Vol. 98, No. 3 DUKE MATHEMATICAL JOURNAL

POSITIVITY OF DUNKL'S INTERTWINING OPERATOR

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1. Introduction and results. In recent years, the theory of Dunkl operators has found a wide area of applications in mathematics and mathematical physics. Besides their use in the study of multivariable orthogonality structures associated with root systems (see, for example, [D1], [D2], [He], [vD], and [R]), these operators are closely related to certain representations of degenerate affine Hecke algebras (see [C], [O2], and, for some background, [Ki]). Moreover, they have been successfully involved in the description and solution of Calogero-Moser-Sutherland–type quantum many-body systems; among the wide literature in this context, we refer to [P], [LV] and [BF].

Let $G \subset O(N, \mathbb{R})$ be a finite reflection group on $\mathbb{R}^N$. For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote by σ_α the reflection in the hyperplane orthogonal to α; that is,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $\langle .,. \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^N$ and $|x| := \sqrt{\langle x, x\rangle}$. We also use the notation $\langle .,. \rangle$ for the bilinear extension of the Euclidean scalar product to $\mathbb{C}^N \times \mathbb{C}^N$, while $z \mapsto |z|$ is the standard Hermitean norm on $\mathbb{C}^N$. Further, let R be the root system of G, normalized such that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$, and fix a positive subsystem R_+ of R. We recall from the general theory of reflection groups (see, e.g., [Hu]) that the set of reflections in G coincides with $\{\sigma_\alpha, \alpha \in R_+\}$ and that the orbits in R under the natural action of G correspond to the conjugacy classes of reflections in G. A function $k : R \to \mathbb{C}$ is called a multiplicity function on R if it is G-invariant. We write $\operatorname{Re} k \geq 0$ if $\operatorname{Re} k(\alpha) \geq 0$ for all $\alpha \in R$, and $k \geq 0$ if $k(\alpha) \geq 0$ for all $\alpha \in R$.

The Dunkl operators associated with G are first-order differential-difference operators on $\mathbb{R}^N$ which are parametrized by some multiplicity function k on R. For $\xi \in \mathbb{R}^N$, the corresponding Dunkl operator $T_\xi(k)$ is given by

$$T_\xi(k) f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}, \quad f \in C^1\big(\mathbb{R}^N\big).$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. As k is G-invariant, the above definition is independent of the choice of R_+. In case $k = 0$, the $T_\xi(k)$ reduce to the corresponding directional derivatives. The operators $T_\xi(k)$ were introduced and

Received 29 December 1997. Revision received 24 April 1998.
1991 *Mathematics Subject Classification*. Primary 33C80; Secondary 44A15, 33C50.

first studied by Dunkl in a series of papers [D1], [D2], [D3], and [D4] in connection with a generalization of the classical theory of spherical harmonics. Here the uniform surface measure on the $(N-1)$-dimensional unit sphere is modified by a weight function that is invariant under the action of a given reflection group G and associated with a multiplicity function $k \geq 0$, namely,

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}. \tag{1.1}$$

The most important properties of the operators $T_\xi(k)$ are as follows. Let $\Pi^N = \mathbb{C}[\mathbb{R}^N]$ denote the algebra of polynomial functions on $\mathbb{R}^N$, and let $\mathcal{P}_n^N$ ($n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$) denote the subspace of homogeneous polynomials of (total) degree n. Then,

(1) the $T_\xi(k), \xi \in \mathbb{R}^N$, generate a commuting family of linear operators on Π^N;
(2) each $T_\xi(k)$ is homogeneous of degree -1 on Π^N, that is, $T_\xi(k)(p) \in \mathcal{P}_{n-1}^N$ for $p \in \mathcal{P}_n^N$;
(3) for all but a singular set of multiplicity functions, in particular for $k \geq 0$, there exists a unique linear isomorphism V_k of Π^N such that

$$V_k\big(\mathcal{P}_n^N\big) = \mathcal{P}_n^N, \qquad V_k|_{\mathcal{P}_0^N} = id, \qquad \text{and} \qquad T_\xi(k)V_k = V_k\partial_\xi \quad \text{for all } \xi \in \mathbb{R}^N.$$

Properties (1) and (2) were shown in [D1], while the existence of an intertwining operator according to (3) was first shown in [D2] under the assumption $k \geq 0$. An abstract and extended treatment of the above items is given in [DJO].

The intertwining operator V_k plays a central part in Dunkl's theory and its applications. In particular, it is involved in the definition of Dunkl's kernel $K_G(x, y)$ (see (1.6)), which generalizes the usual exponential kernel $e^{\langle x,y\rangle}$ and arises as the integral kernel of the Dunkl transform (see [D4] and [dJ]). An explicit form of V_k is known so far only in the following very special cases.

(1) The 1-dimensional case associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Here the multiplicity function is given by a single parameter $k \geq 0$; for $k > 0$, the intertwining operator V_k has the integral representation (see [D3, Theorem 5.1])

$$V_k p(x) = c_k \int_{-1}^{1} p(xt)(1-t)^{k-1}(1+t)^k \, dt \text{ with } c_k = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)}. \tag{1.2}$$

(2) The direct product case associated with the reflection group $\mathbb{Z}_2^N$ on $\mathbb{R}^N$. Here a closed form of the intertwining operator was determined in [X2].

(3) The case of the symmetric group S_3 on $\mathbb{R}^3$, which has been studied in [D5].

In [D3], the intertwining operator V_k is extended, for $k \geq 0$, to a bounded linear operator on a suitably normed algebra of series of homogeneous polynomials on the unit ball. To allow a more convenient formulation of our statements, we introduce a slightly extended notation: For $r > 0$, let $K_r := \{x \in \mathbb{R}^N : |x| \leq r\}$ denote the ball of radius r, and define

(1.3)

$$A_r := \Big\{ f : K_r \to \mathbb{C},\, f = \sum_{n=0}^{\infty} f_n \text{ with } f_n \in \mathcal{P}_n^N \text{ and } \|f\|_{A_r} := \sum_{n=0}^{\infty} \|f_n\|_{\infty, K_r} < \infty \Big\}.$$

It is easily checked that A_r is a commutative Banach-$*$-algebra with complex conjugation as involution (see Section 4). Moreover, it follows from [D3, Theorem 2.7] that V_k extends to a continuous linear operator on A_r by $V_k f := \sum_{n=0}^{\infty} V_k f_n$ for $f = \sum_{n=0}^{\infty} f_n \in A_r$. Up to now, it has been an open question whether, for $k \geq 0$, the intertwining operator V_k is always positive, that is, $V_k p \geq 0$ on $\mathbb{R}^N$ for each nonnegative polynomial $p \in \Pi^N$. More generally, we may ask whether, for every $x \in \mathbb{R}^N$ with $|x| \leq r$, the functional $f \mapsto V_k f(x)$ is positive on A_r. This property, which was first conjectured (in a slightly different setting) in [D3], is obvious in the above special cases (1) and (2) from the explicit representation of V_k. In the S_3 case, however, the integral representations derived in [D5] failed to infer this result—at least for a large range of k. It is the aim of this paper to prove the above conjecture for general reflection groups and nonnegative multiplicity functions. Our first central result establishes positivity of V_k on polynomials.

THEOREM 1.1. *Assume that $k \geq 0$ and let $p \in \Pi^N$ with $p(x) \geq 0$ for all $x \in \mathbb{R}^N$. Then also $V_k p(x) \geq 0$ for all $x \in \mathbb{R}^N$.*

More detailed information about V_k is then obtained by its extension to the algebras A_r. This leads to the following theorem, which is the main result of this paper.

THEOREM 1.2. *Assume that $k \geq 0$. Then, for each $x \in \mathbb{R}^N$, there exists a unique probability measure μ_x on the Borel-σ-algebra of $\mathbb{R}^N$ such that*

$$V_k f(x) = \int_{\mathbb{R}^N} f(\xi)\, d\mu_x(\xi) \quad \text{for all } f \in A_{|x|}. \tag{1.4}$$

The representing measures μ_x are compactly supported with $\operatorname{supp} \mu_x \subseteq \{\xi \in \mathbb{R}^N : |\xi| \leq |x|\}$. *Moreover, they satisfy*

$$\mu_{rx}(B) = \mu_x\big(r^{-1}B\big), \qquad \mu_{gx}(B) = \mu_x\big(g^{-1}(B)\big) \tag{1.5}$$

for each $r > 0$, $g \in G$, and each Borel set $B \subseteq \mathbb{R}^N$.

An important consequence of Theorem 1.2 concerns the generalized exponential kernel K_G, which is defined by

$$K_G(x, y) := V_k\Big(e^{\langle \cdot, y \rangle}\Big)(x) \qquad \big(x, y \in \mathbb{R}^N\big) \tag{1.6}$$

(see [D3]). The function K_G has a holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$ and is symmetric in its arguments. According to [O1], the function $x \mapsto K_G(x, y)(y \in \mathbb{C}^N$

fixed) may be characterized as the unique analytic solution of the system $T_\xi(k)f = \langle \xi, y\rangle f$ ($\xi \in \mathbb{R}^N$) with $f(0) = 1$. This makes it possible to translate invariance properties of the Dunkl operators to corresponding properties of K_G (see [dJ, Theorem 2.8]); in particular, $K_G(x, \lambda y) = K_G(\lambda x, y)$ for all $\lambda \in \mathbb{C}$ and $x, y \in \mathbb{C}^N$. Theorem 1.2 now implies that, for fixed $y \in \mathbb{R}^N$, the kernel $K_G(x, iy)$ is positive-definite as a function of x on $\mathbb{R}^N$, and the same holds for the "generalized Bessel function"

$$J_G(x, iy) := \frac{1}{|G|} \sum_{g \in G} K_G(gx, iy) \qquad \big(x, y \in \mathbb{R}^N\big).$$

As noted in [O1], the kernel J_G allows in some cases (for Weyl groups G and certain discrete sets of multiplicity functions) an interpretation as the spherical function for some Euclidean symmetric space; in these cases, positive-definiteness of J_G is obvious. There are no similar interpretations known for the kernel K_G. Nevertheless, the conjecture that it should be positive-definite has been confirmed by several of its properties (see [dJ]) and, in particular, by the fact that $K_G(x, y) > 0$ for all $x, y \in \mathbb{R}^N$. This was proved in [R] in connection with the study of a generalized heat semigroup for Dunkl operators.

The main parts of Theorem 1.2 are obtained by a standard argumentation from Theorem 1.1. The proof of Theorem 1.1, however, is much more involved. Its crucial step is a reduction from the N-dimensional to a 1-dimensional problem, using semigroup techniques for linear operators on spaces of polynomials. The generators of the semigroups under consideration are certain differential-difference operators whose common decisive property is that they are "degree-lowering." This setting is introduced in Section 2, together with a Hille-Yosida–type theorem, which characterizes positivity of such semigroups by means of their generator. Theorem 1.1 is then proved in Section 3. Section 4 is introduced with a short discussion of the algebras A_r and their spectral properties, which is the basis for the subsequent proof of Theorem 1.2. In the last section we discuss some implications of our results in the theory of Dunkl operators and related applications.

2. Semigroups generated by degree-lowering operators on polynomials. We start with some general notation. Let $\Pi_+^N := \{p \in \Pi^N : p(x) \geq 0 \text{ for all } x \in \mathbb{R}^N\}$ denote the cone of nonnegative polynomials on $\mathbb{R}^N$. Let $\Pi_n^N := \bigoplus_{k=0}^n \mathcal{P}_k^N$ ($n \in \mathbb{Z}_+$) denote the space of polynomials of (total) degree at most n. The action of a subgroup $H \subseteq O(N, \mathbb{R})$ on Π^N will always be the natural one, given by $hp(x) := p(h^{-1}x)(h \in H, p \in \Pi^N)$. Finally, for a locally compact Hausdorff space X, $M_b(X)$ is the space of all regular bounded Borel measures on X and $M_b^+(X)$ is the subspace of those that are nonnegative.

Definition 2.1. A linear operator A on Π^N is called *positive* if $Ap \in \Pi_+^N$ for each $p \in \Pi_+^N$ and *degree-lowering* if $A\big(\Pi_n^N\big) \subseteq \Pi_{n-1}^N$ for all $n \in \mathbb{Z}_+$.

Important examples of degree-lowering operators are linear operators that are homogeneous of some degree $-n$, $n \geq 1$, on Π^N. This includes, in particular, usual partial derivatives and Dunkl operators, as well as products and linear combinations of those. If A is degree-lowering on Π^N, then for every analytic function $f : \mathbb{R} \to \mathbb{C}$ with power series $f(x) = \sum_{k=0}^{\infty} c_k x^k$, there is a linear operator $f(A)$ on Π^N defined by the terminating series

$$f(A)p(x) := \sum_{k=0}^{\infty} c_k A^k p(x).$$

Notice that $f(A)(\Pi_n^N) \subseteq \Pi_n^N$ for each $n \in \mathbb{Z}_+$. This yields a natural restriction of $f(A)$ to a linear operator on the finite-dimensional vector space Π_n^N. In particular, the well-known product and exponential formulas for linear operators on finite-dimensional vector spaces (see, for example, [Ka, §4.7]) imply corresponding exponential formulas for degree-lowering operators on Π^N, where the topology may be chosen to be the one of pointwise convergence. We note two results of this type, which are used later on.

LEMMA 2.2. *Let A and B be degree-lowering linear operators on Π^N. Then, for all $p \in \Pi^N$ and $x \in \mathbb{R}^N$,*

(i) $e^A p(x) = \lim_{n\to\infty} (I - A/n)^{-n} p(x)$.
(ii) $e^{A+B} p(x) = \lim_{n\to\infty} (e^{A/n} e^{B/n})^n p(x)$ *(Trotter product formula).*

Each degree-lowering operator A on Π^N generates a semigroup $(e^{tA})_{t\geq 0}$ of linear operators on Π^N and, in fact, on each of the Π_n^N. The generator A is uniquely determined from the semigroup by

$$Ap(x) = \lim_{t\downarrow 0} t^{-1}\big(e^{tA} - I\big)p(x) \quad \text{for all } p \in \Pi^N.$$

The following key result characterizes positive semigroups generated by degree-lowering operators; it is an adaption of a well-known Hille-Yosida–type characterization theorem for Feller-Markov semigroups on $C(K)$, K being a compact Hausdorff space (see, for example, [GS, Section 2.4]).

THEOREM 2.3. *Let A be a degree-lowering linear operator on Π^N. Then the following statements are equivalent:*

(1) *e^{tA} is positive on Π^N for all $t \geq 0$;*
(2) *A satisfies the "positive minimum principle"*
   (M) *for every $p \in \Pi_+^N$ and $x_0 \in \mathbb{R}^N$, $p(x_0) = 0$ implies $Ap(x_0) \geq 0$.*

Proof. (1) implies (2). Let $p \in \Pi_+^N$ with $p(x_0) = 0$. Then,

$$Ap(x_0) = \lim_{t\downarrow 0} \frac{e^{tA}p(x_0) - p(x_0)}{t} = \lim_{t\downarrow 0} \frac{1}{t} e^{tA} p(x_0) \geq 0.$$

(2) implies (1). Notice first that for each $\lambda \neq 0$, the operator $\lambda I - A$ is bijective on Π^N. In fact, $\lambda I - A$ is injective on Π^N, because otherwise there would exist some

$p \in \Pi^N$, $p \neq 0$, with $Ap = \lambda p$, in contradiction to the degree-lowering character of A. As $(\lambda I - A)(\Pi_n^N) \subseteq \Pi_n^N$, this already proves the bijectivity of $\lambda I - A$ on each Π_n^N and hence on Π^N as well. We next claim that for every $\lambda > 0$, the resolvent operator $R(\lambda; A) := (\lambda I - A)^{-1}$ is positive on Π^N. For this, let $p \in \Pi_+^N$ and $q := R(\lambda; A)p$. If p is constant, then $q = (1/\lambda)p \geq 0$. We may therefore restrict ourselves to the case that the total degree n of p (which must be even) is greater than zero. Suppose first that $p(x) \geq c|x|^n$ for all $x \in \mathbb{R}^N$, with some constant $c > 0$. As A lowers the degree, we may write $q = (1/\lambda)p + r$ with a polynomial r of total degree less than n. Hence, $\lim_{|x|\to\infty} q(x) = \infty$, which shows that q takes an absolute minimum, let us say, in $x_0 \in \mathbb{R}^N$. Put $\widetilde{q}(x) := q(x) - q(x_0)$. Then $\widetilde{q} \in \Pi_+^N$ with $\widetilde{q}(x_0) = 0$, and property (M) assures that $Aq(x_0) = A\widetilde{q}(x_0) \geq 0$. For $\lambda > 0$ and $x \in \mathbb{R}^N$, we therefore obtain

$$\lambda q(x) \geq \lambda q(x_0) = (\lambda I - A)q(x_0) + Aq(x_0) \geq p(x_0) \geq 0.$$

If $p \in \Pi_+^N$ is arbitrary, then consider the polynomials $p_\epsilon(x) := p(x) + \epsilon|x|^n$ for $\epsilon > 0$, where n is the degree of p. As A is degree-lowering, by the above result, we obtain

$$R(\lambda; A)p(x) = \lim_{\epsilon\to 0} R(\lambda; A)p_\epsilon(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N.$$

This proves the stated positivity of $R(\lambda; A)$ for $\lambda > 0$. Now let $p \in \Pi_+^N$ and $t > 0$. Then, according to Lemma 2.2(i),

$$e^{tA}p(x) = \lim_{n\to\infty}\left(I - \frac{tA}{n}\right)^{-n} p(x) = \lim_{n\to\infty}\left(\frac{n}{t}R\left(\frac{n}{t}; A\right)\right)^n p(x) \geq 0$$

for all $x \in \mathbb{R}^N$. This finishes the proof. □

3. Positivity of V_k on polynomials. This section is devoted to the proof of Theorem 1.1. The outline of this proof is as follows. In the first step, we consider the (1-dimensional) differential-difference operators

$$\Lambda_s := e^{-sD^2}\delta e^{sD^2}, \quad s \geq 0,$$

on Π^1. Here D denotes the usual first derivative, that is, $Dp(x) = p'(x)$ for $x \in \mathbb{R}$, and δ is the linear operator on Π^1 given by

$$\delta p(x) := \frac{p'(x)}{x} - \frac{p(x) - p(-x)}{2x^2} = \frac{1}{2}\int_{-1}^{1}\big(D^2 p\big)(tx)(1+t)\,dt. \tag{3.1}$$

This operator is related to the Dunkl operator $T(k)$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ and the multiplicity parameter $k \geq 0$ by

$$T(k)^2 = D^2 + 2k\delta.$$

As both D^2 and δ are homogeneous of degree -2 on Π^1, the operators Λ_s are well defined and degree-lowering on Π^1. We prove that they have the following decisive property.

PROPOSITION 3.1. *The operators* Λ_s, $s \geq 0$, *satisfy the positive minimum principle (M) on* Π^1.

We next turn to the general N-dimensional setting: Here G is an arbitrary finite reflection group on $\mathbb{R}^N$ with multiplicity function $k \geq 0$. We consider the generalized Laplacian associated with G and k, which is defined by

$$\Delta_k := \sum_{i=1}^{N} T_{\xi_i}(k)^2$$

with an arbitrary orthonormal basis $(\xi_1, \dots, \xi_N)$ of $\mathbb{R}^N$ (see [D1]). It is homogeneous of degree -2 on Π^N and (with our convention $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R_+$) is given explicitly by

$$\Delta_k = \Delta + 2 \sum_{\alpha \in R_+} k(\alpha)\delta_\alpha \quad \text{with } \delta_\alpha f(x) = \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}. \tag{3.2}$$

Here Δ and ∇ denote the usual Laplacian and gradient, respectively. Theorem 2.3 is the key to infer from the 1-dimensional setting of Proposition 3.1 to a general multivariable extension.

PROPOSITION 3.2. *Let* $L_k := \Delta_k - \Delta$. *Then, for* $k \geq 0$, *the operators*

$$e^{-s\Delta} e^{tL_k} e^{s\Delta} \quad (s, t \geq 0)$$

are positive on Π^N.

The statement of Theorem 1.1 is then finally reduced to the following consequence of Proposition 3.2.

COROLLARY 3.3. *The operator* $e^{-\Delta/2} e^{\Delta_k/2}$ *is positive on* Π^N.

Proof. Applying Trotter's product formula of Lemma 2.2, we obtain

$$\begin{aligned}
&e^{-\Delta/2} e^{\Delta_k/2} p(x) \\
&\quad = e^{-\Delta/2} e^{\Delta/2 + L_k/2} p(x) = \lim_{n\to\infty} e^{-\Delta/2} \big(e^{\Delta/2n} e^{L_k/2n}\big)^n p(x) \\
&\quad = \lim_{n\to\infty} \prod_{j=1}^{n} \big(e^{-(1-j/n)\cdot\Delta/2} e^{L_k/2n} e^{(1-j/n)\cdot\Delta/2}\big) p(x) \quad \big(p \in \Pi^N, x \in \mathbb{R}^N\big).
\end{aligned}$$

By Proposition 3.2, each of the n factors in the above product is a positive operator on Π^N. Hence, $e^{-\Delta/2} e^{\Delta_k/2}$ is also positive on Π^N. □

We now turn to the proof of Proposition 3.1. We start with two elementary auxiliary results.

LEMMA 3.4. *For each* $p \in \Pi^1$ *and* $c \in \mathbb{R}$,

$$e^{cD^2}\big(xp(x)\big) = x e^{cD^2} p(x) + 2c e^{cD^2} p'(x).$$

Proof. Power series expansion of e^{cD^2} yields

$$
\begin{aligned}
&e^{cD^2}\big(xp(x)\big)\\
&\quad=\sum_{n=0}^{\infty}\frac{c^n}{n!}D^{2n}\big(xp(x)\big)=xp(x)+\sum_{n=1}^{\infty}\frac{c^n}{n!}\big(xD^{2n}p(x)+2nD^{2n-1}p(x)\big)\\
&\quad=xe^{cD^2}p(x)+2c\sum_{n=1}^{\infty}\frac{c^{n-1}}{(n-1)!}D^{2n-1}p(x)=xe^{cD^2}p(x)+2ce^{cD^2}p'(x).
\end{aligned}
$$ □

LEMMA 3.5. *Let $p\in\Pi^1_{2n+1}$, $n\in\mathbb{Z}_+$, be an odd polynomial. Then the differential equation*

$$cy'-xy=p \qquad (c>0) \tag{3.3}$$

has exactly one polynomial solution (which belongs to Π^1_{2n}), namely,

$$y_p(x)=\frac{1}{c}e^{x^2/2c}\int_{-\infty}^{x}e^{-t^2/2c}p(t)\,dt.$$

Proof. The general solution of (3.3) is

$$y(x)=ae^{x^2/2c}+\frac{1}{c}e^{x^2/2c}\int_{-\infty}^{x}e^{-t^2/2c}p(t)\,dt,\quad a\in\mathbb{R}.$$

It therefore remains to prove that

$$x\mapsto e^{x^2/2c}\int_{-\infty}^{x}e^{-t^2/2c}p(t)\,dt \tag{3.4}$$

is a polynomial. We use induction by n. For $n=0$, the statement is obvious. For $n\geq 1$, write $p(x)=-c^{-1}xr(x)$ with $r\in\Pi^N_{2n}$. Partial integration then yields

$$\int_{-\infty}^{x}e^{-t^2/2c}p(t)\,dt=-\frac{1}{c}\int_{-\infty}^{x}te^{-t^2/2c}r(t)\,dt=e^{-x^2/2c}r(x)-\int_{-\infty}^{x}e^{-t^2/2c}r'(t)\,dt.$$

By our induction hypothesis, this equals $e^{-x^2/2c}(r(x)-\widetilde{r}(x))$ with some polynomial $\widetilde{r}\in\Pi^N_{2n-2}$. This finishes the proof. □

Proof of Proposition 3.1. The case $s=0$ is easy and may be treated separately. Let $p\in\Pi^1_+$ with $p(x_0)=0$. Then $p'(x_0)=0$ and $p''(x_0)\geq 0$. Thus, if $x_0\neq 0$, then $\delta p(x_0)=p(-x_0)/(2x_0^2)\geq 0$. In case $x_0=0$, it is seen from the integral representation (3.1) that $\delta p(0)=p''(0)\geq 0$. From now on, we may therefore assume that $s>0$.

We first derive an explicit representation of the operator $\Lambda_s\,(s>0)$, which allows us to check property (M) easily. We claim that

$$\Lambda_s p(x)=-\frac{1}{2s}p(x)-\frac{1}{8s^2}e^{x^2/4s}\left(\int_{-\infty}^{x}g_{p,x}(t)\,dt-\int_{-x}^{\infty}g_{p,x}(t)\,dt\right), \tag{3.5}$$

$$\text{for } p\in\Pi^1,\ \text{with } g_{p,x}(t)=e^{-t^2/4s}(t+x)p(t).$$

This of course, may be verified by a (tedious) direct computation of $\Lambda_s(x^k), k \in \mathbb{Z}_+$, and an explicit evaluation of the corresponding integrals on the right side by series expansions of the involved exponentials. We prefer, however, to give a more instructive proof.

Note first that the operators D^2 and δ map even polynomials to even ones and odd polynomials to odd ones, and

$$\delta p(x) = \begin{cases} \dfrac{1}{x} p'(x) & \text{if } p \text{ is even,} \\ \left(\dfrac{1}{x} p(x)\right)' & \text{if } p \text{ is odd.} \end{cases}$$

Now fix $s > 0$ and suppose that $p \in \Pi^1$ is even. Then the polynomials $e^{sD^2}p$ and $q := \Lambda_s p$ are also even, and we obtain the following equivalences:

$$q = \Lambda_s p \Longleftrightarrow \delta\big(e^{sD^2}p\big) = e^{sD^2}q \Longleftrightarrow p'(x) = e^{-sD^2}\big(xe^{sD^2}q\big)(x).$$

By use of Lemma 3.4, this becomes

$$p'(x) = xq(x) - 2sq'(x), \tag{3.6}$$

which is a differential equation of type (3.3) for q. Lemma 3.5, together with a further partial integration, now implies that

$$\begin{aligned} \Lambda_s p(x) &= -\frac{1}{2s} e^{x^2/4s} \int_{-\infty}^{x} e^{-t^2/4s} p'(t)\,dt \\ &= -\frac{1}{2s} p(x) - \frac{1}{4s^2} e^{x^2/4s} \int_{-\infty}^{x} e^{-t^2/4s} tp(t)\,dt \qquad (p \text{ even}). \end{aligned} \tag{3.7}$$

In a similar way, we calculate $q = \Lambda_s p$ for odd $p \in \Pi^1$. In this case, $e^{sD^2}p$ and $q = \Lambda_s p$ are odd as well, and we have the equivalence

$$q = \Lambda_s p \Longleftrightarrow \frac{d}{dx}\Big(\frac{1}{x} e^{sD^2} p(x)\Big) = e^{sD^2} q(x).$$

Hence, there exists a constant $c_1 \in \mathbb{R}$ such that

$$e^{sD^2} p(x) = x\big(c_1 + h(x)\big), \qquad \text{with } h(x) = \int_0^x e^{sD^2} q(t)\,dt.$$

Applying Lemma 3.4 again, we obtain

$$\begin{aligned} p(x) &= c_1 e^{-sD^2}(x) + xe^{-sD^2}h(x) - 2se^{-sD^2}h'(x) \\ &= c_1 x + xe^{-sD^2}h(x) - 2sq(x). \end{aligned} \tag{3.8}$$

In order to determine $e^{-sD^2}h$, note that

$$\frac{d}{dx}\big(e^{-sD^2}h(x)\big) = e^{-sD^2}h'(x) = q(x).$$

Consequently, there exists a constant $c_2 \in \mathbb{R}$ such that

$$e^{-sD^2}h(x) = c_2 + \int_0^x q(t)\,dt. \tag{3.9}$$

Now write $p(x) = xP(x)$ and $q(x) = xQ(x)$ with even $P, Q \in \Pi^1$. Then, by (3.8) and (3.9),

$$P(x) = c_1 + c_2 + \int_0^x t\,Q(t)\,dt - 2s\,Q(x),$$

and therefore,

$$P'(x) = x\,Q(x) - 2s\,Q'(x).$$

This is exactly the same differential equation as we had in the even case before, and the transfer of (3.7) gives

$$\Lambda_s p(x) = -\frac{1}{2s}p(x) - \frac{1}{4s^2}e^{x^2/4s}x\int_{-\infty}^x e^{-t^2/4s}p(t)\,dt \qquad (p \text{ odd}). \tag{3.10}$$

Finally, if $p \in \Pi^1$ is arbitrary, then write $p = p_e + p_o$ with even part $p_e(x) = (p(x) + p(-x))/2$ and odd part $p_o(x) = (p(x) - p(-x))/2$. The combination of (3.7) for p_e with (3.10) for p_o then leads to

$$\Lambda_s p(x) = -\frac{1}{2s}p(x) - \frac{1}{4s^2}e^{x^2/4s}\int_{-\infty}^x e^{-t^2/4s}\Big(\frac{t+x}{2}p(t) + \frac{t-x}{2}p(-t)\Big)dt,$$

and an easy reformulation yields the stated representation (3.5).

In order to prove that Λ_s satisfies the positive minimum principle (M), define

$$F_p(x) := \int_{-\infty}^x g_{p,x}(t)\,dt - \int_{-x}^\infty g_{p,x}(t)\,dt, \quad \text{for } p \in \Pi^1 \text{ and } x \in \mathbb{R}.$$

Now let $p \in \Pi^1_+$ with $p(x_0) = 0$. Then, in view of (3.5),

$$\Lambda_s p(x_0) = -\frac{1}{8s^2}e^{x_0^2/4s}F_p(x_0),$$

and it remains to check that $F_p(x_0) \le 0$. For this, we rewrite F_p as

$$F_p(x) = \int_{-\infty}^{-|x|} g_{p,x}(t)\,dt - \int_{|x|}^\infty g_{p,x}(t)\,dt.$$

As p is nonnegative, the sign of $g_{p,x}(t)$ coincides with the sign of $(x+t)$ for all $x, t \in \mathbb{R}$. This shows that, in fact, $F_p(x) \le 0$ for all $x \in \mathbb{R}$, which completes the proof. □

Proof of Proposition 3.2. For fixed $s \geq 0$, the operators $(e^{-s\Delta}e^{tL_k}e^{s\Delta})_{t\geq 0}$ form a semigroup on Π^N with generator $e^{-s\Delta}L_k e^{s\Delta}$. According to Theorem 2.3, it therefore suffices to prove that this generator satisfies the positive minimum principle (M) on Π^N. With the notation of (3.2), we have

$$L_k = 2\sum_{\alpha\in R_+} k(\alpha)\delta_\alpha \quad \text{ with } k(\alpha) \geq 0, \text{ for all } \alpha \in R_+.$$

It is therefore enough to make sure that each of the operators

$$\rho_\alpha^s := e^{-s\Delta}\delta_\alpha e^{s\Delta} \quad \big(\alpha \in R_+\big)$$

satisfies (M). (Here the assumption $k \geq 0$ is crucial!) Now fix $\alpha \in R_+$. An easy calculation shows that δ_α, and hence ρ_α^s, are rotation-equivariant, that is,

$$g \circ \rho_\alpha^s \circ g^{-1} = \rho_{g(\alpha)}^s \quad \text{for } g \in SO(N,\mathbb{R}).$$

We may therefore assume that $\alpha = \sqrt{2}e_1 = (\sqrt{2}, 0, \dots, 0)$. As $\delta_{\sqrt{2}e_1}$ obviously commutes with each of the partial derivatives $\partial_2, \dots, \partial_N$ on $\mathbb{R}^N$, we obtain

$$\rho_{\sqrt{2}e_1}^s = e^{-s\partial_1^2}\delta_{\sqrt{2}e_1}e^{s\partial_1^2}.$$

But this operator acts on the first variable only, namely, via Λ_s:

$$\rho_{\sqrt{2}e_1}^s p(x_1,\dots,x_N) = \Lambda_s p_{x_2,\dots,x_N}(x_1),$$
$$\text{where } p_{x_2,\dots,x_N}(x_1) := p(x_1,x_2,\dots,x_N) \quad \text{for } p \in \Pi^N.$$

The assertion now follows from Proposition 3.1. □

In order to complete the proof of Theorem 1.1, we employ the following bilinear form on Π^N associated with G and k, which was introduced in [D3] (for a further discussion, see also [DJO]):

$$[p,q]_k := \big(p(T_k)q\big)(0) \quad \text{for } p,q \in \Pi^N.$$

Here $p(T_k)$ is the differential-difference operator which is obtained from $p(x)$ by replacing each x_i by the corresponding Dunkl operator $T_{e_i}(k)$. The case $k = 0$ is distinguished by the notation $p(\partial)$. Notice that $[p,q]_k = 0$ for $p \in \mathcal{P}_n^N$ and $q \in \mathcal{P}_m^N$ with $n \neq m$. It was shown in [D3] that for $k \geq 0$ and for all $p,q \in \Pi^N$,

$$[p,q]_k = c_k \int_{\mathbb{R}^N} e^{-\Delta_k/2}p(x)e^{-\Delta_k/2}q(x)e^{-|x|^2/2}w_k(x)\,dx, \tag{3.11}$$

where w_k is the weight function defined in (1.1) and $c_k := (\int_{\mathbb{R}^N} e^{-|x|^2/2}w_k(x)dx)^{-1}$. We remark that (3.11) can also be proved in a completely independent way by using certain biorthogonal polynomial systems (Appell characters and cocharacters) in

$L^2(\mathbb{R}^N, e^{-|x|^2/2}w_k(x)dx)$; see [RV2]. Another useful identity for $[.,.]_k$ is

$$[V_k p, q]_k = [p, q]_0, \qquad \text{for all } p, q \in \Pi^N. \tag{3.12}$$

In fact, for $p, q \in \mathcal{P}_n^N$ with $n \in \mathbb{Z}_+$, one obtains

$$[V_k p, q]_k = [q, V_k\, p]_k = q(T_k)(V_k p) = V_k\big(q(\partial)p\big) = q(\partial)(p) = [p, q]_0.$$

Here the characterizing properties of V_k and the fact that $q(\partial)p$ is a constant have been used. For general $p, q \in \Pi^N$, (3.12) then follows from the orthogonality of the spaces $\mathcal{P}_n^N$, $n \in \mathbb{Z}_+$, with respect to both scalar products.

Finally, we need the following positivity criterion for polynomials.

LEMMA 3.6. *Let $\alpha > 0$ and suppose that $h \in C_b(\mathbb{R}^N)$ satisfies*

$$\int_{\mathbb{R}^N} h(x)p(x)e^{-\alpha|x|^2}w_k(x)\,dx \ge 0, \quad \textit{for all } p \in \Pi_+^N. \tag{3.13}$$

Then $h(x) \ge 0$ for all $x \in \mathbb{R}^N$.

Proof. For abbreviation, put

$$dm_k(x) := e^{-\alpha|x|^2}w_k(x)\,dx \in M_b^+\big(\mathbb{R}^N\big).$$

We use that Π^N is dense in $L^2(\mathbb{R}^N, dm_k)$. This is proved (with $\alpha = 1/2$) in [D4, Theorem 2.5] by referring to a well-known theorem of Hamburger for 1-dimensional distributions, but it can also be seen directly as follows. Suppose that Π^N is not dense in $L^2(\mathbb{R}^N, dm_k)$. Then there exists some $f \in L^2(\mathbb{R}^N, dm_k)$, $f \neq 0$, with $\int_{\mathbb{R}^N} fp\,dm_k = 0$ for all $p \in \Pi^N$. Now consider the measure $\nu := fm_k \in M_b(\mathbb{R}^N)$ and its (classical) Fourier-Stieltjes transform

$$\widehat{\nu}(\lambda) = \int_{\mathbb{R}^N} e^{-i\langle\lambda,x\rangle}\,d\nu(x) = \int_{\mathbb{R}^N} f(x)e^{-i\langle\lambda,x\rangle}\,dm_k(x).$$

As $x \mapsto e^{|\lambda||x|}$ belongs to $L^2(\mathbb{R}^N, dm_k)$ for all $\lambda \in \mathbb{R}^N$, the dominated convergence theorem yields

$$\widehat{\nu}(\lambda) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^N} f(x)\langle\lambda, x\rangle^n\,dm_k(x) = 0.$$

By injectivity of the Fourier-Stieltjes transform on $M_b(\mathbb{R}^N)$, it follows that $\nu = 0$, and hence $f = 0$ a.e., a contradiction.

Now assume that $h \in C_b(\mathbb{R}^N)$ satisfies (3.13). In order to prove $h \ge 0$, it suffices to check that

$$\int_{\mathbb{R}^N} fh\,dm_k \ge 0, \quad \text{for all } f \in C_c^+\big(\mathbb{R}^N\big). \tag{3.14}$$

For this, let $f \in C_c^+(\mathbb{R}^N)$ and $\epsilon > 0$. By density of Π^N in $L^2(\mathbb{R}^N, dm_k)$, there exists some $p = p_\epsilon \in \Pi^N$ with $\|\sqrt{f} - p\|_{2,m_k} < \epsilon$. With $M := \|h\|_{\infty,\mathbb{R}^N}$, it follows that

$$\left|\int_{\mathbb{R}^N} f h\, dm_k - \int_{\mathbb{R}^N} p^2 h\, dm_k\right| \le M \int_{\mathbb{R}^N} \left|f - p^2\right| dm_k$$
$$\le M \cdot \big\|\sqrt{f} - p\big\|_{2,m_k} \big\|\sqrt{f} + p\big\|_{2,m_k} \le M\epsilon \cdot \big(2\big\|\sqrt{f}\big\|_{2,m_k} + \epsilon\big),$$

which tends to 0 with $\epsilon \to 0$. This proves (3.14) and yields the assertion. □

The proof of Theorem 1.1 is now easily accomplished.

Proof of Theorem 1.1. Combining formulas (3.12) and (3.11), we obtain for all $p, q \in \Pi^N$ the identity

$$c_k \int_{\mathbb{R}^N} e^{-\Delta_k/2}(V_k p)(x) e^{-\Delta_k/2} q(x) e^{-|x|^2/2} w_k(x)\, dx$$
$$= c_0 \int_{\mathbb{R}^N} e^{-\Delta/2} p(x) e^{-\Delta/2} q(x) e^{-|x|^2/2}\, dx.$$

As $e^{-\Delta_k/2}(V_k p) = V_k(e^{-\Delta/2} p)$ and as we may also replace p by $e^{\Delta/2} p$ and q by $e^{\Delta_k/2} q$ in the above identity, it follows that for all $p, q \in \Pi^N$,

$$c_k \int_{\mathbb{R}^N} V_k p(x) q(x) e^{-|x|^2/2} w_k(x) dx = c_0 \int_{\mathbb{R}^N} p(x) e^{-\Delta/2} e^{\Delta_k/2} q(x) e^{-|x|^2/2} dx.$$

Corollary 3.3 now implies that

$$\int_{\mathbb{R}^N} V_k p(x) q(x) e^{-|x|^2/2} w_k(x) dx \ge 0, \qquad \text{for all } p, q \in \Pi_+^N.$$

For fixed $p \in \Pi_+^N$, we may therefore apply Lemma 3.6 with, let us say, $\alpha = 1/4$ to the function $h(x) := e^{-|x|^2/4} V_k p(x) \in C_b(\mathbb{R}^N)$. This shows that $V_k p(x) \ge 0$ for all $x \in \mathbb{R}^N$ and yields the assertion. □

4. Proof of the main result. We start this section with a short discussion of the algebras A_r ($r > 0$) introduced in (1.3). We should first point out that these are complex algebras, whereas in [D3] only series of real-valued polynomials are considered. It is easily checked that A_r is a subalgebra of the space of functions that are continuous on the ball K_r and real-analytic in its interior. In fact, for real-valued $p \in \mathcal{P}_n^N$ and $i = 1, \ldots, N$, the inequality $\|\partial_i p\|_{\infty,K_1} \le n\|p\|_{\infty,K_1}$ holds as a consequence of the Van der Corput–Schaake inequality (see [D3]). This allows us to differentiate $f = \sum_{n=0}^\infty f_n \in A_r$ termwise and arbitrarily often. The topology of A_r is stronger than the topology induced by the uniform norm on K_r. Notice also that A_r is not closed with respect to $\|\cdot\|_{\infty,K_r}$ and that $A_r \subseteq A_s$ with $\|\cdot\|_{A_r} \ge \|\cdot\|_{A_s}$ for $s \le r$. The following observation is straightforward.

LEMMA 4.1. *$(A_r, \|\cdot\|_{A_r})$ is a commutative Banach-$*$-algebra with the pointwise multiplication of functions, complex conjugation as involution, and unit* 1.

Proof. To show completeness, let $(f^m)_{m\in\mathbb{Z}_+}$ be a Cauchy sequence in A_r. Then for $\epsilon > 0$ there exists an index $m(\epsilon) \in \mathbb{Z}_+$ such that

$$\sum_{n=0}^{\infty} \big\| f_n^m - f_n^{m'} \big\|_{\infty, K_r} < \epsilon, \qquad \text{for } m, m' > m(\epsilon). \tag{4.1}$$

In particular, for each degree n, the homogeneous parts $(f_n^m)_{m\in\mathbb{Z}_+}$ converge uniformly on K_r and hence within $\mathcal{P}_n^N$ to some $g_n \in \mathcal{P}_n^N$. It further follows from (4.1) that

$$\sum_{n=0}^{\infty} \big\| g_n - f_n^m \big\|_{\infty, K_r} < \epsilon, \qquad \text{for } m > m(\epsilon).$$

Therefore, $g := \sum_{n=0}^{\infty} g_n$ belongs to A_r with $\|g - f^m\|_{A_r} \to 0$ for $m \to \infty$. It is also easily checked by a Cauchy-product argument that A_r is an algebra with $\|fg\|_{A_r} \leq \|f\|_{A_r} \cdot \|g\|_{A_r}$ for all $f, g \in A_r$. The rest is clear. $\square$

We next determine the symmetric spectrum of A_r, that is, the subspace of the spectrum $\Delta(A_r)$ given by

$$\Delta_S(A_r) := \big\{\varphi \in \Delta(A_r) : \varphi(\overline{f}) = \overline{\varphi(f)} \quad \text{for all } f \in A_r\big\}.$$

As usual, $\Delta_S(A_r)$ is equipped with the Gelfand topology. For $x \in K_r$, the evaluation homomorphism at x is defined by $\varphi_x : A_r \to \mathbb{C}$, $\varphi_x(f) := f(x)$.

LEMMA 4.2. *$\Delta_S(A_r) = \{\varphi_x : x \in K_r\}$, and the mapping $x \mapsto \varphi_x$ is a homeomorphism from K_r onto $\Delta_S(A_r)$.*

Proof. It is obvious that φ_x belongs to $\Delta_S(A_r)$ for each $x \in K_r$, with $\varphi_x \neq \varphi_y$ for $x \neq y$, and that the mapping $x \mapsto \varphi_x$ is continuous on K_r. It remains to show that each $\varphi \in \Delta_S(A_r)$ is of the form φ_λ with some $\lambda \in K_r$. To this end, put $\lambda_i := \varphi(x_i)$ for $i = 1, \ldots, N$. By symmetry of φ, we have $\lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$. Moreover,

$$|\lambda|^2 = \varphi\big(|x|^2\big) \leq \big\| |x|^2 \big\|_{A_r} = r^2.$$

This shows that $\lambda \in K_r$. By definition of λ, the identity $p(\lambda) = \varphi(p)$ holds for all polynomials $p \in \Pi^N$. The assertion now follows from the density of Π^N in $(A_r, \|\cdot\|_{A_r})$. $\square$

Proof of Theorem 1.2. Fix $x \in \mathbb{R}^N$ and put $r = |x|$. Then the mapping

$$\Phi_x : f \mapsto V_k f(x)$$

is a bounded linear functional on A_r, and Theorem 1.1 implies that it is positive on the dense subalgebra Π^N of A_r, that is, $\Phi_x(|p|^2) \geq 0$ for all $p \in \Pi^N$. Consequently,

Φ_x is a positive functional on the whole Banach-$*$-algebra A_r. Now, by a well-known Bochner-type representation theorem for positive functionals on commutative Banach-$*$-algebras (see, for example, [FD, Theorem 21.2]), there exists a unique measure $\nu_x \in M_b^+(\Delta_S(A_r))$ such that

$$\Phi_x(f) = \int_{\Delta_S(A_r)} \widehat{f}(\varphi)\, d\nu_x(\varphi), \qquad \text{for all } f \in A_r \tag{4.2}$$

with $\widehat{f}$ the Gelfand transform of f. Denote by μ_x the image measure of ν_x under the homeomorphism $\Delta_S(A_r) \to K_r$, $\varphi_x \to x$. Equation (4.2) then becomes

$$V_k f(x) = \int_{\{|\xi|\le|x|\}} f(\xi)\, d\mu_x(\xi), \qquad \text{for all } f \in A_{|x|}.$$

The normalization $V_k 1 = 1$ implies that μ_x is a probability measure on $\{\xi \in \mathbb{R}^N : |\xi| \le |x|\}$. The uniqueness of μ_x among the representing probability measures on $\mathbb{R}^N$ is clear, because identity (1.4) in particular determines the (classical) Fourier-Stieltjes transform of μ_x. Finally, the transformation properties (1.5) follow immediately from the homogeneity-preserving character of V_k on Π^N and the invariance property $V_k \circ g = g \circ V_k$ for all $g \in G$ (see [D3, Theorem 2.3]). This finishes the proof of Theorem 1.2. □

5. Some consequences and applications. In this final section, we discuss only a short selection of implications that arise from the positivity of Dunkl's intertwining operator. We expect that several more useful applications can be found, and it would also be of interest to have an explicit form for V_k for larger classes of reflection groups. In what follows, it is always assumed that $k \ge 0$. The most prominent consequence of Theorem 1.2, as already mentioned in the introduction, is positive-definiteness of Dunkl's generalized exponential kernel. Up to now, this has been known only in the special cases where positivity of V_k is visible from an explicit integral representation. In particular, for the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$ and multiplicity parameter $k > 0$, formula (1.2) shows that

$$K_G(x, iy) = c_k \int_{-1}^{1} e^{itxy}(1-t)^{k-1}(1+t)^k dt = e^{ixy}\, {}_1F_1\big(k, 2k+1, -2ixy\big).$$

The following general result is an immediate consequence of Theorem 1.2 with $f(x) = e^{\langle x,z\rangle}$ and Bochner's theorem.

PROPOSITION 5.1. *For each $z \in \mathbb{C}^N$, the function $x \mapsto K_G(x,z)$ has the Bochner-type representation*

$$K_G(x,z) = \int_{\mathbb{R}^N} e^{\langle \xi, z\rangle} d\mu_x(\xi); \tag{5.1}$$

here the μ_x are the representing measures from Theorem 1.2. In particular, $K_G(x, y) > 0$ for all $x, y \in \mathbb{R}^N$, and for each $x \in \mathbb{R}^N$ the function $y \mapsto K_G(x, iy)$ is positive-definite on $\mathbb{R}^N$.

COROLLARY 5.2. *For each $x \in \mathbb{R}^N$, the generalized Bessel function $y \mapsto J_G(x, iy)$ is positive-definite on $\mathbb{R}^N$.*

We mention that for the group $G = S_3$, this corollary follows from the integral representations in [D5]. From the integral representation (5.1), together with [dJ, Corollary 3.3], we obtain further knowledge about the support of the representing measures μ_x.

COROLLARY 5.3. *The measures μ_x, $x \in \mathbb{R}^N$, satisfy*

(i) $\operatorname{supp}\mu_x$ *is contained in* $\operatorname{co}\{gx,\, g \in G\}$, *the convex hull of the orbit of x under G;*

(ii) $\operatorname{supp}\mu_x \cap \{gx,\, g \in G\} \neq \emptyset$.

Proof. The proof of (i) follows from [dJ, Corollary 3.3]. For the proof of (ii), it is therefore enough to show that

$$\operatorname{supp}\,\mu_x \cap \big\{\xi \in \mathbb{R}^N : |\xi| = |x|\big\} \neq \emptyset.$$

Suppose, to the contrary, that $\operatorname{supp}\mu_x \cap \{\xi \in \mathbb{R}^N : |\xi| = |x|\} = \emptyset$ for some $x \in \mathbb{R}^N$. Then there exists a constant $\sigma \in]0, 1[$ such that $\operatorname{supp}\mu_x \subseteq \{\xi \in \mathbb{R}^N : |\xi| \le \sigma|x|\}$. This leads to the estimation

$$K_G(x, y) = \int_{\{|\xi| \le \sigma|x|\}} e^{\langle \xi, y\rangle} d\mu_x(\xi) \le e^{\sigma|x||y|}$$

for all $y \in \mathbb{R}^N$. On the other hand, [D3, Theorem 3.2] with $z = 0$ says that

$$c_k \int_{\mathbb{R}^N} K_G(x, y) e^{-(|x|^2+|y|^2)/2} w_k(y)\, dy = 1.$$

Now let $r > 0$. As both formulas above remain valid if x is replaced by rx, it follows that

$$1 \le c_k \int_{\mathbb{R}^N} e^{-(|rx|^2+|y|^2)/2} e^{\sigma|rx||y|} w_k(y)\, dy \le c_k \int_{\mathbb{R}^N} e^{(\sigma-1)(r^2|x|^2+|y|^2)/2} w_k(y)\, dy,$$

which tends to 0 with $r \to \infty$, a contradiction. □

This result implies useful estimates for K_G and its derivatives, which partially sharpen those of [dJ].

COROLLARY 5.4. *Let $\nu \in \mathbb{Z}_+^N$ and $|\nu| = \nu_1 + \cdots + \nu_N$. Then, for all $x \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$,*

$$\left|\partial_z^\nu K_G(x, z)\right| \le |x|^{|\nu|} \cdot e^{\max_{g\in G}\langle gx, \operatorname{Re} z\rangle}.$$

In particular, $|K_G(x, iy)| \le 1$ for all $x, y \in \mathbb{R}^N$.

Proof. This is immediate by differentiation under the integral in (5.1) and the fact that, for $\xi \in \text{co}\{gx,\, g \in G\}$ and $z \in \mathbb{C}^N$, the estimation $\text{Re}\langle \xi, z\rangle \le \max_{g\in G}\langle gx, \text{Re}z\rangle$ holds. □

Remark. Our proof of Theorem 1.2 did not involve any results from [dJ]. Without referring to [dJ], the following weaker estimates for K_G and its derivatives are an immediate consequence of (5.1). Let $\nu \in \mathbb{Z}_+^N$. Then, for all $x \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$,

$$\left|\partial_z^\nu K_G(x,z)\right| \le |x|^{|\nu|} \cdot e^{|x|\cdot|\text{Re}\, z|}.$$

We give two further applications of our positivity results. The first one concerns a question from approximation theory stated in [X1], namely, the summability of orthogonal series in generalized spherical harmonics. In fact, the study of generalized spherical harmonics associated with a finite reflection group and a multiplicity function $k \ge 0$ was one of the starting points of Dunkl's theory in [D3] and has been extended in [X1] and [X3]. Many results for classical spherical harmonics carry over to these spherical k-harmonics, where harmonizity is now meant with respect to Δ_k. In particular, there is a natural decomposition of $\mathcal{P}_n^N|_{S^{N-1}}$ into subspaces of k-spherical harmonics, which are orthogonal in $L^2(S^{N-1}, w_k(x)dx)$. In [X1], Xu studies the Cesàro summability of generalized Fourier expansions with respect to an orthonormal basis of spherical k-harmonics. Recall that a sequence $\{s_n\}_{n\in\mathbb{Z}_+}$ is called Cesàro-summable of order δ to s, ((C,δ)-summable to s, for short) if

$$\frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k \longrightarrow s \qquad \text{with } n \to \infty.$$

The following result is proven in [X1] under the requirement that the intertwining operator V_k is positive on Π^N. Theorem 1.1 now assures its validity for all $k \ge 0$.

THEOREM 5.5. *Let $f : S^{N-1} \to \mathbb{C}$ be continuous, and let $\{s_n\}$ denote the sequence of partial sums in the expansion of f as a Fourier series with respect to a fixed orthonormal basis of spherical k-harmonics. Then $\{s_n\}$ is uniformly (C,δ)-summable over S^{N-1} to f, provided $\delta > \gamma + N/2 - 1$ with $\gamma = \sum_{\alpha\in R_+} k(\alpha)$.*

Another application of our positivity result is related with probabilistic aspects of Dunkl's theory and concerns generalizations of the classical moment functions to the Dunkl setting. The definition of the classical moments of probability measures on $\mathbb{R}^N$ is based on the monomial "moment functions" $m_\nu(x) = x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_N^{\nu_N}$, $x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}_+^N$. They have many applications in the study of sums of independent random variables. Recently, a concept of Markov kernels and Markov processes that are homogeneous with respect to a given Dunkl transform was developed in [RV1]. In this context, generalized moment functions on $\mathbb{R}^N$ provide a useful tool. They generalize the classical monomial moment functions $m_\nu(x)$ and are defined as the unique analytic coefficients in the expansion

$$K_G(x,y) = \sum_{\nu \in \mathbb{Z}_+^N} \frac{m_{k,\nu}(x)}{\nu!} y^\nu \qquad \big(x \in \mathbb{R}^N, y \in \mathbb{C}^N\big).$$

From the definition of K_G, it follows that

$$m_{k,\nu}(x) = V_k\big(x^\nu\big), \qquad \text{for } \nu \in \mathbb{Z}_+^N.$$

Theorem 1.2, in particular, implies the following useful relations for the generalized moment functions, which are obvious only in the classical case (again, we assume $k \geq 0$):

$$\big|m_{k,\nu}(x)\big| \leq |x|^{|\nu|} \quad \text{and} \quad 0 \leq m_{k,\nu}(x)^2 \leq m_{k,2\nu}(x), \quad \text{for all } x \in \mathbb{R}^N,\ \nu \in \mathbb{Z}_+^N.$$

The first inequality is clear from the support properties of the measures μ_x, while the second one follows from Jensen's inequality. Among the applications of these moments, we mention the construction of martingales from Dunkl-type Markov processes (for details, refer to [RV1]).

Acknowledgements. It is a great pleasure to thank Charles F. Dunkl, Marcel de Jeu, and Michael Voit for several valuable comments and discussions. The main part of this paper was written while I held a Forschungsstipendium of the DFG at the University of Virginia, Charlottesville, Virginia, U.S.A. I would also like to express my gratitude to members of the Department of Mathematics at the University of Virginia in Charlottesville for their hospitality.

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