POSITIVE INTERTWINERS FOR BESSEL FUNCTIONS OF TYPE B

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Abstract. Let V_k denote Dunkl's intertwining operator for the root sytem B_n with multiplicity $k = (k_1, k_2)$ with $k_1 \geq 0, k_2 > 0$. It was recently shown that the positivity of the operator $V_{k',k} = V_{k'} \circ V_k^{-1}$ which intertwines the Dunkl operators associated with k and $k' = (k_1 + h, k_2)$ implies that $h \in [k_2(n-1), \infty[\cup (\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{Z}_+)$. This is also a necessary condition for the existence of positive Sonine formulas between the associated Bessel functions. In this paper we present two partial converse positive results: For $k_1 \geq 0$, $k_2 \in \{1/2, 1, 2\}$ and $h > k_2(n-1)$, the operator $V_{k',k}$ is positive when restricted to functions which are invariant under the Weyl group, and there is an associated positive Sonine formula for the Bessel functions of type B_n. Moreover, the same positivity results hold for arbitrary $k_1 \geq 0, k_2 > 0$ and $h \in k_2 \cdot \mathbb{Z}_+$. The proof is based on a formula of Baker and Forrester on connection coefficients between multivariate Laguerre polynomials and an approximation of Bessel functions by Laguerre polynomials.

1. Introduction

Let R be a reduced root system in a finite-dimensional Euclidean space $(\mathfrak{a}, \langle\, .\, ,\, .\, \rangle)$ with finite Coxeter group W. Fix a multiplicity function, i.e., a W-invariant function $k : R \to [0, \infty[$ and denote by $\{T_\xi(k), \xi \in \mathfrak{a}\}$ the associated commuting family of rational Dunkl operators as introduced in [D1], see also [D2, DX]. Then there is a unique isomorphism on the vector space $\mathbb{C}[\mathfrak{a}]$ of polynomial functions on $\mathfrak{a}$ which preserves the degree of homogeneity and satisfies

$$V_k(1) = 1, \quad T_\xi(k)V_k = V_k\partial_\xi \quad \text{for all } \xi \in \mathfrak{a}.$$

By [R1], V_k is positive on $\mathbb{C}[\mathfrak{a}]$, i.e. for $p \in \mathbb{C}[\mathfrak{a}]$ with $p \geq 0$ we have $V_k p \geq 0$ on $\mathfrak{a}$. This is equivalent to the fact that for each $x \in \mathfrak{a}$ there is a unique compactly supported probability measure $\mu_x^k \in M^1(\mathfrak{a})$ such that the Dunkl kernel E_k associated with R and k has the positive integral representation

$$E_k(x, z) = \int_{\mathfrak{a}} e^{\langle \xi, z \rangle} d\mu_x^k(\xi), \quad \forall\, x \in \mathfrak{a}, z \in \mathfrak{a}_{\mathbb{C}}. \tag{1.1}$$

The support of μ_x^k is contained in the convex hull of the W-orbit of x. Eq. (1.1) generalizes the Harish-Chandra integral representation of spherical functions on symmetric spaces of Euclidean type (Ch.IV of [Hel]), as for certain multiplicities k,

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the Bessel functions

$$J_k(x,z) := \frac{1}{|W|}\sum_{w\in W} E_k(wx,z), \quad z\in \mathfrak{a}_{\mathbb{C}}, \tag{1.2}$$

coincide with the spherical functions of a Cartan motion group, where R and k depend on the root space data of the underlying symmetric space, see [O, dJ] for details. The Bessel function J_k is W-invariant in both arguments, and in the geometric cases, the integral formula for J_k obtained from (1.1) by taking W-means is just the Harish-Chandra formula [Hel, Prop. IV.4.8].

We now consider two multiplicities k, k' on R with $k'\geq k\geq 0$, i.e., $k'(\alpha)\geq k(\alpha)\geq 0$ for all $\alpha\in R$. In [RV4] we studied the operator

$$V_{k',k} := V_{k'}\circ V_k^{-1}$$

which intertwines the Dunkl operators with multiplicities k and k':

$$T_\xi(k')V_{k',k} = V_{k',k}T_\xi(k) \quad (\xi\in\mathfrak{a}).$$

It had been conjectured until recently that $V_{k',k}$ is also positive. This is equivalent to the statement that for each $x\in\mathfrak{a}$, there is a unique compactly supported probability measure $\mu_x^{k',k}\in M^1(\mathfrak{a})$ such that the Sonine formula

$$E_{k'}(x,z) = \int_{\mathfrak{a}} E_k(\xi,z)\, d\mu_x^{k',k}(\xi) \quad \text{for all } z\in\mathfrak{a}_{\mathbb{C}} \tag{1.3}$$

holds. Notice that (1.3) yields a corresponding formula for the Bessel function:

$$J_{k'}(x,z) = \int_{\mathfrak{a}} J_k(\xi,z)\, d\widetilde{\mu}_x^{k',k}(\xi) \quad (z\in\mathfrak{a}_{\mathbb{C}}) \tag{1.4}$$

with some W-invariant probability measure $\widetilde{\mu}_x^{k',k}$. Denote by $\widetilde{V}_{k',k}$ the restriction of $V_{k',k}$ to W-invariant functions. As in [RV4], it is easy to see that the existence of a positive Sonine formula (1.4) for the Bessel functions is equivalent to the positivity of $\widetilde{V}_{k',k}$.

In the rank-one case with $R=\{\pm 1\}\subset\mathbb{R}$, we have $J_k(x,y) = j_{k-1/2}(ixy)$ with the one-dimensional Bessel function

$$j_\alpha(z) = {}_0F_1(\alpha+1; -z^2/4) \qquad (\alpha\in\mathbb{C}\setminus\{-1,-2,\dots\}). \tag{1.5}$$

Here (1.4) is just the well-known classical Sonine formula (see e.g. [A]):

$$j_{\alpha+\beta}(z) = 2\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1 j_\alpha(zx)x^{2\alpha+1}(1-x^2)^{\beta-1}dx \tag{1.6}$$

for $\alpha\in]-1,\infty[$ and $\beta\in]0,\infty[$. It was proven in [X] that the operator $V_{k',k}$ with $k'>k\geq 0$ is also positive in the rank-one setting.

On the other hand, for the root systems

$$B_n = \{\pm e_i,\, \pm e_i\pm e_j,\, 1\leq i<j\leq n\}\subset\mathbb{R}^n$$

with $n\geq 2$ and certain parameters $k'\geq k\geq 0$, (1.3) and (1.4) were disproved in [RV4]. For these root systems we write the multiplicities as $k=(k_1,k_2)$ with k_1,k_2 the values of k on the roots $\pm e_i$ and $\pm e_i\pm e_j$, respectively. The following result is shown in Section 3 of [RV4] for the Bessel functions J_k^B of type B_n.

Theorem 1.1. *Let $k=(k_1,k_2)$ with $k_1\geq 0$, $k_2>0$, and consider $k'=(k_1+h,k_2)$ with $h>0$. Assume that there exists a probability measure $m\in M_b(\mathbb{R}^n)$ such that the restricted Sonine formula*

$$J_{k'}^B(\mathbf{1},iy)=\int_{\mathbb{R}^n} J_k^B(\xi,iy)\,dm(\xi)\quad \textit{for all } y\in\mathbb{R}^n \tag{1.7}$$

holds, where $\mathbf{1}=(1,\ldots,1)\in\mathbb{R}^n$. Then h is contained in the set

$$\Sigma(k_2):=\,]k_2(n-1),\infty[\,\cup\big(\{0,k_2,\ldots,k_2(n-1)\}-\mathbb{Z}_+\big).$$

Therefore, positivity of $\widetilde{V}_{k',k}$ or $V_{k',k}$ requires that $h\in\Sigma(k_2)$. Theorem 1.1 is related to a classical result of Gindikin [G] about the Wallach set which parametrizes those Riesz distributions on a Euclidean Jordan algebra which are actually positive measures, see [FK]. Related results for Riesz and Beta distributions on symmetric cones and in the Dunkl setting are given in [RV3, R3].

On the other hand, there are some positive Sonine formulas for $J_{k'}^B$ in terms of J_k^B with $k'=(k_1+h,k_2)$ if $k_1\geq 0$ and h is large enough; these will be treated in Section 3. Namely, if $k_2\in\{1/2,1,2\}$ and $h>k_2(n-1)$, then a Sonine formula follows from the fact that in this case the Bessel functions J_k^B are closely related to Bessel functions on matrix cones over $\mathbb{R},\mathbb{C},\mathbb{H}$ where positive Sonine formulas are available by [H, RV3]. Moreover, a restricted explicit Sonine formula of the form (1.7) was obtained in [RV4] for arbitrary $k_1\geq 0, k_2>0$ and $h>k_2(n-1)$ as a consequence of Kadell's [Kad] generalization of the Selberg integral.

The main results of this paper are contained in Sections 4 and 5. In particular, in Theorem 4.1 we prove that the Bessel function J_k^B satisfies a Sonine formula for all $k_1,k_2>0$ and $h\in k_2\cdot\mathbb{Z}_+$, which includes all parameters h in the discrete part of the generalized Wallach set

$$\{0,k_2,\ldots,k_2(n-1)\}\,\cup\,[k_2(n-1),\infty[\,.$$

The proof of this result is based on a formula of Baker and Forrester [BF1] on positive connection coefficients between multivariate Laguerre polynomial systems and an approximation of Bessel functions by Laguerre polynomials. Finally in Section 5, we consider the limit $h\to\infty$ under which the Bessel functions of type B tend to those of type A. This leads to a positive integral representation of Bessel functions of type A in terms of such of type B.

2. Basic facts on Bessel functions

We start with some background on rational Dunkl theory from [D1, D2, DJO, R1]. Let R be a reduced root system in a finite-dimensional Euclidean space $(\mathfrak{a},\langle\,.\,,\,.\,\rangle)$ and W the associated finite Coxeter group. The Dunkl operators associated with R and multiplicity k are

$$T_\xi(k)=\partial_\xi+\frac{1}{2}\sum_{\alpha\in R}k(\alpha)\langle\alpha,\xi\rangle\frac{1}{\langle\alpha,\,.\,\rangle}(1-\sigma_\alpha),\quad \xi\in\mathfrak{a}$$

where the action of W on functions $f:\mathfrak{a}\to\mathbb{C}$ is given by $w.f(x)=f(w^{-1}x)$. It was shown in [D1] that the $T_\xi(k)$, $\xi\in\mathfrak{a}$, commute. Multiplicities $k\geq 0$ are regular, i.e., the joint kernel of the $T_\xi(k)$, considered as linear operators on polynomials, consists of the constants only. This is equivalent to the existence of a necessarily unique intertwining operator V_k as described in the introduction; see [DJO]. Moreover, for each $y \in \mathfrak{a}_{\mathbb{C}}$, there is a unique solution $f = E_k(\,.\,, y)$ of the joint eigenvalue problem

$$T_\xi(k)f = \langle \xi, y\rangle f \quad \forall\, \xi \in \mathfrak{a},\ f(0) = 1.$$

The function E_k is called the Dunkl kernel. The mapping $(x, y) \mapsto E_k(x, y)$ is analytic on $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$ with $E_k(x, y) = E_k(y, x)$, $E_k(x, 0) = 1$, and

$$E_k(\lambda x, y) = E_k(x, \lambda y), \quad E_k(wx, wy) = E_k(x, y) \quad (\lambda \in \mathbb{C}, w \in W).$$

In this paper, we mainly consider the Dunkl kernel E_k^B and the Bessel function J_k^B associated with the root system B_n on $\mathbb{R}^n$ with its usual inner product, c.f. (1.2). The associated reflection group is the hyperoctahedral group $W(B_n) = S_n \ltimes \mathbb{Z}_2^n$, and the multiplicity has the form $k = (k_1, k_2)$ as explained in the introduction. In Section 5, we shall also consider the Bessel function J_k^A associated with the root system

$$A_{n-1} = \{\pm(e_i - e_j), 1 \le i < j \le n\} \subset \mathbb{R}^n.$$

Here the multiplicity is given by a single parameter $k \in [0, \infty[$. The Bessel functions J_k^B and J_k^A have simple expressions in terms of multivariate hypergeometric functions in the sense of [K, M2]. To recall these we need some more notation. Let

$$\Lambda_n^+ = \{\lambda \in \mathbb{Z}_+^n : \lambda_1 \ge \cdots \ge \lambda_n\}$$

be the set of partitions of length at most n. We denote by C_λ^α, $\lambda \in \Lambda_n^+$ the Jack polynomials of index $\alpha > 0$ in n variables (see [Sta]), normalized such that

$$(z_1 + \cdots + z_n)^m = \sum_{|\lambda| = m} C_\lambda^\alpha(z) \quad (m \in \mathbb{Z}_+). \tag{2.1}$$

Following [K], we define for $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > \frac{1}{\alpha}(n-1)$ and arguments $z, w \in \mathbb{C}^n$ the hypergeometric function

$${}_0F_1^\alpha(\mu; z, w) := \sum_{\lambda \in \Lambda_n^+} \frac{1}{[\mu]_\lambda^\alpha\, |\lambda|!} \cdot \frac{C_\lambda^\alpha(z) C_\lambda^\alpha(w)}{C_\lambda^\alpha(\mathbf{1})}, \quad \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n, \tag{2.2}$$

with the generalized Pochhammer symbol

$$[\mu]_\lambda^\alpha = \prod_{j=1}^n \big(\mu - \frac{1}{\alpha}(j-1)\big)_{\lambda_j}. \tag{2.3}$$

If $n = 1$, then the C_λ^α are independent of α and given by $C_\lambda^\alpha(z) = z^\lambda$, $\lambda \in \mathbb{Z}_+$, and

$${}_0F_1^\alpha(\mu; -\frac{z^2}{4}, 1) = j_{\mu-1}(z).$$

Let $k = (k_1, k_2)$ on B_n with $k_1 \ge 0$ and $k_2 > 0$. Then according to [R2, Prop. 4.5] (c.f. also [BF1, Sect. 6]),

$$J_k^B(z, w) = {}_0F_1^\alpha\Big(\mu; \frac{z^2}{2}, \frac{w^2}{2}\Big) \tag{2.4}$$

with $\alpha = \frac{1}{k_2}$ and $\mu = \mu(k) = k_1 + k_2(n-1) + \dfrac{1}{2}.$

Moreover, for $n \geq 2$ and $k \in]0,\infty[$, the Bessel function of type A_{n-1} has the following hypergeometric expansion ([BF2, Sect. 3]):

$$J_k^A(z,w) = {}_0F_0^\alpha(z,w) = \sum_{\lambda\in\Lambda_n^+} \frac{1}{|\lambda|!}\cdot\frac{C_\lambda^\alpha(z)C_\lambda^\alpha(w)}{C_\lambda^\alpha(\mathbf{1})} \quad \text{with } \alpha = 1/k\,. \tag{2.5}$$

3. Explicit Sonine formulas for large parameters

We first recapitulate a restricted Sonine formula of the form (1.7) for J_k^B from [RV4]. Let $k = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 > 0$. For a real parameter $h > k_2(n-1)$ consider the probability density

$$f_{k,h}(x) := c_{k,h}^{-1}\cdot\prod_{i=1}^n (x_i^2)^{k_1}(1-x_i^2)^{h-k_2(n-1)-1}\prod_{i<j}|x_i^2 - x_j^2|^{2k_2}$$

on $[0,1]^n$ with the normalization constant (a Selberg-type integral)

$$c_{k,h} = \int_{[0,1]^n}\prod_{i=1}^n (x_i^2)^{k_1}(1-x_i^2)^{h-k_2(n-1)-1}\prod_{i<j}|x_i^2 - x_j^2|^{2k_2}dx.$$

Then according to Eq. (3.4) in [RV4],

$$J_{(k_1+h,k_2)}^B(\mathbf{1}, z) = \int_{[0,1]^n} J_k^B(x,z)f_{k,h}(x)dx \quad (z\in\mathbb{C}^n). \tag{3.1}$$

The proof of this formula is based on (2.4) and Kadell's [Kad] generalization of the Selberg integral, which implies a restricted Sonine formula for ${}_0F_1^\alpha$. The restricted Sonine formula (3.1) can be extended to a complete Sonine formula, i.e. a Sonine formula for Bessel functions with arbitrary second argument, in the cases $k_2 = 1/2, 1, 2$ via Bessel functions on matrix cones. To explain this, we recapitulate some notations from [FK, R2, RV3]. For the (skew) fields $\mathbb{F} := \mathbb{R}, \mathbb{C}, \mathbb{H}$ with real dimension $d = 1, 2, 4$ consider the vector spaces $H_n := H_n(\mathbb{F})$ of all Hermitian $n\times n$ matrices over $\mathbb{F}$ as well as the cones $\Pi_n := \Pi_n(\mathbb{F}) \subset H_n$ of all positive semidefinite matrices. Let Ω_n be the interior of Π_n consisting of all strictly positive definite matrices. The Bessel functions associated with the symmetric cone Ω_n (in the sense of [FK]) are defined in terms of the spherical polynomials

$$\Phi_\lambda(a) = \int_{U_n}\Delta_\lambda(uau^{-1})du \quad (\lambda\in\Lambda_n^+,\ a\in H_n).$$

Here du is the normalized Haar measure of the compact group $U_n = U_n(\mathbb{F})$ and Δ_λ denotes the power function

$$\Delta_\lambda(a) := \Delta_1(a)^{\lambda_1-\lambda_2}\Delta_2(a)^{\lambda_2-\lambda_3}\cdot\ldots\cdot\Delta_q(a)^{\lambda_q}$$

with the principal minors $\Delta_i(a)$. We renormalize the spherical polynomials according to $Z_\lambda := c_\lambda\Phi_\lambda$ with $c_\lambda > 0$ such that

$$(tr\, a)^k = \sum_{|\lambda|=k} Z_\lambda(a) \qquad \text{for} \quad k\in\mathbb{N}_0\,;$$

c.f. Section XI.5. of [FK]. The Z_λ depend only on the eigenvalues of their argument and are given in terms of Jack polynomials: for $a\in H_n$ with eigenvalues $x = (x_1,\ldots,x_n)\in\mathbb{R}^n$,

$$Z_\lambda(a) = C_\lambda^\alpha(x) \quad \text{with } \alpha = \frac{2}{d}\,,$$

see [FK, M1]. Let

$$\mu_0 := \frac{d}{2}(n-1).$$

Then for $\mu > \mu_0$, the Bessel function J_μ associated with Ω_n is defined according to [FK] as the ${}_0F_1$-hypergeometric series

$$J_\mu(a) = \sum_{\lambda \in \Lambda_n^+} \frac{(-1)^{|\lambda|}}{[\mu]_\lambda^{2/d}\,|\lambda|!}\, Z_\lambda(a) \quad (a \in H_n). \tag{3.2}$$

Notice that here $[\mu]_\lambda^{2/d} \neq 0$ for all λ. Comparing formulas (3.2) and (2.4), one obtains for $a \in H_n$ with eigenvalues $x \in \mathbb{R}^n$ and $\mu \geq \mu_0 + \frac{1}{2}$ the identity

$$J_\mu(a^2) = J_k^B(2ix, \mathbf{1}) \quad \text{ with } k = k(\mu, d) = \big(\mu - \mu_0 - \frac{1}{2}, \frac{d}{2}\big); \tag{3.3}$$

see [R2, Cor. 4.4]. Recall that J_k^B is invariant under $W(B_n)$ in both variables. We denote the associated standard Weyl chamber by

$$C_n^B := \{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n \geq 0\}.$$

Now consider $r, s \in \Pi_n$ with ordered eigenvalues $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in C_n^B$. Then by Corollary 4.6. of [R2],

$$J_k^B(2ix, y) = \int_{U_n} J_\mu(rus^2u^{-1}r)\, du \quad \text{ with } k = k(\mu, d). \tag{3.4}$$

For indices $\mu, \nu > \mu_0$ we introduce the Beta-Riesz probability measures

$$d\beta_{\mu,\nu}(r) := \frac{1}{B_\Omega(\mu, \nu)} \Delta(r)^{\mu-1-\mu_0} \Delta(I-r)^{\nu-1-\mu_0} dr\Big|_{\Omega_n^I} \tag{3.5}$$

on the relatively compact set $\Omega_n^I := \{r \in \Omega_n : \ I_n - r \in \Omega_n\}$, where

$$B_\Omega(\mu, \nu) := \int_{\Omega_n^I} \Delta(r)^{\mu-1-\mu_0} \Delta(I-r)^{\nu-1-\mu_0} dr.$$

We recall the following Sonine formula from [RV3, Theorem 1]; see also (2.6') of [H] for $\mathbb{F} = \mathbb{R}$.

Proposition 3.1. *For all $\mu, \nu > \mu_0$ and $r \in \Pi_n$,*

$$J_{\mu+\nu}(r) = \int_{\Omega_n^I} J_\mu(rs) d\beta_{\mu,\nu}(s).$$

Here we adopt a common notation in the literature and consider the Bessel function J_μ as a function of the eigenvalues of the (not necessarily Hermitian) matrix rs, which coincide with those of the Hermitian matrices $\sqrt{s}\, r \sqrt{s}$ and $\sqrt{r}\, s \sqrt{r}$.

From Proposition 3.1 and formula (3.4) we now derive a Sonine representation for J_k^B. For this, consider the map $\sigma : \Pi_n \to C_n^B$ where $\sigma(r)$ is the ordered spectrum of $r \in \Pi_n$. We shall identify $x \in \mathbb{R}^n$ with the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_n) \in H_n$. Theorem VI.2.3 of [FK] states that

$$\int_{\Pi_n} f(r)\, dr \; = \; c_0 \int_{U_n} \int_{C_n^B} f(uxu^{-1}) \cdot \prod_{i<j} (x_i - x_j)^d \, dx \, du \tag{3.6}$$

for integrable functions $f : \Pi_n \to \mathbb{C}$, with some constant $c_0 > 0$. Let

$$C_{n,1}^B := \{x \in C_n^B : \ x_1 \leq 1\} = \{x \in \mathbb{R}^n : 1 \geq x_1 \geq \cdots \geq x_n \geq 0\}.$$

Then by (3.6), the image measure of $\beta_{\mu,\nu}$ under σ is the probability measure

$$d\rho_{k_1,k_2,h}(x) = \frac{1}{B_{k_1,k_2,h}}\prod_{i=1}^n x_i^{k_1-1/2}(1-x_i)^{h-1-\mu_0}\cdot\prod_{i<j}(x_i-x_j)^{2k_2}dt\Big|_{C_{n,1}^B} \tag{3.7}$$

on $C_{n,1}^B$ with the normalization constant $B_{k_1,k_2,h} = B_\Omega(\mu,\nu)/c_0$ where $(k_1,k_2) = k(\mu,d)$ as in (3.3) and $h=\nu$. We obtain

Theorem 3.2. *Let $\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and $k_2 = d/2$ with $d = dim_{\mathbb{R}}\mathbb{F}$. Let further $k_1\geq 0$ and $h>\mu_0 = \frac{d}{2}(n-1)$. Then for all $z\in\mathbb{C}^n$ and $x\in C_n^B$,*

$$J_{(k_1+h,k_2)}^B(x,z) = \int_{C_{n,1}^B}\int_{U_n(\mathbb{F})} J_{(k_1,k_2)}^B\big(\sqrt{\sigma(xu\xi u^{-1}x)}\,,z\big)\,du\,d\rho_{k_1,k_2,h}(\xi).$$

Proof. Let $x,y\in C_n^B$ and put $\mu = k_1+\mu_0+\frac{1}{2}$. Formulas (3.4), (3.6) and Proposition 3.1 imply that

$$\begin{aligned} J_{(k_1+h,k_2)}^B(x,2iy) &= \int_{U_n}\int_{\Omega_n^I} J_\mu\big(xuy^2u^{-1}xs\big)\,d\beta_{\mu,h}(s)\,du \\ &= \int_{U_n}\int_{\Omega_n^I} J_\mu\big(uy^2u^{-1}xsx\big)\,d\beta_{\mu,h}(s)\,du \\ &= \int_{U_n}\int_{U_n}\int_{C_{n,1}^B} J_\mu\big(uy^2u^{-1}\cdot xv\xi v^{-1}x\big)\,d\rho_{k_1,k_2,h}(\xi)\,dv\,du \\ &= \int_{C_{n,1}^B}\int_{U_n} J_{(k_1,k_2)}^B\big(\sqrt{\sigma(xv\xi v^{-1}x)}\,,2iy\big)\,dv\,d\rho_{k_1,k_2,h}(\xi). \end{aligned}$$

Analytic extension in the first argument now yields the assertion. □

Note that for $k_2 = 1/2, 1, 2$ and $\eta = \mathbf{1}$, the integral representation in Theorem 3.2 coincides with formula (3.1).

4. Positive Sonine formulas for discrete parameters

Besides the Sonine formulas in Theorem 3.2 and Eq. (3.1), Theorem 1.1 admits the following partial converse statement:

Theorem 4.1. *Let $k=(k_1,k_2)$ with $k_1\geq 0, k_2>0$ and $h\in k_2\cdot\mathbb{Z}_+ = \{0,k_2,2k_2,\ldots\}$. Then for each $x\in C_n^B$ there exists a unique probability measure $m_x\in M^1(C_n^B)$ such that*

$$J_{(k_1+h,k_2)}^B(x,z) = \int_{C_n^B} J_k^B(\xi,z)\,dm_x(\xi) \tag{4.1}$$

for all $z\in\mathbb{C}^n$. The support of m_x is contained in $[0,x] := [0,x_1]\times\ldots\times[0,x_n]$.

The uniqueness statement in Theorem 4.1 follows from the injectivity of the Dunkl transform of bounded measures, see [RV1, Theorem 2.6]. For the proof of the existence part, it suffices to consider $z=iy$ with $y\in\mathbb{R}^n$ and prove the case $h=k_2$. In this case, the theorem can be derived from an explicit formula of Baker and Forrester [BF1, formula (4.23)] on the connection coefficients between multivariate Laguerre polynomial systems. To start with, recall that according to

[BF1, Proposition 4.3], the Laguerre polynomials in n variables associated with parameters $\alpha > 0$ and $a > -1$ are given by

$$L_\kappa^a(x;\alpha) = \frac{[a+q]_\kappa^\alpha}{|\kappa|!}\sum_{\lambda\subseteq\kappa}\binom{\kappa}{\lambda}\frac{(-1)^{|\lambda|}}{[a+q]_\lambda^\alpha}\,\frac{C_\lambda^\alpha(x)}{C_\lambda^\alpha(\mathbf{1})} \quad (\kappa\in\Lambda_n^+\,,\ x\in\mathbb{R}^n) \tag{4.2}$$

where $q = 1+(n-1)/\alpha$, the notion $\lambda\subseteq\kappa$ means that the diagram of λ is contained in that of κ, i.e. $\lambda_i\le\kappa_i$ for all $i=1,\ldots,n$, and the generalized binomial coefficients $\binom{\kappa}{\lambda} = \binom{\kappa}{\lambda}_\alpha$ are defined by the binomial formula for the Jack polynomials,

$$\frac{C_\kappa^\alpha(x+\mathbf{1})}{C_\kappa^\alpha(\mathbf{1})} = \sum_{\lambda\subseteq\kappa}\binom{\kappa}{\lambda}\frac{C_\lambda^\alpha(x)}{C_\lambda^\alpha(\mathbf{1})}.$$

We consider the renormalized Laguerre polynomials

$$\widetilde{L}_\kappa^a(x;\alpha) := \frac{L_\kappa^a(x;\alpha)}{L_\kappa^a(0;\alpha)} = \sum_{\lambda\subseteq\kappa}\binom{\kappa}{\lambda}\frac{(-1)^{|\lambda|}}{[a+q]_\lambda^\alpha}\,\frac{C_\lambda^\alpha(x)}{C_\lambda^\alpha(\mathbf{1})}$$

and choose the parameters according to those in (2.4), namely $\alpha := \frac{1}{k_2},\ a := k_1 - \frac{1}{2}$. Then $a+q = k_1 + k_2(n-1) + \frac{1}{2} = \mu(k) =: \mu$.

The first step towards the proof of Theorem 4.1 is the following limit result for Laguerre polynomials which generalizes Proposition 3.3 of [F] for $\alpha = 1$. In the one-dimensional case $n=1$, this limit transition is well-known.

Lemma 4.2. *Let $k = (k_1,k_2)$ with $k_1\ge 0, k_2 > 0$, put $\alpha := \frac{1}{k_2}$, $a := k_1 - \frac{1}{2}$ and fix $y\in\mathbb{R}^n$. For $x\in C_n^B$ consider the sequence of partitions $(\lambda_j(x))_{j\in\mathbb{N}}\subseteq\Lambda_n^+$ with $\lambda_j(x) := \lfloor j\cdot x\rfloor$ where Gaussian brackets are taken componentwise. Then $\lim_{j\to\infty}\lambda_j(x)/j = x$, and*

$$\lim_{j\to\infty}\widetilde{L}_{\lambda_j(x)}^a(y^2/j;\alpha) = J_k^B(iy, 2\sqrt{x}\,)$$

locally uniformly in x (here the square root is also taken componentwise).

Proof. We adopt the method from [F]. First, recall the monic renormalization of the Jack polynomials C_λ^α, which is determined by

$$P_\lambda^\alpha = m_\lambda + \sum_{\mu<\lambda} d_{\lambda\mu}(\alpha)m_\mu$$

where the m_λ are the monomial symmetric functions $m_\lambda(z) = \sum_{w\in S_n} z^{w.\lambda}$ and $\mu<\nu$ means that μ is strictly lower than ν in the usual dominance order on Λ_n^+. According to [K, formula (16)] and the identities on p. 15 of [KS],

$$P_\lambda^\alpha = \frac{c'_\lambda}{\alpha^{|\lambda|}|\lambda|!}\,C_\lambda^\alpha \quad\text{with}\quad c'_\lambda = \prod_{s\in\lambda}(\alpha(a_\lambda(s)+1)+l_\lambda(s)),$$

where $a_\lambda(s)$ and $l_\lambda(s)$ denote the arm-length and leg-length of $s\in\lambda$, c.f. [KS, S1] for the definitions. According to [OO], the generalized binomial coefficients $\binom{\kappa}{\lambda}$ can be written in terms of the so-called shifted Jack polynomials $P_\lambda^*(x,1/\alpha)$ as

$$\binom{\kappa}{\lambda} = \frac{P_\lambda^*(\kappa;1/\alpha)}{P_\lambda^*(\lambda;1/\alpha)}.$$

Moreover,

$$P_\lambda^*(\lambda;1/\alpha) = \frac{1}{\alpha^{|\lambda|}}c'_\lambda.$$

The shifted Jack polynomials are of the form (c.f. [OO])

$$P_\lambda^*(z;1/\alpha) = P_\lambda^\alpha(z) + \text{terms of lower degree in dominance order.}$$

Therefore

$$\lim_{j\to\infty} \frac{1}{j^{|\lambda|}}\binom{\lambda_j(x)}{\lambda} = \frac{\alpha^{|\lambda|}}{c'_\lambda}\cdot \lim_{j\to\infty} P_\lambda^\alpha\Big(\frac{\lambda_j(x)}{j}\Big) = \frac{\alpha^{|\lambda|}}{c'_\lambda}\cdot P_\lambda^\alpha(x) = \frac{C_\lambda^\alpha(x)}{|\lambda|!}, \tag{4.3}$$

and the convergence is locally uniform in x. With $\alpha = \frac{1}{k_2}$, $a = k_1 - \frac{1}{2}$ and $\mu = \mu(k) = k_1 + k_2(n-1) + \frac{1}{2}$ we thus obtain

$$\begin{aligned}\lim_{j\to\infty} \widetilde{L}^a_{\lambda_j(x)}(y^2/j;\alpha) &= \lim_{j\to\infty} \sum_{\lambda\subseteq\lambda_j(x)} \frac{1}{j^{|\lambda|}}\binom{\lambda_j(x)}{\lambda}\frac{(-1)^{|\lambda|}}{[\mu]_\lambda^\alpha}\frac{C_\lambda^\alpha(y^2)}{C_\lambda^\alpha(\mathbf{1})}\\ &= \sum_{\lambda\in\Lambda_n^+} \frac{(-1)^{|\lambda|}}{[\mu]_\lambda^\alpha}\cdot\frac{C_\lambda^\alpha(x)\, C_\lambda^\alpha(y^2)}{|\lambda|!\, C_\lambda^\alpha(\mathbf{1})} \;=\; J_k^B(iy, 2\sqrt{x}). \end{aligned}\tag{4.4}$$

For the uniformity statement, we proceed as in [V]. We use that the binomial coefficients satisfy $\binom{\kappa}{\lambda}\geq 0$ for $\lambda\subseteq\kappa$ (Theorem 5 of [S2]) as well as

$$\sum_{|\lambda|=m}\binom{\kappa}{\lambda} = \binom{|\kappa|}{m}, \quad m\in\mathbb{Z}_+$$

from [L]. Note also that $[\mu]_\lambda^\alpha \geq (\frac{1}{2})^{|\lambda|}$. As the coefficients of the C_λ^α in their monomial expansion are nonnegative, we may estimate

$$\Big|\frac{C_\lambda^\alpha(y^2)}{C_\lambda^\alpha(\mathbf{1})}\Big| \leq \|y\|_\infty^{2|\lambda|}.$$

We therefore obtain

$$\begin{aligned}\sum_{\lambda\subseteq\lambda_j(x)} \frac{1}{j^{|\lambda|}}\binom{\lambda_j(x)}{\lambda}\frac{1}{[\mu]_\lambda^\alpha}\Big|\frac{C_\lambda^\alpha(y^2)}{C_\lambda^\alpha(\mathbf{1})}\Big| &\leq \sum_{m=0}^\infty \frac{2^m}{j^m}\|y\|_\infty^{2m}\cdot\sum_{|\lambda|=m}\binom{\lambda_j(x)}{\lambda}\\ &= \sum_{m=0}^\infty\Big(\frac{2\|y\|_\infty^2}{j}\Big)^m\cdot\binom{|\lambda_j(x)|}{m} \leq \sum_{m=0}^\infty \frac{(2n\|y\|_\infty^2\|x\|_\infty)^m}{m!}.\end{aligned}$$

This estimate shows that the convergence in (4.4) is locally uniform in $x\in C_n^B$. $\square$

The following observation is the central ingredient for the proof of Theorem 4.1. It follows immediately from identity (4.23) of [BF1].

Lemma 4.3. *For each partition $\kappa\in\Lambda_n^+$ there exist connection coefficients $c_{\kappa,\lambda} = c_{\kappa,\lambda}(a,\alpha)\geq 0$ with $\sum_{\lambda\subseteq\kappa} c_{\kappa,\lambda} = 1$ such that*

$$\widetilde{L}_\kappa^{a+1/\alpha}(x;\alpha) \;=\; \sum_{\lambda\subseteq\kappa} c_{\kappa,\lambda}\,\widetilde{L}_\lambda^a(x;\alpha) \qquad (x\in\mathbb{R}^n).$$

Proof of Theorem 4.1. Let $k = (k_1,k_2)$, $a := k_1 - \frac{1}{2}$ and $\alpha = \frac{1}{k_2}$. Fix $y\in\mathbb{R}^n$. For $x\in C_n^B$ consider the sequence of partitions $(\lambda_j = \lambda_j(x))_{j\in\mathbb{N}}$ as described in Lemma 4.2 with $\lim_{j\to\infty}\lambda_j/j = x$. We proceed as in [RV4] and introduce the discrete probability measures

$$\mu_j := \mu_j(x) := \sum_{\lambda\subseteq\lambda_j} c_{\lambda_j,\lambda}\,\delta_{\lambda/j}\in M^1(\mathbb{R}^n), \quad j\in\mathbb{N};$$

here δ_ξ denotes the point measure in $\xi \in \mathbb{R}^n$. The supports of the μ_j are contained in the compact set $[0,x] \cap C_n^B$. In terms of these measures, Lemma 4.3 implies that

$$\widetilde{L}_{\lambda_j}^{a+k_2}(\xi;\alpha) = \int_{[0,\eta]} \widetilde{L}_{jw}^a(\xi;\alpha)d\mu_j(w), \quad \xi \in \mathbb{R}^n.$$

By Prohorov's theorem (see e.g. [Kal]), the set $\{\mu_j : j \in \mathbb{N}\}$ is relatively sequentially compact. After passing to a subsequence if necessary, we obtain that the μ_j tend weakly to a probability measure $m_x \in M^1([0,x])$ as $j \to \infty$. Using Lemma 4.2, we conclude that

$$\begin{aligned} J^B_{(k_1+k_2,k_2)}\big(iy, 2\sqrt{x}\,\big) &= \lim_{j\to\infty} \widetilde{L}_{\lambda_j}^{a+k_2}(y^2/j;\alpha) = \lim_{j\to\infty} \int_{[0,x]} \widetilde{L}_{jw}^a(y^2/j;\alpha)\, d\mu_j(w) \\ &= \int_{[0,x]} J_k^B\big(iy, 2\sqrt{w}\,\big)\, dm_x(w). \end{aligned}$$

This readily implies the assertion. □

Remark 4.4. Our results on complete Sonine formulas for J_k^B in Theorems 3.2 and 4.1 do not cover the case $k = (k_1, k_2)$ with arbitrary $k_1 \geq 0, k_2 > 0$ and $h > k_2(n-1)$. We conjecture that in this case, a positive Sonine formula exists as well.

We conclude this section with an immediate consequence of Theorem 1.1, Lemma 4.2, and the proof of Theorem 4.1:

Corollary 4.5. *Let $a \geq -1/2$, $\alpha > 0$ and $h > 0$. Assume that for each partition $\kappa \in \Lambda_n^+$ there exist nonnegative connection coefficients $c_{\kappa,\lambda} \geq 0$ such that*

$$L_\kappa^{a+h}(x;\alpha) = \sum_{\lambda \subseteq \kappa} c_{\kappa,\lambda} L_\lambda^a(x;\alpha) \qquad (x \in \mathbb{R}^n).$$

Then h is contained in the set $\Sigma(\alpha^{-1})$ of Theorem 1.1.

5. The limit $h \to \infty$

We finally turn to some application of Theorem 4.1 for $h \to \infty$ which is based on the fact that the Bessel function $J^B_{(k_1,k_2)}$ of type B_n tends to a Bessel function of type A_{n-1} as $k_1 \to \infty$.

Indeed, the following limit relation follows easily from a comparison of the coefficients in (2.5) and (2.2) and is well-known; see e.g. [RV2] where also estimations for the rate of convergence are given.

Lemma 5.1. *Let $n \geq 2$ and $k_2 > 0$. Then, locally uniformly in $x, y \in \mathbb{R}^n$,*

$$\lim_{k_1\to+\infty} J^B_{(k_1,k_2)}(2\sqrt{k_1}\,x, iy) = J^A_{k_2}(x^2, -y^2).$$

Using Prohorov's theorem as in the proof of Theorem 4.1, we obtain the following integral representation from Lemma 5.1 and Theorem 4.1:

Theorem 5.2. *Let $k = (k_1, k_2)$ with $k_1, k_2 > 0$. Then for each $x \in C_n^B$ there exists a unique probability measure $\mu_x = \mu_x(k) \in M^1(C_n^B)$ such that*

$$J^A_{k_2}(x^2, -y^2) = \int_{C_n^B} J^B_{(k_1,k_2)}(\xi, iy)\, d\mu_x(\xi) \quad \textit{for all } y \in \mathbb{R}^n. \tag{5.1}$$

Proof. The uniqueness again follows from the injectivity of the Dunkl transform of measures. For the existence, fix k_1, k_2 as well as $x \in C_n^B$. Theorem 4.1 shows that for each $j \in \mathbb{N}$ there is a probability measure $\mu_j \in M^1(C_n^B)$ such that

$$J^B_{(k_1+jk_2,k_2)}\big(2\sqrt{k_1+jk_2}\cdot x\,,iy\big) \;=\int_{C_n^B} J^B_{(k_1,k_2)}(\xi,iy)\,d\mu_j(\xi) \quad \forall\, y\in\mathbb{R}^n, \tag{5.2}$$

where the support of μ_j is contained in $\big[0,\,2\sqrt{k_1+jk_2}\cdot x\big]$. We prove that the sequence (μ_j) is tight, which implies that it has a subsequence which converges weakly to some probability measure $\mu_x \in M^1(C_n^B)$; for tightness and the existence of convergent subsequences we refer to Section 4 of [Kal]. The arguments at the end of the proof of Theorem 4.1 in combination with Lemma 5.1 will then lead to

$$J^A_{k_2}(x^2,-y^2) = \int_{C_n^B} J^B_{(k_1,k_2)}(\xi,iy)\,d\mu_x(\xi)$$

as claimed. In order to check the tightness of (μ_j), we recapitulate from formulas (2.2) and (2.4) that for $k=(k_1,k_2)$,

$$J_k^B(z,w) = \sum_{\lambda\in\Lambda_n^+}\frac{1}{[\mu(k)]^\alpha_\lambda\,|\lambda|!\,4^{|\lambda|}}\cdot\frac{C^\alpha_\lambda(z^2)C^\alpha_\lambda(w^2)}{C^\alpha_\lambda(\mathbf{1})} \quad (z,w\in\mathbb{C}^n) \tag{5.3}$$

with $\mu(k) = k_1 + k_2(n-1)+\frac12$. Hence, by (5.2),

$$\begin{aligned}&\sum_{\lambda\in\Lambda_n^+}\frac{(k_1+jk_2)^{|\lambda|}}{[\mu(k_1+jk_2,k_2)]^\alpha_\lambda\,|\lambda|!}\cdot\frac{C^\alpha_\lambda(x^2)\,C^\alpha_\lambda(-y^2)}{C^\alpha_\lambda(\mathbf{1})}\\ &\qquad=\sum_{\lambda\in\Lambda_n^+}\frac{1}{[\mu(k)]^\alpha_\lambda\,|\lambda|!\,4^{|\lambda|}}\cdot\frac{C^\alpha_\lambda(-y^2)}{C^\alpha_\lambda(\mathbf{1})}\cdot\int_{C_n^B} C^\alpha_\lambda(\xi^2)\,d\mu_j(\xi)\end{aligned}$$

for all $y\in[0,\infty[^n$. If we compare the coefficients of this power series in y for $\lambda=(1,0,\ldots,0)$ and use that $C^\alpha_{(1,0,\ldots,0)}(z) = z_1+\ldots+z_n$ (c.f. the normalization (2.1)), we obtain from (2.3) and a straightforward computation that

$$\int_{C_n^B}(\xi_1^2+\ldots+\xi_n^2)\,d\mu_j(\xi) \;=\; \frac{4(k_1+jk_2)(k_1+k_2(n-1)+1/2)}{k_1+k_2(n+j-1)+1/2} \tag{5.4}$$

which remains bounded as $j\to\infty$. By Exercise 4 in Section 4 of [Kal] this implies the tightness of (μ_j) and thus the claim. $\square$

We have no general explicit formula for the measures μ_η in Theorem 5.2. However, in certain cases explicit formulas are known. For example, Lemma 5.1 and Theorem 3.2 lead for $k_2 = d/2\in\{1/2,1,2\}$ to the following

Corollary 5.3. *Let $\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and $d = dim_{\mathbb{R}}\mathbb{F}$. Then for $k_1\ge 0$, $x\in C_n^B$ and all $z\in\mathbb{C}^n$,*

$$J^A_{d/2}(x^2,-z^2) = \int_{C_n^B}\int_{U_n(\mathbb{F})} J^B_{(k_1,d/2)}\big(\sqrt{\sigma(xu\xi u^{-1}x)}\,,z\big)\,du\,d\mu_{k_1,d}(\xi) \tag{5.5}$$

with the probability measure

$$d\mu_{k_1,d}(\xi) = c_{k_1,d}\prod_{i=1}^n \xi_i^{k_1-1/2}\prod_{i<j}(\xi_i-\xi_j)^d\cdot e^{-(\xi_1+\ldots+\xi_n)/2}\,d\xi$$

with suitable normalizing constant $c_{k_1,d}>0$.

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