Beta Distributions and Sonine Integrals for Bessel Functions on Symmetric Cones

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There exist several multivariate extensions of the classical Sonine integral representation for Bessel functions of some index $\mu + \nu$ with respect to such functions of lower index μ. For Bessel functions on matrix cones, Sonine formulas involve beta densities $\beta_{\mu,\nu}$ on the cone and trace already back to Herz. The Sonine representations known so far on symmetric cones are restricted to continuous ranges $\Re\mu, \Re\nu > \mu_0$, where the involved beta densities are probability measures and the limiting index $\mu_0 \geq 0$ depends on the rank of the cone. It is zero only in the one-dimensional case, but larger than zero in all multivariate cases. In this paper, we study the extension of Sonine formulas for Bessel functions on symmetric cones to values of ν below the critical limit μ_0. This is achieved by an analytic extension of the involved beta measures as tempered distributions. Following recent ideas by A. Sokal for Riesz distributions on symmetric cones, we analyze for which indices the obtained beta distributions are still measures. At the same time, we characterize the indices for which a Sonine formula between the related Bessel functions exists. As for Riesz distributions, there occur gaps in the admissible range of indices, which are determined by the so-called Wallach set.

1. Introduction

Consider the one-variable normalized Bessel functions

$$j_\alpha(z) := {}_0F_1(\alpha + 1; -z^2/4) \qquad (\alpha \in \mathbb{C} \setminus \{-1, -2, \ldots\}),$$

which for $\alpha > -1/2$ have the well-known Laplace integral representation

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DOI: 10.1111/sapm.12217

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$$j_\alpha(z) := \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{izx}(1-x^2)^{\alpha-1/2}\, dx \qquad (z \in \mathbb{C}). \tag{1}$$

For half integers $\alpha = p/2 - 1$ with $p \geq 2$, formula (1) may be regarded as a Harish-Chandra integral representation for the spherical functions of the Euclidean space $\mathbb{R}^p$ with $SO(p)$-action. It is also well known that for $\alpha > -1$ and $\beta > 0$, $j_{\alpha+\beta}$ can be expressed in terms of j_α as a Sonine integral (formula (3.4) in Askey [1]):

$$j_{\alpha+\beta}(z) = 2\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 j_\alpha(zx)x^{2\alpha+1}(1-x^2)^{\beta-1}\, dx. \tag{2}$$

This follows easily by power series expansion of both sides and is a particular case of classical integral representations for one-variable hypergeometric functions. Notice that for $\beta = 0$, formula (2) degenerates in a trivial way. As $j_{-1/2}(z) = \cos z$, formula (1) is actually a special case of (2). For some background on these classical formulas we also refer to the monograph [2].

We now ask for which indices $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\alpha + \beta > -1$ there actually exists a Sonine integral representation

$$j_{\alpha+\beta}(z) = \int_0^\infty j_\alpha(zx)d\mu_{\alpha,\beta}(x)$$

with a positive measure $\mu_{\alpha,\beta}$. It is easily seen that this is only possible if $\beta \geq 0$. Indeed, if such a representation with $\beta < 0$ would exist, we could combine it with (2) for the parameter pairs $(\alpha+\beta, -\beta)$ instead of (α, β). This would lead to a Sonine integral representation of j_α in terms of j_α with a measure different from the point measure δ_1, which is impossible by the injectivity of the Hankel transform of bounded measures. In particular, a Laplace representation such as (1) with a positive representing measure exists precisely for $\alpha \geq -1/2$.

In this paper we study extensions of Sonine-type integral representations for Bessel functions of matrix argument and more generally, on Euclidean Jordan algebras and the associated symmetric cones. The general Jordan algebra setting includes the Jordan algebras of Hermitian matrices over the (skew) fields $\mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$ as important special cases. Bessel functions in this setting trace back to the fundamental work of Herz [3], which was motivated by questions in number theory and multivariate statistics. For example, Bessel functions of matrix argument occur naturally in the explicit expression of noncentral Wishart distributions ([4, 5]). They are imbedded in a theory of hypergeometric functions on Euclidean Jordan algebras, which are defined as hypergeometric series in terms of so-called spherical polynomials. Integral representations of Bessel functions play an important role in the analysis on symmetric cones and are closely related to Laplace transforms. For details and a general background see Refs. [3, 6, 7].

For various aspects concerning the rich harmonic analysis associated with Bessel functions on symmetric cones, we also refer to [8–12].

Let us now describe our results in more detail. To avoid abstract notation, we restrict in this introduction to the case where the underlying Jordan algebra is the space $V = H_q(\mathbb{F})$ of $q \times q$ Hermitian matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. The (real) dimension of V is

$$n = q + \frac{d}{2}q(q-1) \text{ with } d = dim_{\mathbb{R}}\mathbb{F} \in \{1, 2, 4\},$$

and V is associated with the symmetric cone $\Omega = \Omega_q(\mathbb{F})$ of positive definite matrices over $\mathbb{F}$. The Bessel functions on V are defined by

$$\mathcal{J}_\mu(x) := {}_0F_1(\mu; -x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(x), \qquad x \in V.$$

Here, the sum is over all partitions of length q, $(\mu)_\lambda$ is a generalized Pochhammer symbol, and the Z_λ are the (renormalized) spherical polynomials of Ω, see Section 2 for the details.

For indices $\mu, \nu \in \mathbb{C}$ with $\Re\mu, \Re\nu > n/q - 1 =: \mu_0$, the associated beta measure on Ω is defined by

$$d\beta_{\mu,\nu}(x) := \frac{1}{B_\Omega(\mu,\nu)} \Delta(x)^{\mu - n/q} \Delta(I_q - x)^{\nu - n/q} \cdot 1_{\Omega_e}(x)dx, \tag{3}$$

where $B_\Omega(\mu, \nu)$ is the beta function associated with Ω, Δ denotes the determinant polynomial on V, and $\Omega_e = \{x \in \Omega : x < I_q\}$. For real $\mu, \nu > \mu_0$, $\beta_{\mu,\nu}$ is a probability measure. It is known for $H_q(\mathbb{R})$ and easy to see in the general Jordan setting (Theorem 1) that the Bessel functions $\mathcal{J}_\mu$ have the following integral representation of Sonine type generalizing the one-variable case (2): For indices $\mu, \nu \in \mathbb{C}$ with $\Re\mu, \Re\nu > \mu_0$,

$$\mathcal{J}_{\mu+\nu}(r) = \int_{\Omega_e} \mathcal{J}_\mu(\sqrt{r}s\sqrt{r}) d\beta_{\mu,\nu}(s) \quad \text{ for all } r \in \Omega\,. \tag{4}$$

Notice that $\mu_0 = 0$ if $q = 1$, but μ_0 is larger than zero if $q > 1$, and in this case formula (4) is not available in the range $\Re\nu > 0$. This is to some extent unexpected and makes the situation more interesting in higher dimension than in the one-variable case.

Let us mention at this point that there is a broad literature on beta probability distributions on matrix cones and their relevance in statistics, in particular in relation with Wishart distributions, see [5, 13–16] as well as the survey [17]. For some applications in mathematical physics and representation theory, see, for example, the survey [18] and references therein. To our knowledge, beta distributions have so far only rarely been considered for indices beyond the critical value μ_0. References in this case are [19] and [20], where certain discrete indices are considered for which the associated beta measures become singular.

Our aim in this paper is to study the extendability of the Sonine formula (4) to larger ranges of the index ν. This will be achieved by analytic extension (with respect to ν) of the beta probability measures as distributions, and a detailed analysis when these distributions are still measures.

Our method is motivated by the theory of Gindikin for Riesz distributions associated with symmetric cones (see [21, 22], chapter 7 of [6], and the recent simplifications in [23]). Let us recall the basic facts, again for the case $V = \mathbb{H}_q(\mathbb{F})$. For indices $\alpha \in \mathbb{C}$ with $\Re\alpha > \mu_0 = n/q - 1$, the Riesz probability distributions R_α on V are defined by

$$R_\alpha(\varphi) = \frac{1}{\Gamma_\Omega(\alpha)} \int_\Omega \varphi(x)\Delta(x)^{\alpha - n/q}\, dx,$$

where Γ_Ω is the gamma function associated with Ω and Δ denotes the determinant on V. According to the results by Gindikin, the measures R_α have a (weakly) analytic extension to distributions $R_\alpha \in \mathcal{D}'(V)$ for all $\alpha \in \mathbb{C}$. This means that the mapping $\alpha \mapsto R_\alpha(\varphi)$ is analytic on $\mathbb{C}$ for each $\varphi \in \mathcal{D}(V)$. The distributions R_α are tempered and their support is contained in the closed cone $\overline{\Omega} \subset V$. Moreover, R_α is a positive measure exactly if α belongs to the Wallach set

$$\left\{0, \frac{d}{2}, \ldots, (q-1)\frac{d}{2} = \mu_0\right\} \cup \,]\mu_0, \infty[\,. \tag{5}$$

A simple proof for the necessity of this condition is given in [23].

We consider the beta measures $\beta_{\mu,\nu}$ in (3) also on measures on $\overline{\Omega}$ or as compactly supported distributions on V of order zero. Their extension to a larger range of the index ν is more involved than in the Riesz case. Indeed, the range of extension we are able to obtain depends on $\Re\mu$. To become precise, consider the open half planes

$$E_k := \{\nu \in \mathbb{C} : \Re\nu > \mu_0 - k\}, \qquad k \in \mathbb{N}_0,$$

where $E_0 \subset E_k \subset E_{k+1}$. It is easily checked that for fixed $\mu \in E_0$, the mapping

$$E_0 \to \mathcal{D}'(V), \qquad \nu \mapsto \beta_{\mu,\nu} \tag{6}$$

is (weakly) analytic, i.e., $\nu \mapsto \beta_{\mu,\nu}(\varphi)$ is analytic for each test function $\varphi \in \mathcal{D}(V)$. Recall that compactly supported distributions on V extend continuously to $\mathcal{E}(V)$, the space $C^\infty(V)$ with its usual locally convex topology. In Theorem 3, we prove the following:

Theorem A. For $k \in \mathbb{N}_0$ and $\mu \in \mathbb{C}$ with $\Re\mu > \mu_0 + kq + 1$, the mapping (6) has a unique analytic extension from E_0 to E_k with values in $\mathcal{D}'(V)$. The distributions $\beta_{\mu,\nu}$ obtained in this way are compactly supported with support contained in $\overline{\Omega_e}$. Moreover, the Bessel functions $\mathcal{J}_\mu$ and $\mathcal{J}_{\mu+\nu}$ associated with Ω are related by the Sonine formula

$$\mathcal{J}_{\mu+\nu}(r) = \beta_{\mu,\nu}\left(\mathcal{J}_\mu^r\right) \qquad \textit{for all } r \in \Omega, \tag{7}$$

where $\mathcal{J}_\mu^r(x) = \mathcal{J}_\mu(\sqrt{r}x\sqrt{r}) \in \mathcal{E}(V)$.

We next ask when the distributions $\beta_{\mu,\nu}$ are actually complex Radon measures or even probability measures. The latter requires that $\mu, \nu \in \mathbb{R}$. The following result is contained in Corollary 2:

Theorem B. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $k \in \mathbb{N}$, and $\mu \in \mathbb{R}$ with $\mu > \mu_0 + kq + 3/2$. Then for $\nu \in E_k$, the following statements are equivalent:

(1) ν is contained in the Wallach set (5).
(2) The distribution $\beta_{\mu,\nu}$ is a positive measure.
(3) There exists a probability measure $\beta \in M^1(\overline{\Omega})$ such that

$$\mathcal{J}_{\mu+\nu}(r) = \int_{\overline{\Omega}} \mathcal{J}_\mu(\sqrt{r}s\sqrt{r})d\beta(s) \quad \text{ for all } r \in \overline{\Omega}.$$

In this case, the measure $\beta \in M^1(\overline{\Omega})$ in (3) is unique, and $\beta = \beta_{\mu,\nu}$.

We prove this result, as well as a counterpart for complex measures, actually in the more general setting of symmetric cones with Peirce constant $d = 1$ or 2. This also includes the Lorentz cones in $\mathbb{R} \times \mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R}^3$. Without restriction on the Peirce constant d, our results (contained in Theorem 5) are somewhat less complete, but still give interesting restrictions on the indices, which are necessary to assure that $\beta_{\mu,\nu}$ is a measure. This in particular concerns the case of quaternionic matrix cones.

We finally mention that the spherical polynomials and thus also the Bessel functions on Euclidean Jordan algebras depend only on the eigenvalues of their argument. Considered as functions of the spectra, the spherical polynomials can be identified with Jack polynomials whose index depends on d; this was first observed by Macdonald [24]. There is a natural theory of hypergeometric expansions in terms of Jack polynomials (see [25, 26]), which encompasses the theory on symmetric cones and is closely related with rational Dunkl theory, cf. [10] and Lemma 2. Riesz distributions in this setting are Selberg densities, and their analytic extension and consequences for integral representations of Bessel functions of Dunkl type will be studied in a forthcoming paper.

The organization of this paper is as follows: The next section gives a short survey about Bessel functions on Euclidean Jordan algebras. In Section 3, we discuss several facts concerning the beta measures $\beta_{\mu,\nu}$ and the Sonine formula for $\Re\mu, \Re\nu > \mu_0$. Section 4 contains the main results of this paper on the analytic extension of the beta measures and their consequences for Sonine integral representations of the Bessel functions.

2. Bessel functions on Euclidean Jordan algebras

In this section, we present some basic facts and notions on symmetric cones and associated Bessel functions. We illustrate the general notions by the important example of matrix cones. For a background on symmetric cones and Jordan algebras we refer to the monograph [6].

A real algebra V of finite dimension n is called a (real) Jordan algebra if its multiplication $(x, y) \mapsto x \cdot y = xy$, satisfies

$$xy = yx \text{ and } x(x^2y) = x^2(xy) \qquad \textit{for all } x, y \in V.$$

A real Jordan algebra is called Euclidean, if it has an identity $e \in V$ and a scalar product $(\,.\,|\,.\,)$ such that $(xy|z) = (y|xz)$ for all $x, y, z \in V$. It is called simple if it contains no nontrivial ideals. Let V be a Euclidean Jordan algebra. Then the topological interior Ω of the set $\{x^2 : x \in V\}$ is a symmetric cone. We recall that a symmetric cone Ω in a Euclidean vector space V is an open cone $\Omega \subseteq V$, which is proper (i.e., $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$), self-dual, and homogeneous, in the sense that the automorphism group of Ω acts in a transitive way. Let G denote the identity component of this automorphism group and $K = G \cap O(V)$. Then already G is transitive on Ω, and there exist points $e \in \Omega$ such that K is the stabilizer of e in G. Thus, $\Omega \cong G/K$, which is a Riemannian symmetric space. With e fixed as above, there is a natural product in V for which V becomes a Euclidean Jordan algebra with identity element e and such that $\overline{\Omega} = \{x^2 : x \in V\}$ (see Theorem III.3.1 of [6]). Every symmetric cone is a product of irreducible ones, and in the above way, the simple Euclidean Jordan algebras correspond to the irreducible symmetric cones.

Example. Let $\mathbb{F}$ be one of the (skew) fields $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with real dimension $d = 1, 2,$ or 4, respectively. The usual conjugation in $\mathbb{F}$ is denoted by $t \mapsto \overline{t}$ and the real part of $t \in \mathbb{F}$ by $\Re t = \frac{1}{2}(t + \overline{t})$. Let

$$H_q(\mathbb{F}) := \left\{x \in M_q(\mathbb{F}) : x = x^*\right\}$$

be the space of Hermitian $q \times q$-matrices over $\mathbb{F}$, where $x^* = \overline{x}^t$. We consider $H_q(\mathbb{F})$ as a Euclidean vector space with scalar product $\langle x, y\rangle = \Re\,\mathrm{Tr}(xy)$, where $\mathrm{Tr}(x) = \sum_{i=1}^q x_{ii}$ denotes the usual trace. With this scalar product and the Jordan product $x \cdot y = \frac{1}{2}(xy + yx)$, the space $\mathbb{H}_q(\mathbb{F})$ becomes a simple Euclidean Jordan algebra with identity $e = I_q$. The associated symmetric cone is given by

$$\Omega_q(\mathbb{F}) = \{x \in H_q(\mathbb{F}) : x \text{ positive definite}\}.$$

The pairs (G, K) are in this case $(GL_q^+(\mathbb{R}), SO_q(\mathbb{R}))$, $(GL_q(\mathbb{C}), U_q(\mathbb{C}))$, and $(GL_q(\mathbb{H}), U_q(\mathbb{H}))$, respectively, where the action of G on $\Omega_q(\mathbb{F})$ is given by $r \mapsto grg^*$. Notice that this reduces to conjugation when restricted to K.

Let now V be a simple Euclidean Jordan algebra and Ω the associated symmetric cone. A Jordan frame in V is a complete set $c_1, \dots, c_q \in V$ of orthogonal primitive idempotents, i.e.,

$$c_i^2 = c_i,\, c_i c_j = 0 \quad \text{if } \ i \neq j,\, c_1 + \cdots c_q = e.$$

The group K acts transitively on the set of Jordan frames, and their common cardinality q is called the rank of V (or Ω). The rank of V is related to its real dimension n via

$$n = q + \frac{d}{2}q(q-1),$$

where d is the so-called Peirce constant, see p. 71 of [6]. Each $x \in V$ admits a decomposition $x = k\sum_{i=1}^n \xi_i c_i$ with $k \in K$ and unique real numbers $\xi_1 \geq \ldots \geq \xi_q$, which are called the eigenvalues of x (section VI.2. of [6]). The Jordan trace and determinant of x are defined by

$$\mathrm{tr}(x) = \sum \xi_i, \quad \Delta(x) = \prod \xi_i\, .$$

Both functions are K-invariant.

Example. In the Jordan algebras $H_q(\mathbb{F})$, a natural Jordan frame consists of the matrices $c_i = E_{ii}$, $1 \leq i \leq q$ (having entry 1 in position (i, i) and 0 else). The eigenvalues of $x \in \mathbb{H}_q(\mathbb{F})$ are the usual (right) eigenvalues, and Δ coincides with the usual determinant if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, while for $\mathbb{F} = \mathbb{H}$ it is given by the so-called Moore determinant, see [27].

The simple Euclidean Jordan algebras are classified. Up to isomorphism, there are the series $H_q(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the exceptional Jordan algebra $H_3(\mathbb{O})$ over the octonions, as well as the Jordan algebras $V = \mathbb{R} \times \mathbb{R}^{q-1}$, $q \geq 3$, with Jordan product $(\lambda, u) \cdot (\mu, v) = (\lambda\mu + \langle u, v\rangle, \lambda v + \mu u)$, where $\langle\, .\, ,\, .\, \rangle$ denotes the usual Euclidean scalar product on $\mathbb{R}^{q-1}$. In this case, Ω is the Lorentz cone

$$\Lambda_q = \{(\lambda, u) \in \mathbb{R} \times \mathbb{R}^{q-1} : \lambda^2 - \langle u, u\rangle > 0, \lambda > 0\}.$$

The following table summarizes these Jordan algebras and their structure data.

V	Ω	Rank	d	$n = \dim \mathrm{V}$
$H_q(\mathbb{R})$	$\Omega_q(\mathbb{R})$	q	1	$\frac{1}{2}q(q+1)$
$H_q(\mathbb{C})$	$\Omega_q(\mathbb{C})$	q	2	q^2
$H_q(\mathbb{H})$	$\Omega_q(\mathbb{H})$	q	4	$q(2q-1)$
$H_3(\mathbb{O})$	$\Omega_3(\mathbb{O})$	3	8	27
$\mathbb{R} \times \mathbb{R}^{q-1}$	Λ_q	2	$q-2$	q

In this paper, we always assume that V is a simple Euclidean Jordan algebra with associated symmetric cone Ω and that the scalar product of V is given by

$$\langle x, y \rangle = \mathrm{tr}\,(xy),$$

where xy denotes the Jordan product. (This is no loss of generality, cf. Section III.4 of [6].) We need some further notation: On V we use the partial orderings

$$x < y :\Longleftrightarrow y - x \in \Omega \quad \text{and} \quad x \leq y :\Longleftrightarrow y - x \in \overline{\Omega}.$$

The quadratic representation P of V is defined by

$$P(x) := 2L(x)^2 - L(x^2), \; x \in V,$$

where $L(x) \in End(V)$ denotes the left multiplication by x on V, i.e., $L(x)y = xy$. For the Jordan algebras $H_q(\mathbb{F})$, the quadratic representation is given by

$$P(x)y = xyx,$$

where on the right side, the product is the usual matrix product (see [6], Section II.3). An element $x \in V$ is invertible in V if and only if $P(x)$ is invertible, and in this case $P(x^{-1}) = P(x^{-1})$. We finally mention an important invariance property: Let $r, s \in \Omega$. Then, by Lemma XIV.1.2 of [6], there exists $k \in K$ such that

$$P(\sqrt{r}\,)s = kP(\sqrt{s}\,)r. \tag{8}$$

For normalizations we need the gamma and beta function associated with the cone Ω ([6], chapter VII.1). They are defined by

$$\Gamma_\Omega(z) = \int_\Omega e^{-\mathrm{tr}(x)} \Delta(x)^{z-n/q}\, dx,$$

$$B_\Omega(z, w) = \int_{\Omega_e} \Delta(x)^{z-n/q} \Delta(e-x)^{w-n/q} dx,$$

where dx is the Lebesgue measure on V induced by the scalar product $\langle\, .\, ,\, .\,\rangle$ and

$$\Omega_e = \{x \in \Omega : x < e\}.$$

Both integrals are absolutely convergent for all $z, w \in \mathbb{C}$ with $\Re z, \Re w > \mu_0$, where

$$\mu_0 := \frac{n}{q} - 1 = \frac{d}{2}(q-1). \tag{9}$$

By corollary VII.1.3 of [6], Γ_Ω can be expressed in terms of the classical gamma function as

$$\Gamma_\Omega(z) = (2\pi)^{(n-q)/2} \prod_{j=1}^{q} \Gamma\left(z - \frac{d}{2}(j-1)\right). \tag{10}$$

Moreover,

$$B_\Omega(z, w) = \frac{\Gamma_\Omega(z)\Gamma_\Omega(w)}{\Gamma_\Omega(z+w)},$$

see Theorem VII.1.7 of [6]. Notice that Γ_Ω is meromorphic on $\mathbb{C}$ without zeros, and its set of poles is

$$\left\{0, \frac{d}{2}, \ldots, (q-1)\frac{d}{2} = \mu_0\right\} - \mathbb{N}_0\,.$$

The basic functions for the harmonic analysis on the cone Ω and building blocks for related special functions are the so-called spherical polynomials. For their definition, recall that a q-tuple $(\lambda_1, \ldots, \lambda_q) \in \mathbb{N}_0^q$ is called a partition if $\lambda_1 \geq \ldots \geq \lambda_q$. We write $\lambda \geq 0$ for short. The spherical polynomials associated with Ω are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{N}_0^q$ and are defined by

$$\Phi_\lambda(x) = \int_K \Delta_\lambda(kx)dk, \quad x \in V,$$

where dk is the normalized Haar measure on K and Δ_λ is the generalized power function on V,

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1-\lambda_2}\Delta_2(x)^{\lambda_2-\lambda_3} \cdot \ldots \cdot \Delta_q(x)^{\lambda_q}.$$

Here, the $\Delta_i(x)$ are the principal minors of $\Delta(x)$, see Section VII.1 of [6] for details. For the matrix algebras $V = H_q(\mathbb{F})$, the $\Delta_i(x)$ are just the usual principal minors. The power function Δ_λ is a homogeneous polynomial of degree $|\lambda| = \lambda_1 + \ldots + \lambda_q$, which is positive on Ω. For convenience, we work with renormalized spherical polynomials $Z_\lambda = c_\lambda \Phi_\lambda$, where the constants $c_\lambda > 0$ are such that

$$(\operatorname{tr} x)^k = \sum_{|\lambda|=k} Z_\lambda(x) \qquad \text{for } k \in \mathbb{N}_0\,, \tag{11}$$

see Section XI.5. of [6]. The Z_λ are K-invariant and thus depend only on the eigenvalues of their argument. In view of (8),

$$Z_\lambda(P(\sqrt{r}\,)s) = Z_\lambda(P(\sqrt{s}\,)r) \quad \text{for all } r, s \in \Omega.$$

We mention that for each symmetric cone the associated spherical polynomials are given in terms of Jack polynomials ([28]). More precisely, it was observed by Macdonald in [24] that for $x \in V$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$,

$$Z_\lambda(x) = C_\lambda^\alpha(\xi) \quad \text{with} \quad \alpha = \frac{2}{d}, \tag{12}$$

where the C_λ^α are the Jack polynomials of q variables and index $\alpha > 0$, normalized such that

$$(\xi_1 + \cdots + \xi_q)^k = \sum_{|\lambda|=k} C_\lambda^\alpha(\xi). \tag{13}$$

The C_λ^α are homogeneous of degree $|\lambda|$ and symmetric in their arguments.

For a simple Euclidean Jordan algebra with Peirce constant d, the associated ($\mathcal{J}$-) Bessel functions are defined on $V^{\mathbb{C}}$ as

$$\mathcal{J}_\mu(z) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda\, |\lambda|!} Z_\lambda(z), \tag{14}$$

with the generalized Pochhammer symbol

$$(\mu)_\lambda := \prod_{j=1}^{q} \left(\mu - \frac{d}{2}(j-1)\right)_{\lambda_j},$$

cf. [6]. Here it is assumed that $\mu \in \mathbb{C}$ with $(\mu)_\lambda \neq 0$ for all $\lambda \geq 0$, which is for example satisfied if $\Re\mu > \mu_0$. The function $\mathcal{J}_\mu$ is analytic on $V^{\mathbb{C}}$.

In the rank one case $q = 1$, we have $\Omega =]0, \infty[$, $\mu_0 = 0$, and with the notation of the introduction, the Bessel function $\mathcal{J}_\mu$ satisfies

$$\mathcal{J}_\mu\left(\frac{z^2}{4}\right) = j_{\mu-1}(z).$$

For the matrix cones $\Omega_q(\mathbb{F})$, it is well known (see, e.g., [10] or [11]) that for certain indices μ, the associated Bessel functions $\mathcal{J}_\mu$ occur as spherical functions of flat symmetric spaces. In fact, fix some integer $p \geq q$ and denote by $M_{p,q} = M_{p,q}(\mathbb{F})$ the space of $p \times q$-matrices over $\mathbb{F}$. Consider the Gelfand pair $(M_{p,q} \rtimes U_p, U_p)$, where the unitary group $U_p := U_p(\mathbb{F})$

acts on $M_{p,q}$ by left multiplication. The double coset space $M_{p,q} \rtimes U_p//U_p$ may be naturally identified with the orbit space $M_{p,q}^{U_p}$, which is in turn homeomorphic with the closed cone

$$\overline{\Omega_q(\mathbb{F})} = \{x \in H_q(\mathbb{F}) : x \text{ positive semidefinite}\}$$

via the mapping

$$U_p.x \mapsto \sqrt{x^*x}\,.$$

Considered as functions on the cone $\overline{\Omega_q(\mathbb{F})}$, the bounded spherical functions of the Gelfand pair $(M_{p,q} \rtimes U_p, U_p)$ are precisely given by the Bessel functions

$$\varphi_s^\mu(r) = \mathcal{J}_\mu\left(\frac{1}{4}rs^2r\right), \quad s \in \overline{\Omega_q(\mathbb{F})} \quad \text{with } \mu = pd/2.$$

As spherical functions, these Bessel functions have an integral representation of Harish-Chandra type. Analytic continuation with respect to μ leads to the following integral representation, see [10], Section 3.3 as well as [3] for $\mathbb{F} = \mathbb{R}$.

Proposition 1. *Let $d = dim_{\mathbb{R}}\mathbb{F}$ and $\mu \in \mathbb{C}$ with $\Re\mu > d(q-1/2)$. Then for all $x \in H_q(\mathbb{F})$, the Bessel function of index μ associated with $H_q(\mathbb{F})$ satisfies*

$$\mathcal{J}_\mu(x^2) = \frac{1}{\kappa_\mu}\int_{B_q(\mathbb{F})} e^{-2i\langle v,x\rangle}\Delta(I - v^*v)^{\mu-1-d(q-1/2)}dv \tag{15}$$

with $B_q(\mathbb{F}) = \{v \in M_q(\mathbb{F}) : v^*v < I_q\}$, the scalar product $\langle x,y\rangle = \Re\,\mathrm{Tr}(x^*y)$ on $M_q(\mathbb{F})$, and

$$\kappa_\mu = \int_{B_q(\mathbb{F})} \Delta(I - v^*v)^{\mu-1-d(q-1/2)}dv.$$

Formula (15) generalizes the Laplace representation (1) to higher rank.

3. Beta measures and the Sonine formula on symmetric cones

Throughout this paper, $M_b(X)$ is the set of bounded, regular, complex Borel measures on a locally compact Hausdorff space X and $M^1(X)$ the set of all probability measures in $M_b(X)$.

As before, let V be a simple Euclidean Jordan algebra and Ω the associated symmetric cone. For $\mu, \nu \in \mathbb{C}$ with $\Re\mu, \Re\nu > \mu_0 = \frac{n}{q} - 1$, we introduce the beta measures

$$d\beta_{\mu,\nu}(x) := \frac{1}{B_\Omega(\mu,\nu)}\Delta(x)^{\mu-n/q}\Delta(e-x)^{\nu-n/q}\cdot 1_{\Omega_e}(x)dx \in M_b(\Omega), \tag{16}$$

which we also consider as measures on V with compact support $\overline{\Omega_e}$. The $\beta_{\mu,\nu}$ are probability measures for $\mu,\nu\in]\mu_0,\infty[$. We here do not use the notion "beta distributions," as we study (tempered) distributions and want to avoid any misunderstanding.

Our starting point is the following Sonine formula (2) for Bessel functions on Euclidean Jordan algebras, which generalizes formula (4) announced in the Introduction. For $\mathbb{H}_q(\mathbb{R})$ it goes already back to [3] (formula (2.6')).

THEOREM 1. *Let V be a simple Euclidean Jordan algebra. Then for all $\mu,\nu\in\mathbb{C}$ with $\Re\mu,\Re\nu>\mu_0$ and $x\in V$,*

$$\mathcal{J}_{\mu+\nu}(x)=\int_{\Omega_e}\mathcal{J}_\mu(P(\sqrt{r}\,)x)d\beta_{\mu,\nu}(r).$$

Proof. For indices $\alpha,\beta_1,\beta_2\in\mathbb{C}$ with $\Re\alpha,\Re\beta_i>\mu_0$ consider the hypergeometric function

$${}_1F_2(\alpha;\beta_1,\beta_2;z):=\sum_{\lambda\geq 0}\frac{(\alpha)_\lambda}{(\beta_1)_\lambda(\beta_2)_\lambda\,|\lambda|!}\,Z_\lambda(z),$$

which is holomorphic on $V^{\mathbb{C}}$ (Proposition XV.1.1. of [6]). Consider first $x\in\Omega_e$. As G is transitive on Ω, there exists $g\in G$ with $x=ge$. According to Proposition XV.1.4 of [6],

$${}_1F_2(\mu;\beta,\mu+\nu;ge)=\frac{1}{B_\Omega(\mu,\nu)}\int_{\Omega_e}{}_0F_1(\mu;gr)d\beta_{\mu,\nu}(r).$$

With $\beta=\mu$, this becomes

$$\mathcal{J}_{\mu+\nu}(-x)={}_0F_1(\mu+\nu;ge)=\frac{1}{B_\Omega(\mu,\nu)}\int_{\Omega_e}\mathcal{J}_\mu(-gr)d\beta_{\mu,\nu}(r).$$

By Theorem III.5.1 of [6], g can be written in polar form as $g=P(s)k$ with $s\in\Omega,k\in K$. Thus, $x=ge=P(s)e=s^2$ and $g=P(\sqrt{x}\,)k$. The measures $\beta_{\mu,\nu}$ and the function $\mathcal{J}_\mu$ are K-invariant. Thus, in view of (8),

$$\mathcal{J}_{\mu+\nu}(-x)=c\int_{\Omega_e}\mathcal{J}_\mu(-P(\sqrt{x})r)d\beta_{\mu,\nu}(r)=c\int_{\Omega_e}\mathcal{J}_\mu(-P(\sqrt{r})x)d\beta_{\mu,\nu}(r)$$

with $c=B_\Omega(\mu,\nu)^{-1}$. The last formula extends analytically to all $x\in V$. ■

We conclude with some remarks concerning the matrix cones $\Omega_q(\mathbb{F})$.

Remark 1.

(1) It follows form the analysis in Section 3 of [10] that for $\mu>\mu_0$ and $\nu=\mu_0$, there exist degenerated beta probability measures $\beta_{\mu,\nu}$ on $\Omega_q(\mathbb{F})_e$ such that the mapping $\nu\mapsto\beta_{\mu,\nu}$ becomes weakly continuous

on $[\mu_0, \infty[$. In this way, Theorem 1 extends to indices $\nu \geq \mu_0$ and $\mu > \mu_0$. In [19] and [20] some singular beta measures are studied for $\mathbb{F} = \mathbb{R}$.

(2) Formula (15) may be regarded as a special case of Theorem 1 with the parameters $(qd/2, \mu - qd/2)$ instead of (μ, ν). To check this, we first recall from formula (3.4) of [10] that for $x \in H_q(\mathbb{F})$,

$$\mathcal{J}_{qd/2}(x^*x) = \int_{U_q} e^{-2i\langle u,x\rangle}\, du, \tag{17}$$

where du is the normalized Haar measure on $U_q = U_q(\mathbb{F})$ and the scalar product is that of Proposition 1. We also need the integral formula for the polar decomposition of $M_q(\mathbb{F})$ (see [8] or Section 3.1 of [10]):

$$\int_{M_q(\mathbb{F})} f(x)dx \;=\; C\int_{U_q}\int_{\Omega_q(\mathbb{F})} f(u\sqrt{r})\, \Delta(r)^{qd/2-n/q}\, dr du$$

with some constant $C = C_q > 0$. Let $\Re\mu > d(q-1/2)$. Then identity (15) becomes

$$\begin{aligned}
\mathcal{J}_\mu(x^*x) =&\; C\int_{B_q(\mathbb{F})} e^{-2i\langle v,x\rangle}\,\Delta(I - v^*v)^{\mu-1-d(q-1/2)}dv \\
&= C\int_{\Omega_q(\mathbb{F})_e}\left(\int_{U_q} e^{-2i\langle u\sqrt{r},x\rangle}du\right)\Delta(r)^{qd/2-n/q}\,\Delta(I-r)^{\mu-1-d(q-1/2)}\, dr \\
&= C\int_{\Omega_q(\mathbb{F})_e} \mathcal{J}_{qd/2}(\sqrt{r}\, x^*x\sqrt{r})\,\Delta(r)^{qd/2-n/q}\,\Delta(I-r)^{\mu-1-d(q-1/2)}dr.
\end{aligned}$$

Put $\nu := \mu - qd/2$ and notice that $\Re\nu > \mu_0$. In view of the normalization $\mathcal{J}_\mu(0) = 1$ this is equivalent to

$$\mathcal{J}_{\nu+qd/2}(s) = \int_{\Omega_q(\mathbb{F})_e} \mathcal{J}_{qd/2}(\sqrt{r}\, s\sqrt{r})\, d\beta_{qd/2,\nu}(r)$$

for all $s \in \overline{\Omega_q(\mathbb{F})}$, which is a special case of Theorem 1 as stated.

In the next section we construct an extension of Theorem 1 with respect to the ranges of the indices μ, ν. Before that, we mention a special case in the matrix cone setting, which follows from group theory:

PROPOSITION 2. *Let $\mu = pd/2$ and $\nu = \widetilde{p}d/2$ with integers $p \geq q$ and $\widetilde{p} \geq 0$. Then there exists a unique probability measure $\widetilde{\beta}_{\mu,\nu}$ on the matrix cone $\overline{\Omega_q(\mathbb{F})}$ such that for all $r \in \overline{\Omega_q(\mathbb{F})}$,*

$$\mathcal{J}_{\mu+\nu}(r) = \int_{\overline{\Omega_q(\mathbb{F})}} \mathcal{J}_\mu(\sqrt{r}s\sqrt{r})\, d\widetilde{\beta}_{\mu,\nu}(s).$$

Proof. For brevity we omit $\mathbb{F}$ in the notation of the relevant matrix spaces. Recall that the functions

$$\varphi_s^\mu(r) = \mathcal{J}_\mu\left(\frac{1}{4}rs^2r\right) \qquad s, r \in \overline{\Omega_q}$$

can be naturally identified with the bounded spherical functions of the Gelfand pair $(M_{p,q} \rtimes U_p, U_p)$, and the $\varphi_s^{\mu+\nu}$ with those of the pair $(M_{p+\widetilde{p},q} \rtimes U_{p+\widetilde{p}}, U_{p+\widetilde{p}})$. In Section 6 of [11] it is deduced from this characterization by a positive definiteness argument that the functions $\phi_s^{\mu+\nu}$ have a representation

$$\varphi_s^{\mu+\nu}(r) = \int_{\overline{\Omega_q}} \varphi_t^\mu(r) d\alpha_{\mu,\nu;s}(t),$$

with a unique probability measure $\alpha_{\mu,\nu;s}$, see formula (6.1) of [11]. (There is a misprint in [11]: the indices p_1, p_2 of the groups G_{p_1}, G_{p_2} are mixed up). With $s = 2I_q$ this immediately yields our claim. ■

4. Beta distributions and extension of the Sonine formula

In this section, we present an analytic extension of Theorem 1 with respect to the parameters μ, ν by distributional methods. Let us first fix some notation. For an open subset U of some finite dimensional vector space V over $\mathbb{R}$, denote by $\mathcal{D}(U)$ the space of compactly supported C^∞-functions on U and by $\mathcal{D}'(U)$ the space of distributions on U. We write $\mathcal{E}(U)$ for the space $C^\infty(U)$ with its usual Fréchet space topology; its dual $\mathcal{E}'(U)$ coincides with the space of compactly supported distributions on U. Further, $\mathcal{D}'^k(U)$ denotes the space of distributions of order $\leq k$. Recall that $\mathcal{D}'^0(U)$ consists of those distributions, which are given by a (not necessarily bounded) complex Radon measure on U. In particular, each locally integrable function $f \in L^1_{loc}(U)$ defines a regular distribution $T_f \in \mathcal{D}'^0(U)$ via $T_f(\varphi) = \int_U \varphi(x)f(x)\,dx$.

We also consider regular distributions associated with functions $f_\lambda \in L^1_{loc}(U)$ where f_λ depends analytically on some parameter $\lambda \in D$ with some open, connected set $D \subseteq \mathbb{C}$. We ask for which parameters the associated distributions $T_\lambda := T_{f_\lambda} \in \mathcal{D}'^0(U)$ admit extensions to distributions of order 0 on V. We need the following observation from Sokal [23]; see Lemmas 2.1, 2.2, and Proposition 2.3 there.

LEMMA 1. *Let $U \subseteq V$ and $D \subseteq \mathbb{C}$ be as above, and let*

$$F : U \times D \to \mathbb{C}, \qquad (x, \lambda) \to f_\lambda(x) := F(x, \lambda)$$

be a continuous function such that $F(x, .)$ is analytic in D for each $x \in U$. Define $T_\lambda \in \mathcal{D}'^0(U)$ by

$$T_\lambda(\varphi) = \int_U \varphi(x) f_\lambda(x) dx.$$

Then the following hold:

(1) The map $\lambda \mapsto T_\lambda$, $D \to \mathcal{D}'(U)$ is (weakly) analytic in the sense that $\lambda \mapsto T_\lambda(\varphi)$ is analytic for all $\varphi \in \mathcal{D}(U)$.
(2) Let $D_0 \subseteq D$ be a nonempty open set, and let $\lambda \mapsto \widetilde{T}_\lambda$, $D \to \mathcal{D}'(V)$ be an analytic map such that for each $\lambda \in D_0$, the distribution $\widetilde{T}_\lambda$ extends the distribution T_λ from U to V. Then $\widetilde{T}_\lambda$ extends T_λ for each $\lambda \in D$. Moreover, for each $\lambda \in D$ with $\widetilde{T}_\lambda \in \mathcal{D}'^0(V)$ one has $f_\lambda \in L^1_{loc}(\overline{U})$, that is f_λ is integrable over each sufficiently small neighborhood in V of any point $x \in \overline{U}$. In particular, if $\overline{U}$ is compact, then f_λ is the density of a bounded measure.

We start our considerations on symmetric cones with an injectivity result, which is of interest on its own and will be of importance in the sequel. Let again Ω be an irreducible symmetric cone and V the associated simple Euclidean Jordan algebra. Following [9], we consider the Schwartz space of the closed cone $\overline{\Omega} \subset V$,

$$\mathcal{S}(\overline{\Omega}) := \left\{ f \in C^\infty(\overline{\Omega}) : \|f\|_{\alpha,\beta,\overline{\Omega}} := \|x^\beta \partial^\alpha f\|_{\infty,\overline{\Omega}} < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^q \right\}.$$

Here $C^\infty(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$, which are smooth on Ω and whose partial derivatives extend continuously to $\overline{\Omega}$. We note that each $f \in C^\infty(\overline{\Omega})$ can be extended to a smooth function on V. This follows by the Whitney extension theorem (see [29], Theorem 2.6 and Proposition 2.16), because $\overline{\Omega}$ is a semialgebraic (and hence subanalytic) subset of V with dense interior. Therefore,

$$\mathcal{S}(\overline{\Omega}) = \left\{ f\big|_{\overline{\Omega}} : f \in C^\infty(V), \ \|f\|_{\alpha,\beta,\overline{\Omega}} < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^q \right\}.$$

The same approximation argument as for the density of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ (see for instance [30]) shows that the space

$$\mathcal{D}(\overline{\Omega}) := \{ f\big|_{\overline{\Omega}} : f \in \mathcal{D}(V) \}$$

is dense in $\mathcal{S}(\overline{\Omega})$ with respect to the seminorms $\|.\|_{\alpha,\beta,\overline{\Omega}}$. We denote by $\mathcal{S}'(\overline{\Omega})$ the dual of the locally convex space $\mathcal{S}(\overline{\Omega})$, i.e., the space of tempered distributions on $\overline{\Omega}$. Let $\Re\mu > d(q-1)+1 = 2\mu_0+1$. Then, according to Theorem 2.2. of [9], the Hankel transform

$$f \mapsto \widehat{f}^{\mu}, \qquad \widehat{f}^{\mu}(r) = \int_{\Omega} f(s)\mathcal{J}_\mu(P(\sqrt{s}\,)r)\Delta(s)^{\mu-n/q}ds$$

is a homeomorphism of $\mathcal{S}(\overline{\Omega})$. Actually, this is stated in [9] for $\Re\mu > \mu_0$, but the proof requires absolute convergence of the inverse Laplace integral representing the Bessel function, which is guaranteed only for $\Re\mu > d(q-1)+1$, see [6], Proposition XV.2.2. The stated homeomorphism property allows to deduce the following injectivity result.

THEOREM 2. *Let $\mu \in \mathbb{C}$ with $\Re\mu > 2\mu_0+1$. For $r \in \overline{\Omega}$ define*

$$\mathcal{J}_\mu^r(x) := \mathcal{J}_\mu(P(\sqrt{r}\,)x) \in \mathcal{E}(V).$$

Suppose that $T \in \mathcal{E}'(V)$ has compact support which is contained in $\overline{\Omega}$. Then the following hold:

(1) If $T(\mathcal{J}_\mu^r) = 0$ for all $r \in \overline{\Omega}$, then $T = 0$.
(2) Suppose that $\mathcal{J}_\mu^r$ is bounded for each $r \in \overline{\Omega}$, and that there is a bounded measure $\beta \in M_b(\overline{\Omega})$ (also considered as a measure on V) such that

$$T(\mathcal{J}_\mu^r) = \int_{\overline{\Omega}} \mathcal{J}_\mu^r(s)\,d\beta(s) \qquad \textit{for all } r \in \overline{\Omega}\,.$$

Then, $T = \beta$.

Proof. We first observe that T belongs to $\mathcal{S}'(\overline{\Omega})$. Indeed, choose a compact, convex and connected subset $K \subset \overline{\Omega}$ containing the support of T, and let k denote the order of T. Then, according to Theorem 2.3.10 of [31], there exists a constant $C > 0$ such that for all $\varphi \in \mathcal{E}(V)$,

$$|T(\varphi)| \leq C \sum_{|\alpha|\leq k} \|\partial^\alpha\varphi\|_{\infty,K}\,. \tag{18}$$

This shows that $T \in \mathcal{S}'(\overline{\Omega})$ and that the inclusion

$$\{T \in \mathcal{E}'(V) : \operatorname{supp} T \subset \overline{\Omega}\} \to \mathcal{S}'(\overline{\Omega})$$

is injective. Now let $\varphi \in \mathcal{D}(\overline{\Omega})$. It is easy to check that the mapping $r \mapsto \mathcal{J}_\mu^r$, $\overline{\Omega} \to \mathcal{E}(V)$, is continuous. Therefore,

$$\int_{\operatorname{supp}\varphi} \mathcal{J}_\mu^r\,\varphi(r)\Delta(r)^{\mu-n/q}dr$$

is well defined as an integral with values in $\mathcal{E}(V)$ (see, e.g., Section 3 of [30]), and we obtain

$$T(\widehat{\varphi}^{\mu}) = T\left(\int_{\mathrm{supp}\varphi} \mathcal{J}_{\mu}^{r}\,\varphi(r)\Delta(r)^{\mu-n/q}dr\right) = \int_{\mathrm{supp}\varphi} T\left(\mathcal{J}_{\mu}^{r}\right)\varphi(r)\Delta(r)^{\mu-n/q}dr. \tag{19}$$

In the situation of part (1), it follows that $T(\widehat{\varphi}^{\mu}) = 0$. As $\mathcal{D}(\overline{\Omega})$ is dense in $\mathcal{S}(\overline{\Omega})$ and the Hankel transform is a homeomorphism of $\mathcal{S}(\overline{\Omega})$, this implies that $T = 0$ as an element of $\mathcal{S}'(\overline{\Omega})$, which yields assertion (1). In the situation of part (2), identity (19) leads to

$$T(\widehat{\varphi}^{\mu}) = \int_{\overline{\Omega}} \widehat{\varphi}^{\mu}(s)\,d\beta(s),$$

and the same argument as above shows that $T = \beta$. ■

The following estimate implies that already for $\Re\mu \geq \mu_0 + 1/2$, the Bessel functions $\mathcal{J}_{\mu}^{r}$ with $r \in \overline{\Omega}$ are indeed bounded on Ω as required in part (2) of the above theorem.

LEMMA 2. *Let $\Re\mu \geq \mu_0 + 1/2$. Then,*

$$|\mathcal{J}_{\mu}(x)| \leq \sqrt{2^q q!} \qquad \textit{for all } x \in V.$$

For further bounds on $\mathcal{J}$-Bessel functions see [12] and [32]; they do however not cover Lemma 2 above. Our proof of this Lemma will be based on the connection between $\mathcal{J}_{\mu}$ and Bessel functions of Dunkl type associated with the root system

$$B_q = \{\pm e_i, 1 \leq i \leq q\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq q\} \subset \mathbb{R}^q$$

as established in [10]. The reflection group associated with B_q is the hyperoctahedral group $G = S_q \ltimes \mathbb{Z}_2^q$. For a general background on Dunkl theory see [33, 34] and the references cited there. Let $E_k^B : \mathbb{C}^q \times \mathbb{C}^q \to \mathbb{C}$ denote the Dunkl kernel associated with B_q and multiplicity $k = (k_1, k_2)$, where k_1 and k_2 are the values of k on the roots $\pm e_i$ and $\pm e_i \pm e_j$, respectively. Here, k belongs to a regular multiplicity set $K^{reg} \subset \mathbb{C}^2$ which contains those k with $\Re k \geq 0$, i.e., $\Re k_i \geq 0$. The associated Bessel function is given by

$$J_k^B(z, w) = \frac{1}{|G|}\sum_{g\in G} E_k^B(z, gw).$$

It is G-invariant in both arguments and satisfies $J_k^B(\lambda z, w) = J_k^B(z, \lambda w)$ for all $\lambda \in \mathbb{C}$. If $\Re k \geq 0$, then by [33],

$$\left|J_k^B(i\xi, \eta)\right| \leq \sqrt{|G|} \qquad \text{for all } \xi, \eta \in \mathbb{R}^q. \tag{20}$$

Proof of Lemma 2. Let $x \in \overline{\Omega}$ with eigenvalues $\xi = (\xi_1, \dots, \xi_q) \in \mathbb{R}^q$ and suppose that $\Re\mu \geq \mu_0 + 1/2$. According to Corollary 4.6 of [10],

$$\mathcal{J}_\mu(x^2) = J_k^B(2i\xi, \mathbf{1}) \quad \text{with } k = (\mu - \mu_0 - 1/2, d/2), \mathbf{1} = (1, \dots, 1).$$

Estimate (20) implies the stated estimate of $\mathcal{J}_\mu(x)$ with $x \in \overline{\Omega}$. By the K-invariance of $\mathcal{J}_\mu$ it extends to all $x \in V$. ■

Theorem 2 together with the integral representation of Theorem 1 can be used to derive the following composition result for beta measures.

LEMMA 3. *Let $\mu, \nu_1, \nu_2 \in \mathbb{C}$ with $\Re\mu > 2\mu_0 + 1$ and $\Re\nu_i > \mu_0$. Then, for the mapping*

$$C : \Omega_e \times \Omega_e \to \Omega_e, \qquad (r, s) \mapsto P(\sqrt{s}\,)r$$

the push forward (or image measure)

$$\beta_{\mu,\nu_1} \circ \beta_{\mu+\nu_1,\nu_2} := C\big(\beta_{\mu,\nu_1} \otimes \beta_{\mu+\nu_1,\nu_2}\big) \in M_b(\Omega_e)$$

satisfies

$$\beta_{\mu,\nu_1} \circ \beta_{\mu+\nu_1,\nu_2} = \beta_{\mu,\nu_1+\nu_2}.$$

Proof. We recall that for $r \in \Omega$, $P(r)$ is a positive operator and contained in $G = G(\Omega)_0$. (The latter follows from Proposition III.2.2. of [6] and the continuity of P.) Thus, for $r, s \in \Omega_e$, we have $0 < P(\sqrt{s}\,)r < P(\sqrt{s}\,)e = s < e$, which confirms that $C(r, s) \in \Omega_e$. By Theorem 1 we obtain for $r \in \Omega$

$$\begin{aligned}\mathcal{J}_{\mu+\nu_1+\nu_2}(r) &= \int_{\Omega_e} \mathcal{J}_{\mu+\nu_1}(P(\sqrt{s}\,)r)\, d\beta_{\mu+\nu_1,\nu_2}(s) \\ &= \int_{\Omega_e}\int_{\Omega_e} \mathcal{J}_\mu(P(\sqrt{t}\,)P(\sqrt{s}\,)r) d\beta_{\mu,\nu_1}(t) d\beta_{\mu+\nu_1,\nu_2}(s).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{\Omega_e} \mathcal{J}_\mu(P(\sqrt{r})y) d\beta_{\mu,\nu_1} \circ \beta_{\mu+\nu_1,\nu_2}(y) = \\ &\qquad = \int_{\Omega_e}\int_{\Omega_e} \mathcal{J}_\mu(P(\sqrt{r})P(\sqrt{s})t) d\beta_{\mu,\nu_1}(t) d\beta_{\mu+\nu_1,\nu_2}(s).\end{aligned}$$

Now consider the argument of $\mathcal{J}_\mu$. By the polar decomposition of G (Theorem III.5.1 of [6]), there exist $k \in K$ and $x \in \Omega$ such that $P(\sqrt{r})P(\sqrt{s})P(\sqrt{t}) = P(x)k$ and therefore

$$P(\sqrt{r})P(\sqrt{s})t = P(\sqrt{r})P(\sqrt{s})P(\sqrt{t})e = P(x)ke = P(x)e = x^2$$

and

$$P(\sqrt{t})P(\sqrt{s})r = P(\sqrt{t})P(\sqrt{s})P(\sqrt{r})e = (P(x)k)^* e = k^{-1}P(x)e = k^{-1}x^2.$$

As $\mathcal{J}_\mu$ is K-invariant, we obtain

$$\mathcal{J}_{\mu+\nu_1+\nu_2}(r) = \int_{\Omega_e} \mathcal{J}_\mu(P(\sqrt{r})y)d\beta_{\mu,\nu_1} \circ \beta_{\mu+\nu_1,\nu_2}(y).$$

If we compare this with Theorem 1 and use Theorem 2(1), the result follows. ■

We now turn to the distributional extension of beta measures on symmetric cones. We apply Lemma 1 to the Jordan algebra V, the relatively compact set $U := \Omega_e$, $\lambda = \nu$ and the densities

$$f_\nu(x) := \frac{\Gamma_\Omega(\mu+\nu)}{\Gamma_\Omega(\mu)\Gamma_\Omega(\nu)} \Delta(x)^{\mu-n/q} \Delta(e-x)^{\nu-n/q} \tag{21}$$

of the beta measures $\beta_{\mu,\nu}$ from (16) on U, where the index μ is suppressed. We consider the open half planes

$$E_k := \{\nu \in \mathbb{C} : \Re\nu > \mu_0 - k\}, \qquad k \in \mathbb{N}_0.$$

Note that $E_0 \subset E_k \subset E_{k+1}$. It is clear that for fixed μ with $\Re\mu > \max(\mu_0, k)$ and $x \in U$, the function $\nu \mapsto f_\nu(x)$ is analytic on E_k. Moreover, by Lemma 1(1), the mapping

$$E_0 \to \mathcal{D}'(V), \qquad \nu \mapsto \beta_{\mu,\nu} \tag{22}$$

is analytic for fixed μ with $\Re\mu > \mu_0$. To apply the approach of Sokal [23] and Lemma 1(2), we construct distributions $\beta_{\mu,\nu} \in \mathcal{D}'(V)$ for $\nu \in E_k$. We here use ideas of Gindikin [21], [22] for Riesz distributions; see Chapter VII of [6].

THEOREM 3. *Fix $k \in \mathbb{N}_0$ and an index $\mu \in \mathbb{C}$ with $\Re\mu > \mu_0 + kq + 1$.*

(1) For $\nu \in E_k$ there exists a unique distribution $\beta_{\mu,\nu} \in \mathcal{D}'(V)$ such that the mapping

$$E_k \to \mathcal{D}'(V), \qquad \nu \mapsto \beta_{\mu,\nu}$$

is a (weakly) analytic extension of the mapping (22) from E_0 to E_k.

(2) The distributions $\beta_{\mu,\nu}$ from part (1) belong to $\mathcal{D}'^{kq}(V)$ and have compact support, which is contained in $\overline{\Omega_e}$. In particular, $\beta_{\mu,\nu}(\varphi)$ is well defined for each $\varphi \in \mathcal{E}(V)$ and $\nu \to \beta_{\mu,\nu}(\varphi)$ is analytic on E_k for fixed $\varphi \in \mathcal{E}(V)$.

(3) For each $\nu \in E_k$, the Bessel function $\mathcal{J}_{\mu+\nu}$ satisfies

$$\mathcal{J}_{\mu+\nu}(r) = \beta_{\mu,\nu}(\mathcal{J}_\mu^r) \qquad \textit{for all } r \in \overline{\Omega}. \tag{23}$$

Proof. We first note that for $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ with $\Re\alpha > \mu_0 + m + 1 = m + n/q$, the function on V defined by

$$g_\alpha(x) := \begin{cases} \Delta(x)^{\alpha - n/q} & \text{for} \quad x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

is contained in $C^m(V)$. Moreover, by Proposition VII.1.4 and the arguments on p. 133 of [6], the functions g_α are related to the linear differential operator $\Delta(\frac{\partial}{\partial x})$ of order q via

$$\Delta\left(\frac{\partial}{\partial x}\right) g_\alpha = \frac{\Gamma_\Omega(\alpha)}{\Gamma_\Omega(\alpha - 1)} g_{\alpha-1}\,. \tag{24}$$

This leads to part (1) as follows: The case $k = 0$ is trivial. For $k \geq 1$ and $\nu \in E_k$ we define a distribution $\beta_{\mu,\nu} \in \mathcal{D}'(V)$ by

$$\beta_{\mu,\nu}(\varphi) := \frac{\Gamma_\Omega(\mu+\nu)}{\Gamma_\Omega(\mu)\Gamma_\Omega(\nu+k)} \int_V \Delta\left(\frac{\partial}{\partial x}\right)^k \big(\varphi(x) g_\mu(x)\big) \cdot g_{\nu+k}(e - x)\, dx. \tag{25}$$

Notice for this definition that $g_\mu \in C^{kq}(V)$ by our assumptions. Moreover, the above expression is analytic in $\nu \in E_k$. It is now easy to see from (24) that definition (25) is consistent with (22). Indeed, for $\Re\nu + k > \mu_0 + qk + 1$ we may carry out integration by parts. As

$$\Delta\left(-\frac{\partial}{\partial x}\right)^k g_{\nu+k}(e - x) = \frac{\Gamma_\Omega(\nu+k)}{\Gamma_\Omega(\nu)} g_\nu(e - x), \tag{26}$$

we obtain that (25) coincides with the beta measure $\beta_{\mu,\nu}$ for such ν, and by analyticity with respect to ν, it coincides for all $\nu \in E_0$. Part (2) is clear from formula (25). Finally, identity (23) holds for all $\nu \in E_0$ according to Theorem 1, and as both sides are analytic in $\nu \in E_k$, it extends to all $\nu \in E_k$. This proves part (3). ∎

Similar to Riesz distributions in Theorem VII.2.2 of [6], one can extend analytic relations for the beta measures (16) to distributions with parameters μ, ν as in Theorem 3. For instance, (16) immediately leads to:

LEMMA 4. *Let $\mu, \nu \in \mathbb{C}$ be as in Theorem 3 for some $k \in \mathbb{N}_0$. Then*

$$\Delta(e - x) \cdot \beta_{\mu,\nu} = \left(\prod_{j=0}^{q-1} \frac{\nu - jd/2}{\mu + \nu - jd/2}\right) \cdot \beta_{\mu,\nu+1}\,, \tag{27}$$

$$\Delta(x) \cdot \beta_{\mu,\nu} = \left(\prod_{j=0}^{q-1} \frac{\mu - jd/2}{\mu + \nu - jd/2}\right) \cdot \beta_{\mu+1,\nu}\,.$$

The following result concerning the existence of Sonine representations is an immediate consequence of Theorem 2, Lemma 2, and Theorem 3(3).

COROLLARY 1. *Let $k \in \mathbb{N}_0$ and $\Re\mu > \max(\mu_0 + kq + 1, 2\mu_0 + 1)$. Then for $\nu \in E_k$, the following are equivalent:*

(1) The distribution $\beta_{\mu,\nu}$ is a complex measure.
(2) There exists a bounded complex measure $\beta \in M_b(\overline{\Omega})$ such that $\mathcal{J}_{\mu+\nu}$ has the Sonine representation

$$\mathcal{J}_{\mu+\nu}(r) = \int_{\overline{\Omega}} \mathcal{J}_\mu(P\sqrt{s})r)d\beta(s) \quad \text{for all } r \in \overline{\Omega}.$$

In this case, the measure β in (2) is unique and given by $\beta = \beta_{\mu,\nu}$.

We now investigate for which $\nu \in E_k$ the distribution $\beta_{\mu,\nu}$ (with $\Re\mu > \mu_0 + kq + 1$) is actually a complex measure, i.e., contained in $\mathcal{D}'^0(V)$, or even a positive measure.

It is well known (see Section VII.3 of [6]) that the Riesz distributions, which are given for $\Re\alpha > \mu_0$ by

$$R_\alpha(\varphi) = \frac{1}{\Gamma_\Omega(\alpha)} \int_V \varphi(x) g_\alpha(x)\, dx,$$

have a (weakly) analytic extension with respect to α to distributions R_α for all $\alpha \in \mathbb{C}$. These distributions are tempered and supported in $\overline{\Omega}$. Moreover, R_α is a positive measure exactly if α belongs to the Wallach set

$$\left\{0, \frac{d}{2}, \ldots, (q-1)\frac{d}{2} = \mu_0\right\} \cup \,]\mu_0, \infty[\,.$$

A simple proof for the necessity of this condition is given in [23]. By the same method, it is also shown there that R_α is a locally finite complex Borel measure exactly if α belongs to the set

$$W_{q,d} := \left\{0, \frac{d}{2}, \ldots, (q-1)\frac{d}{2}\right\} \cup E_0.$$

The following sufficient condition for beta distributions is a consequence of the known results for Riesz distributions.

THEOREM 4. *Let $k \in \mathbb{N}_0$, $\Re\mu > \mu_0 + kq + 1$, and let $\nu \in E_k \cap W_{q,d}$. Then $\beta_{\mu,\nu}$ belongs to $\mathcal{D}'^0(V)$, i.e., $\beta_{\mu,\nu}$ is a compactly supported complex Borel measure. In particular, $\beta_{\mu,0} = \delta_e$, provided that $0 \in E_k$.*

If in addition μ and ν are real, then $\beta_{\mu,\nu}$ is a probability measure.

Proof. For the normalization, recall from Theorem 3 that $\nu \to \beta_{\mu,\nu}(1)$ is analytic on E_k. Therefore, $\beta_{\mu,\nu}(1) = 1$ for all $\nu \in E_k$.

Now let $\nu \in E_k \cap W_{q,d}$. Then the distribution

$$\Delta\left(\frac{\partial}{\partial x}\right)^k g_{\nu+k} = \Gamma_\Omega(\nu+k)R_\nu$$

is a locally finite complex Borel measure. We claim that for $\varphi \in \mathcal{D}(V)$,

$$\beta_{\mu,\nu}(\varphi) = \frac{\Gamma_\Omega(\mu+\nu)}{\Gamma_\Omega(\mu)} \cdot R_\nu^\theta(\varphi g_\mu), \tag{28}$$

where R_ν^θ denotes the image measure (pushforward) of the Riesz measure R_ν under the mapping $\theta : V \to V, x \mapsto e - x$. Indeed, for $\psi \in \mathcal{D}(V)$ we have

$$\begin{aligned}\Gamma_\Omega(\nu+k)R_\nu^\theta(\psi) &= \left(\Delta\left(\frac{\partial}{\partial x}\right)^k g_{\nu+k}\right)(\psi\circ\theta)\\ &= \int_V \Delta\left(\frac{\partial}{\partial x}\right)^k \psi(x)\cdot g_{\nu+k}(e-x)dx.\end{aligned}$$

An approximation argument shows that this identity also holds for $\psi \in C_c^{kq}(V)$, as we may approximate ψ by a net $(\psi_\epsilon)_{\epsilon>0} \subseteq \mathcal{D}(V)$ such that $\partial^\alpha\psi_\epsilon \to \partial^\alpha\psi$ uniformly on V for all $|\alpha| \leq kq$ and the supports of the ψ_ϵ stay in a fixed relatively compact neighborhood of $\operatorname{supp}\psi$. Putting $\psi = \varphi g_\mu \in C_c^{kq}(V)$ and using formula (25), we thus obtain (28). From identity (28) it is now obvious that $\beta_{\mu,\nu}$ is a complex measure, which is even positive if μ, ν are real. As $R_0 = \delta_0$, it is also immediate that $\beta_{\mu,0} = \delta_e$. ■

Remark 2.

(1) The supports of the Riesz measures R_ν with $\nu \in W_{q,d}$, are known (see Propositions VII.2.3 of [6]). Identity (28) then easily gives the supports of the corresponding measures $\beta_{\mu,\nu}$. In particular, $\beta_{\mu,\nu}$ is a point measure only if $\nu = 0$.
(2) Theorem 4 is in accordance with Proposition 2 in the group cases for μ sufficiently large. It is not clear whether for small parameters $\mu = pd/2$ and $\nu = \widetilde{p}d/2$, the probability measures $\widetilde{\beta}_{\mu,\nu}$ from Proposition 2 can be obtained as distributions via analytic extension as above. Nevertheless, we from now on denote $\widetilde{\beta}_{\mu,\nu}$ by $\beta_{\mu,\nu}$.

We are now aiming at necessary conditions on the indices under which the beta distributions $\beta_{\mu,\nu}$ on a symmetric cone are actually measures. Such conditions will also imply that the existence of an integral representation as in the above corollary requires nontrivial restrictions on the indices of the Bessel functions involved. As a first step, we extend Lemma 3 for beta measures to a larger set of parameters for which the involved beta distributions are measures according to Theorem 4 or Proposition 2. The same proof as in Lemma 3 implies:

LEMMA 5. *Let* $\Re\mu > 2\mu_0 + 1$ *and* $\nu_1, \nu_2 \in \mathbb{C}$ *be such that the beta measures* $\beta_{\mu,\nu_1}, \beta_{\mu+\nu_1,\nu_2}$, $\beta_{\mu,\nu_1+\nu_2}$ *exist. Then, in notation of Lemma 3,*

$$\beta_{\mu,\nu_1} \circ \beta_{\mu+\nu_1,\nu_2} := C\big(\beta_{\mu,\nu_1} \otimes \beta_{\mu+\nu_1,\nu_2}\big) = \beta_{\mu,\nu_1+\nu_2}.$$

We do not know whether it is possible to derive a converse statement of Theorem 4 by following the approach of Gindikin for Riesz distributions; see Section VII.3 of [6]. We use a different approach by Sokal [23] (specifically, Lemma 1), by which we easily obtain the following result:

THEOREM 5. *Let* $k \in \mathbb{N}_0$, $\Re\mu > \mu_0 + kq + 1$, *and* $\nu \in E_k$. *If* $\beta_{\mu,\nu} \in \mathcal{D}'^0(V)$, *i.e.,* $\beta_{\mu,\nu}$ *is a complex measure, then*

$$\nu \in \left(\left\{0, \frac{d}{2}, \ldots, (q-1)\frac{d}{2}\right\} - \mathbb{N}_0\right) \cup E_0\,.$$

In particular, $\nu + l \in W_{q,d}$ *for some* $l \in \mathbb{N}_0$.

Proof. We apply Lemma 1(2) to $D_0 := E_0$, $D := E_k$ and $U = \Omega_e$ and obtain that the beta density f_ν given by (21) belongs to $L^1_{loc}(\overline{\Omega_e})$. It is well known that

$$x \mapsto \Delta(x)^{\mu - n/q}\Delta(e-x)^{\nu - n/q}$$

is contained in $L^1_{loc}(\overline{\Omega_e})$ precisely for $\nu \in E_0$; see, for instance, Lemma 3.4 of [23]. Therefore, either $\nu \in E_0$ or

$$\frac{\Gamma_\Omega(\mu+\nu)}{\Gamma_\Omega(\mu)\Gamma_\Omega(\nu)} = 0,$$

where the latter just means that Γ_Ω has a pole in ν. ∎

We conjecture that under the conditions of Theorem 5, it should be even true, similar as for Riesz distributions, that $\nu \in W_{q,d}$. Our next statement confirms this conjecture under the assumption that $d \in \{1, 2\}$. This covers the important case of the matrix cones $\Omega_q(\mathbb{F})$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, as well as the Lorentz cones Λ_3 and Λ_4.

THEOREM 6. *Suppose that* $d \in \{1, 2\}$. *Let* $k \in \mathbb{N}$, $\Re\mu > \mu_0 + kq + 3/2$, *and* $\nu \in E_k$. *If* $\beta_{\mu,\nu}$ *is a complex measure, then* $\Re\nu \geq 0$. *If in addition* μ *is real and* $\beta_{\mu,\nu}$ *is a positive measure, then* $\nu \in [0, \infty[$.

Proof. Notice first in the present situation, $\mu_0 + kq + 3/2 > 2\mu_0 + 1$. Now, suppose that $\beta_{\mu,\nu}$ is a complex measure. In view of Theorem 5 it suffices to consider $\nu = rd/2 - l$ with $r = 0, \ldots, q-1$ and $l > 0$ an integer. We may also assume that $\mu_0 - k < \nu \leq \mu_0 - k + 1$, and therefore

$\nu = \mu_0 - k + \alpha$ with $\alpha \in \{1/2, 1\}$. We now assume that $\nu < 0$ and claim that $\beta_{\mu+\nu,-\nu}$ is a complex measure. In fact, our assumptions imply that

$$\Re(\mu + \nu) > (\mu_0 + kq + 3/2) + (\mu_0 - k + 1/2) = 2\mu_0 + k(q-1) + 2 > \mu_0 .$$

If $-\nu > \mu_0$, our claim is obvious. Let us consider the case $-\nu \leq \mu_0$. Then $-\nu \in E_{2\mu_0+2-k}$ with $2\mu_0 + 2 - k \in \mathbb{N}$. As $\nu < 0$ and $\mu_0, \nu \in \frac{1}{2}\mathbb{N}_0$, it follows that $k \geq \mu_0 + 1$ and, therefore

$$\Re(\mu+\nu) > 2\mu_0 + k(q-1) + 2 \geq 2\mu_0 + (\mu_0 + 1)(q-1) + 2 \geq \mu_0 + q(2\mu_0 + 2 - k) + 1 .$$

Moreover, as $0 < -\nu \leq \mu_0 = (q-1)d/2$ and $d = 1$ or $d = 2$, we conclude that $-\nu \in \{d/2, \ldots, (q-1)d/2\} \subset W_{q,d}$. (Here the assumption $d \in \{1, 2\}$ has been used for the first time.) We may now apply Theorem 4 to the pair $(\mu + \nu, -\nu)$ and obtain again that $\beta_{\mu+\nu,-\nu}$ is a complex measure. Notice also that $\beta_{\mu,0}$ is a complex measure because $0 \in E_k$ according to our assumptions. Thus, by Lemma 5,

$$\beta_{\mu,\nu} \circ \beta_{\mu+\nu,-\nu} = \beta_{\mu,0} = \delta_e . \tag{29}$$

On the other hand, the support of the measure $\beta_{\mu,\nu} \circ \beta_{\mu+\nu,-\nu}$ is given by

$$\{P(\sqrt{s})r : r \in \operatorname{supp} \beta_{\mu+\nu,-\nu} , \ s \in \operatorname{supp} \beta_{\mu,\nu}\}.$$

If $P(\sqrt{s})r = e$ with $0 \leq r, s \leq e$, then $r = s = e$. Identity (29) therefore implies that $\operatorname{supp} \beta_{\mu,\nu} = \operatorname{supp} \beta_{\mu+\nu,-\nu} = \{e\}$, which is possible only if $\nu = 0$. This contradicts our assumption and proves the first statement.

If in addition μ is real and $\beta_{\mu,\nu}$ is a positive measure, then it is clear from Theorem 3(3) that ν is real. This shows the second statement. ■

The argument above relies on the condition $d \in \{1, 2\}$, and we do not know whether Theorem 6 extends to larger Peirce constants. Let us summarize our results for $d \in \{1, 2\}$.

COROLLARY 2. *Suppose $d \in \{1, 2\}$. Let $k \in \mathbb{N}$ and $\Re\mu > \mu_0 + kq + 3/2$. Then, for $\nu \in E_k$, the following statements are equivalent:*

(1) $\beta_{\mu,\nu}$ is a complex measure;
(2) $\nu \in W_{q,d}$;
(3) There exists a bounded complex measure $\beta \in M_b(\overline{\Omega})$ such that

$$\mathcal{J}_{\mu+\nu}(r) = \int_{\overline{\Omega}} \mathcal{J}_\mu(rs)d\beta(s) \quad \textit{for all } r \in \overline{\Omega} .$$

If μ is real with $\mu > \mu_0 + kq + 3/2$, then for $\nu \in E_k$ the following are equivalent:

(1) $\beta_{\mu,\nu}$ is a positive measure;

(2) ν is contained in the Wallach set

$$\left\{0, \frac{d}{2}, \ldots, (q-1)\frac{d}{2} = \mu_0\right\} \cup]\mu_0, \infty[\,;$$

(3) There exists a probability measure $\beta \in M^1(\overline{\Omega})$ such that

$$\mathcal{J}_{\mu+\nu}(r) = \int_{\overline{\Omega}} \mathcal{J}_\mu(rs)d\beta(s) \quad \textit{for all } r \in \overline{\Omega}\,.$$

In both cases, the measure β in (3) *is unique and given by $\beta_{\mu,\nu}$.*

Proof. In both cases, implication (1)⇒(2) follows from Theorem 5 in combination with Theorem 6. The remaining parts are immediate from Corollary 1, Theorem 4 and Theorem 2(2). ■

Corollary 2 implies in particular that for $q > 1$ and sufficiently large $\mu > 0$, there exist indices $\nu > 0$ such that $\mathcal{J}_{\mu+\nu}$ admits no positive integral representation with respect to $\mathcal{J}_\mu$. So there exists no Sonine-type formula in these cases. This is a surprising contrast compared to the one-variable case.

Remark 3. The Jack polynomials C_λ^α have nonnegative coefficients in their expansion with respect to the monomial symmetric functions ([35]). In view of formula (14), this implies that

$$\mathcal{J}_\mu(-r) > 0 \qquad \text{for } \mu > \mu_0 \text{ and all } r \in \overline{\Omega}. \tag{30}$$

Similar to an argument in the appendix of [23], this observation together with Theorem 3(3) and identity (27) leads for $d = 2$ to an alternative proof that for $\mu > \mu_0 + kq + 1$ and indices $\nu \in [0, \infty[$, which do not belong to the Wallach set, the distribution $\beta_{\mu,\nu}$ cannot be a positive measure. In fact, otherwise identity (27) would imply that $\beta_{\mu,\nu+l}$ is a negative measure for $l = 1$ or $l = 2$, because the product on the right side of formula (27) will be negative for either ν or $\nu + 1$. (Here $d = 2$ is relevant.) On the other hand, Theorem 3(3) immediately implies that

$$\mathcal{J}_{\mu+\nu+l}(-r) = \int_{\overline{\Omega_e}} \mathcal{J}_\mu\left(-P(\sqrt{s})r\right) d\beta_{\mu,\nu+l}(s)$$

for all $r \in \overline{\Omega}$, in contradiction to (30).

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(Received January 22, 2018)