Dunkl Operators: Theory and Applications

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Summary. These lecture notes are intended as an introduction to the theory of rational Dunkl operators and the associated special functions, with an emphasis on positivity and asymptotics. We start with an outline of the general concepts: Dunkl operators, the intertwining operator, the Dunkl kernel and the Dunkl transform. We point out the connection with integrable particle systems of Calogero-Moser-Sutherland type, and discuss some systems of orthogonal polynomials associated with them. A major part is devoted to positivity results for the intertwining operator and the Dunkl kernel, the Dunkl-type heat semigroup, and related probabilistic aspects. The notes conclude with recent results on the asymptotics of the Dunkl kernel.

1 Introduction

While the theory of special functions in one variable has a long and rich history, the growing interest in special functions of several variables is comparatively recent. During the last years, there has in particular been a rapid development in the area of special functions with reflection symmetries and the harmonic analysis related with root systems. The motivation for this subject comes to some extent from the theory of Riemannian symmetric spaces, whose spherical functions can be written as multi-variable special functions depending on certain discrete sets of parameters. A key tool in the study of special functions with reflection symmetries are Dunkl operators. Generally speaking, these are commuting differential-difference operators, associated to a finite reflection group on a Euclidean space. The first class of such operators, now often called "rational" Dunkl operators, were introduced by C.F. Dunkl in the late 80ies. In a series of papers ([11]-[15]), he built up the framework for a theory of special functions and integral transforms in several variables related with reflection groups. Since then, various other classes of Dunkl operators have become important, in the first place the trigonometric Dunkl operators of Heckman, Opdam and the Cherednik operators. These will not be discussed in our notes; for an overview, we refer to [27]. An important motivation to study Dunkl operators originates in their relevance for the analysis of quantum many body systems of Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics, especially in conformal field theory. A good bibliography is contained in [10].

The aim of these lecture notes is an introduction to rational Dunkl theory, with an emphasis on the author's results in this area. Rational Dunkl operators bear a rich analytic structure which is not only due to their commutativity, but also to the existence of an intertwining operator between Dunkl operators and usual partial derivatives. We shall first give an overview of the general concepts, including an account on the relevance of Dunkl operators in the study of Calogero-Moser-Sutherland models. We also discuss some of the special functions related with them. A major topic will be positivity results; these concern the intertwining operator as well as the kernel of the Dunkl transform, and lead to a variety of positive semigroups in the Dunkl setting with possible probabilistic interpretations. We make this explicit at hand of the most important example: the Dunkl-type heat semigroup, which is generated by the analog of the Laplacian in the Dunkl setting. The last section presents recent results on the asymptotics of the Dunkl kernel and the short-time behavior of heat kernels associated with root systems.

2 Dunkl operators and the Dunkl transform

The aim of this section is to provide an introduction to the theory of rational Dunkl operators, which we shall call Dunkl operators for short, and to the Dunkl transform. General references are [11]-[15], [18], [30] and [44]; for a background on reflection groups and root systems the reader is referred to [29] and [23]. We do not intend to give a complete survey, but rather focus on those aspects which will be important in the context of this lecture series.

2.1 Root systems and reflection groups

The basic ingredient in the theory of Dunkl operators are root systems and finite reflection groups, acting on some Euclidean space $(E, \langle \,.\,,.\,\rangle)$ of finite dimension N. We shall always assume that $E = \mathbb{R}^N$ with the standard Euclidean scalar product $\langle x, y\rangle = \sum_{j=1}^N x_j y_j$. For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote by σ_α the reflection in the hyperplane $\langle \alpha\rangle^\perp$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\,\frac{\langle \alpha, x\rangle}{|\alpha|^2}\,\alpha\,,$$

where $|x| := \sqrt{\langle x, x\rangle}$. Each reflection σ_α is contained in the orthogonal group $O(N, \mathbb{R})$. We start with the basic definitions:

Definition 2.1. *Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a finite set. Then R is called a root system, if*

(1) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ *for all* $\alpha \in R$;
(2) $\sigma_\alpha(R) = R$ *for all* $\alpha \in R$.

The subgroup $G = G(R) \subseteq O(N, \mathbb{R})$ which is generated by the reflections $\{\sigma_\alpha,\, \alpha \in R\}$ is called the reflection group (or Coxeter-group) associated with R. The dimension of $span_{\mathbb{R}} R$ is called the rank of R.

Property (1) is called reducedness. It is often not required in Lie-theoretic contexts, where instead the root systems under consideration are assumed to be crystallographic. This means that

$$\frac{2\langle \alpha, \beta\rangle}{\langle \beta, \beta\rangle} \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in R.$$

If R is crystallographic and has full rank, then $span_{\mathbb{Z}} R$ forms a lattice in $\mathbb{R}^N$ (called the *root-lattice*) which is stabilized by the action of the associated reflection group.

Lemma 2.1. (1) *For any root system R in $\mathbb{R}^N$, the reflection group $G = G(R)$ is finite.*
(2) *The set of reflections contained in $G(R)$ is exactly $\{\sigma_\alpha\,,\, \alpha \in R\}$.*

Proof. As R is left invariant by G, we have a natural homomorphism $\varphi : G \to S(R)$ of G into the symmetric group of R, given by $\varphi(g)(\alpha) := g\alpha \in R$. This homomorphism is injective: indeed, each reflection s_α, and therefore also each element $g \in G$ fixes pointwise the orthogonal complement of the subspace spanned by R. If also $g(\alpha) = \alpha$ for all $\alpha \in R$, then g must be the identity. This implies assertion (1) because the order of $S(R)$ is finite. Property (2) is more involved. An elegant proof can be found in Section 4.2 of [19].

Exercise 1. If $g \in O(N, \mathbb{R})$ and $\alpha \in \mathbb{R}^N \setminus \{0\}$, then $g\,\sigma_\alpha g^{-1} = \sigma_{g\alpha}$.

Together with part (2) of the previous lemma, this shows that there is a bijective correspondence between the conjugacy classes of reflections in G and the orbits in R under the natural action of G. We shall need some more concepts: Each root system can be written as a disjoint union $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by a hyperplane through the origin. Such a set R_+ is called a *positive subsystem.* Of course, its choice is not unique. The set of hyperplanes $\{\langle\alpha\rangle^\perp,\, \alpha \in R\}$ divides $\mathbb{R}^N$ into connected open components, called the *Weyl chambers* of R. It can be shown that the topological closure $\overline{C}$ of any chamber C is a fundamental domain for G, i.e. $\overline{C}$ is naturally homeomorphic with the space $(\mathbb{R}^N)^G$ of all G-orbits in $\mathbb{R}^N$, endowed with the quotient topology. G permutes the reflecting hyperplanes as well as the chambers.

Exercise 2. *Dihedral groups.* In the Euclidean plane $\mathbb{R}^2$, let $d \in O(2, \mathbb{R})$ denote the rotation around $2\pi/n$ with $n \geq 3$ and s the reflection at the y-axis. Show that the group $\mathcal{D}_n$ generated by d and s consists of all orthogonal transformations which preserve a regular n-sided polygon centered at the origin. (Hint: $dsd = s$.) Show that $\mathcal{D}_n$ is a finite reflection group and determine its root system. Can the crystallographic condition always be satisfied?

Examples 2.1. **(1) Type** A_{N-1}. Let S_N denote the symmetric group in N elements. It acts faithfully on $\mathbb{R}^N$ by permuting the standard basis vectors $e_1, \ldots, e_N$. Each transposition (ij) acts as a reflection σ_{ij} sending $e_i - e_j$ to its negative. Since S_N is generated by transpositions, it is a finite reflection group. A root system of S_N is given by

$$R = \{\pm(e_i - e_j),\, 1 \leq i < j \leq N\}.$$

Its span is $(e_1 + \ldots + e_N)^\perp$, and thus the rank is $N - 1$.
(2) Type B_N. Here G is the reflection group in $\mathbb{R}^N$ generated by the transpositions σ_{ij} as above, as well as the sign changes $\sigma_i : e_i \mapsto -e_i$, $i = 1, \ldots, N$. The group of sign changes is isomorphic to $\mathbb{Z}_2^N$, intersects S_N trivially and is normalized by S_N, so G is isomorphic with the semidirect product $S_N \ltimes \mathbb{Z}_2^N$. The corresponding root system has rank N; it is given by

$$R = \{\pm e_i,\, 1 \leq i \leq N,\, \pm(e_i \pm e_j), 1 \leq i < j \leq N\}.$$

A root system R is called *irreducible*, if it cannot be written as the orthogonal disjoint union $R = R_1 \cup R_2$ of two root systems R_1 , R_2. Any root system can be uniquely written as an orthogonal disjoint union of irreducible root systems. There exists a classification of all irreducible root systems in terms of Coxeter graphs. The crystallographic ones are made up by 4 infinite families A_n, B_n (those discussed above), C_n , D_n , as well as 5 exceptional root systems. For details, we refer to [29].

2.2 Dunkl operators

Let R be a fixed root system in $\mathbb{R}^N$ and G the associated reflection group. From now on we assume that R is normalized in the sense that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$; this simplifies formulas, but is no loss of generality for our purposes. The Dunkl operators attached with R can be considered as perturbations of the usual partial derivatives by reflection parts. These reflection parts are coupled by parameters, which are given in terms of a multiplicity function:

Definition 2.2. *A function $k : R \to \mathbb{C}$ on the root system R is called a multiplicity function on R, if it is invariant under the natural action of G on R. The $\mathbb{C}$-vector space of multiplicity functions on R is denoted by K.*

Notice that the dimension of K is equal to the number of G-orbits in R. We write $k \geq 0$ if $k(\alpha) \geq 0$ for all $\alpha \in R$.

Definition 2.3. *Let $k \in K$. Then for $\xi \in \mathbb{R}^N$, the Dunkl operator $T_\xi := T_\xi(k)$ is defined (for $f \in C^1(\mathbb{R}^N)$) by*

$$T_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \, \langle \alpha, \xi \rangle \, \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Here ∂_ξ denotes the directional derivative corresponding to ξ, and R_+ is a fixed positive subsystem of R. For the i-th standard basis vector $\xi = e_i \in \mathbb{R}^N$ we use the abbreviation $T_i = T_{e_i}$.

The above definition does not depend on the special choice of R_+, thanks to the G-invariance of k. In case $k = 0$, the T_ξ reduce to the corresponding directional derivatives. The operators T_ξ were introduced and first studied for $k \geq 0$ by C.F. Dunkl ([11]-[15]). They enjoy regularity properties similar to usual partial derivatives on various spaces of smooth functions on $\mathbb{R}^N$. We shall use the following notations:

Notation 2.1. 1. $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$.
2. $\Pi := \mathbb{C}[\mathbb{R}^N]$ is the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$. It has a natural grading

$$\Pi = \bigoplus_{n \geq 0} \mathcal{P}_n \,,$$

where $\mathcal{P}_n$ is the subspace of homogeneous polynomials of (total) degree n.

3. $\mathscr{S}(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$,
$$\mathscr{S}(\mathbb{R}^N) := \{f \in C^\infty(\mathbb{R}^N) : \|x^\beta \partial^\alpha f\|_{\infty,\mathbb{R}^N} < \infty \quad \text{for all } \alpha, \beta \in \mathbb{Z}_+^N\}.$$
It is a Fréchet space with the usual locally convex topology.

The Dunkl operators T_ξ have the following regularity properties:

Lemma 2.2. (1) *If $f \in C^m(\mathbb{R}^N)$ with $m \geq 1$, then $T_\xi f \in C^{m-1}(\mathbb{R}^N)$.*
(2) *T_ξ leaves $C_c^\infty(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$ invariant.*
(3) *T_ξ is homogeneous of degree -1 on Π, that is, $T_\xi\, p \in \mathcal{P}_{n-1}$ for $p \in \mathcal{P}_n$.*

Proof. All statements follow from the representation
$$\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle} = \int_0^1 \partial_\alpha f(x - t\langle \alpha, x\rangle \alpha)dt \quad \text{for } f \in C^1(\mathbb{R}^N),\ \alpha \in R$$
(recall our normalization $\langle \alpha, \alpha\rangle = 2$). (1) and (3) are immediate; the proof of (2) (for $\mathscr{S}(\mathbb{R}^N)$) is also straightforward but more technical; it can be found in [30].

Due to the G-invariance of k, the Dunkl operators T_ξ are G-equivariant: In fact, consider the natural action of $O(N,\mathbb{R})$ on functions $f : \mathbb{R}^N \to \mathbb{C}$, given by
$$h \cdot f(x) := f(h^{-1}x), \quad h \in O(N, \mathbb{R}).$$
Then an easy calculation shows:

Exercise 3. $g \circ T_\xi \circ g^{-1} = T_{g\xi}$ for all $g \in G$.

Moreover, there holds a product rule:

Exercise 4. If f is G-invariant then $T_\xi f = \partial_\xi f$. If $f, g \in C^1(\mathbb{R}^N)$ and at least one of them is G-invariant, then
$$T_\xi(fg) = T_\xi(f) \cdot g + f \cdot T_\xi(g). \tag{2.1}$$

The most striking property of the Dunkl operators, which is the foundation for rich analytic structures related with them, is the following

Theorem 2.1. *For fixed k, the associated $T_\xi = T_\xi(k)$, $\xi \in \mathbb{R}^N$ commute.*

This result was obtained in [12] by a clever direct argumentation. An alternative proof, relying on Koszul complex ideas, is given in [18]. As a consequence of Theorem 2.1 there exists an algebra homomorphism $\Phi_k : \Pi \to \operatorname{End}_{\mathbb{C}}(\Pi)$ which is defined by

$$\Phi_k : x_i \mapsto T_i,\ 1 \mapsto id.$$

For $p \in \Pi$ we write

$$p(T) := \Phi_k(p).$$

The classical case $k = 0$ will be distinguished by the notation $\Phi_0(p) =: p(\partial)$. Of particular importance is the *Dunkl Laplacian*, which is defined by

$$\Delta_k := p(T) \quad \text{with } p(x) = |x|^2.$$

Theorem 2.2.

$$\Delta_k = \Delta + 2\sum_{\alpha\in R_+} k(\alpha)\delta_\alpha \qquad \textit{with} \qquad \delta_\alpha f(x) = \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}; \tag{2.2}$$

here Δ *and* ∇ *denote the usual Laplacian and gradient respectively.*

This representation is obtained by a direct calculation (recall again our convention $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$) by use of the following Lemma:

Lemma 2.3. [12] *For* $\alpha \in R$, *define*

$$\rho_\alpha f(x) := \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle} \quad (f \in C^1(\mathbb{R}^N)).$$

Then

$$\sum_{\alpha,\beta\in R_+} k(\alpha)k(\beta)\langle \alpha, \beta\rangle \rho_\alpha\rho_\beta = 0.$$

It is not difficult to check that

$$\Delta_k = \sum_{i=1}^N T_{\xi_i}^2$$

for any orthonormal basis $\{\xi_1, \dots, \xi_N\}$ of $\mathbb{R}^N$, see [12] for the proof. Together with the G-equivariance of the Dunkl operators, this immediately implies that Δ_k is G-invariant, i.e.

$$g \circ \Delta_k = \Delta_k \circ g \quad (g \in G).$$

Examples 2.2. **(1) The rank-one case.** In case $N = 1$, the only choice of R is $R = \{\pm\sqrt{2}\}$, which is the root system of type A_1. The corresponding reflection group is $G = \{id, \sigma\}$ acting on $\mathbb{R}$ by $\sigma(x) = -x$. The Dunkl operator $T := T_1$ associated with the multiplicity parameter $k \in \mathbb{C}$ is given by

$$Tf(x) = f'(x) + k\,\frac{f(x) - f(-x)}{x}.$$

Its square T^2, when restricted to the even subspace $C^2(\mathbb{R})^e := \{f \in C^2(\mathbb{R}) : f(x) = f(-x)\}$, is given by a singular Sturm-Liouville operator:

$$T^2|_{C^2(\mathbb{R})^e} f \;=\; f'' + \frac{2k}{x}\cdot f'\,.$$

(2) Dunkl operators of type A_{N-1}**.** Suppose $G = S_N$ with root system of type A_{N-1}. (In contrast to the above example, G now acts on $\mathbb{R}^N$). As all transpositions are conjugate in S_N, the vector space of multiplicity functions is one-dimensional. The Dunkl operators associated with the multiplicity parameter $k \in \mathbb{C}$ are given by

$$T_i^S \;=\; \partial_i \;+\; k\cdot\sum_{j\neq i}\frac{1-\sigma_{ij}}{x_i - x_j} \quad (i = 1,\ldots,N),$$

and the Dunkl Laplacian is

$$\Delta_k^S \;=\; \Delta + 2k\sum_{1\leq i<j\leq N}\frac{1}{x_i - x_j}\big[(\partial_i - \partial_j) - \frac{1-\sigma_{ij}}{x_i - x_j}\big]\,.$$

(3) Dunkl operators of type B_N**.** Suppose R is a root system of type B_N, corresponding to $G = S_N \ltimes \mathbb{Z}_2^N$. There are two conjugacy classes of reflections in G, leading to multiplicity functions of the form $k = (k_0, k_1)$ with $k_i \in \mathbb{C}$. The associated Dunkl operators are given by

$$T_i^B \;=\; \partial_i \;+\; k_1\frac{1-\sigma_i}{x_i} \;+\; k_0\cdot\sum_{j\neq i}\big[\frac{1-\sigma_{ij}}{x_i - x_j} + \frac{1-\tau_{ij}}{x_i + x_j}\big] \quad (i = 1,\ldots,N),$$

where $\tau_{ij} := \sigma_{ij}\sigma_i\sigma_j$.

2.3 A formula of Macdonald and its analog in Dunkl theory

In the classical theory of spherical harmonics (see for instance [28]) the following bilinear pairing on Π, sometimes called Fischer product, plays an important role:

$$[p,q]_0 := (p(\partial)q)(0), \quad p,q \in \Pi.$$

This pairing is closely related to the scalar product in $L^2(\mathbb{R}^N, e^{-|x|^2/2}dx)$; in fact, in his short note [39] Macdonald observed the following identity:

$$[p,q]_0 \;=\; (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{-\Delta/2}p(x)\, e^{-\Delta/2}q(x)\, e^{-|x|^2/2}\,dx.$$

Here $e^{-\Delta/2}$ is well-defined as a linear operator on Π by means of the terminating series

$$e^{-\Delta/2}p = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\Delta^n p.$$

Both the Fischer product as well as Macdonald's identity have a useful generalization in the Dunkl setting. In the following, we shall always restrict to the case $k \geq 0$.

Definition 2.4. *For* $p, q \in \Pi$ *define*

$$[p, q]_k := (p(T)q)(0).$$

This bilinear form was introduced in [14]. We collect some of its basic properties:

Lemma 2.4. (1) *If* $p \in \mathcal{P}_n$ *and* $q \in \mathcal{P}_m$ *with* $n \neq m$, *then* $[p, q]_k = 0$.
(2) $[x_i\, p\,, q]_k = [p, T_i\, q]_k \quad (p, q \in \Pi,\ i = 1, \ldots, N)$.
(3) $[g \cdot p\,, g \cdot q]_k = [p, q]_k \quad (p, q \in \Pi,\ g \in G)$.

Proof. (1) follows from the homogeneity of the Dunkl operators, (2) is clear from the definition, and (3) follows from Exercise 3.

Let w_k denote the weight function on $\mathbb{R}^N$ defined by

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}. \tag{2.3}$$

It is G-invariant and homogeneous of degree 2γ, with the index

$$\gamma := \gamma(k) := \sum_{\alpha \in R_+} k(\alpha). \tag{2.4}$$

Notice that by G-invariance of k, we have $k(-\alpha) = k(\alpha)$ for all $\alpha \in R$. Hence this definition does again not depend on the special choice of R_+. Further, we define the constant

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x) dx,$$

a a so-called Macdonald-Mehta-Selberg integral. There exists a closed form for it which was conjectured and proved by Macdonald [40] for the infinite series of root systems. An extension to arbitrary crystallographic reflection groups is due to Opdam [44], and there are computer-assisted proofs for some non-crystallographic root systems. As far as we know, a general proof for arbitrary root systems has not yet been found.

We shall need the following anti-symmetry of the Dunkl operators:

Proposition 2.1. [15] *Let* $k \geq 0$. *Then for every* $f \in \mathscr{S}(\mathbb{R}^N)$ *and* $g \in C_b^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} T_\xi f(x) g(x) w_k(x) dx = -\int_{\mathbb{R}^N} f(x) T_\xi g(x) w_k(x) dx\,.$$

Proof. A short calculation. In order to have the appearing integrals well defined, one has to assume $k \geq 1$ first, and then extend the result to general $k \geq 0$ by analytic continuation.

Proposition 2.2. *For all* $p, q \in \Pi$,

$$[p,q]_k \,=\, c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}p(x)\, e^{-\Delta_k/2}q(x)\, e^{-|x|^2/2}\, w_k(x)dx. \qquad (2.5)$$

This result is due to Dunkl ([14]). As the Dunkl Laplacian is homogeneous of degree -2, the operator $e^{-\Delta_k/2}$ is well-defined and bijective on Π, and it preserves the degree. We give here a direct proof which is partly taken from an unpublished part of M. de Jeu's thesis ([31], §3.3). It involves the following commutator results in $\mathrm{End}_{\mathbb{C}}(\Pi)$, where as usual, $[A,B] = AB - BA$ for $A, B \in \mathrm{End}_{\mathbb{C}}(\Pi)$.

Lemma 2.5. *For* $i = 1, \ldots, N$,

(1) $[x_i\,,\, \Delta_k/2] \,=\, -T_i\,;$
(2) $[x_i\,,\, e^{-\Delta_k/2}] \,=\, T_i\, e^{-\Delta_k/2}\,.$

Proof. (1) follows by direct calculation, c.f. [12]. Induction then yields that

$$[x_i\,, (\Delta_k/2)^n] \,=\, -n\, T_i(\Delta_k/2)^{n-1} \quad \text{for } n \geq 1,$$

and this implies (2).

Proof (of Proposition 2.2). Let $i \in \{1, \ldots, N\}$, and denote the right side of (2.5) by $(p,q)_k$. Then by the anti-symmetry of T_i in $L^2(\mathbb{R}^N, w_k)$, the product rule for T_i, see Exercise 4, and the above Lemma,

$$\begin{aligned}
(p, T_i\, q)_k \,&= c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}p\cdot (T_i\, e^{-\Delta_k/2}q)\, e^{-|x|^2/2}\, w_k\, dx \\
&= \,-\, c_k^{-1}\int_{\mathbb{R}^N} T_i(e^{-|x|^2/2}e^{-\Delta_k/2}p)\cdot (e^{-\Delta_k/2}q)\, w_k\, dx \\
&= c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}(x_i\, p)\cdot (e^{-\Delta_k/2}q)\, e^{-|x|^2/2}\, w_k\, dx \,=\, (x_i\, p, q)_k.
\end{aligned}$$

But the form $[.,.]_k$ has the same property by Lemma 2.4(2). It is now easily checked that the assertion is true if p or q is constant, and then, by induction on $\max(\deg p, \deg q)$, for all homogeneous p, q. This suffices by the linearity of both forms.

Corollary 2.1. *Let again* $k \geq 0$. *Then the pairing* $[\,.\,,\,.\,]_k$ *on* Π *is symmetric and non-degenerate, i.e.* $[p,q]_k = 0$ *for all* $q \in \Pi$ *implies that* $p = 0$.

Exercise 5. Check the details in the proofs of Proposition 2.2 and Corollary 2.1.

2.4 Dunkl's intertwining operator

It was first shown in [14] that for non-negative multiplicity functions, the associated commutative algebra of Dunkl operators is intertwined with the algebra of usual partial differential operators by a unique linear and homogeneous isomorphism on polynomials. A thorough analysis in [18] subsequently revealed that for general k, such an intertwining operator exists if and only if the common kernel of the T_ξ, considered as linear operators on Π, contains no "singular" polynomials besides the constants. More precisely, the following characterization holds:

Theorem 2.3. [18] *Let* $K^{reg} := \{k \in K : \bigcap_{\xi\in\mathbb{R}^N} Ker T_\xi(k) = \mathbb{C}\cdot 1\}$ *(the regular parameter set). Then the following statements are equivalent*

(1) $k \in K^{reg}$;
(2) *There exists a unique linear isomorphism ("intertwining operator")* V_k *of* Π *such that*

$$V_k(\mathcal{P}_n) = \mathcal{P}_n\,,\;\; V_k\,|_{\mathcal{P}_0} = id \quad \textit{and} \;\; T_\xi V_k = V_k\,\partial_\xi \quad \textit{for all } \xi \in \mathbb{R}^N.$$

The proof of this result is by induction on the degree of homogeneity and requires only linear algebra.

The intertwining operator V_k commutes with the G-action:

Exercise 6. $g^{-1}\circ V_k \circ g = V_k \quad (g \in G)$.
Hint: Use the G-equivariance of the T_ξ and the defining properties of V_k.

Proposition 2.3. $\{k \in K : k \geq 0\} \subseteq K^{reg}$.

Proof. Suppose that $p \in \oplus_{n\geq 1}\mathcal{P}_n$ satisfies $T_\xi(k)p = 0$ for all $\xi \in \mathbb{R}^N$. Then $[q,p]_k = 0$ for all $q \in \oplus_{n\geq 1}\mathcal{P}_n$, and hence also $[q,p]_k = 0$ for all $q \in \Pi$. Thus $p = 0$, by the non-degeneracy of $[\,.\,,\,.\,]_k$, see Corollary 2.1.

The complete singular parameter set $K \setminus K^{reg}$ is explicitly determined in [18]. K^{reg} is an open subset of K which is invariant under complex conjugation, and contains $\{k \in K : \text{Re } k \geq 0\}$. Later in these lectures, we will in fact restrict our attention to non-negative multiplicity functions. These are of particular interest concerning our subsequent positivity results, which could not be expected for non-positive multiplicities. Though the intertwining operator plays an important role in Dunkl's theory, an explicit "closed" form for it is known so far only in some special cases. Among these are

1. *The rank-one case.* Here

$$K^{reg} = \mathbb{C}\setminus\{-1/2 - n,\; n \in \mathbb{Z}_+\}.$$

The associated intertwining operator is given explicitly by

$$V_k(x^{2n}) \;=\; \frac{(\frac{1}{2})_n}{(k+\frac{1}{2})_n}\, x^{2n}\,; \quad V_k(x^{2n+1}) \;=\; \frac{(\frac{1}{2})_{n+1}}{(k+\frac{1}{2})_{n+1}}\, x^{2n+1}\,,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer-symbol. For $\mathrm{Re}\, k > 0$, this amounts to the following integral representation (see [14], Th. 5.1):

$$V_k\, p(x) \;=\; \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)} \int_{-1}^{1} p(xt)\,(1-t)^{k-1}(1+t)^k\, dt. \tag{2.6}$$

2. *The case $G = S_3$.* This was studied in [16]. Here

$$K^{reg} = \mathbb{C} \setminus \{-1/2 - n, -1/3 - n, -2/3 - n,\, n \in \mathbb{Z}_+\}.$$

In order to bring V_k into action in a further development of the theory, it is important to extend it to larger function spaces. For this we shall always assume that $k \geq 0$. In a first step, V_k is extended to a bounded linear operator on suitably normed algebras of homogeneous series on a ball. This concept goes back to [14].

Definition 2.5. *For $r > 0$, let $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$ denote the closed ball of radius r, and let A_r be the closure of Π with respect to the norm*

$$\|p\|_{A_r} := \sum_{n=0}^{\infty} \|p_n\|_{\infty, B_r} \quad \textit{for } p = \sum_{n=0}^{\infty} p_n\,,\; p_n \in \mathcal{P}_n.$$

Clearly A_r is a commutative Banach-$*$-algebra under the pointwise operations and with complex conjugation as involution. Each $f \in A_r$ has a unique representation $f = \sum_{n=0}^{\infty} f_n$ with $f_n \in \mathcal{P}_n$, and is continuous on the ball B_r and real-analytic in its interior. The topology of A_r is stronger than the topology induced by the uniform norm on B_r. Notice also that A_r is not closed with respect to $\|.\|_{\infty, B_r}$ and that $A_r \subseteq A_s$ with $\|\,.\,\|_{A_r} \geq \|\,.\,\|_{A_s}$ for $s \leq r$.

Theorem 2.4. $\|V_k p\|_{\infty, B_r} \leq \|p\|_{\infty, B_r}$ *for each $p \in \mathcal{P}_n$.*

The proof of this result is given in [14] and can also be found in [19]. It uses the *van der Corput-Schaake inequality* which states that for each *real-valued* $p \in \mathcal{P}_n$,

$$\sup\{|\langle \nabla p(x), y\rangle| : x, y \in B_1\} \;\leq\; n\|p\|_{\infty, B_1}.$$

Notice that here the converse inequality is trivially satisfied, because for $p \in \mathcal{P}_n$ we have $\langle \nabla p(x), x\rangle = np(x)$. The following is now immediate:

Corollary 2.2. $\|V_k f\|_{A_r} \leq \|f\|_{A_r}$ *for every $f \in \Pi$, and V_k extends uniquely to a bounded linear operator on A_r via*

$$V_k f := \sum_{n=0}^{\infty} V_k f_n \quad \textit{for } f = \sum_{n=0}^{\infty} f_n\,.$$

Formula (2.6) shows in particular that in the rank-one case with $k > 0$, the operator V_k is positivity-preserving on polynomials. It was conjectured by Dunkl in [14] that for arbitrary reflection groups and non-negative multiplicity functions, the linear functional $f \mapsto V_k f(x)$ on A_r should be positive. We shall see in Section 4.1 that this is in fact true. As a consequence, we shall obtain the existence of a positive integral representation generalizing (2.6), which in turn allows to extend V_k to larger function spaces. This positivity result also has important consequences for the structure of the Dunkl kernel, which generalizes the usual exponential function in the Dunkl setting. We shall introduce it in the following section.

Exercise 7. The *symmetric spectrum* $\Delta_S(A)$ of a (unital) commutative Banach-$*$-algebra A is defined as the set of all non-zero algebra homomorphisms $\varphi : A \to \mathbb{C}$ satisfying the $*$-condition $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in A$. It is a compact Hausdorff space with the weak-*-topology (sometimes called the Gelfand topology). Prove that the symmetric spectrum of the algebra A_r is given by $\Delta_S(A_r) = \{\varphi_x : x \in B_r\}$, where φ_x is the evaluation homomorphism $\varphi_x(f) := f(x)$. Show also that the mapping $x \mapsto \varphi_x$ is a homeomorphism from B_r onto $\Delta_S(A_r)$.

2.5 The Dunkl kernel

Throughout this section we assume that $k \geq 0$. Moreover, we denote by $\langle \,.\,,\,.\,\rangle$ not only the Euclidean scalar product on $\mathbb{R}^N$, but also its *bilinear* extension to $\mathbb{C}^N \times \mathbb{C}^N$. For fixed $y \in \mathbb{C}^N$, the exponential function $x \mapsto e^{\langle x,y\rangle}$ belongs to each of the algebras A_r, $r > 0$. This justifies the following

Definition 2.6. [14] *For $y \in \mathbb{C}^N$, define*

$$E_k(x,y) := V_k(e^{\langle\,.\,,y\rangle})(x), \quad x \in \mathbb{R}^N.$$

The function E_k is called the *Dunkl kernel* associated with G and k. It can alternatively be characterized as the solution of a joint eigenvalue problem for the associated Dunkl operators.

Proposition 2.4. *Let $k \geq 0$ and $y \in \mathbb{C}^N$. Then $f = E_k(\,.\,,y)$ is the unique solution of the system*

$$T_\xi f \,=\, \langle \xi, y\rangle f \quad \text{for all } \xi \in \mathbb{R}^N \tag{2.7}$$

which is real-analytic on $\mathbb{R}^N$ and satisfies $f(0) = 1$.

Proof. $E_k(\,.\,y)$ is real-analytic on $\mathbb{R}^N$ by our construction. Define

$$E_k^{(n)}(x,y) := \frac{1}{n!} V_k \langle\,.\,,y\rangle^n(x), \quad x \in \mathbb{R}^N,\, n = 0,1,2,\ldots.$$

Then $E_k(x,y) = \sum_{n=0}^\infty E_k^{(n)}(x,y)$, and the series converges uniformly and absolutely with respect to x. The homogeneity of V_k immediately implies $E_k(0,y) = 1$. Further, by the intertwining property,

$$T_\xi E_k^{(n)}(\,.\,,y) \,=\, \frac{1}{n!} V_k\,\partial_\xi \langle\,.\,,y\rangle^n \,=\, \langle \xi, y\rangle E_k^{(n-1)}(\,.\,,y) \tag{2.8}$$

for all $n \geq 1$. This shows that $E_k(\,.\,,y)$ solves (2.7). To prove uniqueness, suppose that f is a real-analytic solution of (2.7) with $f(0) = 1$. Then T_ξ can be applied termwise to the homogeneous expansion $f = \sum_{n=0}^\infty f_n$, $f_n \in \mathcal{P}_n$, and comparison of homogeneous parts shows that

$$f_0 = 1, \quad T_\xi f_n = \langle \xi, y\rangle f_{n-1} \quad \text{for } n \geq 1.$$

As $\{k \in K : k \geq 0\} \subseteq K^{reg}$, it follows by induction that all f_n are uniquely determined.

While this construction has been carried out only for $k \geq 0$, there is a more general result by Opdam which assures the existence of a general exponential kernel with properties according to the above lemma for arbitrary regular multiplicity parameters. The following is a weakened version of [44], Proposition 6.7; it in particular implies that E_k has a holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$:

Theorem 2.5. *For each $k \in K^{reg}$ and $y \in \mathbb{C}^N$, the system*

$$T_\xi\, f \,=\, \langle \xi, y\rangle f \quad (\xi \in \mathbb{R}^N)$$

has a unique solution $x \mapsto E_k(x,y)$ which is real-analytic on $\mathbb{R}^N$ and satisfies $f(0) = 1$. Moreover, the mapping $(x,k,y) \mapsto E_k(x,y)$ extends to a meromorphic function on $\mathbb{C}^N \times K \times \mathbb{C}^N$ with pole set $\mathbb{C}^N \times (K \setminus K^{reg}) \times \mathbb{C}^N$

We collect some further properties of the Dunkl kernel E_k.

Proposition 2.5. *Let $k \geq 0$, $x, y \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$ and $g \in G$.*

(1) $E_k(x,y) = E_k(y,x)$
(2) $E_k(\lambda x, y) = E_k(x, \lambda y)$ *and* $E_k(gx, gy) = E_k(x,y)$.
(3) $\overline{E_k(x,y)} = E_k(\overline{x}, \overline{y})$.

Proof. (1) This is shown in [14]. (2) is easily obtained from the definition of E_k together with the homogeneity and equivariance properties of V_k. For (3), notice that $f := \overline{E_k(\,.\,,y)}$, which is again real-analytic on $\mathbb{R}^N$, satisfies $T_\xi f = \langle \xi, \overline{y}\rangle\, f$, $f(0) = 1$. By the uniqueness part of the above Proposition, $\overline{E_k(x,y)} = E_k(x,\overline{y})$ for all real x. Now both $x \mapsto \overline{E_k(\overline{x},y)}$ and $x \mapsto E_k(x,\overline{y})$ are holomorphic on $\mathbb{C}^N$ and agree on $\mathbb{R}^N$. Hence they coincide.

Just as with the intertwining operator, the kernel E_k is explicitly known for some particular cases only. An important example is again the rank-one situation:

Example 2.1. In the *rank-one case* with $\operatorname{Re} k > 0$, the integral representation (2.6) for V_k implies that for all $x, y \in \mathbb{C}$,

$$\begin{aligned} E_k(x,y) &= \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)} \int_{-1}^{1} e^{txy}(1-t)^{k-1}(1+t)^k\,dt \\ &= e^{xy}\cdot {}_1F_1(k, 2k+1, -2xy). \end{aligned}$$

This can also be written as

$$E_k(x,y) = j_{k-1/2}(ixy) + \frac{xy}{2k+1}\, j_{k+1/2}(ixy),$$

where for $\alpha \geq -1/2$, j_α is the normalized spherical Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\cdot \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\cdot \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}\,. \tag{2.9}$$

This motivates the following

Definition 2.7. [44] *The generalized Bessel function is defined by*

$$J_k(x,y) := \frac{1}{|G|}\sum_{g\in G} E_k(gx,y) \quad (x,y\in\mathbb{C}^N). \tag{2.10}$$

Thanks to Proposition 2.5 J_k is G-invariant in both arguments and therefore naturally considered on Weyl chambers of G (or their complexifications). In the rank-one case, we have

$$J_k(x,y) = j_{k-1/2}(ixy).$$

It is a well-known fact from classical analysis that for fixed $y \in \mathbb{C}$, the function $f(x) = j_{k-1/2}(ixy)$ is the unique analytic solution of the differential equation

$$f'' + \frac{2k}{x} f' = y^2 f$$

which is even and normalized by $f(0) = 1$. In order to see how this can be generalized to the multivariable case, consider the algebra of G-invariant polynomials on $\mathbb{R}^N$,

$$\Pi^G = \{p \in \Pi : g\cdot p = p \quad \text{for all } g \in G\}.$$

If $p \in \Pi^G$, then it follows from the equivariance of the Dunkl operators (Exercise 3) that $p(T)$ commutes with the G-action; a detailed argument for this is given in [26]. Thus $p(T)$ leaves Π^G invariant, and we obtain in particular that for fixed $y \in \mathbb{C}^N$, the generalized Bessel function $J_k(\,.\,,y)$ is a solution to the following Bessel-system:

$$p(T)f = p(y)f \quad \text{for all } p\in \Pi^G,\ f(0) = 1.$$

According to [44], it is in fact the only G-invariant and analytic solution. We mention that there exists a group theoretic context in which, for a certain parameters k, generalized Bessel functions occur in a natural way: namely as the spherical functions of a Euclidean type symmetric space, associated with a so-called Cartan motion group. We refer to [44] for this connection and to [28] for the necessary background in semisimple Lie theory.

The Dunkl kernel is of particular interest as it gives rise to an associated integral transform on $\mathbb{R}^N$ which generalizes the Euclidean Fourier transform in a natural way. This transform will be discussed in the following section. Its definition and essential properties rely on suitable growth estimates for E_k. In our case $k \geq 0$, the best ones to be expected are available:

Proposition 2.6. [51] *For all $x \in \mathbb{R}^N$, $y \in \mathbb{C}^N$ and all multi-indices $\alpha \in \mathbb{Z}_+^N$,*

$$|\partial_y^\alpha E_k(x,y)| \,\leq\, |x|^{|\alpha|} \max_{g\in G} e^{Re\langle gx,y\rangle}.$$

In particular, $\;|E_k(-ix,y)| \,\leq\, 1\;$ *for all $x,y \in \mathbb{R}^N$.*

This result will be obtained later from a positive integral representation of Bochner-type for E_k, c.f. Corollary 4.1. M. de Jeu had slightly weaker bounds in [30], differing by an additional factor $\sqrt{|G|}$.

We conclude this section with two important reproducing properties for the Dunkl kernel. Notice that the above estimate on E_k assures the convergence of the involved integrals.

Proposition 2.7. *Let $k \geq 0$. Then for $p \in \Pi$, $y, z \in \mathbb{C}^N$,*

(1) $$\int_{\mathbb{R}^N} e^{-\Delta_k/2} p\,(x)\, E_k(x,y)\, e^{-|x|^2/2} w_k(x)dx \,=\, c_k\, e^{\langle y,y\rangle/2} p(y),$$

(2) $$\int_{\mathbb{R}^N} E_k(x,y)\, E_k(x,z)\, e^{-|x|^2/2} w_k(x)dx \,=\, c_k\, e^{(\langle y,y\rangle + \langle z,z\rangle)/2} E_k(y,z).$$

Proof. (c.f. [15].) We shall use the Macdonald-type formula (2.5) for the pairing $[\,.\,,\,.\,]_k$. First, we prove that

$$[E_k^{(n)}(x,\,.\,),\,.\,]_k \,=\, p(x) \quad \text{for all } p \in \mathcal{P}_n,\; x \in \mathbb{R}^N. \tag{2.11}$$

In fact, if $p \in \mathcal{P}_n$, then

$$p(x) = (\langle x, \partial_y\rangle^n / n!)p(y) \quad \text{and} \quad V_k^x p(x) = E_k^{(n)}(x, \partial_y)p(y).$$

Here the uppercase index in V_k^x denotes the relevant variable. Application of V_k^y to both sides yields $V_k^x p(x) = E_k^{(n)}(x, T^y)V_k^y p(y)$. As V_k is bijective on $\mathcal{P}_n$, this implies (2.11). For fixed y, let $L_n(x) := \sum_{j=0}^n E_k^{(j)}(x,y)$. If n is larger than the degree of p, it follows from (2.11) that $[L_n, p]_k \,=\, p(y)$. Thus in view of the Macdonald formula,

$$c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}L_n(x)e^{-\Delta_k/2}p(x)e^{-|x|^2/2}w_k(x)dx \,=\, p(y).$$

On the other hand, it is easily checked that

$$\lim_{n\to\infty} e^{-\Delta_k/2}L_n(x) \,=\, e^{-\langle y,y\rangle/2}E_k(x,y).$$

This gives (1). Identity (2) then follows from (1), again by homogeneous expansion of E_k.

2.6 The Dunkl transform

The Dunkl transform was introduced in [15] for non-negative multiplicity functions and further studied in [30] in the more general case $\mathrm{Re}\, k \geq 0$. In these notes, we again restrict ourselves to $k \geq 0$.

Definition 2.8. *The Dunkl transform associated with G and $k \geq 0$ is given by*

$$\widehat{.}^{\,k} \,:\, L^1(\mathbb{R}^N, w_k) \,\to\, C_b(\mathbb{R}^N);$$
$$\widehat{f}^{\,k}(\xi) := c_k^{-1}\int_{\mathbb{R}^N} f(x)E_k(-i\xi, x)\, w_k(x)dx \qquad (\xi \in \mathbb{R}^N).$$

The inverse transform is defined by $f^{\vee k}(\xi) = \widehat{f}^{\,k}(-\xi)$.

Notice that $\widehat{f}^{\,k} \in C_b(\mathbb{R}^N)$ results from our bounds on E_k. The Dunkl transform shares many properties with the classical Fourier transform. Here are the most basic ones:

Lemma 2.6. *Let $f \in \mathscr{S}(\mathbb{R}^N)$. Then for $j = 1, \ldots, N$,*

(1) $\widehat{f}^{\,k} \in C^\infty(\mathbb{R}^N)$ *and* $T_j(\widehat{f}^{\,k}) \,=\, -(ix_jf)^{\wedge k}$.
(2) $(T_jf)^{\wedge k}(\xi) = i\xi_j\widehat{f}^{\,k}(\xi)$.
(3) *The Dunkl transform leaves $\mathscr{S}(\mathbb{R}^N)$ invariant.*

Proof. (1) is obvious from (2.7), and (2) follows from the anti-symmetry relation (Proposition 2.1) for the Dunkl operators. For (3), notice that it suffices to prove that $\partial_\xi^\alpha(\xi^\beta\widehat{f}^{\,k}(\xi))$ is bounded for arbitrary multi-indices α, β. By the previous Lemma, we have $\xi^\beta\widehat{f}^{\,k}(\xi) = \widehat{g}^{\,k}(\xi)$ for some $g \in \mathscr{S}(\mathbb{R}^N)$. Using the growth bounds of Proposition 2.6 yields the assertion.

Exercise 8.

(1) $C_c^\infty(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$ are dense in $L^p(\mathbb{R}^N, w_k)$, $p = 1, 2$.
(2) Conclude the *Lemma of Riemann-Lebesgue* for the Dunkl transform:

$$f \in L^1(\mathbb{R}^N, w_k) \implies \widehat{f}^{\,k} \in C_0(\mathbb{R}^N).$$

Here $C_0(\mathbb{R}^N)$ denotes the space of continuous functions on $\mathbb{R}^N$ which vanish at infinity.

The following are the main results for the Dunkl transform; we omit the proofs but refer the reader to [15] and [30]:

Theorem 2.6. (1) *The Dunkl transform* $f \mapsto \widehat{f}^k$ *is a homeomorphism of* $\mathscr{S}(\mathbb{R}^N)$ *with period* 4.
(2) *(Plancherel theorem) The Dunkl transform has a unique extension to an isometric isomorphism of* $L^2(\mathbb{R}^N, w_k)$. *We denote this isomorphism again by* $f \mapsto \widehat{f}^k$.
(3) *(L^1-inversion) For all* $f \in L^1(\mathbb{R}^N, w_k)$ *with* $\widehat{f}^k \in L^1(\mathbb{R}^N, w_k)$,

$$f = (\widehat{f}^k)^{\vee k} \quad a.e.$$

3 CMS models and generalized Hermite polynomials

3.1 Quantum Calogero-Moser-Sutherland models

Quantum Calogero-Moser-Sutherland (CMS) models describe quantum mechanical systems of N identical particles on a circle or line which interact pairwise through long range potentials of inverse square type. They are exactly solvable and have gained considerable interest in theoretical physics during the last years. Among the broad literature in this area, we refer to [10], [36], [33], [5], [2]-[4], [46], [47], [61], [17]. CMS models have in particular attracted some attention in conformal field theory, and they are being used to test the ideas of fractional statistics ([24], [25]). While explicit spectral resolutions of such models were already obtained by Calogero and Sutherland ([6], [57]), a new aspect in the understanding of their algebraic structure and quantum integrability was much later initiated by [48] and [26]. The Hamiltonian under consideration is hereby modified by certain exchange operators, which allow to write it in a decoupled form. These exchange modifications can be expressed in terms of Dunkl operators of type A_{N-1}. The Hamiltonian of the *linear CMS model with harmonic confinement* in $L^2(\mathbb{R}^N)$ is given by

$$\mathcal{H}_C = -\Delta + g \sum_{1 \le i < j \le N} \frac{1}{(x_i - x_j)^2} + \omega^2 |x|^2; \tag{3.1}$$

here $\omega > 0$ is a frequency parameter and $g \ge -1/2$ is a coupling constant. In case $\omega = 0$, (3.1) describes the free Calogero model. On the other hand, if $g = 0$, then $\mathcal{H}_C$ coincides with the Hamiltonian of the N-dimensional isotropic harmonic oscillator,

$$\mathcal{H}_0 = -\Delta + \omega^2 |x|^2.$$

The spectral decomposition of this operator in $L^2(\mathbb{R}^N)$ is well-known: The spectrum is discrete, $\sigma(\mathcal{H}_0) = \{(2n+N)\omega,\, n \in \mathbb{Z}_+\}$, and the classical multivariable Hermite functions (tensor products of one-variable Hermite functions, c.f. Examples 3.1), form a complete set of eigenfunctions. The study of the Hamiltonian $\mathcal{H}_C$ was initiated by Calogero ([6]); he computed its spectrum and determined the structure of the bosonic eigenfunctions and scattering states in the confined and free case, respectively. Perelomov [47] observed that (3.1) is completely quantum integrable, i.e. there exist N commuting, algebraically independent symmetric linear operators in $L^2(\mathbb{R}^N)$ including $\mathcal{H}_C$. We mention that the complete integrability of the classical Hamiltonian systems associated with (3.1) goes back to Moser [41]. There exist generalizations of the classical Calogero-Moser-Sutherland models in the context of abstract root systems, see for instance [42], [43]. In particular, if R is an arbitrary root system on $\mathbb{R}^N$ and k is a nonnegative multiplicity function on it, then the corresponding abstract Calogero Hamiltonian with harmonic confinement is given by

$$\widetilde{\mathcal{H}}_k = -\widetilde{\mathcal{F}}_k + \omega^2|x|^2$$

with the formal expression

$$\widetilde{\mathcal{F}}_k = \Delta - 2\sum_{\alpha\in R_+} k(\alpha)(k(\alpha)-1)\frac{1}{\langle\alpha,x\rangle^2}.$$

If R is of type A_{N-1}, then $\widetilde{\mathcal{H}}_k$ just coincides with $\mathcal{H}_C$. For both the classical and the quantum case, partial results on the integrability of this model are due to Olshanetsky and Perelomov [42], [43]. A new aspect in the understanding of the algebraic structure and the quantum integrability of CMS systems was initiated by Polychronakos [48] and Heckman [26]. The underlying idea is to construct quantum integrals for CMS models from differential-reflection operators. Polychronakos introduced them in terms of an “exchange-operator formalism” for (3.1). He thus obtained a complete set of commuting observables for (3.1) in an elegant way. In [26] it was observed in general that the complete algebra of quantum integrals for free, abstract Calogero models is intimately connected with the corresponding algebra of Dunkl operators. Let us briefly describe this connection: Consider the following modification of $\widetilde{\mathcal{F}}_k$, involving reflection terms:

$$\mathcal{F}_k = \Delta - 2\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle\alpha,x\rangle^2}\,(k(\alpha)-\sigma_\alpha)\,. \tag{3.2}$$

In order to avoid singularities in the reflecting hyperplanes, it is suitable to carry out a gauge transform by $w_k^{1/2}$. A short calculation, using again results from [12], gives

$$w_k^{-1/2}\mathcal{F}_k w_k^{1/2} = \Delta_k\,,$$

c.f. [52]. Here Δ_k is the Dunkl Laplacian associated with G and k. Now consider the algebra of Π^G of G-invariant polynomials on $\mathbb{R}^N$. By a classical theorem of Chevalley (see e.g. [29]), it is generated by N homogeneous, algebraically independent elements. For $p\in\Pi^G$ we denote by $\operatorname{Res}(p(T))$ the restriction of the Dunkl operator $p(T)$ to Π^G (Recall that $p(T)$ leaves Π^G invariant!). Then

$$\mathcal{A} := \{\operatorname{Res} p(T) : \, p \in \Pi^G\}$$

is a commutative algebra of differential operators on Π^G containing the operator

$$\operatorname{Res}(\Delta_k) \, = \, w_k^{-1/2} \widetilde{\mathcal{F}}_k \, w_k^{1/2},$$

and $\mathcal{A}$ has N algebraically independent generators, called quantum integrals for the free Hamiltonian $\widetilde{\mathcal{F}}_k$.

3.2 Spectral analysis of abstract CMS Hamiltonians

This section is devoted to a spectral analysis of abstract linear CMS operators with harmonic confinement. We follow the expositions in [50], [53]. To simplify formulas, we fix $\omega = 1/2$; corresponding results for general ω can always be obtained by rescaling. We again work with the gauge-transformed version with reflection terms,

$$\mathcal{H}_k := \, w_k^{-1/2}(-\mathcal{F}_k + \frac{1}{4}|x|^2) \, w_k^{1/2} \, = \, -\Delta_k + \frac{1}{4}|x|^2.$$

Due to the anti-symmetry of the first order Dunkl operators (Proposition 2.1), this operator is symmetric and densely defined in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathcal{D}(\mathcal{H}_k) := \mathscr{S}(\mathbb{R}^N)$. Notice that in case $k = 0$, $\mathcal{H}_k$ is just the Hamiltonian of the N-dimensional isotropic harmonic oscillator. We further consider the Hilbert space $L^2(\mathbb{R}^N, m_k)$, where m_k is the probability measure

$$dm_k \, := \, c_k^{-1} e^{-|x|^2/2} w_k(x) dx \tag{3.3}$$

and the operator

$$\mathcal{J}_k \, := \, -\Delta_k + \sum_{i=1}^N x_i \partial_i$$

in $L^2(\mathbb{R}^N, m_k)$, with domain $\mathcal{D}(\mathcal{J}_k) := \Pi$. It can be shown by standard methods that Π is dense in $L^2(\mathbb{R}^N, m_k)$. We do not carry this out; a proof can be found in [51] or in [32], where a comprehensive treatment of density questions in several variables is given.

The next theorem contains a complete description of the spectral properties of $\mathcal{H}_k$ and $\mathcal{J}_k$ and generalizes the already mentioned well-known facts for the classical harmonic oscillator Hamilonian. For the proof, we shall employ the $sl(2)$-commutation relations of the operators

$$E := \frac{1}{2}|x|^2, \;\; F := -\frac{1}{2}\Delta_k \quad \text{and} \;\; H := \sum_{i=1}^N x_i \partial_i + (\gamma + N/2)$$

on Π (with the index $\gamma = \gamma(k)$ as defined in (2.4)) which can be found in [26]. They are

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H. \tag{3.4}$$

Notice that the first two relations are immediate consequences of the fact that the Euler operator

$$\rho := \sum_{i=1}^{N} x_i \partial_i \tag{3.5}$$

satisfies $\rho(p) = np$ for each homogeneous $p \in \mathcal{P}_n$. We start with the following

Lemma 3.1. *On* $\mathcal{D}(\mathcal{J}_k) = \Pi$,

$$\mathcal{J}_k = e^{|x|^2/4}(\mathcal{H}_k - (\gamma + N/2))\, e^{-|x|^2/4}.$$

In particular, $\mathcal{J}_k$ *is symmetric in* $L^2(\mathbb{R}^N, m_k)$.

Proof. From (3.4) it is easily verified by induction that

$$[\Delta_k, E^n] = 2nE^{n-1}H + 2n(n-1)E^{n-1} \quad \text{for all} \quad n \in \mathbb{N},$$

and therefore $[\Delta_k, e^{-E/2}] = -e^{-E/2}H + \frac{1}{2}Ee^{-E/2}$. Thus on Π,

$$\begin{aligned}\mathcal{H}_k\, e^{-E/2} &= -\Delta_k e^{-E/2} + \frac{1}{2}Ee^{-E/2} = -e^{-E/2}\Delta_k + e^{-E/2}H \\ &= e^{-E/2}(\mathcal{J}_k + \gamma + N/2).\end{aligned}$$

Theorem 3.1. *The spaces* $L^2(\mathbb{R}^N, m_k)$ *and* $L^2(\mathbb{R}^N, w_k)$ *admit orthogonal Hilbert space decompositions into eigenspaces of the operators* $\mathcal{J}_k$ *and* $\mathcal{H}_k$ *respectively. More precisely, define*

$$V_n := \{e^{-\Delta_k/2}\, p : p \in \mathcal{P}_n\} \subset \Pi, \quad W_n := \{e^{-|x|^2/4}\, q(x),\ q \in V_n\} \subset \mathscr{S}(\mathbb{R}^N).$$

Then V_n *is the eigenspace of* $\mathcal{J}_k$ *corresponding to the eigenvalue* n, W_n *is the eigenspace of* $\mathcal{H}_k$ *corresponding to the eigenvalue* $n + \gamma + N/2$, *and*

$$L^2(\mathbb{R}^N, m_k) = \bigoplus_{n \in \mathbb{Z}_+} V_n, \quad L^2(\mathbb{R}^N, w_k) = \bigoplus_{n \in \mathbb{Z}_+} W_n.$$

Remark 3.1. A densely defined linear operator $(A, \mathcal{D}(A))$ in a Hilbert space H is called essentially self-adjoint, if it satisfies

(i) A is symmetric, i.e. $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in \mathcal{D}(A)$;
(ii) The closure $\overline{A}$ of A is selfadjoint.

In fact, every symmetric operator A in H has a unique closure $\overline{A}$ (because $A \subseteq A^*$, and the adjoint A^* is closed). If H has a countable orthonormal basis $\{v_n,\, n \in \mathbb{Z}_+\} \subset \mathcal{D}(A)$ consisting of eigenvectors of A corresponding to eigenvalues $\lambda_n \in \mathbb{R}$, then it is straightforward that A is essentially self-adjoint, and that the spectrum of the self-adjoint operator $\overline{A}$ is given by $\sigma(\overline{A}) = \{\lambda_n, n \in \mathbb{Z}_+\}$. (See for instance Lemma 1.2.2 of [8]).

In our situation, the operator $\mathcal{H}_k$ is densely defined and symmetric in $L^2(\mathbb{R}^N, w_k)$ (the first order Dunkl operators being anti-symmetric), and the same holds for $\mathcal{J}_k$ in $L^2(\mathbb{R}^N, m_k)$. The above theorem implies that $\mathcal{H}_k$ and $\mathcal{J}_k$ are essentially self-adjoint and that

$$\sigma(\overline{\mathcal{H}_k}) = \{n + \gamma + N/2, n \in \mathbb{Z}_+\}, \quad \sigma(\overline{\mathcal{J}_k}) = \mathbb{Z}_+ .$$

Proof (of Theorem 3.1). Equation (3.4) and induction yield the commuting relations $[\rho, \Delta_k^n] = -2n\Delta_k^n$ for all $n \in \mathbb{Z}_+$, and hence

$$[\rho, e^{-\Delta_k/2}] \,=\, \Delta_k\, e^{-\Delta_k/2}.$$

If $q \in \Pi$ is arbitrary and $p := e^{\Delta_k/2}q$, it follows that

$$\rho(q) \,=\, (\rho e^{-\Delta_k/2})(p) \,=\, e^{-\Delta_k/2}\rho(p) + \Delta_k e^{-\Delta_k/2}\,p \,=\, e^{-\Delta_k/2}\rho(p) + \Delta_k q.$$

Hence for $a \in \mathbb{C}$ there are equivalent:

$$(-\Delta_k + \rho)(q) = aq \iff \rho(p) = ap \iff a = n \in \mathbb{Z}_+ \text{ and } p \in \mathcal{P}_n.$$

Thus each function from V_n is an eigenfunction of $\mathcal{J}_k$ corresponding to the eigenvalue n, and $V_n \perp V_m$ for $n \neq m$ by the symmetry of $\mathcal{J}_k$. This proves the statements for $\mathcal{J}_k$ because $\Pi = \bigoplus V_n$ is dense in $L^2(\mathbb{R}^N, m_k)$. The statements for $\mathcal{H}_k$ are then immediate by the Lemma 3.1.

3.3 Generalized Hermite polynomials

The eigenvalues of the CMS Hamiltonians $\mathcal{H}_k$ and $\mathcal{J}_k$ are highly degenerate if $N > 1$. In this section, we construct natural orthogonal bases for them. They are made up by generalizations of the classical N-variable Hermite polynomials and Hermite functions to the Dunkl setting. We follow [50], but change our normalization by a factor 2.

The starting point for our construction is the Macdonald-type identity: if $p, q \in \Pi$, then

$$[p,q]_k \,=\, \int_{\mathbb{R}^N} e^{-\Delta_k/2}p(x)e^{-\Delta_k/2}q(x)dm_k(x), \tag{3.6}$$

with the probability measure m_k defined according to (3.3). Notice that $[\,.\,,\,.\,]_k$ is a scalar product on the $\mathbb{R}$- vector space $\Pi_\mathbb{R}$ of polynomials with real coefficients. Let $\{\varphi_\nu\,, \nu \in \mathbb{Z}_+^N\}$ be an orthonormal basis of $\Pi_\mathbb{R}$ with respect to the scalar product $[\,.\,,.]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$. As homogeneous polynomials of different degrees are orthogonal, the φ_ν with fixed $|\nu| = n$ can for example be constructed by Gram-Schmidt orthogonalization within $\mathcal{P}_n \cap \Pi_\mathbb{R}$ from an arbitrary ordered real-coefficient basis. If $k = 0$, the canonical choice of the basis $\{\varphi_\nu\}$ is just $\varphi_\nu(x) := (\nu!)^{-1/2}x^\nu$.

Definition 3.1. *The generalized Hermite polynomials* $\{H_\nu\,,\,\nu\in\mathbb{Z}_+^N\}$ *associated with the basis* $\{\varphi_\nu\}$ *on* $\mathbb{R}^N$ *are given by*

$$H_\nu(x) := e^{-\Delta_k/2}\varphi_\nu(x). \tag{3.7}$$

Moreover, we define the generalized Hermite functions on $\mathbb{R}^N$ *by*

$$h_\nu(x) := e^{-|x|^2/4}H_\nu(x), \quad \nu\in\mathbb{Z}_+^N. \tag{3.8}$$

H_ν is a polynomial of degree $|\nu|$ satisfying $H_\nu(-x) = (-1)^{|\nu|}H_\nu(x)$ for all $x\in\mathbb{R}^N$. By virtue of (3.6), the H_ν, $\nu\in\mathbb{Z}_+^N$ form an orthonormal basis of $L^2(\mathbb{R}^N, m_k)$.

Examples 3.1. (1) *Classical multivariable Hermite polynomials.* Let $k=0$, and choose the standard orthonormal system $\varphi_\nu(x) = (\nu!)^{-1/2}x^\nu$, with respect to $[\,.\,,.\,]_0$. The associated Hermite polynomials are given by

$$H_\nu(x) \;=\; \frac{1}{\sqrt{\nu!}}\prod_{i=1}^N e^{-\partial_i^2/2}(x_i^{\nu_i}) \;=\; \frac{2^{-|\nu|/2}}{\sqrt{\nu!}}\prod_{i=1}^N \widehat{H}_{\nu_i}(x_i/\sqrt{2}), \tag{3.9}$$

where the $\widehat{H}_n$, $n\in\mathbb{Z}_+$ are the classical Hermite polynomials on $\mathbb{R}$ defined by

$$\widehat{H}_n(x) \;=\; (-1)^n\, e^{x^2}\,\frac{d^n}{dx^n}e^{-x^2}\,.$$

(2) *The one-dimensional case.* Up to sign changes, there exists only one orthonormal basis with respect to $[\,.\,,\,.\,]_k$. The associated Hermite polynomials are given, up to multiplicative constants, by the generalized Hermite polynomials $H_n^k(x/\sqrt{2})$ on $\mathbb{R}$. These polynomials can be found in [7] and were further studied in [56] in connection with a Bose-like oscillator calculus. The H_n^k are orthogonal with respect to $|x|^{2k}e^{-|x|^2}$ and can be written as

$$\begin{cases} H_{2n}^k(x) = (-1)^n 2^{2n} n!\, L_n^{k-1/2}(x^2),\\ H_{2n+1}^k(x) = (-1)^n 2^{2n+1} n!\, x L_n^{k+1/2}(x^2); \end{cases}$$

here the L_n^α are the usual Laguerre polynomials of index $\alpha\geq -1/2$, given by

$$L_n^\alpha(x) = \frac{1}{n!}x^{-\alpha}e^x\,\frac{d^n}{dx^n}\big(x^{n+\alpha}e^{-x}\big).$$

(3) *The A_{N-1}-case.* There exists a natural orthogonal system $\{\varphi_\nu\}$, made up by the so-called *non-symmetric Jack polynomials.* For a multiplicity parameter $k>0$, the associated non-symmetric Jack polynomials E_ν, $\nu\in\mathbb{Z}_+^N$, as introduced in [45] (see also [35]), are uniquely defined by the following conditions:

(i) $E_\nu(x) \;=\; x^\nu \;+\; \sum_{\mu<_P\,\nu} c_{\nu,\,\mu}x^\mu$ with $c_{\nu,\mu}\in\mathbb{R}$;

(ii) For all $\mu <_P \nu$, $(E_\nu(x), x^\mu)_k = 0$.
Here $<_P$ is a dominance order defined within multi-indices of equal total length (see [45]), and the inner product $(.,.)_k$ on $\Pi \cap \Pi_{\mathbb{R}}$ is given by

$$(f,g)_k := \int_{\mathbb{T}^N} f(z)g(\overline{z}) \prod_{i<j} |z_i - z_j|^{2k} dz$$

with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and dz being the Haar measure on $\mathbb{T}^N$. If f and g have different total degrees, then $(f,g)_k = 0$. The set $\{E_\nu, |\nu| = n\}$ forms a vector space basis of $\mathcal{P}_n \cap \Pi_{\mathbb{R}}$. It can be shown (by use of A_{N-1}-type Cherednik operators) that the Jack polynomials E_ν are also orthogonal with respect to the Dunkl pairing $[\,.\,,\,.\,]_k$; for details see [50]. The corresponding generalized Hermite polynomials and their symmetric counterparts have been studied in [37], [38] and in [2] - [4].

As an immediate consequence of Theorem 3.1 we obtain analogues of the classical second order differential equations for generalized Hermite polynomials and Hermite functions:

Corollary 3.1. (i) $(-\Delta_k + \sum_{i=1}^N x_i\partial_i)H_\nu = |\nu| H_\nu$.
(ii) $(-\Delta_k + \frac{1}{4}|x|^2)h_\nu = (|\nu| + \gamma + N/2)h_\nu$.

Various further useful properties of the classical Hermite polynomials and Hermite functions have extensions to our general setting. We conclude this section with a list of them. The proofs can be found in [50]. For further results on generalized Hermite polynomials, one can also see for instance [9].

Theorem 3.2. *Let $\{H_\nu\}$ be the Hermite polynomials associated with the basis $\{\varphi_\nu\}$ on $\mathbb{R}^N$ and let $x, y \in \mathbb{R}^N$. Then*

(1) *(Rodrigues formula)* $H_\nu(x) = (-1)^{|\nu|} e^{|x|^2/2} \varphi_\nu(T) e^{-|x|^2/2}$
(2) *(Generating relation)* $e^{-|y|^2/2} E_k(x,y) = \sum_{\nu \in \mathbb{Z}_+^N} H_\nu(x)\varphi_\nu(y)$
(3) *(Mehler formula) For all* $0 < r < 1$,

$$\sum_{\nu \in \mathbb{Z}_+^N} H_\nu(x)H_\nu(y) r^{|\nu|} = \frac{1}{(1-r^2)^{\gamma+N/2}} \exp\left\{-\frac{r^2(|x|^2 + |y|^2)}{2(1-r^2)}\right\} E_k\Big(\frac{rx}{1-r^2}, y\Big).$$

The sums are absolutely convergent in both cases.

The Dunkl kernel E_k in (2) and (3) replaces the usual exponential function. It comes in via the following relation with the (arbitrary!) basis $\{\varphi_\nu\}$:

$$E_k(x,y) = \sum_{\nu\in\mathbb{Z}_+^N} \varphi_\nu(x)\varphi_\nu(y) \quad (x,y\in\mathbb{R}^N).$$

Proposition 3.1. *The generalized Hermite functions* $\{h_\nu\,,\,\nu\in\mathbb{Z}_+^N\}$ *are a basis of eigenfunctions of the Dunkl transform on* $L^2(\mathbb{R}^N, w_k)$ *with*

$$h_\nu^{\wedge k} = (-i)^{|\nu|}h_\nu.$$

4 Positivity results

4.1 Positivity of Dunkl's intertwining operator

In this section it is always assumed that $k\geq 0$. The reference is [51].

We shall say that a linear operator A on Π is *positive*, if A leaves the positive cone

$$\Pi_+ := \{p\in\Pi : p(x)\geq 0 \quad \text{for all } x\in\mathbb{R}^N\}$$

invariant. The following theorem is the central result of this section:

Theorem 4.1. V_k *is positive on* Π.

Once this is known, more detailed information about V_k can be obtained by its extension to the algebras A_r, which were introduced in Definition 2.5. This leads to

Theorem 4.2. *For each* $x\in\mathbb{R}^N$ *there exists a unique probability measure* μ_x^k *on the Borel-*σ*-algebra of* $\mathbb{R}^N$ *such that*

$$V_kf(x) = \int_{\mathbb{R}^N} f(\xi)\,d\mu_x^k(\xi) \quad \text{for all } f\in A_{|x|}. \tag{4.1}$$

The representing measures μ_x^k *are compactly supported with* $\operatorname{supp}\mu_x^k\subseteq \operatorname{co}\{gx,\; g\in G\}$, *the convex hull of the orbit of* x *under* G. *Moreover, they satisfy*

$$\mu_{rx}^k(B) = \mu_x^k(r^{-1}B), \quad \mu_{gx}^k(B) = \mu_x^k(g^{-1}(B)) \tag{4.2}$$

for each $r>0$, $g\in G$ *and each Borel set* $B\subseteq\mathbb{R}^N$.

The proof of Theorem 4.1 affords several steps, the crucial one being a reduction from the N-dimensional to a one-dimensional problem. We shall give an outline, but beforehand we turn to the proof of Theorem 4.2.

Proof (of Theorem 4.2). Fix $x\in\mathbb{R}^N$ and put $r=|x|$. Then the mapping

$$\Phi_x : f\mapsto V_kf(x)$$

is a bounded linear functional on A_r, and Theorem 4.1 implies that it is positive on the dense subalgebra Π of A_r, i.e. $\Phi_x(|p|^2) \geq 0$. Consequently, Φ_x is a positive functional on the full Banach-$*$-algebra A_r. There exists a representation theorem of Bochner for positive functionals on commutative Banach-$*$-algebras (see for instance Theorem 21.2 of [21]). It implies in our case that there exists a unique measure $\nu_x \in M_b^+(\Delta_S(A_r))$ such that

$$\Phi_x(f) \,=\, \int_{\Delta_S(A_r)} \widehat{f}(\varphi)\, d\nu_x(\varphi) \qquad \text{for all } f \in A_r,$$

with $\widehat{f}$ the Gelfand transform of f. Keeping Exercise 7 in mind, one obtains representing measures μ_x^k supported in the ball B_r; the sharper statement on the support is obtained by results of [30]. The remaining statements are easy.

The key for the proof of Theorem 4.1 is a characterization of positive semigroups on polynomials which are generated by degree-lowering operators. We call a linear operator A on Π *degree-lowering*, if $deg(Ap) < deg(p)$ for all Π. Again, the exponential $e^A \in \mathrm{End}(\Pi^N)$ is defined by a terminating power-series, and it can be considered as a linear operator on each of the finite dimensional spaces $\{p \in \Pi : deg(p) \leq m\}$. Important examples of degree-lowering operators are linear operators which are homogeneous of some degree $-n < 0$, such as Dunkl operators. The following key result characterizes positive semigroups generated by degree-lowering operators; it is an adaption of a well-known Hille-Yosida type characterization theorem for so called Feller-Markov semigroups which will be discussed a little later in our course, see Theorem 4.7.

Theorem 4.3. *Let A be a degree-lowering linear operator on Π. Then the following statements are equivalent:*
(1) *e^{tA} is positive on Π for all $t \geq 0$.*
(2) *A satisfies the "positive minimum principle"*

(M) *For every $p \in \Pi_+$ and $x_0 \in \mathbb{R}^N$, $p(x_0) = 0$ implies $Ap\,(x_0) \geq 0$.*

Exercise 9.

(1) Prove implication (1) $\Rightarrow$ (2) of this theorem.
(2) Verify that the (usual) Laplacian Δ satisfies the positive minimum principle (M). Can you extend this result to the Dunkl Laplacian Δ_k? (C.f. Exercise 12!)

Let us now outline the proof of Theorem 4.1. We consider the generalized Laplacian Δ_k associated with G and k, which is homogeneous of degree -2 on Π. With the notation introduced in (2.2), it can be written as

$$\Delta_k \,=\, \Delta + L_k \quad \text{ with } L_k \,=\, 2\sum_{\alpha\in R_+} k(\alpha)\delta_\alpha\,. \tag{4.3}$$

Here δ_α acts in direction α only.

Theorem 4.4. *The operator* $e^{-\Delta/2}e^{\Delta_k/2}$ *is positive on* Π *.*

Proof. We shall deduce this statement from a positivity result for a suitable semigroup. For this, we employ Trotter's product formula, which works for degree-lowering operators just as on finite-dimensional vector spaces: If A, B are degree-lowering linear operators on Π, then

$$e^{A+B}p\,(x) \;=\; \lim_{n\to\infty}(e^{A/n}e^{B/n})^n p\,(x).$$

Thus, we can write

$$\begin{aligned} e^{-\Delta/2}e^{\Delta_k/2}\,p\,(x) \;&=\; e^{-\Delta/2}e^{\Delta/2+L_k/2}\,p\,(x) \;=\; \lim_{n\to\infty} e^{-\Delta/2}\big(e^{\Delta/2n}\,e^{L_k/2n}\big)^n p\,(x) \\ &=\; \lim_{n\to\infty}\prod_{j=1}^{n}\big(e^{-(1-j/n)\cdot\Delta/2}\,e^{L_k/2n}\,e^{(1-j/n)\cdot\Delta/2}\big)p\,(x). \end{aligned}$$

It therefore suffices to verify that the operators

$$e^{-s\Delta}e^{tL_k}e^{s\Delta} \qquad (s,t\geq 0)$$

are positive on Π. Consider s fixed, then

$$e^{-s\Delta}e^{tL_k}e^{s\Delta} \;=\; e^{tA} \quad \text{with } A = e^{-s\Delta}L_k e^{s\Delta}\,.$$

It is easily checked that A is degree-lowering. Hence, in view of Theorem 4.3, it remains to show that A satisfies the positive minimum principle (M). We may write

$$A = \;e^{-s\Delta}L_k e^{s\Delta} \;=\; 2\sum_{\alpha\in R_+} k(\alpha)e^{-s\partial_\alpha^2}\delta_\alpha\, e^{s\partial_\alpha^2};$$

here it was used that δ_α acts in direction α only. It can now be checked by direct computation that the one-dimensional operators $e^{-s\partial_\alpha^2}\delta_\alpha e^{s\partial_\alpha^2}$ satisfy (M), and as the $k(\alpha)$ are non-negative, this must be true for A as well.

Proof (of Theorem 4.1). Notice first that

$$[V_k\,p,q]_k \;=\; [p,q]_0 \quad \text{ for all } p,q\in\Pi. \tag{4.4}$$

In fact, for $p,q\in\mathcal{P}_n$ with $n\in\mathbb{Z}_+$, one obtains

$$[V_k\,p,q]_k \;=\; [q,V_k\,p]_k \;=\; q(T)(V_k\,p) \;=\; V_k(q(\partial)p) \;=\; q(\partial)(p) \;=\; [p,q]_0\,;$$

here the characterizing properties of V_k and the fact that $q(\partial)(p)$ is a constant have been used. For general $p,q\in\Pi$, (4.4) then follows from the orthogonality of the spaces $\mathcal{P}_n$ with respect to both pairings.

Combining the Macdonald-type identity (2.5) with part (4.4), we obtain for all $p,q\in\Pi$ the identity

$$c_k^{-1}\int_{\mathbb{R}^N} e^{-\Delta_k/2}(V_kp)e^{-\Delta_k/2}q\, e^{-|x|^2/2}\, w_k(x)dx \;=\; c_0^{-1}\int_{\mathbb{R}^N} e^{-\Delta/2}p\, e^{-\Delta/2}q\, e^{-|x|^2/2}dx.$$

As $e^{-\Delta_k/2}(V_k\, p) \,=\, V_k(e^{-\Delta/2}p)$, and as we may also replace p by $e^{\Delta/2}p$ and q by $e^{\Delta_k/2}q$ in the above identity, it follows that for all $p,q\in \Pi$

$$c_k^{-1}\int_{\mathbb{R}^N} V_kp\, q\, e^{-|x|^2/2}\, w_k(x)dx \;=\; c_0^{-1}\int_{\mathbb{R}^N} p\, e^{-\Delta/2}e^{\Delta_k/2}q\, e^{-|x|^2/2}dx. \tag{4.5}$$

Due to Theorem 4.4, the right side of (4.5) is non-negative for all $p,q\in\Pi_+$. From this, the assertion can be deduced by standard density arguments (Π is dense in $L^2(\mathbb{R}^N, e^{-|x|^2/4}w_k(x)dx)$).

Corollary 4.1. *For each $y\in\mathbb{C}^N$, the function $x\mapsto E_k(x,y)$ has the Bochner-type representation*

$$E_k(x,y) \;=\; \int_{\mathbb{R}^N} e^{\langle\xi,\, y\rangle}d\mu_x^k(\xi), \tag{4.6}$$

where the μ_x^k are the representing measures from Theorem 4.2. In particular, E_k satisfies the estimates stated in Proposition 2.6, and

$$E_k(x,y) > 0 \quad \textit{for all } x,y\in\mathbb{R}^N.$$

Analogous statements hold for the generalized Bessel function J_k.

In those cases where the generalized Bessel functions $J_k(.,y)$ allow an interpretation as the spherical functions of a Cartan motion group, the Bochner representation of these functions is an immediate consequence of Harish-Chandra's theory ([28]). There are, however, no group-theoretical interpretations known for the kernel E_k so far.

We conclude this section with an open conjecture which would yield a nice extension of Corollary 4.1: Let $k'\leq k$ be two multiplicity functions on the same root system R, i.e. $k'(\alpha)\leq k(\alpha)$ for all $\alpha\in R$. Prove that for each $x\in\mathbb{R}^N$, there exists a compactly supported probability measure $\mu_x^{k,k'}$ on $\mathbb{R}^N$ such that

$$E_k(x,y) = \int_{\mathbb{R}^N} E_{k'}(\xi,y)d\mu_x^{k,k'}(\xi) \quad \text{for all } y\in\mathbb{R}^N.$$

4.2 Heat kernels and heat semigroups

We start with a motivation: Consider the following initial-value problem for the classical heat equation in $\mathbb{R}^N$:

$$\begin{cases}\Delta u - \partial_t u \,=0 & \text{on } \mathbb{R}^N\times(0,\infty),\\ u(\,.\,,0) = f\end{cases} \tag{4.7}$$

with initial data $f \in C_0(\mathbb{R}^N)$, the space of continuous functions on $\mathbb{R}^N$ which vanish at infinity. (We could equally take data from $C_b(\mathbb{R}^N)$, but $C_0(\mathbb{R}^N)$ is more convenient in the following considerations). The basic idea to solve (4.7) is to carry out a Fourier transform with respect to x. This yields the candidate

$$u(x,t) = g_t * f(x) = \int_{\mathbb{R}^N} g_t(x-y)f(y)dy \quad (t>0), \tag{4.8}$$

where g_t is the Gaussian kernel

$$g_t(x) = \frac{1}{(4\pi t)^{N/2}}\, e^{-|x|^2/4t}.$$

It is a well-known fact from classical analysis that (4.8) is in fact the unique bounded solution within the class $C^2(\mathbb{R}^N \times (0,\infty)) \cap C(\mathbb{R}^N \times [0,\infty))$.

Exercise 10. Show that $H(t)f(x) := g_t * f(x)$ for $t > 0$, $H(0) := id$ defines a strongly continuous contraction semigroup on the Banach space $(C_0(\mathbb{R}^N), \|.\|_\infty)$ in the sense of the definition given below.
Hint: Once contractivity is shown, it suffices to check the continuity for functions from the Schwartz space $\mathscr{S}(\mathbb{R}^N)$. For this, use the Fourier inversion theorem.

Definition 4.1. *Let X be a Banach space. A one-parameter family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a strongly continuous semigroup on X, if it satisfies*

(i) $T(0) = id_X$, $\quad T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$
(ii) *The mapping $t \mapsto T(t)x$ is continuous on $[0,\infty)$ for all $x \in X$.*

A strongly continuous semigroup is called a contraction semigroup, if $\|T(t)\| \leq 1$ for all $t \geq 0$.

Let $L(X)$ denote the space of bounded linear operators in X. If $A \in L(X)$, then

$$e^{tA} = \sum_{n=0}^\infty \frac{t^n}{n!}A^n \in L(X)$$

defines a strongly continuous semigroup on X (this one is even continuous with respect to the uniform topology on $L(X)$). We obviously have

$$A = \lim_{t\downarrow 0} \frac{1}{t}(e^{tA} - id) \quad \text{in } L(X).$$

Definition 4.2. *The generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in X is defined by*

$$\begin{aligned} Ax :=& \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x), \quad \textit{with domain} \\ \mathcal{D}(A) :=& \{x \in X : \lim_{t\downarrow 0}\frac{1}{t}(T(t)x - x) \textit{ exists in } X\}. \end{aligned} \tag{4.9}$$

Theorem 4.5. *The generator A of $(T(t))_{t\geq 0}$ is densely defined and closed.*

An important issue in the theory of operator semigroups and evolution equations are criteria which characterize generators of strongly continuous semigroups.

Let us return to the Dunkl setting. As before, Δ_k denotes the Dunkl Laplacian associated with a finite reflection group on $\mathbb{R}^N$ and some multiplicity function $k\geq 0$, and the index γ is defined according to (2.4). We are going to consider the following initial-value problem for the Dunkl-type heat operator $\Delta_k - \partial_t$:

Find $u \in C^2(\mathbb{R}^N \times (0,\infty))$ which is continuous on $\mathbb{R}^N \times [0,\infty)$ and satisfies

$$\begin{cases} (\Delta_k - \partial_t)\, u \,=\, 0 & \text{on } \mathbb{R}^N \times (0,\infty), \\ u(\,.\,,0) \,=\, f & \in C_b(\mathbb{R}^N). \end{cases} \tag{4.10}$$

The solution of this problem is given, just as in the classical case $k=0$, in terms of a positivity-preserving semigroup. We shall essentially follow the treatment of [50].

Lemma 4.1. *The function*

$$F_k(x,t) := \frac{1}{(2t)^{\gamma+N/2}c_k}e^{-|x|^2/4t}$$

solves the generalized heat equation $\Delta_k u - \partial_t u = 0$ on $\mathbb{R}^N \times (0,\infty)$.

Proof. A short calculation. Use the product rule (2.1) as well as the identity $\sum_{i=1}^N T_i(x_i) = N+2\gamma$.

F_k generalizes the fundamental solution for the classical heat equation which is given by $F_0(x,t) = g_t(x)$ (as defined above). It is easily checked that

$$\int_{\mathbb{R}^N} F_k(x,t)\, w_k(x)dx \,=\, 1 \quad \text{for all } t>0.$$

In order to solve (4.10), it suggests itself to apply the Dunkl transform under suitable decay assumptions on the initial data. In the classical case, the heat kernel $g_t(x-y)$ on $\mathbb{R}^N$ is obtained from the fundamental solution simply by translations. In the Dunkl setting, it is still possible to define a generalized translation which matches the action of the Dunkl transform, i.e. makes it a homomorphism on suitable function spaces.

The notion of a *generalized translation* in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is as follows (c.f. [50]):

$$\tau_y f(x) := \frac{1}{c_k}\int_{\mathbb{R}^N} \widehat{f}^k(\xi)\, E_k(ix,\xi)E_k(iy,\xi)\, w_k(\xi)d\xi; \quad y\in\mathbb{R}^N. \tag{4.11}$$

In the same way, this could be done in $L^2(\mathbb{R}^N, w_k)$. A powerful extension to $C^\infty(\mathbb{R}^N)$ is due to Trimèche [59]. Note that in case $k=0$, we simply have $\tau_y f(x) = f(x+y)$. In the rank-one case, the above translation coincides with the convolution on a so-called signed hypergroup structure which was defined in [49]; see also [56]. Similar structures are not yet known in higher rank cases. Clearly, $\tau_y f(x) = \tau_x f(y)$; moreover, the inversion theorem for the Dunkl transform assures that $\tau_0 f = f$ and

$$(\tau_y f)^{\wedge k}(\xi) = E_k(iy, \xi)\widehat{f}^{\,k}(\xi).$$

From this it is not hard to see that $\tau_y f$ belongs to $\mathscr{S}(\mathbb{R}^N)$ again. Let us now consider the "fundamental solution" $F_k(.\,, t)$ for $t > 0$. A short calculation, using the reproducing property Proposition 2.7(2), shows that

$$\widehat{F}_k^k(\xi, t) \;=\; c_k^{-1} e^{-t|\xi|^2}. \tag{4.12}$$

By the quoted reproducing formula one therefore obtains from (4.11) the representation

$$\tau_{-y} F_k(x,t) \;=\; \frac{1}{(2t)^{\gamma+N/2} c_k} e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big).$$

This motivates the following

Definition 4.3. *The Dunkl type heat kernel* Γ_k *is defined by*

$$\Gamma_k(t,x,y) := \frac{1}{(2t)^{\gamma+N/2} c_k}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x, y \in \mathbb{R}^N,\ t > 0.$$

Notice in particular that $\Gamma_k > 0$ (thanks to Corollary 4.1) and that $y \mapsto \Gamma_k(t,x,y)$ belongs to $\mathscr{S}(\mathbb{R}^N)$ for fixed x and t. We collect a series of further fundamental properties of this kernel which are all more or less straightforward.

Lemma 4.2. *The Dunkl type heat kernel* Γ_k *has the following properties:*

(1) $\Gamma_k(t,x,y) \;=\; c_k^{-2} \displaystyle\int_{\mathbb{R}^N} e^{-t|\xi|^2}\, E_k(ix,\xi)\, E_k(-iy,\xi)\, w_k(\xi) d\xi\,.$

(2) $\displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, w_k(y) dy \;=\; 1.$

(3) $\Gamma_k(t,x,y) \;\leq\; \dfrac{1}{(2t)^{\gamma+N/2} c_k} \displaystyle\max_{g\in G} e^{-|gx-y|^2/4t}\,.$

(4) $\Gamma_k(t+s,x,y) \;=\; \displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,z)\, \Gamma_k(s,y,z)\, w_k(z) dz.$

(5) *For fixed* $y \in \mathbb{R}^N$, *the function* $u(x,t) := \Gamma_k(t,x,y)$ *solves the generalized heat equation* $\Delta_k u = \partial_t u$ *on* $\mathbb{R}^N \times (0,\infty)$.

Proof. (1) is clear from the definition of generalized translations. For details concerning (2) see [50]. (3) follows from our estimates on E_k, while (4) is obtained by inserting (1) for one of the kernels in the integral. Finally, (5) is obtained from differentiating (1) under the integral. For details see again [50].

Definition 4.4. *For $f \in C_b(\mathbb{R}^N)$ and $t \geq 0$ set*

$$H(t)f(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,y)f(y)\, w_k(y)dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases} \tag{4.14}$$

Notice that the decay of Γ_k assures the convergence of the integral. The properties of the operators $H(t)$ are most easily described on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$. The following theorem is completely analogous to the classical case.

Theorem 4.6. *Let $f \in \mathscr{S}(\mathbb{R}^N)$. Then $u(x,t) := H(t)f(x)$ solves the initial-value problem* (4.10)*. Moreover, $H(t)f$ has the following properties:*

(1) *$H(t)f \in \mathscr{S}(\mathbb{R}^N)$ for all $t > 0$.*
(2) *$H(t+s)\,f \,=\, H(t)H(s)f$ for all $s,t \geq 0$.*
(3) *$\|H(t)f - f\|_{\infty,\mathbb{R}^N} \to 0$ as $t \to 0$.*

Proof. (Sketch) By use of Lemma 4.2 (1) and Fubini's theorem, we write

$$u(x,t) \,=\, H(t)f(x) \,=\, c_k^{-1}\int_{\mathbb{R}^N} e^{-t|\xi|^2}\,\widehat{f}^{\,k}(\xi)\,E_k(ix,\xi)\,w_k(\xi)d\xi \quad (t>0). \tag{4.15}$$

In view of the inversion theorem for the Dunkl transform, this holds for $t = 0$ as well. Properties (1) and (3) as well as the differential equation are now easy consequences. Part (2) follows from the reproducing formula for Γ_k (Lemma 4.2 (4)).

Exercise 11. Carry out the details in the proof of Theorem 4.6.

We know that the Dunkl type heat kernel Γ_k is positive; this implies that $H(t)f \geq 0$ if $f \geq 0$.

Definition 4.5. *Let Ω be a locally compact Hausdorff space. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on $(C_0(\Omega), \|.\|_\infty)$ is called a Feller-Markov semigroup, if it is contractive and positive, i.e. $f \geq 0$ on Ω implies that $T(t)f \geq 0$ on Ω for all $t \geq 0$.*

We shall prove that the linear operators $H(t)$ on $\mathscr{S}(\mathbb{R}^N)$ extend to a Feller-Markov semigroup on the Banach space $(C_0(\mathbb{R}^N), \|.\|_\infty)$. This could be done by direct calculations similar to the usual procedure for the classical heat semigroup, relying on the positivity of the kernel Γ_k. We do however prefer to give a proof which does not require this rather deep result, but works on the level of the tentative generator. The tool is the following useful variant of the Lumer-Phillips theorem, which characterizes Feller-Markov semigroups in terms of a "positive maximum principle", see e.g. [34], Theorem 17.11. In fact, this Theorem motivated the positive minimum principle of Theorem 4.3 in the positivity-proof for V_k.

Theorem 4.7. *Let $(A, \mathcal{D}(A))$ be a densely defined linear operator in $(C_0(\Omega), \|.\|_\infty)$. Then A is closable, and its closure $\overline{A}$ generates a Feller-Markov semigroup on $C_0(\Omega)$, if and only if the following conditions are satisfied:*

(i) *If $f \in \mathcal{D}(A)$ then also $\overline{f} \in \mathcal{D}(A)$ and $A(\overline{f}) = \overline{A(f)}$.*
(ii) *The range of $\lambda id - A$ is dense in $C_0(\Omega)$ for some $\lambda > 0$.*
(iii) *If $f \in \mathcal{D}(A)$ is real-valued with a non-negative maximum in $x_0 \in \Omega$, i.e. $0 \leq f(x_0) = \max_{x \in \Omega} f(x)$, then $Af(x_0) \leq 0$. (Positive maximum principle).*

We consider the Dunkl Laplacian Δ_k as a densely defined linear operator in $C_0(\mathbb{R}^N)$ with domain $\mathscr{S}(\mathbb{R}^N)$. The following Lemma implies that it satisfies the positive maximum principle:

Lemma 4.3. *Let $\Omega \subseteq \mathbb{R}^N$ be open and G-invariant. If a real-valued function $f \in C^2(\Omega)$ attains an absolute maximum at $x_0 \in \Omega$, i.e. $f(x_0) = \sup_{x \in \Omega} f(x)$, then*

$$\Delta_k f(x_0) \leq 0.$$

Exercise 12. Prove this lemma in the case that $\langle \alpha, x_0 \rangle \neq 0$ for all $\alpha \in R$. (If $\langle \alpha, x_0 \rangle = 0$ for some $\alpha \in R$, one has to argue more carefully; for details see [50].)

Theorem 4.8. *The operators $(H(t))_{t \geq 0}$ define a Feller-Markov semigroup on $C_0(\mathbb{R}^N)$. Its generator is the closure $\overline{\Delta}_k$ of $(\Delta_k, \mathscr{S}(\mathbb{R}^N))$. This semigroup is called the heat semigroup, generalized on $C_0(\mathbb{R}^N)$.*

Proof. In the first step, we check that Δ_k (with domain $\mathscr{S}(\mathbb{R}^N)$) satisfies the conditions of Theorem 4.7: Condition (i) is obvious and (iii) is an immediate consequence of the previous lemma. Condition (ii) is also satisfied, because $\lambda id - \Delta_k$ maps $\mathscr{S}(\mathbb{R}^N)$ onto itself for each $\lambda > 0$; this follows from the fact that the Dunkl transform is a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$ and $((\lambda I - \Delta_k) f)^{\wedge k}(\xi) = (\lambda + |\xi|^2)\widehat{f}^k(\xi)$. Theorem 4.7 now implies that Δ_k is closable, and that its closure $\overline{\Delta}_k$ generates a Feller-Markov semigroup $(T(t))_{t \geq 0}$. It remains to show that $T(t) = H(t)$ on $C_0(\mathbb{R}^N)$. Let first $f \in \mathscr{S}(\mathbb{R}^N)$. From basic facts in semigroup theory, it follows that the function $t \mapsto T(t)f$ is the unique solution of the so-called abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt} u(t) = \overline{\Delta}_k u(t) & \text{for } t > 0, \\ u(0) = f \end{cases} \tag{4.16}$$

within the class of all (strongly) continuously differentiable functions u on $[0, \infty)$ with values in $(C_0(\mathbb{R}^N), \|.\|_\infty)$. It is easily seen from Theorem 4.6, and in particular from formula (4.15), that $t \mapsto H(t)f$ satisfies these conditions. Hence $T(t) = H(t)$ on $\mathscr{S}(\mathbb{R}^N)$. This easily implies that $\Gamma_k \geq 0$ (which we did not presuppose for the proof!), and therefore the operators $H(t)$ are also contractive on $C_0(\mathbb{R}^N)$. A density argument now finishes the proof.

Based on this result, it is checked by standard arguments that for data $f \in C_b(\mathbb{R}^N)$, the function $u(x,t) := H(t)f(x)$ solves the initial-value problem (4.10). Uniqueness results are established by means of maximum principles, just as with the classical heat equation. Moreover, the heat semigroup $(H(t))_{t\geq 0}$ can also be defined (by means of (4.14)) on the Banach spaces $L^p(\mathbb{R}^N, w_k), 1 \leq p < \infty$. In case $p = 2$, the following is easily seen by use of the Dunkl transform:

Proposition 4.1. [50] *The operator $(\Delta_k, \mathscr{S}(\mathbb{R}^N))$ in $L^2(\mathbb{R}^N, w_k)$ is densely defined and closable. Its closure generates a strongly continuous and positivity-preserving contraction semigroup on $L^2(\mathbb{R}^N, w_k)$ which is given by*

$$H(t)f(x) = \int_{\mathbb{R}^N} \Gamma_k(t,x,y)f(y)w_k(y)dy\,, \quad (t>0).$$

Theorem 4.8 was the starting point in [55] to construct an associated Feller-Markov process on $\mathbb{R}^N$ which can be considered a generalization of the usual Brownian motion. The transition probabilities of this process are defined in terms of a semigroup of Markov kernels of $\mathbb{R}^N$, as follows: For $x \in \mathbb{R}^N$ and a Borel set $A \in \mathscr{B}(\mathbb{R}^N)$ put

$$P_t(x,A) := \int_A \Gamma_k(t,x,y)w_k(y)dy \quad (t>0), \quad P_0(x,A) := \delta_x(A),$$

with δ_x denoting the point measure in $x \in \mathbb{R}^N$. Then $(P_t)_{t\geq 0}$ is a semigroup of Markov kernels on $\mathbb{R}^N$ in the following sense:

(1) Each P_t is a Markov kernel, and for all $s,t \geq 0$, $x \in \mathbb{R}^N$ and $A \in \mathscr{B}(\mathbb{R}^N)$,

$$P_s \circ P_t(x,A) := \int_{\mathbb{R}^N} P_t(z,A)\,P_s(x,dz) \;=\; P_{s+t}(x,A).$$

(2) The mapping $[0,\infty) \to M^1(\mathbb{R}^N)$, $t \mapsto P_t(0,\,.\,)$, is continuous with respect to the $\sigma(M^1(\mathbb{R}^N, C_b(\mathbb{R}^N))$-topology.

Moreover, the semigroup $(P_t)_{t\geq 0}$ has the following particular property:

(3) $P_t(x,.)^{\wedge k}(\xi) = E_k(-ix,\xi)\,P_t(0,\,.\,)^{\wedge k}(\xi)$ for all $\xi \in \mathbb{R}^N$,

hereby the Dunkl transform of the probability measures $P_t(x,\,.\,)$ is defined by

$$P_t(x,\,.\,)^{\wedge k}(\xi) := \int_{\mathbb{R}^N} E_k(-i\xi,x)\,P_t(x,d\xi).$$

The proof of (1) – (3) is straightforward by the properties of Γ_k and Theorem 4.8.

In the classical case $k=0$, property (3) is equivalent to $(P_t)_{t\geq 0}$ being translation-invariant, i.e.

$$P_t(x+y, A+y) \;=\; P_t(x,A) \quad \text{for all } y \in \mathbb{R}^N.$$

In our general setting, a positivity-preserving translation on $M^1(\mathbb{R}^N)$ cannot be expected (and does definitely not exist in the rank-one case according to [49]). Property (3) thus serves as a substitute for translation-invariance. The reader can see [55] for a study of the semigroup $(P_t)_{t\geq 0}$ and the associated Feller-Markov process.

5 Asymptotic analysis for the Dunkl kernel

This final section deals with the asymptotic behavior of the Dunkl kernel E_k with $k \geq 0$ when one of its arguments is fixed and the other tends to infinity either within a Weyl chamber of the associated reflection group, or within a suitable complex domain. These results are contained in [54]. They generalize the well-known asymptotics of the confluent hypergeometric function ${}_1F_1$ to the higher-rank setting. One motivation to study the asymptotics of E_k is to determine the asymptotic behavior of the Dunkl type heat kernel Γ_k for short times. Partial results in this direction were obtained in [52].

Recall from Proposition 4.1 that Γ_k is the kernel of the generalized heat semigroup in the weighted $L^2(\mathbb{R}^N, w_k)$. We want to compare it with the free Gaussian kernel Γ_0. For this, it is appropriate to transfer the semigroup $(H(t))_{t\geq 0}$ from $L^2(\mathbb{R}^N, w_k)$ to the unweighted space $L^2(\mathbb{R}^N)$, which leads to the strongly continuous contraction semigroup

$$\widetilde{H}(t)f := w_k^{1/2} H(t)(w_k^{-1/2} f), \quad f \in L^2(\mathbb{R}^N).$$

The corresponding renormalized heat kernel is given by

$$\widetilde{\Gamma}_k(t,x,y) := \sqrt{w_k(x)w_k(y)}\,\Gamma_k(t,x,y).$$

The generator of $(\widetilde{H}(t))_{t\geq 0}$ is the gauge-transformed version of the Dunkl Laplacian discussed in connection with CMS-models,

$$\mathcal{F}_k \,=\, \Delta - 2\sum_{\alpha\in R_+} \frac{k(\alpha)}{\langle \alpha, x\rangle^2}(k(\alpha) - \sigma_\alpha)$$

(with suitable domain). $\mathcal{F}_k$ can be considered a perturbation of the Laplacian Δ. This suggests that within the Weyl chambers of G, the heat kernel $\widetilde{\Gamma}_k(t,x,y)$ should not "feel" the reflecting hyperplanes and behave for short times like the free Gaussian kernel $\Gamma_0(t,x,y) \,=\, g_t(x-y)$, in other words, we have the conjecture

$$\lim_{t\downarrow 0} \frac{\sqrt{w_k(x)w_k(y)}\,\Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} \,=\, 1 \tag{5.1}$$

provided x and y belong to the same (open) Weyl chamber. In [52], this could be proven true only for a restricted range of arguments x, y, and by rather technical methods (completely different from those below).

Example 5.1. **The rank-one case.** Here E_k is explicitly known. According to Example 2.1,

$$E_k(z,w) \,=\, e^{zw}\cdot {}_1F_1(k,2k+1,-2zw), \quad z,w\in\mathbb{C}.$$

The confluent hypergeometric function ${}_1F_1$ has well-known asymptotic expansions in the sectors

$$\begin{aligned} S_+ &= \{z\in\mathbb{C}: -\pi/2<\arg(z)<3\pi/2\},\\ S_- &= \{z\in\mathbb{C}: -3\pi/2<\arg(z)<\pi/2\}, \end{aligned}$$

see for instance [1]. They are of the form

$$\begin{aligned} {}_1F_1(k,2k+1,z) \,=&\, \frac{\Gamma(2k+1)}{\Gamma(k)}e^z z^{-k-1}(1+\mathcal{O}(\frac{1}{|z|}))\\ &+\frac{\Gamma(2k+1)}{\Gamma(k+1)}e^{\pm i\pi k}z^{-k}(1+\mathcal{O}(\frac{1}{|z|})), \end{aligned}$$

with $\pm$ for $z\in S_\pm$. Specializing to the right half plane $H=\{z\in\mathbb{C}:\mathrm{Re}z\geq 0\}$ we thus obtain

$$\lim_{zw\to\infty, zw\in H}(zw)^k e^{-zw}E_k(z,w) \,=\, \frac{\Gamma(2k+1)}{2^k\Gamma(k+1)}.$$

Let us now turn to the general case of an arbitrary reflection group and multiplicity parameter $k\geq 0$. Let C denote the Weyl chamber attached with the positive subsystem R_+,

$$C=\{x\in\mathbb{R}^N:\langle\alpha,x\rangle>0 \;\text{ for all }\; \alpha\in R_+\},$$

and for $\delta>0$,

$$C_\delta:=\{x\in C:\langle\alpha,x\rangle\,>\,\delta|x| \;\text{ for all }\; \alpha\in R_+\}.$$

The main result given here is the following asymptotic behavior, uniform for the variable tending to infinity in cones C_δ:

Theorem 5.1. *There exists a constant non-zero vector $v=(v_g)_{g\in G}\in\mathbb{C}^{|G|}$ such that for all $y\in C$, $g\in G$ and each $\delta>0$,*

$$\lim_{|x|\to\infty,\, x\in C_\delta}\sqrt{w_k(x)w_k(y)}\,e^{-i\,\langle x,gy\rangle}E_k(ix,gy) \,=\, v_g.$$

Notice that one variable is being fixed. A locally uniform result with respect to both variables should be true, but is open yet. Also, the explicit values of the constants v_g – apart from v_e – are not known. We come back to this point later. An immediate consequence of Theorem 5.1 is the following ray asymptotic for the Dunkl kernel (already conjectured in [15]):

Corollary 5.1. *For all* $x, y \in C$ *and* $g \in G$,

$$\lim_{t\to\infty} t^\gamma \, e^{-it\,\langle x, gy\rangle} E_k(itx, gy) \;=\; \frac{v_g}{\sqrt{w_k(x)w_k(y)}}\,,$$

the convergence being locally uniform with respect to the parameter x.

In the particular case $g = e$ (the unit of G), this latter result can be extended to a larger range of complex arguments by use of the Phragmén-Lindelöf principle for the right half plane H (see e.g. [58]):

Proposition 5.1. *Suppose* $f : H \to \mathbb{C}$ *is analytic and regular in* $H \cap \{z \in \mathbb{C} : |z| > R\}$ *for some* $R > 0$ *with* $\lim_{t\to\infty} f(it) = a$, $\lim_{t\to\infty} f(-it) = b$ *and for each* $\delta > 0$,

$$f(z) = \mathcal{O}(e^{\delta|z|}) \quad \textit{as } z \to \infty \textit{ within } H.$$

Then $a = b$ *and* $f(z) \to a$ *uniformly as* $z \to \infty$ *in* H.

Theorem 5.2. *Let* $x, y \in C$. *Then*

$$\lim_{z\to\infty, z\in H} z^\gamma e^{-z\langle x,y\rangle} E_k(zx, y) \;=\; \frac{i^\gamma v_e}{\sqrt{w_k(x)w_k(y)}}.$$

Here z^γ *is the holomorphic branch in* $\mathbb{C} \setminus \{x \in \mathbb{R} : x \le 0\}$ *with* $1^\gamma = 1$.

Proof. Consider

$$G(z) := z^\gamma e^{-z\langle x,y\rangle} E_k(zx, y).$$

The estimate of Proposition 2.6 on E_k implies that G satisfies the required growth bound in Proposition 5.1 (here it is of importance that x and y lie in the same Weyl chamber), and Corollary 5.1 assures that G has limits along the boundary lines of H.

When restricted to real arguments, Theorem 5.2 implies the above stated short-time asymptotic for the Dunkl type heat kernel Γ_k:

Corollary 5.2. *For all* $x, y \in C$,

$$\lim_{t\downarrow 0} \frac{\sqrt{w_k(x)\,w_k(y)}\;\Gamma_k(t,x,y)}{\Gamma_0(t,x,y)} \;=\; 1\,.$$

Hereby the precise value of the limit follows from the results of [52]. Along with it, we thus obtain the value of v_e:

$$v_e \;=\; i^{-\gamma}\frac{c_k}{c_0}.$$

We shall now give an outline of the proof of Theorem 5.1. It is based on the analysis of an associated system of first order differential equations, which is derived from the eigenfunction characterization (2.7) of E_k. This approach goes back to [30], where it was used to obtain exponential estimates for the Dunkl kernel. Put

$$\mathbb{R}^N_{reg} := \mathbb{R}^N \setminus \{\langle\alpha\rangle^\perp, \alpha \in R\}$$

and define

$$\varphi(x,y) = \sqrt{w_k(x)w_k(y)}\, e^{-i\langle x,y\rangle} E_k(ix,y), \quad x,y \in \mathbb{R}^N.$$

Observe that φ is symmetric in its arguments. We have to study the asymptotic behavior of $x \mapsto \varphi(x,y)$ along curves in C, with the second component $y \in \mathbb{R}^N_{reg}$ being fixed. Let us introduce the auxiliary vector field $F = (F_g)_{g\in G}$ on $\mathbb{R}^N \times \mathbb{R}^N$ by

$$F_g(x,y) := \varphi(x, gy).$$

For fixed y, we consider F along a differentiable curve $\kappa : (0,\infty) \to C$. The eigenfunction characterization of E_k then translates into a first order ordinary differential equation for $t \mapsto F(\kappa(t), y)$. Below, we shall determine the asymptotic behavior of its solutions, provided κ is admissible in the following sense:

Definition 5.1. *A C^1-curve $\kappa : (0,\infty) \to C$ is called* admissible, *if it satisfies the subsequent conditions:*

(1) *There exists a constant $\delta > 0$ such that $\kappa(t) \in C_\delta$ for all $t > 0$.*
(2) $\lim_{t\to\infty} |\kappa(t)| = \infty$ *and $\kappa'(t) \in C$ for all $t > 0$.*

An important class of admissible curves are the rays $\kappa(t) = tx$ with some fixed $x \in C$. In a first step, it is shown that $t \mapsto F(\kappa(t), y)$ is asymptotically constant as $t \to \infty$ for arbitrary admissible curves:

Theorem 5.3. *If $\kappa : (0,\infty) \to C$ is admissible, then for every $y \in C$, the limit*

$$\lim_{t\to\infty} F(\kappa(t), y)$$

exists in $\mathbb{C}^{|G|}$, and is different from 0.

Proof. (Sketch) It is easily calculated that (2.7) translates into the following system of differential equations for $(F_g)_{g\in G}$, where $\xi,\, y \in \mathbb{R}^N$ are fixed:

$$\partial_\xi F_g(x,y) = \sum_{\alpha\in R_+} k(\alpha)\, \frac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}\, e^{-i\langle\alpha,\,x\rangle\langle\alpha,\,gy\rangle} \cdot F_{\sigma_\alpha g}(x,y) \quad (x \in \mathbb{R}^N_{reg}). \quad (5.2)$$

From this, one obtains a differential equation for $x(t) := F(\kappa(t), y)$ of the form $x'(t) = A(t)x(t)$, with a continuous matrix function $A : (0, \infty) \to \mathbb{C}^{|G|\times|G|}$. The proof of Theorem 5.3 is accomplished by verifying that A satisfies the conditions of the following classical theorem on the asymptotic integration of ordinary differential equations (hereby of course, the admissibility conditions on κ come in).

Theorem 5.4. [20], [60]. *Consider the linear differential equation*

$$x'(t) = A(t)x(t), \tag{5.3}$$

where $A : [t_0, \infty) \to \mathbb{C}^{n\times n}$ is a continuous matrix-valued function satisfying the following integrability conditions:

(1) *The matrix-valued improper Riemann integral $\int_{t_0}^\infty A(t)dt$ converges.*
(2) $t \mapsto A(t)\int_t^\infty A(s)ds$ *belongs to* $L^1([t_0, \infty), \mathbb{C}^{n\times n})$.

Then (5.3) *has a fundamental system Φ of solutions which satisfies* $\lim_{t\to\infty} \Phi(t) = Id.$

Notice that in the situation of this theorem, for each solution x of $x'(t) = A(t)x(t)$ the limit $\lim_{t\to\infty} x(t)$ exists, and is different from zero, unless $x \equiv 0$.

Proof (of Theorem 5.1). It remains to show that the limit value according to Theorem 5.3 is actually independent of y and κ; the assertion is then easily obtained. The stated independence is accomplished as follows: In a first step, we show that there exists a non-zero vector $v(y) = (v_g(y))_{g\in G} \in \mathbb{C}^{|G|}$ such that for each admissible κ,

$$\lim_{t\to\infty} F(\kappa(t), y) = v(y). \tag{5.4}$$

This can be achieved via interpolation of the admissible curves κ_1, κ_2 by a third admissible curve κ, such that equality of all three limits is being enforced. Next, we focus on admissible rays. Observe that $F_g(tx, y) = F_{g^{-1}}(ty, x)$ for all $g \in G$ and $x, y \in C$. Together with (5.4), this implies that $v_g(y) = v_{g^{-1}}(x)$, and therefore also $v_g(x) = v_{g^{-1}}(x) = v_g(y) =: v_g$. Put $v = (v_g)_{g\in G}$. Then

$$\lim_{t\to\infty} F(\kappa(t), y) = v$$

for every admissible κ and every $y \in C$.

The asymptotic result of Theorem 5.1 also allows to deduce at least a certain amount of information about the structure of the intertwining operator V_k and its representing measures μ_x^k according to formula (4.1). The key for our approach is the following simple observation: according to Corollary 4.1, one may write

$$E_k(x, -i\xi) = \int_{\mathbb{R}^N} e^{-i\langle \xi, y\rangle} d\mu_x^k(y) = \widehat{\mu_x^k}(\xi) \quad (x, \xi \in \mathbb{R}^N),$$

where $\widehat{\mu}$ stands for the (classical) Fourier-Stieltjes transform of $\mu \in M^1(\mathbb{R}^N)$,

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^N} e^{-i\langle \xi, y\rangle} d\mu(y).$$

Recall that a measure $\mu \in M^1(\mathbb{R}^N)$ is called continuous, if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^N$. There is a well-known criterion of Wiener which characterizes Fourier-Stieltjes transforms of continuous measures on locally compact abelian groups, here $(\mathbb{R}^N, +)$; see for instance Lemma 8.3.7 of [22]:

Lemma 5.1. *(Wiener) For $\mu \in M^1(\mathbb{R}^N)$ the following properties are equivalent:*

(1) *μ is continuous.*
(2) $\displaystyle\lim_{n\to\infty} \frac{1}{n^N} \int_{\{\xi \in \mathbb{R}^N : |\xi| \leq n\}} |\widehat{\mu}(\xi)|^2 d\xi = 0.$

This yields the following result:

Theorem 5.5. *Let $k \geq 0$. Then apart from the case $k = 0$ (i.e. the classical Fourier case), the measure μ_x^k is continuous for all $x \in \mathbb{R}^N_{reg}$.*

We conclude with two open problems:

(a) In the situation of the last theorem, prove that the measures μ_x^k are even absolutely continuous with respect to Lebesgue measure, provided $\{\alpha \in R : k(\alpha) > 0\}$ spans $\mathbb{R}^N$.
(b) Determine the values of the constants v_g, $g \in G$.

6 Notation

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integer, real and complex numbers respectively. Further, $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. For a locally compact Hausdorff space X, we denote by $C(X), C_b(X), C_c(X), C_0(X)$ the spaces of continuous complex-valued functions on X, those which are bounded, those with compact support, and those which vanish at infinity, respectively. Further, $M_b(X)$, $M_b^+(X)$, $M^1(X)$ are the spaces of regular bounded Borel measures on X, those which are positive, and those which are probability-measures, respectively. Finally, $\mathscr{B}(X)$ stands for the σ-algebra of Borel sets on X.

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