathbb{C}_- := \mathbb{C}\setminus\{w\in\mathbb{R}: w\leq 0\}$; here $z^{\gamma+N/2}$ is the holomorphic branch in $\mathbb{C}_-$ with $1^{\gamma+N/2}=1$. We next determine the Schrödinger semigroup on a sufficiently large subset of $\mathscr{S}(\mathbb{R}^N)$.

**4.3.10 Lemma.** *If $f(x)=e^{-b|x|^2}$ with a parameter $b>0$, then*

$$e^{it\overline{\Delta}_k}f = \int_{\mathbb{R}^N}\Gamma_k(it,.,y)\,f(y)\,w_k(y)dy \quad \textit{for all } t>0. \tag{4.22}$$

*Proof.* Consider the function

$$u(t,x) := \frac{1}{(1+4ibt)^{\gamma+N/2}}\, e^{-b|x|^2/(1+4ibt)} \quad (t \geq 0,\ \ x \in \mathbb{R}^N).$$

The same calculation as in Lemma 4.1.1 shows that $u$ satisfies the generalized Schrödinger equation

$$\partial_t u \,=\, i\Delta_k u \quad \text{on } (0,\infty)\times\mathbb{R}^N,$$

with $u(0,x) = e^{-b|x|^2}$. It is also easily verified that the function $t \mapsto u(t,\,.\,)$ belongs to $C^1([0,\infty), L^2(\mathbb{R}^N, w_k))$. This shows that $e^{it\overline{\Delta}_k}f = u(t,\,.\,)$ for $t \geq 0$. The reproducing identity in Proposition 1.4.3(2) for $E_k$ implies that for $t \geq 0$,

$$\frac{1}{(1+4bt)^{\gamma+N/2}}\, e^{-b|x|^2/(1+4bt)} \,=\, \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, e^{-b|y|^2} w_k(y)dy.$$

By analytic continuation, this identity remains true if $t$ is replaced by $it$. This completes the proof. □

The following statement is obtained exactly as its classical analogue in [Ka1], by using the Plancherel formula for the Dunkl transform, Lemma 1.5.2(7) for the generalized translation in $L^2(\mathbb{R}^N, w_k)$, and the injectivity of the Dunkl transform.

**4.3.11 Lemma.** *The* $\mathbb{C}$*-linear hull* $\langle M\rangle$ *of the set*

$$M := \big\{x \mapsto L_k^a\, e^{-b|x|^2}, \quad a \in \mathbb{R}^N,\, b>0\big\}$$

*is dense in* $L^2(\mathbb{R}^N, w_k)$.

We thus have shown that on the dense subspace $\langle M\rangle$ of $L^2(\mathbb{R}^N, w_k)$, the linear operators

$$S_k(t)f := \int_{\mathbb{R}^N} \Gamma_k(it,\,.\,,y) f(y)\, w_k(y)dy\,, \quad t>0,$$

coincide with the unitary operators $e^{it\overline{\Delta}_k}$. They can therefore be extended uniquely to unitary operators on $L^2(\mathbb{R}^N, w_k)$, which are written in the same way, the integral now being understood in the $L^2$-sense. In this sense, we have for all $f \in L^2(\mathbb{R}^N, w_k)$,

$$e^{it\overline{\Delta}_k} f \,=\, \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Gamma_k(it,\,.\,,y) f(y)\, w_k(y)dy & \text{if } \ t>0, \\ f & \text{if } \ t=0. \end{cases} \tag{4.23}$$

## 4.4 Short–time estimates for the heat kernel

In this section we study the asymptotic behaviour of the generalized heat kernel $\Gamma_k$ for short times. It will be appropriate to transfer the semigroup $(H_k(t))_{t\geq 0}$ from $L^2(\mathbb{R}^N, w_k)$ to the unweighted space $L^2(\mathbb{R}^N)$, which leads to the strongly continuous contraction semigroup

$$\widetilde{H}_k(t)f := w_k^{1/2} H_k(t)\big(w_k^{-1/2} f\big), \quad f \in L^2(\mathbb{R}^N).$$

The corresponding renormalized heat kernel is given by

$$\widetilde{\Gamma}_k(t,x,y) := \sqrt{w_k(x)w_k(y)}\,\Gamma_k(t,x,y) \quad (x,y\in\mathbb{R}^N,\; t>0).$$

According to our results from Section 4.1 and Lemma 3.1.1, the generator of the semigroup $(\widetilde{H}_k(t))_{t\geq 0}$ is given by $w_k^{1/2}\,\overline{\Delta}_k w_k^{-1/2}$, which is the closure of the operator

$$\mathcal{F}_k \,=\, \Delta - 2\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}(k(\alpha)-\sigma_\alpha) \tag{4.24}$$

with domain

$$\mathcal{D}(\mathcal{F}_k) := \{w_k^{1/2}g:\, g\in\mathscr{S}(\mathbb{R}^N)\}\,\subset L^2(\mathbb{R}^N).$$

Thus $\mathcal{F}_k$ can be considered to be a perturbation of the Laplacian $\Delta$, with singularities of inverse square type in the reflecting hyperplanes. We conjecture that within the Weyl chambers of $G$, the heat kernel $\widetilde{\Gamma}_k(t,x,y)$ behaves for short times like the free Gaussian heat kernel

$$\Gamma_0(t,x,y) \,=\, \frac{M_0}{t^{N/2}}\,e^{-|x-y|^2/4t}\,.$$

In the following, $W$ is an arbitrary fixed Weyl chamber of $G$.

**4.4.1 Conjecture.** *For all $x,y\in W$,*

$$\lim_{t\downarrow 0}\frac{\sqrt{w_k(x)\,w_k(y)}\;\Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} \,=\, 1\,. \tag{4.25}$$

In case of the symmetric group, (4.25) was stated in [B-F3]; there was, however, no rigorous proof given for it. At present, we are not able to prove the above conjecture in full generality. We present two partial results; the first one gives the precise asymptotics under a certain restriction on the arguments, the second one provides a sharp and global lower bound.
For $x,y\in W$, we introduce the notation

$$C(x,y) := \,\max\{\mathrm{dist}\,(x,\partial W),\,\mathrm{dist}\,(y,\partial W)\}\,.$$

Our first result is the following

**4.4.2 Theorem.** *For all $x,y\in W$ with $|x-y|\,<\,C(x,y)$,*

$$\lim_{t\downarrow 0}\frac{\sqrt{w_k(x)\,w_k(y)}\;\Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} \,=\, 1\,.$$

Writing the involved heat kernels in their explicit form, this implies the following ray asymptotics for the Dunkl kernel:

**4.4.3 Corollary.** *If $x,y\in W$ with $|x-y|\,<C(x,y)$, then*

$$\lim_{t\to\infty} t^\gamma\,e^{-\langle tx,\,y\rangle}E_k(tx,y) \,=\, \frac{M_0}{2^\gamma M_k}\cdot\frac{1}{\sqrt{w_k(x)w_k(y)}}\,. \tag{4.26}$$

*4.4.4 Remark.* In the one-dimensional case, the condition on the distance $|x-y|$ is always satisfied; (4.26) then becomes a special case of the well-known asymptotics of Kummer's function ${}_1F_1$ (recall the representation of the Dunkl kernel $E_k^{\mathbb{Z}_2}$ according to Example 1.3.3), namely

$${}_1F_1(k, 2k+1, -x) \sim \frac{(2k)!}{k!}\, x^{-k} \quad (x \to +\infty).$$

Our subsequent proof of Theorem 4.4.2 is based on the maximum principle for the classical heat equation in bounded domains of $\mathbb{R}^N$. It is elementary, but unfortunately somewhat technical with respect to details. By similar methods and a successive extension of the admissible range of arguments, we also obtain

**4.4.5 Theorem.** *Let $x, y \in W$. Then*

$$\liminf_{t\downarrow 0} \frac{\sqrt{w_k(x)\, w_k(y)}\; \Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} \geq 1\,.$$

To start with the proofs of these theorems, we first introduce some notations. For $f \in \mathcal{D}(\mathcal{F}_k)$, define

$$u_f(t,x) \,=\, w_k(x)^{1/2} H_k(t)\big(w_k^{-1/2} f\big)(x),$$

with the heat semigroup $(H_k(t))_{t\geq 0}$ as defined in (4.5). In particular,

$$u_f(t,x) \,=\, \int_{\mathbb{R}^N} \widetilde{\Gamma}_k(t,x,y)\, f(y)\, dy \quad \text{for } t>0,\, x\in\mathbb{R}^N. \tag{4.27}$$

Theorem 4.1.6 shows that $u_f$ belongs to $C([0,\infty)\times\mathbb{R}^N)\cap C^2\big((0,\infty)\times(\mathbb{R}^N\setminus H)\big)$ and satisfies the generalized heat equation

$$(\mathcal{F}_k - \partial_t)\, u_f \,=\, 0 \quad \text{on } (0,\infty)\times(\mathbb{R}^N\setminus H), \quad u_f(0,.) \,=\, f \quad \text{on } \mathbb{R}^N. \tag{4.28}$$

For a parameter $\mu\in\mathbb{R}$ we further set

$$u_{f,\mu}(t,x) \,:=\, e^{-t\mu}\, u_f(t,x) \quad (t\geq 0,\, x\in\mathbb{R}^N).$$

We shall compare $u_{f,\mu}$ with the function

$$v_f(t,x) := \, H_0(t)f(x) \quad (t\geq 0,\, x\in\mathbb{R}^N),$$

which satisfies

$$v_f(t,x) \,=\, \int_{\mathbb{R}^N} \Gamma_0(t,x,y) f(y)\, dy \quad \text{for } t>0,\, x\in\mathbb{R}^N. \tag{4.29}$$

Since $\mathcal{D}(\mathcal{F}_k)\subset C_b(\mathbb{R}^N)$, the function $v_f$ belongs to $C_b([0,\infty)\times\mathbb{R}^N)\cap C^2((0,\infty)\times\mathbb{R}^N)$ and solves the classical heat equation

$$(\Delta - \partial_t)\, v_f \,=\, 0 \quad \text{on } (0,\infty)\times\mathbb{R}^N, \quad v_f(0,.) \,=\, f. \tag{4.30}$$

Finally, we fix the following notations: Let $\text{dist}(\,.\,,.)$ be the Euclidean distance in $\mathbb{R}^N$ and define

$$\begin{aligned} K_r(x) &:= \{y \in \mathbb{R}^N : \, |y-x| < r\} \quad \text{for } r>0, x \in \mathbb{R}^N; \\ W_\delta &:= \{x \in W : \text{dist}(x, \partial W) > \delta, \; |x| < \delta^{-1}\} \quad \text{for } \delta > 0; \\ C_\delta(x,y) &:= \max\{\text{dist}\,(x, \partial W_\delta), \; \text{dist}\,(y, \partial W_\delta)\}. \end{aligned} \tag{4.31}$$

Notice that $W_{\delta'} \subset W_\delta$ for $\delta' > \delta$. Moreover, for an open subset $U \subset \mathbb{R}^N$, let

$$\mathcal{A}(U) := \{f \in \mathcal{D}(\mathcal{F}_k) : \, \text{supp}\, f \subseteq U, \; f \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^N} f\,dx = 1\}.$$

We start with the following auxiliary result concerning certain means of the Dunkl kernel; it is based on the positive integral representation for $V_k$ according to Theorem 2.4.1.

**4.4.6 Lemma.** *For all $x, y \in \mathbb{R}^N$,*

$$\sum_{\alpha \in R_+} k(\alpha) \, \frac{\langle \alpha, y\rangle}{\langle \alpha, x\rangle} \big(E_k(x,y) - E_k(\sigma_\alpha x, y)\big) \, \geq \, 0.$$

*Proof.* In the following, $y \in \mathbb{R}^N$ is considered to be a fixed parameter, and differentiations are carried out with respect to the variable $x$. The kernel $E_k(\,.\,,y)$ being an eigenfunction of every Dunkl operator (see Theorem 1.3.1), we have

$$|y|^2 \, E_k(x,y) \, = \, T_y E_k(x,y) \, = \, \partial_y E_k(x,y) \, + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, y\rangle \, \frac{E_k(x,y) - E_k(\sigma_\alpha x, y)}{\langle \alpha, x\rangle}\,. \tag{4.32}$$

Employing the integral representation (2.19) for $E_k$, we see that

$$\partial_y E_k(x,y) \, = \, \partial_y \int_{|\eta| \leq |y|} e^{\langle x, \eta\rangle} \, d\mu_y^k(\eta) \, = \, \int_{|\eta| \leq |y|} \langle y, \eta\rangle \cdot e^{\langle x, \eta\rangle} \, d\mu_y^k(\eta) \, \leq \, |y|^2 E_k(x,y).$$

Together with (4.32), this proves the assertion. □

**4.4.7 Corollary.** *For all $x, y \in \mathbb{R}^N$ and $t > 0$,*

$$\sum_{\alpha \in R_+} k(\alpha) \, \frac{\langle \alpha, y\rangle}{\langle \alpha, x\rangle} \big(\widetilde{\Gamma}_k(t,x,y) - \widetilde{\Gamma}_k(t, \sigma_\alpha x, y)\big) \, \geq \, 0.$$

The next result is the fundamental ingredient for the proofs of both theorems stated above.

**4.4.8 Lemma.** *Let $\delta > 0$. Then there exist constants $m_\delta, M_\delta \in \mathbb{R}$ such that for all $f \in \mathcal{A}(W_\delta)$,*

$$(\Delta - \partial_t)(u_{f,\mu} - v_f) \begin{cases} \geq 0 & \text{on } (0,\infty) \times W_\delta \;\; \text{for } \mu \geq M_\delta, \\ \leq 0 & \text{on } (0,\infty) \times W_\delta \;\; \text{for } \mu \leq m_\delta. \end{cases} \tag{4.33}$$

*Proof.* Putting together (4.28) and (4.30) and observing (4.24), we obtain that for $t > 0$ and $x \in \mathbb{R}^N \setminus H$,

$$(\Delta \, - \, \partial_t)(u_{f,\mu} - v_f)(t,x) \, = \, \Big(\mu + 2 \sum_{\alpha \in R_+} \frac{k(\alpha)}{\langle \alpha, x\rangle^2} (k(\alpha) \, - \, \sigma_\alpha^x)\Big) \, u_{f,\mu}(t,x); \tag{4.34}$$

here the superscript in $\sigma_\alpha^x$ indicates the operation in the variable $x$. Now let $x, y \in W_\delta$. Then $2\delta^2 \leq \langle \alpha, x\rangle\langle\alpha, y\rangle \leq 2/\delta^2$ (recall that $|\alpha| = \sqrt{2}$). Together with Corollary 4.4.7, this leads to the estimate

$$\begin{aligned}\sum_{\alpha\in R_+} \frac{k(\alpha)}{\langle\alpha,x\rangle^2}\,\widetilde{\Gamma}_k(t,\sigma_\alpha x, y) &\leq \frac{1}{2\delta^2}\sum_{\alpha\in R_+} k(\alpha)\frac{\langle\alpha,y\rangle}{\langle\alpha,x\rangle}\,\widetilde{\Gamma}_k(t,\sigma_\alpha x,y)\\ &\leq \frac{1}{2\delta^2}\sum_{\alpha\in R_+} k(\alpha)\frac{\langle\alpha,y\rangle}{\langle\alpha,x\rangle}\,\widetilde{\Gamma}_k(t,x,y) \leq \frac{1}{\delta^4}\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}\,\widetilde{\Gamma}_k(t,x,y)\,.\end{aligned}$$

It follows that for $x \in W_\delta$, $t > 0$ and $f \in \mathcal{A}(W_\delta)$,

$$\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}\cdot\sigma_\alpha^x\, u_{f,\mu}(t,x) \leq \frac{1}{\delta^4}\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}\cdot u_{f,\mu}(t,x).$$

The assertion is now immediate. □

For each $\delta > 0$ fix now constants $m_\delta, M_\delta \in \mathbb{R}$ according to Lemma 4.4.8. We may in addition assume that $M_\delta > 0$. This will simplify some estimates.

**4.4.9 Lemma.** *Let $\delta > 0$. Then for all $x, y \in W_\delta$ and $t > 0$,*

(1) $\Gamma_0(t,x,y) - e^{-tm_\delta}\,\widetilde{\Gamma}_k(t,x,y) \leq \dfrac{M_0}{t^{N/2}}\,e^{-C_\delta(x,y)^2/4t}$.

(2) $e^{-tM_\delta}\,\widetilde{\Gamma}_k(t,x,y) - \Gamma_0(t,x,y) \leq (2\delta^{-2})^\gamma\cdot\dfrac{M_k}{t^{\gamma+N/2}}\,e^{-C_\delta(x,y)^2/4t}$.

*Proof.* (1) Fix $y \in W_\delta$. For abbreviation, put $\mu := m_\delta$ and choose $h > 0$ sufficiently small such that $\overline{K_h}(y) \subset W_\delta$. Take an arbitrary test function $f \in \mathcal{A}\big(K_h(y)\big)$ and consider the function $v_f - u_{f,\mu}$ on $[0,\infty)\times W_\delta$. First of all,

$$(v_f - u_{f,\mu})(0,x) = 0 \quad \text{for all } x \in W_\delta.$$

Put $C_\delta := \operatorname{dist}(y, \partial W_\delta)$. Since $u_{f,\mu} \geq 0$, and by the definition of $v_f$, we further obtain for all $t > 0$ and $x \in \partial W_\delta$ the estimate

$$(v_f - u_{f,\mu})(t,x) \leq v_f(t,x) \leq \frac{M_0}{t^{N/2}}\,e^{-(C_\delta-h)^2/4t}.$$

Moreover, Lemma 4.4.8 ensures that

$$(\Delta - \partial_t)(v_f - u_{f,\mu}) \geq 0 \quad \text{on } (0,\infty)\times W_\delta.$$

The maximum principle for $\Delta - \partial_t$ on $(0,\infty)\times W_\delta$ therefore implies that

$$v_f - u_{f,\mu} \leq \frac{M_0}{t^{N/2}}\,e^{-(C_\delta-h)^2/4t} \quad \text{on } (0,\infty)\times W_\delta.$$

Since $f \in \mathcal{A}\big(K_h(y)\big)$ was arbitrary, we conclude that

$$\Gamma_0(t,x,\eta) - e^{-t\mu}\widetilde{\Gamma}_k(t,x,\eta) \leq \frac{M_0}{t^{N/2}}e^{-(C_\delta-h)^2/4t} \quad \text{for all } (t,x,\eta) \in (0,\infty)\times W_\delta\times K_h(y).$$

In particular, by taking the limit $h \to 0$,

$$\Gamma_0(t,x,y) \,-\, e^{-t\mu}\widetilde{\Gamma}_k(t,x,y) \,\leq\, \frac{M_0}{t^{N/2}} e^{-C_\delta^2/4t} \quad \text{for all } (t,x) \in (0,\infty)\times W_\delta\,.$$

The assertion now follows by the symmetry of (1) in $x$ and $y$.
(2) The proof is very similar to the previous one; we therefore restrict ourselves to a short outline. We put $\mu := M_\delta > 0$ and consider now, for $f \in \mathcal{A}\big(K_h(y)\big)$, the function $u_{f,\mu} - v_f$ on $[0,\infty)\times W_\delta$. We use the fact that for all $x,\, y \in W$ and all $g \in G$, the inequality $|gx-y| \geq |x-y|$ holds; see, for instance, Theorem 3.1.2 of [G-B]. Together with the bounds on $\Gamma_k$ according to Lemma 4.1.3(3), we obtain the following estimate, valid for all $t>0$ and $x \in W_\delta$:

$$(u_{f,\mu} - v_f)(t,x) \,\leq\, u_f(t,x) \,\leq\, (2\delta^{-2})^\gamma \cdot \frac{M_k}{t^{\gamma+N/2}}\, e^{-(C_\delta - h)^2/4t}\,.$$

Moreover, Lemma 4.4.8 implies that

$$(\Delta - \partial_t)(u_{f,\mu} - v_f) \,\geq\, 0 \quad \text{on } (0,\infty)\times W_\delta.$$

The assertion is now obtained in the same way as above. □

Theorem 4.4.2 is an easy consequence of this lemma.

*Proof of Theorem 4.4.2.* Suppose that $x,y \in W$ satisfy the stated condition. Choose $\delta > 0$ sufficiently small such that $x,y \in W_\delta$. Then Lemma 4.4.9 implies the relations

$$\begin{aligned} 1 \,-\, e^{-tm_\delta}\,\frac{\widetilde{\Gamma}_k(t,x,y)}{\Gamma_0(t,x,y)} \,&\leq e^{\,(|x-y|^2 - C_\delta(x,y)^2)/4t}\,, \\ e^{-tM_\delta}\,\frac{\widetilde{\Gamma}_k(t,x,y)}{\Gamma_0(t,x,y)} \,-\, 1 \,&\leq (2\delta^{-2})^\gamma \cdot \frac{M_k}{M_0\, t^\gamma}\, e^{\,(|x-y|^2 - C_\delta(x,y)^2)/4t}\,. \end{aligned}$$

The expressions on the right-hand side tend to $0$ as $t \downarrow 0$, provided that $|x-y| < C_\delta(x,y)$. Under this condition on $x,\, y \in W_\delta$, we therefore obtain

$$\lim_{t\downarrow 0} \frac{\widetilde{\Gamma}_k(t,x,y)}{\Gamma_0(t,x,y)} \,=\, 1\,. \tag{4.35}$$

The statement now follows by taking the limit $\delta \to 0$. □

The proof of Theorem 4.4.5 is based on the following iteration:

**4.4.10 Lemma.** *Let $\epsilon, \delta > 0$, $n \in \mathbb{N}_0$, and*

$$H_{n,\epsilon,\delta}(t,x,y) := (1-\epsilon)^{(2^n-1)} \cdot \Gamma_0(2^n t,x,y) \,-\, \frac{2^n M_0}{t^{N/2}}\, e^{-\delta^2/4t}$$

*for $x,y \in \mathbb{R}^N$ and $t>0$. Then there is a constant $t_{\epsilon,\delta,n} > 0$ such that for all $0 < t < t_{\epsilon,\delta,n}$ and all $x,\, y \in W_{(n+2)\delta}$ with $|x-y| < \sqrt{2}^{\,n-3}\delta$,*

$$e^{-2^n t m_\delta}\, \widetilde{\Gamma}_k(2^n t,x,y) \,\geq\, H_{n,\epsilon,\delta}(t,x,y) \,\geq\, 0\,. \tag{4.36}$$

*Proof.* We start with some introductory remarks concerning the kernels $\Gamma_0$ and $\widetilde{\Gamma}_k$.

1. $\Gamma_0$ satisfies

$$\int_{\mathbb{R}^N} \Gamma_0(t,x,z)\,dz \,=\, 1 \quad (t>0,\, x\in\mathbb{R}^N). \tag{4.37}$$

2. The reproducing identity for $\Gamma_k$ (Lemma 4.1.3(4)) implies that

$$\widetilde{\Gamma}_k(2t,x,y) \,=\, \int_{\mathbb{R}^N} \widetilde{\Gamma}_k(t,x,z)\,\widetilde{\Gamma}_k(t,y,z)\,dz \quad (t>0,\, x,y\in\mathbb{R}^N). \tag{4.38}$$

3. A short calculation shows that for all $x,y,z\in\mathbb{R}^N$ and $t>0$,

$$\Gamma_0(2t,x,z)\,\Gamma_0(2t,y,z) \,=\, \Gamma_0(4t,x,y)\,\Gamma_0(t,z,(x+y)/2). \tag{4.39}$$

4. The localizing property and translation invariance of the Gaussian kernel $\Gamma_0$ assure that for all $\epsilon,\delta>0$, there exists a constant $t_{\epsilon,\delta}>0$ such that

$$\int_{|z-x|\le\delta/4} \Gamma_0(t,x,z)\,dz \,\ge\, 1-\epsilon \quad \text{for } 0<t<t_{\epsilon,\delta}\,,\; x\in\mathbb{R}^N. \tag{4.40}$$

5. For $\epsilon,\delta>0$, $n\in\mathbb{N}_0$ there exist constants $t'_{\epsilon,\delta,n}>0$ such that

$$H_{n,\,\epsilon,\delta}(t,x,y) \,\ge\, 0 \tag{4.41}$$

for all $0<t<t'_{\epsilon,\delta,n}$ and $x,\, y\in\mathbb{R}^N$ with $|x-y|<\sqrt{2}^{\,n-1}\delta$. In fact, by the definition of $\Gamma_0$, (4.41) is equivalent to

$$(1-\epsilon)^{(2^n-1)} \,-\, 2^{n(1+N/2)}\exp\Big\{\frac{|x-y|^2}{2^{n+2}\,t} \,-\, \frac{\delta^2}{4t}\Big\} \ge 0\,,$$

and this can be clearly achieved, under the assumption above on $x$ and $y$, for all sufficiently small times $t>0$. We may also assume that $t'_{\epsilon,\delta,n+1} \,\le\, t'_{\epsilon,\delta,n}$ for all $n$.

We now turn to the main part of the proof. We put

$$t_{\epsilon,\delta,n} := \min\,(t'_{\epsilon,\delta,n}\,,\, 2^{2-n}t_{\epsilon,\delta}).$$

The proof of (4.36) will be carried out by induction with respect to $n$. If $n=0$, then the statement follows directly from part (1) of Lemma 4.4.9 (because $C_\delta(x,y)\ge\delta$ for all $x,y\in W_\delta$), and by (4.41). Now suppose that (4.36) is true for $n\in\mathbb{N}_0$. Let further $0<t<t_{\epsilon,\delta,n+1}$ and $x,y\in W_{(n+3)\delta}$ with $|x-y|<\sqrt{2}^{\,n-2}\delta$. Then (4.41) ensures that $H_{n+1,\,\epsilon,\delta}(t,x,y)\,\ge\,0$ holds. To prove the first inequality in (4.36) for $n+1$, we put $\mu=m_\delta$ and

$$K_{x,y,\delta} := \{z\in\mathbb{R}^N :\, |z-(x+y)/2|<\delta/4\}\,.$$

In particular, $K_{x,y,\delta}\subset W_{(n+2)\delta}$. Moreover, we observe that for all $z\in K_{x,y,\delta}$,

$$|z-x| \,\le\, |z-(x+y)/2| + |x-y|/2 \,\le\, \delta/4 + \sqrt{2}^{\,n-2}\delta \,\le\, \sqrt{2}^{\,n-1}\delta\,,$$

and, by (4.41), $H_{n,\epsilon,\delta}(t,x,z) \geq 0$. In the same way we obtain $H_{n,\epsilon,\delta}(t,y,z) \geq 0$. Using the results from the beginning of the proof and the induction hypothesis, we therefore conclude

$$\begin{aligned}
e^{-2^{n+1}t\mu}\,\widetilde{\Gamma}_k(2^{n+1}t,x,y) &\underset{(4.38)}{=} \int_{\mathbb{R}^N} e^{-2^n t\mu}\,\widetilde{\Gamma}_k(2^n t,x,z)\, e^{-2^n t\mu}\,\widetilde{\Gamma}_k(2^n t,y,z)\,dz \\
&\geq \int_{K_{x,y,\delta}} e^{-2^n t\mu}\,\widetilde{\Gamma}_k(2^n t,x,z)\, e^{-2^n t\mu}\,\widetilde{\Gamma}_k(2^n t,y,z)\,dz \\
&\geq \int_{K_{x,y,\delta}} H_{n,\epsilon,\delta}(t,x,z)\,H_{n,\epsilon,\delta}(t,y,z)\,dz \\
&\underset{(4.37)}{\geq} (1-\epsilon)^{2(2^n-1)}\int_{K_{x,y,\delta}} \Gamma_0(2^n t,x,z)\,\Gamma_0(2^n t,y,z)\,dz \;-\; \frac{2^{n+1}M_0}{t^{N/2}}\,e^{-\delta^2/4t} \\
&\underset{(4.39)}{=} (1-\epsilon)^{2(2^n-1)}\,\Gamma_0(2^{n+1}t,x,y)\int_{K_{x,y,\delta}} \Gamma_0(2^{n-1}t,(x+y)/2,z)dz \;-\; \frac{2^{n+1}M_0}{t^{N/2}}\,e^{-\delta^2/4t}\,.
\end{aligned}$$

Since $2^{n-1}t < t_{\epsilon,\delta}$, property (4.40) implies that

$$\int_{K_{x,y,\delta}} \Gamma_0(2^{n-1}t,(x+y)/2,z)dz \;>\; 1-\epsilon\,.$$

This finally leads to

$$e^{-2^{n+1}t\mu}\,\widetilde{\Gamma}_k(2^{n+1}t,x,y) \;\geq\; H_{n+1\,\epsilon,\delta}(t,x,y),$$

which finishes the proof. □

*Proof of Theorem 4.4.5.* By choosing $\delta > 0$ small and $n \in \mathbb{N}$ large enough, we can achieve that

$$x,\, y \in W_{(n+2)\delta} \quad \text{and} \quad |x-y| \;<\; \sqrt{2}^{\,n-3}\delta\,.$$

Let $\epsilon > 0$ be arbitrary and fix $t_{\epsilon,\delta,n} > 0$ according to Lemma 4.4.10. Then (4.36) implies that for $0 < t < t_{\epsilon,\delta,n}$,

$$e^{-2^n t m_\delta}\,\frac{\widetilde{\Gamma}_k(2^n t,x,y)}{\Gamma_0(2^n t,x,y)} \;\geq\; (1-\epsilon)^{(2^n-1)} \;-\; 2^{n(1+N/2)}\exp\Big\{\frac{|x-y|^2}{2^{n+2}\,t} \;-\; \frac{\delta^2}{4t}\Big\},$$

and the subtracted term tends to $0$ as $t \downarrow 0$. This yields the assertion. □

# Appendix: Notation

**General**

| | |
|---|---|
| $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ | the sets of integer, real and complex numbers, respectively |
| $\mathbb{N}$ | $\{n \in \mathbb{Z} : n > 0\}$ |
| $\mathbb{Z}_+$ | $\{n \in \mathbb{Z} : n \geq 0\}$ |
| $\langle z, w\rangle$ | $\sum_{i=1}^N z_i\, w_i\,; \quad z, w \in \mathbb{C}^N$ |
| $\lvert x\rvert$ | $\langle x, x\rangle^{1/2}\,; \quad x \in \mathbb{R}^N$ |
| $\lvert z\rvert$ | $\big(\lvert z_1\rvert^2 + \ldots + \lvert z_N\rvert^2\big)^{1/2}\,; \quad z \in \mathbb{C}^N$ |

*For* $\nu = (\nu_1\,, \ldots\,, \nu_N) \in \mathbb{Z}_+^N$ *and* $z = (z_1\,, \ldots\,, z_n) \in \mathbb{C}^N$:

| | |
|---|---|
| $\nu!$ | $\nu_1! \cdot \ldots \cdot \nu_N!$ |
| $\lvert\nu\rvert$ | $\nu_1 + \ldots + \nu_N$ |
| $z^\nu$ | $z_1^{\nu_1} \cdot \ldots \cdot z_N^{\nu_N}$ |
| $\partial M$, $\overline{M}$, $M^\circ$ | the topological boundary, closure and interior of a set $M \subset \mathbb{R}^N$ |
| $S^{N-1}$ | $\{x \in \mathbb{R}^N : \lvert x\rvert = 1\}$ |
| $O(N, \mathbb{R})$ | group of orthogonal linear transformations in $\mathbb{R}^N$ |
| $\mathbb{Z}_2$ | $\mathbb{Z}\,/\,2\mathbb{Z}$ |
| $S_N$ | symmetric group in $N$ elements |

**Spaces of polynomials**

| | |
|---|---|
| $\Pi^N = \mathbb{C}[\mathbb{R}^N]$ | $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$ $(N \in \mathbb{N})$ |
| $\Pi_{\mathbb{R}}^N = \mathbb{R}\,[\mathbb{R}^N]$ | $\mathbb{R}$-algebra of polynomial functions on $\mathbb{R}^N$ |
| $\mathbb{C}\,[\mathbb{C}^N]$ | $\mathbb{C}$-algebra of polynomial functions on $\mathbb{C}^N$ |
| $\mathcal{P}_n^N$ | $\{p \in \Pi^N : p(\lambda x) = \lambda^n p(x)$ for all $\lambda > 0,\, x \in \mathbb{R}^N\}$, $n \in \mathbb{Z}_+$ |
| | the homogeneous polynomials of total degree $n$ |
| $\Pi_n^N$ | $\bigoplus_{k=0}^n \mathcal{P}_k^N\,;$ the polynomials in $\Pi^N$ of total degree at most $n$ |
| $\mathcal{P}_n^N(\mathbb{R})$ | $\Pi_{\mathbb{R}}^N \cap \mathcal{P}_n^N$ |
| $\Pi_+^N$ | $\{p \in \Pi^N : p(x) \geq 0$ for all $x \in \mathbb{R}^N\}$ |

**Spaces of measures and functions**

*For a locally compact Hausdorff space* $X$:

| | |
|---|---|
| $C(X)$ | space of complex-valued, continuous functions on $X$ |
| $C_b(X)$ | the bounded functions in $C(X)$ |

| | |
|---|---|
| $C_c(X)$ | the compactly supported functions in $C(X)$ |
| $C_0(X)$ | the functions in $C(X)$ which vanish at infinity |
| $C^+(X)$ | the real-valued, nonnegative functions in $C(X)$ |
| $C_c^+(X)$ | $C_c(X) \cap C^+(X)$ |
| $C_b^+(X)$ | $C_b(X) \cap C^+(X)$ |
| $\mathcal{B}(X)$ | Borel-$\sigma$-algebra on $X$ |
| $M^+(X)$ | space of positive Radon measures on $X$ |
| $M_b(X)$ | space of bounded Radon measures on $X$ |
| $M_b^+(X)$ | $M_b(X) \cap M^+(X)$ |
| $M^1(X)$ | subspace of $M_b(X)$ consisting of probability measures |
| $\sigma(V, W)$ | the weak topology on $V$ induced by $W$; |
| | $V$ a vector space and $W$ a subspace of its algebraic dual |
| $\sigma(M^1(X), C_b(X))$ | the weak topology on $M^1(X)$ |
| $\delta_x$ | point measure in $x \in X$ |
| $\operatorname{supp} \mu$ | support of a Radon measure $\mu$ |
| $\operatorname{supp} f$ | support of a function $f : X \to \mathbb{C}$ |
| $L^p(X, \mu)$ | $L^p$-space on $X$ with respect to the measure $\mu \in M^+(X)$; $1 \leq p \leq \infty$ |
| $\\|f\\|_{p,\mu}$ | $\left(\int_X \|f\|^p \, d\mu\right)^{1/p}$; $f \in \mathcal{B}(X)$, $1 \leq p < \infty$ |
| $\\|f\\|_{\infty,\mu}$ | $\inf\{C > 0 : \|f(x)\| \leq C$ locally $\mu$- almost everywhere$\}$ |

*If $X$ is a locally compact subspace of $\mathbb{R}^N$:*

| | |
|---|---|
| $L^p(X)$ | $L^p(X, dx)$; $dx$ the Lebesgue mesure on $X$ |
| $\\|f\\|_{p,w}$ | $\\|f\\|_{p,w(x)dx}$ for $w \in \mathcal{B}(X)$, $w \geq 0$ |

*For an open set $U \subseteq \mathbb{R}^N$:*

| | |
|---|---|
| $C^k(U)$ | space of $k$-times continuously differentiable functions on $U$; $k \in \mathbb{Z}_+ \cup \{\infty\}$ |
| $C_c^k(U)$ | subspace of compactly supported functions in $C^k(U)$ |
| $\mathscr{S}(\mathbb{R}^N)$ | Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$ |

**Particular symbols**

(The page number refers to the first occurence in the text)

| | |
|---|---|
| $H_\alpha$ | hyperplane in $\mathbb{R}^N$ orthogonal to $\alpha \in \mathbb{R}^N$; p. 7 |
| $\sigma_\alpha$ | reflection in the hyperplane $H_\alpha$; p. 7 |
| $R$ | a root system in $\mathbb{R}^N$ (usually fixed); p. 8 |
| $R_+$ | a positive subsystem of $R$; p. 8 |
| $G$ | the finite reflection group generated by $R$; p. 8 |
| $M$ | vector space of multiplicity functions on $R$; p. 8 |
| $M^{reg}$ | the regular parameter set; p. 12 |
| $k$ | a multiplicity function on $R$ (usually fixed) |

| | |
|---|---|
| $k \geq 0$ | $k(\alpha) \geq 0$ for all $\alpha \in R$ (always assumed after Chapter 1); p. 8 |
| $\gamma := \gamma(k)$ | $\sum_{\alpha\in R_+} k(\alpha)$; p. 8 |

*For a multiplicity function $k \geq 0$ and a parameter $\omega > 0$:*

| | |
|---|---|
| $w_k(x)$ | $\prod_{\alpha\in R_+} \lvert\langle\alpha, x\rangle\rvert^{2k(\alpha)}$; p. 8 |
| $c_k$ | $\int_{\mathbb{R}^N} e^{-\lvert x\rvert^2/2} w_k(x)dx$; p. 8 |
| $dm_k^\omega(x)$ | $c_k^{-1}(2\omega)^{\gamma+N/2}\, e^{-\omega\lvert x\rvert^2} w_k(x)dx \in M^1(\mathbb{R}^N)$; p. 47 |
| $d_k^\omega$ | $c_k^{-1/2}(2\omega)^{\gamma/2+N/4}$; p. 53 |
| $M_k$ | $c_k^{-1}\, 2^{-\gamma-N/2}$; p. 66 |

*Dunkl operators associated with $G$ and $k$:*

| | |
|---|---|
| $T_\xi = T_\xi(k)$ | Dunkl operator in direction $\xi \in \mathbb{R}^N$ associated with the (fixed) reflection group $G$ and multiplicity function $k$; p. 8 |
| $T_i = T_i(k)$ | $T_{e_i}(k)$; $\{e_1, \ldots, e_N\}$ the standard basis of $\mathbb{R}^N$ |
| $\Delta_k$ | $\sum_{i=1}^N T_i(k)^2$; the Dunkl Laplacian associated with $G$ and $k$; p. 10 |
| $\mathcal{F}_k$ | $w_k^{1/2}\Delta_k\, w_k^{-1/2}$ in $L^2(\mathbb{R}^N)$; p. 45 |
| $\mathcal{H}_k$ | $-\Delta_k + \omega^2\lvert x\rvert^2$ in $L^2(\mathbb{R}^N, w_k)$; p. 47 |
| $\mathcal{J}_k$ | $-\Delta_k + 2\omega\sum_{j=1}^N x_j\partial_j$, in $L^2(\mathbb{R}^N, dm_k^\omega)$; p. 47 |

*Further symbols (refering to a fixed reflection group $G$ and multiplicity fuction $k \geq 0$):*

| | |
|---|---|
| $V_k$ | intertwining operator; p. 12 |
| $A$ | Dunkl's algebra of homogeneous series on $\{x \in \mathbb{R}^N : \lvert x\rvert \leq 1\}$; p. 14 |
| $A_r$ | rescaled version of $A$ (scaling factor $r > 0$); p. 37 |
| $E_k$ | generalized exponential kernel (Dunkl kernel); p. 16 |
| $J_k$ | generalized Bessel function; p. 16 |
| $j_\alpha$ | normalized spherical Bessel function of order $\alpha \geq -1/2$; p. 17 |
| $\Gamma_k$ | generalized heat kernel; p. 66 |
| $\Gamma_0$ | classical heat kernel |
| $f \mapsto \widehat{f}^{\,k}$ | Dunkl transform on $L^1(\mathbb{R}^N, w_k)$ or $L^2(\mathbb{R}^N, w_k)$; p. 20 |
| $f \mapsto f^{\vee k}$ | inverse Dunkl transform; p. 20 |
| $L_k^y$ | generalized translation by $y \in \mathbb{R}^N$; p. 24 |
| $[p, q]_k$ | $(p(T(k)\, q)(0)$; $p, q \in \Pi^N$; p. 14 |
| $\{\varphi_\nu\,,\, \psi_\nu\}$ | homogeneous dual system with respect to $[\,.\,,\,.\,]_k$; p. 51 |
| $\{R_\nu(\omega; .),\ S_\nu(\omega; .)\}$ | biorthogonal polynomial system in $L^2(\mathbb{R}^N, dm_k^\omega)$; p. 53 |
| $\{r_\nu(\omega; .),\ s_\nu(\omega; .)\}$ | biorthonormal function system in $L^2(\mathbb{R}^N, w_k)$; p. 53 |
| $\{H_\nu(\omega; .)\}$ | system of generalized Hermite polynomials in $L^2(\mathbb{R}^N, dm_k^\omega)$; p. 54 |
| $\{h_\nu(\omega; .)\}$ | system of generalized Hermite functions in $L^2(\mathbb{R}^N, w_k)$; p. 54 |

# Bibliography


[A] Arendt, W., Characterization of positive semigroups on Banach lattices. In: Nagel, R. (ed.) One-parameter Semigroups of Positive operators. *Lecture Notes in Math.* 1184, Springer, 1986, pp. 247–291.

[B-F1] Baker, T.H., Forrester, P.J., The Calogero-Sutherland model and generalized classical polynomials. *Comm. Math. Phys.* 188 (1997), 175–216.

[B-F2] Baker, T.H., Forrester, P.J., The Calogero-Sutherland model and polynomials with prescribed symmetry. *Nucl. Phys. B* 492 (1997), 682–716.

[B-F3] Baker, T.H., Forrester, P.J., Non-symmetric Jack polynomials and integral kernels. *Duke Math. J.* 95 (1998), 1–50.

[dB] DiBenedetto, E., *Partial Differential Equations.* Birkhäuser, 1995.

[Ba] Bauer, H., *Wahrscheinlichkeitstheorie.* 4. Auflage, de Gruyter, 1991.

[B-M] Berest, Y.Y., Molchanov, Y., Fundamental solutions for partial differential equations with reflection group invariance. *J. Math. Phys.* 36 (1995), 4324–4339.

[Be-F] Berg, C., Forst, G., *Potential Theory on Locally Compact Abelian Groups.* Springer, 1975.

[B-H] Bloom, W., Heyer, H., *Harmonic Analysis of Probability Measures on Hypergroups.* De Gruyter, 1994.

[dBr] de Bruijn, N.G., Uncertainty principles in Fourier analysis. In: *Inequalities.* O. Shisha (ed.), Academic Press 1967, pp. 57–71.

[BHV] Brink, L., Hansson, T.H., Vasiliev, M.A., Explicit solution for the $N$-body Calogero problem. *Phys. Lett. B* 286 (1992), 109–111.

[BHKV] Brink, L., Hansson, T.H., Konstein, S., Vasiliev, M.A., The Calogero model - anyonic representation, fermionic extension and supersymmetry. *Nucl. Phys. B* 401 (1993), 591–612.

[Ca] Calogero, F., Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. *J. Math. Phys.* 12 (1971), 419–436.

[CRM] Calogero, F., Ragnisco, O., Marchioro, C., Exact solution of the classical and quantal one-dimensional many-body problems with the two-body potential $V_\alpha(x) = g^2a^2/sinh^2(ax)$. *Lett. Nuovo Cimento* 13 (1975), 383–387.

[Che] Cherednik, I., A unification of the Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras. *Invent. Math.* 106 (1991), 411–432.

[Chi] Chihara, T.S., *An Introduction to Orthogonal Polynomials.* Gordon and Breach, 1978.

[Da1] Davies, E.B., *One-Parameter Semigroups.* L.M.S. Monographs 15, Academic Press, 1980.

[Da2] Davies, E.B.: *Heat Kernels and Spectral Theory.* Cambridge University Press, 1989.

[Da3] Davies, E.B.: *Spectral Theory and Differential Operators.* Cambridge University Press, 1995.

[vD] van Diejen, J.F., Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement. *Comm. Math. Phys.* 188 (1997), 467–497.

[D-S] Donoho, D.L., Stark, P.B., Uncertainty principle and signal recovery. *SIAM J. Appl. Math.* 49 (1989), 906–931.

[D1] Dunkl, C.F., Reflection groups and orthogonal polynomials on the sphere. *Math. Z.* 197 (1988), 33–60.

[D2] Dunkl, C.F., Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* 311 (1989), 167–183.

[D3] Dunkl, C.F., Operators commuting with Coxeter group actions on polynomials. In: Stanton, D. (ed.), *Invariant Theory and Tableaux*, Springer, 1990, pp. 107–117.

[D4] Dunkl, C.F., Integral kernels with reflection group invariance. *Canad. J. Math.* 43 (1991), 1213–1227.

[D5] Dunkl, C.F., Hankel transforms associated to finite reflection groups. In: *Proc. of the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications.* Proceedings, Tampa 1991, Contemp. Math. 138 (1992), pp. 123–138.

[D6] Dunkl, C.F., Intertwining operators associated to the group $S_3$. *Trans. Amer. Math. Soc.* 347 (1995), 3347–3374.

[D-J-O] Dunkl, C.F., de Jeu M.F.E., Opdam, E.M., Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.* 346 (1994), 237–256.

[F-D] Fell, J.M.G., Doran, R.S., *Representations of $*$-Algebras, Locally Compact Groups , and Banach-$*$-Algebraic Bundles*, Vol. 1. Academic Press, 1988.

[G-B] Grove, L.C., Benson, C.T., *Finite Reflection Groups;* Second edition. Springer, 1985.

[G-S] Gihman, I.I., Skorohod, A.V., *The Theory of Stochastic Processes II.* Springer, 1975.

[Go] Goldstein, J.A., *Semigroups of Linear Operators and Applications.* Oxford University Press, 1985.

[Ha] Ha, Z.N.C., Exact dynamical correlation functions of the Calogero-Sutherland model and one dimensional fractional statistics in one dimension: View from an exactly solvable model. *Nucl. Phys. B* 435 (1995), 604–636.

[Hal] Haldane, D., Physics of the ideal fermion gas: Spinons and quantum symmetries of the integrable Haldane-Shastry spin chain. In: A. Okiji, N. Kamakani (eds.), *Correlation effects in low-dimensional electron systems.* Springer, 1995, pp. 3–20.

[He21] Heckman, G.J. (part I with Opdam, E.M.), Root systems and hypergeometric functions I, II. *Compositio Math.* 64 (1987), 329–352, 353–373.

[He1] Heckman, G.J., An elementary approach to the hypergeometric shift operators of Opdam. *Invent. Math.* 103 (1991), 341–350.

[He2] Heckman, G.J., A remark on the Dunkl differential-difference operators. In: Barker, W., Sally, P. (eds.) *Harmonic analysis on reductive groups.* Progress in Math. **101**, Birkhäuser, 1991. pp. 181 – 191.

[He3] Heckman, G.J., Dunkl operators. Séminaire Bourbaki 828, 1996–97; *Astérisque* 245 (1997), 223–246.

[H-Sc] Heckman, G., Schlichtkrull, H., *Harmonic Analysis and Special Functions on Symmetric Spaces.* Academic Press, 1994.

[H-St] Hewitt, E., Stromberg, K., *Real and Abstract Analysis.* Springer, 1975.

[Hi-K] Hikami, K., Komori, Y., Integrable three-body problems with two- and three-body interactions. *J. Phys. A* 30 (1997), 1913–1923.

[Hu] Humphreys, J.E., *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1990.

[dJ1] de Jeu, M.F.E., The Dunkl transform. *Invent. Math.* 113 (1993), 147–162.

[dJ2] de Jeu, M.F.E., An uncertainty principle for integral operators. *J. Funct. Anal.* 122 (1994), 247–253.

[dJ3] de Jeu, M.F.E., Dunkl operators. Thesis, University of Leiden, 1994.

[Je] Jewett, R.I., Spaces with an abstract convolution of measures. *Adv. Math.* 18 (1975), 1 – 101.

[Jo] John, F., *Partial Differential Equations.* Springer, 1986.

[K1] Kakei, S., Common algebraic structure for the Calogero-Sutherland models. *J. Phys. A* 29 (1996), L619–L624.

[K2] Kakei, S., Intertwining operators for a degenerate double affine Hecke algebra and multivariable orthogonal polynomials. Preprint 1997; q-alg/9706019.

[Ka1] Kato, T. *Perturbation Theory for Linear Operators.* Springer, 1966.

[Ka2] Kato, T., *A Short Introduction to Perturbation Theory for Linear Operators.* Springer, 1982.

[Ki] Kirillov, A.A., Lectures on affine Hecke algebras and Macdonald conjectures. *Bull. Amer. Math. Soc.* 34 (1997), 251–292.

[K-S] Knop, F., Sahi, S., A recursion and combinatorial formula for Jack polynomials. *Invent. Math.* 128 (1997), 9–22.

[L-V] Lapointe L., Vinet, L., Exact operator solution of the Calogero-Sutherland model. *Comm. Math. Phys.* 178 (1996), 425–452.

[La1] Lassalle, M., Polynômes de Laguerre généralisés. *C.R. Acad. Sci. Paris* t. 312 Série I (1991), 725–728.

[La2] Lassalle, M., Polynômes de Hermite généralisés. *C.R. Acad. Sci. Paris* t. 313 Série I (1991), 579–582.

[M1] Macdonald, I.G., Some conjectures for root systems. *SIAM J. Math. Anal.* 13 (1982), 988–1007.

[M2] Macdonald, I.G., Symmetric Functions and Orthogonal Polynomials. *University Lecture Series* 12, American Mathematical Society, 1998.

[Mo] Moser, J., Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. in Math.* 16 (1975), 197–220.

[O-P1] Olshanetsky, M.A., Perelomov, A.M., Completely integrable Hamiltonian systems connected with semisimple Lie algebras. *Invent. Math.* 37 (1976), 93–108.

[O-P2] Olshanetsky, M.A., Perelomov, A.M., Quantum systems related to root systems, and radial parts of Laplace operators. *Funct. Anal. Appl.* 12 (1978), 121–128.

[O1] Opdam, E.M., Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. *Compositio Math.* 85 (1993), 333–373.

[O2] Opdam, E.M., Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* 175 (1995), 75 – 121.

[Pa] Pasquier, V.: A lecture on the Calogero-Sutherland models. In: Integrable models and strings (Espoo, 1993), *Lecture Notes in Phys.* 436, Springer, 1994, pp. 36–48.

[Pe] Perelomov, A.M., Algebraical approach to the solution of a one-dimensional model of $N$ interacting particles. *Teor. Mat. Fiz.* 6 (1971), 364–391.

[Po] Polychronakos, A.P., Exchange operator formalism for integrable systems of particles. *Phys. Rev. Lett.* 69 (1992), 703–705.

[R1] Rösler, M., Convolution algebras which are not necessarily positivity-preserving. In: Applications of hypergroups and related measure algebras (Summer Research Conference, Seattle, 1993). *Contemp. Math.* 183 (1995), 299 – 318.

[R2] Rösler, M., On the dual of a commutative signed hypergroup. *Manuscr. Math.* 88 (1995), 147–163.

[R3] Rösler, M., Bessel-type signed hypergroups on $\mathbb{R}$. In: Heyer, H., Mukherjea, A. (eds.) Probability measures on groups and related structures XI. Proceedings, Oberwolfach 1994. World Scientific 1995, pp. 292–304.

[R4] Rösler, M., Generalized Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys.* 192 (1998), 519–542.

[R5] Rösler, M., Positivity of Dunkl's intertwining operator. *Duke math. J.*, to appear.

[R6] Rösler, M., An uncertainty principle for the Dunkl transform. *Bull. Austral. Math. Soc.*, to appear.

[R-V1] Rösler, M., Voit, M., Biorthogonal polynomials associated with reflection groups and a formula of Macdonald. *J. Comp. Appl. Math.* 99 (1998), 337–351.

[R-V2] Rösler, M., Voit, M., Markov Processes related with Dunkl operators. *Adv. Appl. Math.* 21 (1998), 575–643.

[R-V3] Rösler, M., Voit, M., An uncertainty principle for Hankel transforms. *Proc. Amer. Math. Soc.* 127 (1999), 183–194.

[Roo] Roosenraad, C.T., Inequalities with orthogonal polynomials, thesis, Univ. of Wisconsin, 1969.

[Ros] Rosenblum, M., Generalized Hermite polynomials and the Bose-like oscillator calculus. In: Operator Theory: Advances and Applications, Vol. 73, Basel, Birkhäuser Verlag 1994, 369–396.

[Su] Sutherland, B., Exact results for a quantum many-body problem in one dimension. *Phys. Rep.* A5 (1972), 1372–1376.

[Sz] Szegö, G., *Orthogonal Polynomials.* Amer. Math. Soc., 1959.

[U-W] Ujino, H., Wadati., M., Rodrigues formula for Hi-Jack symmetric polynomials associated with the quantum Calogero model. *J. Phys. Soc. Japan* 65 (1996), 2423–2439.

[W] Watson, G.N., *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, 1966.

[X1] Xu, Y., Orthogonal polynomials for a family of product weight functions on the spheres. *Canad. J. Math.* 49 (1997), 175–192.

[X2] Xu, Y., Integration of the intertwining operator for $h$-harmonic polynomials associated to reflection groups. *Proc. Amer. Math. Soc.* 125 (1997), 2963–2973.

[X3] Xu, Y., Intertwining operator and $h$-harmonics associated with reflection groups. *Canad. J. Math.* 50 (1998), 193–209.