Convolution algebras for multivariable Bessel functions

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In this article we give an overview of recent results on hypergroup algebras which are associated with Bessel functions on cones of positive semidefinite matrices and a class of Dunkl-type Bessel functions for root systems of type B. The Bessel hypergroups on matrix cones are obtained from a discrete series with a group-theoretic background by interpolation with respect to a dimension parameter. Passing to the matrix spectra leads to Bessel functions of Dunkl type and three continuous series of hypergroup algebras matching the Dunkl transform in the Weyl-group invariant case.

Keywords: Bessel functions; symmetric cones; Dunkl operators; hypergroups.

1. Introduction

Bessel functions occur naturally in the analysis of radial problems. The simplest case is the analysis of structures on $\mathbb{R}^n$ which are invariant under the action of the orthogonal group $O(n)$. The starting point of the present article is radiality on matrix spaces $M_{p,q} = M_{p,q}(\mathbb{F})$ over one of the skew-fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. On a first level, we consider radiality on $M_{p,q}$ as invariance under the action of the unitary group $U_p = U_p(\mathbb{F})$ from the left:

$$U_p \times M_{p,q} \to M_{p,q}, \quad (u,x) \mapsto ux.$$

The mapping $U_p.x \mapsto \sqrt{x^*x}$ establishes a homeomorphism between the space of U_p-orbits in $M_{p,q}$ and the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive-semidefinite Hermitian $q \times q$-matrices over $\mathbb{F}$ ($x^* = \overline{x}^t$ denotes the adjoint of x). Radial functions on $M_{p,q}$ can thus be considered as functions on the cone Π_q and the Fourier transform of a radial function can be expressed in terms of a generalized Hankel transform involving Bessel functions with matrix argument. These ideas trace back to the fundamental work of C. Herz, Ref. 6. In Refs. 2,4,5, Bessel functions and associated Hankel transforms were put into the general framework of symmetric cones. The interest in

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Bessel functions of matrix argument is motivated to some extent by questions in number theory and multivariate statistics. For example, they occur in non-central Wishart distributions which generalize non-central χ-squared distributions to the higher rank case. Another interesting aspect is that they can be imbedded into the theory of multivariable hypergeometric functions of Dunkl-type. This fact is closely related to our second level of radiality on $M_{p,q}$: Let $G = U(p,q)$ denote the indefinite unitary group of index (p,q) over $\mathbb{F}$. Its maximal compact subgroup K is naturally isomorphic with $U_p \times U_q$. We may identify $M_{p,q}$ with the tangent space of the Riemannian symmetric space G/K in the coset eK. This identification induces an action of $U_p \times U_q$ on $M_{p,q}$ according to

$$((u,v),x) \mapsto uxv^{-1}, \quad u \in U_p,\, v \in U_q. \tag{1}$$

The associated orbit space is canonically parametrized by the possible singular spectra of matrices from $M_{p,q}$ and is homeomorphic to

$$\Xi_q = \{\xi \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0\}$$

which is a Weyl chamber of type B_q. The semidirect product $(U_p \times U_q) \ltimes M_{p,q}$ is the Cartan motion group G_0 associated with the Grassmann manifold G/K. The Fourier transform of $U_p \times U_q$-invariant functions on $M_{p,q}$ is therefore given by the spherical transform on the flat symmetric space G_0/K, and the spherical functions involved in this case are generalized Bessel functions which can be imbedded into the class of Bessel functions associated with root systems studied in Dunkl theory.

In this article, we present mainly results from Refs. 13,17. In Ref. 13, the radial convolution algebras on the cone Π_q associated with the orbit spaces $M_{p,q}^{U_p}$ are interpolated with respect to the dimension parameter p. This gives a continuous class of commutative hypergroups on Π_q having matrix Bessel functions of continuous index as characters. These hypergroup algebras extend the L^2-harmonic analysis for the Hankel transform on matrix cones as developed in Refs. 2,5. They have interesting and rich automorphism groups which were studied in Ref. 17. Moreover, they can be pushed down to the level of the spectra, leading to three continuous classes of commutative hypergroups on the Weyl chamber Ξ_q (corresponding to $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$). These hypergroups interpolate the convolution algebras of the Gelfand pairs $((U_p \times U_q) \ltimes M_{p,q}, U_p \times U_q)$ with respect to p. Their characters are given by Dunkl-type Bessel functions associated with the root system B_q, where the multiplicity on the short roots $\pm e_i$ runs through a continuous range, while the multiplicity on the roots $\pm e_i \pm e_j$ remains restricted to the values $1/2, 1$ or 2 as in the geometric cases. It is conjectured

that for arbitrary root systems and non-negative multiplicities, Dunkl-type Bessel functions satisfy a positive product formula and can be characterized as the characters of a commutative hypergroup structure on the underlying Weyl chamber. Our three continuous series for B_q provide the first new class of examples beyond the Cartan motion group cases.

2. Bessel functions on matrix cones

Let $H_q = \{x \in M_{q,q}(\mathbb{F}) : x = x^*\}$ denote the space of Hermitian $q \times q$-matrices over $\mathbb{F}$, $d = \dim_{\mathbb{R}}\mathbb{F} \in \{1, 2, 4\}$ and $\Pi_q = \{x^2 : x \in H_q\}$ the cone of positive semidefinite Hermitian matrices as above. Let further $\Omega_q \subset \Pi_q$ be the subset of strictly positive definite matrices. H_q is a Euclidean Jordan algebra in the natural way, and Ω_q is the associated symmetric cone, see Ref. 4 for details. The Bessel functions associated with Ω_q represent a class of hypergeometric series of matrix argument which are defined in terms of the spherical polynomials of Ω_q. The latter are indexed by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_q) \in \mathbb{N}_0^q$ (we write $\lambda \geq 0$ for short) and defined by

$$Z_\lambda(x) = c_\lambda \int_{U_q} \Delta_\lambda(uxu^{-1})du, \quad x \in H_q$$

where du is the normalized Haar measure of U_q and Δ_λ is the power function

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1-\lambda_2}\Delta_2(x)^{\lambda_2-\lambda_3} \cdot \ldots \cdot \Delta_q(x)^{\lambda_q}.$$

The $\Delta_i(x)$ are the principal minors of the determinant $\Delta(x)$ with respect to an arbitrary, but fixed Jordan frame of H_q, see Ref. 4 for details. The normalization constant $c_\lambda > 0$ can be chosen such that for all $x \in H_q$ and $k \in \mathbb{N}_0$,

$$(\operatorname{tr} x)^k = \sum_{\lambda \geq 0, |\lambda|=k} Z_\lambda(x).$$

The $Z_\lambda(x)$ are U_q-invariant and thus depend only on the eigenvalues of x. As such, they are given by Jack polynomials. More precisely, let $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$ denote the set of eigenvalues of $x \in H_q$. Then

$$Z_\lambda(x) = C_\lambda^{2/d}(\xi)$$

where the C_λ^α are the (suitably normalized) Jack polynomials with index $\alpha > 0$ (c.f. Ref. 9). The identification of the spherical polynomials with Jack polynomials follows from their common system of differential equations. This was first observed by Macdonald, Ref. 11; see also Ref. 4. For arbitrary

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$\alpha > 0$ and a parameter $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \frac{1}{\alpha}(q-1)$, the hypergeometric function ${}_0F_1^\alpha(\mu;\,.\,)$ on $\mathbb{R}^q$ is defined by

$$_0F_1^\alpha(\mu;\xi) = \sum_{\lambda \geq 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!} C_\lambda^\alpha(\xi)$$

with the generalized Pochhammer symbol

$$(\mu)_\lambda^\alpha = \prod_{j=1}^q \big(\mu - \frac{1}{\alpha}(j-1)\big)_{\lambda_j}.$$

Later on, we shall also need the ${}_0F_1$ with two arguments,

$$_0F_1^\alpha(\mu;\xi,\eta) = \sum_{\lambda \geq 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!} \frac{C_\lambda^\alpha(\xi)C_\lambda^\alpha(\eta)}{C_\lambda^\alpha(\mathbf{1})}, \quad \mathbf{1} = (1,\ldots,1).$$

The Bessel function associated with the cone Ω_q is defined in a similar way in terms of spherical functions,

$$\mathcal{J}_\mu(x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda^{2/d} |\lambda|!} Z_\lambda(x), \quad x \in H_q.$$

Thus for $x \in H_q$ with eigenvalues $\xi = (\xi_1,\ldots,\xi_q)$, one has

$$\mathcal{J}_\mu(x) = {}_0F_1^{2/d}(\mu;-\xi).$$

If $q = 1$ then J_μ does not depend on d and is given by

$$\mathcal{J}_\mu\big(\frac{x^2}{4}\big) = j_{\mu-1}(x) \quad (x \in \mathbb{R})$$

where $j_{\mu-1}$ is the one-variable Bessel function $j_{\mu-1}(x) = {}_0F_1(\mu;-x^2/4)$.

3. Radial analysis on matrix spaces, integral formulas, and Hankel transforms

We consider the space $M_{p,q}$ of $p \times q$-matrices over $\mathbb{F}$ as a real vector space with scalar product $(x,y) = \operatorname{Re}\operatorname{tr}(x^*y)$ and norm $\|x\| = \sqrt{\operatorname{tr}(x^*x)}$. As explained in the introduction, the space $M_{p,q}^{U_p}$ of U_p-orbits in $M_{p,q}$ is naturally identified with the cone Π_q via $U_p.x \mapsto \sqrt{x^*x}$. We recall from Ref. 4 the polar coordinates in $M_{p,q}$ which are adapted to this setting: Define the Stiefel manifold

$$\Sigma_{p,q} = \{x \in M_{p,q} : x^*x = I\}.$$

Then for $f \in L^1(M_{p,q})$,

$$\int_{M_{p,q}} f(x)dx = \frac{\pi^{dpq/2}}{\Gamma_{\Omega_q}(dp/2)} \int_{\Omega_q} \int_{\Sigma_{p,q}} f(\sigma\sqrt{r})\Delta(r)^\gamma dr d\sigma$$

where Δ denotes again the detminant on the space H_q

$$\Gamma_{\Omega_q}(z) = (2\pi)^{dq(q-1)/4} \prod_{j=1}^{q} \Gamma\Big(z - \frac{d}{2}(j-1)\Big)$$

is the gamma function of the cone Ω_q. Further, $d\sigma$ denotes the unique U_p-invariant measure on $\Sigma_{p,q}$ normalized according to $\int_{\Sigma_{p,q}} d\sigma = 1$, and

$$\gamma = \frac{d}{2}(p - q + 1) - 1.$$

Put $\mu = pd/2$ and let ω_μ denote the measure on Π_q which is obtained as the image measure of the normalized Lebesgue measure $(2\pi)^{-dpq/2}dx$ on $M_{p,q}$ under the mapping $x \mapsto \sqrt{x^*x}$. Calculation in polar coordinates gives

$$\omega_\mu(f) = \frac{2^{-\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta(r)^\gamma dr. \tag{2}$$

Now suppose $F \in L^1(M_{p,q})$ is radial with $F(x) = f(\sqrt{x^*x})$. Then the Fourier transform of F is also radial and given by

$$\widehat{F}(\lambda) = \frac{1}{(2\pi)^{dpq/2}} \int_{M_{p,q}} F(x)e^{-i(\lambda,x)}dx = \int_{\Pi_q} f(r)\Big(\int_{\Sigma_{p,q}} e^{-i(\lambda,\sigma r)}d\sigma\Big)d\omega_\mu(r).$$

The inner integral over the Stiefel manifold can be expressed in terms of the Bessel function $\mathcal{J}_\mu$ on Ω_q with parameter $\mu = pd/2$. Indeed, according to Propos. XVI.2.2. of Ref. 4,

$$\int_{\Sigma_{p,q}} e^{-i(\sigma,x)}d\sigma = \mathcal{J}_\mu\Big(\frac{1}{4}x^*x\Big) \quad \text{for all } x \in M_{p,q}.$$

For $s, r \in \Pi_q$, define

$$\varphi_s^\mu(r) = \mathcal{J}_\mu\Big(\frac{1}{4}sr^2s\Big).$$

Note that $\varphi_s^\mu(r) = \varphi_r^\mu(s)$, because $\mathcal{J}_\mu$ depends only on the eigenvalues of its argument and the eigenvalues of sr^2s and rs^2r are the same. Thus

$$\widehat{F}(\lambda) = \int_{\Pi_q} f(r)\varphi_{|\lambda|}^\mu(r)d\omega_\mu(r), \quad \mu = pd/2.$$

The Fourier transform of F is therefore given by a Hankel transform of f. An L^2-theory for Hankel transforms associated with Bessel functions of general index μ was developed by Herz, Ref. 6 and Faraut and Travaglini, Ref. 5. In our notation, the relevant result of Ref. 5 is as follows:

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Theorem 3.1 (Ref. 5, Theorem 1). *Let $\mu > \frac{d}{2}(q-1)$ and define the measure ω_μ on Π_q by*

$$\omega_\mu(f) = \frac{2^{-\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta(r)^\gamma dr$$

with $\gamma = \mu - \frac{d}{2}(q-1) - 1$. Put

$$\varphi_s^\mu(r) := \mathcal{J}_\mu\left(\frac{1}{4}sr^2s\right) = \varphi_r^\mu(s).$$

Then the Hankel transform

$$f \mapsto \widehat{f}^\mu, \quad \widehat{f}^\mu(s) = \int_{\Pi_q} f(r)\varphi_s^\mu(r)d\omega_\mu(r)$$

is an isometric and self-dual isomorphism of $L^2(\Pi_q, \omega_\mu)$.

It is natural to define a generalized translation on $L^2(\Pi_q, \omega_\mu)$ by

$$f \mapsto \tau_s f, \quad \tau_s f(r) = \int_{\Pi_q} \widehat{f}^\mu(t)\varphi_s^\mu(t)\varphi_r^\mu(t)d\omega_\mu(t); \quad s \in \Pi_q.$$

An L^p-theory for $p \geq 1$ and general indices μ requires boundedness of this translation with respect to $\|.\|_{p,\omega_\mu}$ and is provided by the underlying hypergroup structure which will be described in Section 5.

Let us now turn to the action of $U_p \times U_q$ on $M_{p,q}$ given by (1). We denote the singular spectrum of $x \in M_{p,q}$ by $\sigma_{sing}(x) = \sigma(\sqrt{x^*x})$, where for $s \in \Pi_q$, $\sigma(s) = (\sigma_1, \ldots, \sigma_q) \in \mathbb{R}^q$ is the set of eigenvalues of s ordered by size according to $\sigma_1 \geq \ldots \geq \sigma_q \geq 0$. Two matrices $x, y \in M_{p,q}$ belong to the same orbit under $U_p \times U_q$ if and only if $\sigma_{sing}(x) = \sigma_{sing}(y)$. The orbit space is therefore naturally identified with the set

$$\Xi_q = \{\xi \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0\}$$

via $U_p x U_q \mapsto \sigma_{sing}(x)$. Again, this identification is also topological. Alternatively, we may start on the level of the matrix cone Π_q which was obtained as the orbit space under the action of U_p from the left. Action (1) on $M_{p,q}$ induces an action of U_q on Π_q by conjugation:

$$(v, r) \mapsto vrv^{-1}.$$

The orbits under this action are parametrized by the possible eigenvalues of matrices from Π_q, which is again Ξ_q. The following integration formula is a special case of Theorem VI.2.3 in Ref. 4: For integrable $g : \Pi_q \to \mathbb{C}$,

$$\int_{\Pi_q} g(r)dr = \kappa_q \int_{\Xi_q} \int_{U_q} g(v\xi v^{-1}) \, dv \prod_{i<j} (\xi_i - \xi_j)^d \, d\xi \qquad (3)$$

with a normalization constant $\kappa_q > 0$. Here dv denotes the normalized Haar measure on U_q and $\xi \in \Xi_q$ is identified with the diagonal matrix $\mathrm{diag}(\xi_1, \ldots, \xi_q) \in \Pi_q$. Note that $\xi_i - \xi_j \geq 0$ on Ξ_q for $i < j$, hence the density in the integral is non-negative. Let us consider the canonical mapping

$$\sigma : \Pi_q \to \Xi_q, \ r \mapsto \sigma(r)$$

which is continuous and surjective.

Lemma 3.1.

(a) *The image measure $\widetilde{\omega}_\mu$ of ω_μ under σ is given by*

$$\widetilde{\omega}_\mu = d_\mu h_\mu(\xi) d\xi \quad \text{with} \ \ h_\mu(\xi) = \prod_{i=1}^q \xi_i^{2\gamma+1} \prod_{i<j}(\xi_i^2 - \xi_j^2)^d$$

and a constant $d_\mu > 0$.

(b) *The measure ω_μ on Π_q is given by*

$$d\omega_\mu(r) = d'_\mu \cdot \prod_{i=1}^q \sigma(r)_i^{2\gamma+1} \prod_{i<j}\big(\sigma(r)_i + \sigma(r)_j\big)^d dr$$

with the constant $d'_\mu = \kappa_q / d_\mu$.

Proof. Part (a) is shown in Ref. 13. For the proof of (b), let $g(\xi) := \prod_{i=1}^q \xi_i^{2\gamma+1} \prod_{i<j}(\xi_i + \xi_j)^d$ and consider $\widetilde{g} := g \circ \sigma$. Let $f : \Xi_q \to \mathbb{C}$ be compactly supported. Then by formula (3) and part (a),

$$\int_{\Pi_q} \widetilde{f}(r)\widetilde{g}(r)dr = \kappa_q \int_{\Xi_q} f(\xi)g(\xi)\prod_{i<j}(\xi_i - \xi_j)^d d\xi$$

$$= \kappa_q \int_{\Xi_q} f(\xi)h_\mu(\xi)d\xi \ = \ \frac{\kappa_q}{d_\mu} \int_{\Pi_q} \widetilde{f}(r)d\omega_\mu(r).$$

This implies the assertion. $\square$

The measures ω_μ and $\widetilde{\omega}_\mu$ will become relevant as the Haar (and Plancherel) measures of commutative hypergroups on the matrix cone Π_q and the chamber Ξ_q, respectively. The constant d_μ is given by

$$d_\mu = \left(\int_{\Xi_q} h_\mu(x)e^{-|x|^2/2}dx\right)^{-1}.$$

This is a Selberg integral for the root system B_q and can be evaluated explicitly; see Ref. 10.

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4. Hypergroups

Hypergroups generalize the convolution algebras of locally compact groups, with the convolution product of two point measures δ_x and δ_y being in general not a point measure again but a probability measure depending on x and y. More precisely, a hypergroup is a locally compact Hausdorff space X with a weakly continuous, associative convolution $*$ on the space $M_b(X)$ of regular bounded Borel measures on X, satisfying the following additional properties:

(1) The convolution product $\delta_x * \delta_y$ of two point measures is a compactly supported probability measure on X, and $\mathrm{supp}(\delta_x * \delta_y)$ depends continuously on x and y with respect to a suitable Hausdorff topology on the space of compact subsets of X (see Ref. 8).
(2) There is a neutral element δ_e satisfying $\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$ for all $x \in X$.
(3) There is a continuous involution $x \mapsto \bar{x}$ on X such that for all $x, y \in X$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ is equivalent to $x = \bar{y}$, and $\delta_{\bar{x}} * \delta_{\bar{y}} = (\delta_y * \delta_x)^-$. Here for $\mu \in M_b(X)$, the measure μ^- is given by $\mu^-(A) = \mu(A^-)$ for Borel sets $A \subset X$.

Due to weak continuity, the convolution of measures on a hypergroup is uniquely determined by the convolution of point measures.

We recapitulate some basic facts from hypergroup theory, see Ref. 8 for details: If the convolution is commutative, then $(M_b(X), *)$ becomes a commutative Banach-$*$-algebra with identity δ_e. As proven by Spector (Ref. 15), there exists an (up to a multiplicative factor) unique Haar measure ω, that is a positive Radon measure on X satisfying

$$\int_X f(x*y)d\omega(y) = \int_X f(y)d\omega(y) \quad \text{for all } x \in X,\ f \in C_c(X),$$

where $f(x*y) = (\delta_x * \delta_y)(f)$. A decisive object for harmonic analysis on a commutative hypergroup is its dual space which is defined by

$$\widehat{X} := \{\varphi \in C_b(X) : \varphi \neq 0,\ \varphi(\overline{x} * y) = \overline{\varphi(x)}\varphi(y) \ \forall\, x, y \in X\}.$$

The elements of $\widehat{X}$ are called characters. As in the case of LCA groups, the dual of a commutative hypergroup is a locally compact Hausdorff space with the topology of locally uniform convergence and can be identified with the symmetric part of the spectrum of the convolution algebra $L^1(X, \omega)$. Accordingly, the Fourier transform on $L^1(X, \omega)$ is defined by $\widehat{f}(\varphi) := \int_X f\overline{\varphi}d\omega$. The Fourier transform is injective, and there exists a

unique positive Radon measure π on $\widehat{X}$, called the Plancherel measure of $(X, *)$, such that $f \mapsto \widehat{f}$ extends to an isometric isomorphism from $L^2(X, \omega)$ onto $L^2(\widehat{X}, \pi)$. As for groups, there are convolutions between functions from various classes of L^p-spaces (or measures) on a hypergroup with Haar measure ω. For example, if $1 \leq p \leq \infty$ and $f \in L^1(X, \omega), g \in L^p(X, \omega)$, then the convolution product

$$f * g(x) = \int_X f(x * y) g(\overline{y}) d\omega(y)$$

belongs to $L^p(X, \omega)$ and satisfies $\|f * g\|_{p,\omega} \leq \|f\|_{1,\omega} \|g\|_{p,\omega}$.

Example 4.1 (Orbit hypergroups). *(Ref. 8, Chapt. 8) Let $(G, +)$ be a locally compact abelian group and K a compact subgroup of $\operatorname{Aut} G$. Then the space $G^K = \{K.x : x \in G\}$ of K-orbits in G (equipped with the quotient topology) becomes a commutative hypergroup with convolution*

$$(\delta_{K.x} * \delta_{K.y})(f) = \int_K f(K.(x + ky)) dk.$$

The neutral element of the orbit hypergroup $(G^K, *)$ is 0 and the involution is given by $(K.x)^- = K.(-x)$. For a Haar measure m on G, the image measure of m under the the canonical map $\pi : G \to G^K$ provides a Haar measure of G^K. The dual space of G^K consists of the functions

$$\varphi_\alpha(K.x) := \int_K \alpha(k.x) dk, \quad \alpha \in \widehat{G}$$

where $\varphi_\alpha = \varphi_{\alpha'}$ iff α and α' are in the same orbit under the dual action of K on $\widehat{G}$ given by $k.\alpha(x) = \alpha(k^{-1}.x)$, see Ref. 13.

5. Bessel hypergroups on matrix cones

For each integer $p \geq q$ we interpret radial analysis on $M_{p,q}$ in terms of a commutative orbit hypergroup on the cone Π_q which is derived from the action of U_p by multiplication from the left. The convolution and all further data of this hypergroup can be calculated explicitly according to Example 4.1. We summarize the results obtained in Ref. 13:

Theorem 5.1. *Let $\mu = pd/2$. The convolution of the orbit hypergroup $M_{p,q}^{U_p} \cong \Pi_q$ is given by*

$$(\delta_r *_\mu \delta_s)(f) = \int_{\Sigma_{p,q}} f\left(\sqrt{r^2 + s^2 + r\widetilde{\sigma}s + (r\widetilde{\sigma}s)^*}\right) d\sigma$$

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where $\widetilde{\sigma} \in M_q$ is the truncated $q \times q$-matrix whose rows are given by the first q rows of σ. The neutral element of $(\Pi_q, *_\mu)$ is 0, and the involution is the identity mapping. A Haar measure is given by the measure ω_μ as defined in (2), and the dual space consists of the Bessel functions $\varphi_s^\mu(r) = \mathcal{J}_\mu\!\left(\frac{1}{4} s r^2 s\right),\ s \in \Pi_q.$

In the above convolution formula, the integrand does not depend on the complete matrix σ but depends only the truncation $\widetilde{\sigma}$, which is contained in the closure of the matrix ball

$$D_q := \{v \in M_{q,q} : v^* v < I\}$$

($r < s$ means that $s - r$ is positive definite). By a corresponding splitting of coordinates on the Stiefel manifold, one obtains the convolution formula in a different form with D_q as domain of integration. The dimension parameter p then occurs as an exponent in the density of the integral. Let

$$\varrho := d\!\left(q - \frac{1}{2}\right) + 1 \quad \text{and} \quad \kappa_\mu := \int_{D_q} \Delta(I - v^* v)^{\mu - \varrho} dv$$

for $\mu \in \mathbb{R}$ with $\mu > \varrho - 1$. Then the result is as follows:

Proposition 5.1. *Suppose that $p \geq 2q$. Then the convolution $*_\mu$ with $\mu = pd/2$ can be written as*

$$(\delta_r *_\mu \delta_s)(f) = \frac{1}{\kappa_\mu} \int_{D_q} f\!\left(\sqrt{r^2 + s^2 + rvs + sv^* r}\,\right) \Delta(I - v^* v)^{\mu - \varrho}\, dv.$$

Notice that $\delta_r *_\mu \delta_s$ defines a probability measure on Π_q not only for the discrete series $\mu = pd/2$ but also for arbitrary $\mu > \varrho - 1$. For $\mu = \varrho - 1 = \frac{d}{2}(2q - 1)$, which corresponds to the orbit hypergroup $M_{p,q}^{U_p}$ with $p = 2q - 1$, the integral becomes singular. The degenerate form of the formula in this case can be calculated after a suitable change of coordinates; this is carried out in Ref. 13.

Using the above reformulation of the convolution, the corresponding product formula for the characters, i.e. the Bessel functions, as well as the statements of Theorem 5.1 can be extended to the range $\mu > \rho - 1$. The basic step for this is analytic continuation with respect to μ from the discrete set of values $\mu = pd/2$ into the full half-plane $\{\mu \in \mathbb{C} : \operatorname{Re}\mu > \rho - 1\}$ by use of Carlson's Phragmen-Lindelöf-type Theorem (see e.g. Ref. 16, p.186). This gives three continuous series of commutative hypergroup structures on Π_q (corresponding to $d = 1, 2, 4$) which interpolate those occuring as orbit hypergroups for the indices $\mu = pd/2$. The following theorem contains the main results of Ref. 13:

Theorem 5.2. *Let $\mu \in \mathbb{R}$ with $\mu > \varrho - 1$.*

(a) The assignment

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f\big(\sqrt{r^2 + s^2 + rvs + sv^*r}\,\big)\, \Delta(I - v^*v)^{\mu-\varrho}\, dv$$

defines a commutative hypergroup $X_\mu = (\Pi_q\,, *_\mu)$ with neutral element 0 and the identity mapping as involution. The support of $\delta_r *_\mu \delta_s$ satisfies

$$supp\,(\delta_r *_\mu \delta_s) \subseteq \{t \in \Pi_q : \|t\| \leq \|r\| + \|s\|\}.$$

(b) A Haar measure of X_μ is given by the measure ω_μ from Theorem 3.1.

(c) $\widehat{X_\mu} = \{\varphi_s^\mu(r) = \mathcal{J}_\mu\big(\frac{1}{4}sr^2s\big) : \ s \in \Pi_q\}$.

Part (c) is based on the analytic extension of the product formula for the Bessel functions, but one also has to make sure that there are no further characters apart from the Bessel functions $\varphi_s^\mu, s \in \Pi_q$. For this, the Plancherel Theorem 3.1 as well as subexponential growth of the hypergroup are needed.

Theorem 5.2 implies in particular a positivity-preserving generalized translation and a full L^p-theory beyond the L^2-theory established earlier in Refs. 5,6.

As $\varphi_s^\mu(r) = \varphi_r^\mu(s)$, the convolution $*_\mu$ determines a dual hypergroup convolution on $\widehat{X_\mu}$ by

$$\varphi_r^\mu(t)\varphi_s^\mu(t) = \int_{\Pi_q} \varphi_w^\mu(t)d(\delta_r *_\mu \delta_s)(w).$$

The mapping $X_\mu \mapsto \widehat{X_\mu}$, $r \mapsto \varphi_r^\mu$ is an isomorphism of hypergroups in the sense of the definition below. When X_μ and its dual are identified in this way, the Plancherel measure of X_μ coincides with the Haar measure ω_μ of $\widehat{X_\mu}$ (the normalization of ω_μ is chosen appropriately). Commutative hypergroups with these properties are called self-dual.

Definition 5.1. Let X, Y be hypergroups. A homeomorphism $T : X \to Y$ is called a hypergroup isomorphism, if $T(\delta_x * \delta_{\overline{y}}) = \delta_{T(x)} * \delta_{\overline{T(y)}}$ $\quad \forall x, y \in X$, where the mapping T is extended to bounded Borel measures by taking image measures. An automorphism of the hypergroup X is an isomorphism $T : X \to X$. We denote the group of automorphisms of X by $\mathrm{Aut}(X)$.

The hypergroups $X_\mu = (\Pi_q, *_\mu)$ have interesting algebraic properties. In particular, they have rich groups of automorphisms, which are closely

related to the automorphisms of the underlying symmetric cone. The automorphisms, as well as subhypergroups and quotients, were studied in detail in Ref. 17. For the following, we recall that $GL_q = GL_q(\mathbb{F})$ acts on Π_q as a group of homeomorphisms via

$$T_a(r) := \sqrt{ar^2a^*} \quad \text{for } a \in GL_q, \ r \in \Pi_q.$$

Proposition 5.2 (Ref. 17). $\{T_a : a \in GL_q\}$ *is a subgroup of* $Aut(X_\mu)$.

If $q = 1$, then $\Pi_q = [0, \infty)$ and the hypergroups X_μ are the same for all values of d. They are just the well-known Bessel-Kingman hypergroups. In this case, the automorphism group of X_μ for $\mu > \frac{1}{2}$ was determined already by Zeuner in Ref. 18; it consists of the mappings T_a with $a > 0$.

If $\mathbb{F} = \mathbb{R}$, then the T_a constitute the full automorphism group $Aut(X_\mu)$ also in higher dimensions. If $\mathbb{F} = \mathbb{C}$, one has to add one further generating element: in fact, it is easily seen from the explicit form of the hypergroup convolution on X_μ that for $\mathbb{F} = \mathbb{C}$, the transposition $\tau : x \mapsto x^t$ is also an automorphism of X_μ.

Theorem 5.3 (Ref.17). *Let* $\mu > \rho - 1$.

(1) If $\mathbb{F} = \mathbb{R}$, *then* $Aut(X_\mu) = \{T_a : a \in GL_q\}$.
(2) If $\mathbb{F} = \mathbb{C}$, *then* $Aut(X_\mu) = \{\sigma \circ T_a : a \in GL_q, \ \sigma = id, \tau\}$.

A similar result is expected for $\mathbb{F} = \mathbb{H}$, but so far, the full classification of the automorphism group is still open in this case. The classification of automorphisms of X_μ is closely related to the structure of the possible subhypergroups, which are also completely classified in Ref. 17. For arbitrary $\mathbb{F}$ and $\mu > d(q - \frac{1}{2})$, they are given by

$$X_\mu^{k,v} = \left\{ v \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} v^{-1} : r \in \Pi_k \right\}$$

with $0 \leq k \leq q$ and a unitary matrix $v \in U_q$.

6. Wishart-distributions and probabilistic aspects.

The standard (squared) Wishart distribution on the cone Π_q with shape parameter $\mu > \rho - 1$ is the probability measure

$$dW_\mu(r) = (2\pi)^{-q\mu} e^{-tr(r^2)/2} d\omega_\mu(r).$$

If $\mu = pd/2$ with $p \geq q$, then this is just the image measure of the Gaussian distribution $(2\pi)^{-dpq/2} e^{-\|x\|^2/2} dx$ on $M_{p,q}$ under the mapping $x \mapsto \sqrt{x^*x}$.

In this way, a non-centered Gaussian $(2\pi)^{-dpq/2}e^{-\|x-a\|^2/2}dx$ leads to the non-centered Wishart distribution $W_\mu^{\sqrt{a^*a}}$ with $\mu = pd/2$, where

$$dW_\mu^s(r) = (2\pi)^{-q\mu}\mathcal{J}_\mu\big(-\frac{1}{4}sr^2s\big)e^{-tr(r^2+s^2)/2}d\omega_\mu(r)$$

for $s \in \Pi_q$ and $\mu > \rho - 1$. With the theorem above, non-centered Wishart distributions with arbitrary shape parameter $\mu > \rho - 1$ can be written as hypergroup translates of the non-centered one:

$$W_\mu^s = \delta_s *_\mu W_\mu.$$

For a proof we refer to Ref. 17, where Wishart distributions and their probabilistic properties are discussed from this new point of view. A central limit theorem with Wishart distributions as limits is obtained, as well as strong laws of large numbers for random walks on X_μ. In Ref. 14, we consider the situation where the dimension parameter p (or, more general, the index μ) tends to infinity, while q is fixed. In particular, we obtain a strong law of large numbers. An essential ingredient is the following estimate for the Bessel functions $\mathcal{J}_\mu$ with large indices μ:

Theorem 6.1 (Ref. 14). *There exists a constant C depending on d and q such that for all $\mu > 2\rho$ and $r \in \Pi_q$,*

$$\big|\mathcal{J}_\mu(\mu r) - e^{-tr(r)}\big| \leq \frac{C}{\mu} \cdot min\,(1, tr(r)^2).$$

7. Bessel functions associated with root system B_q

Bessel functions associated with root systems are part of the theory of rational Dunkl operators which was initiated by C.F. Dunkl in the late nineteen-eighties, see Ref. 3. They include the usual one-variable Bessel functions (rank one case), and the spherical functions of Cartan motion groups. The general framework is as follows:

Let G be a finite reflection group on $\mathbb{R}^q$ with the usual Euclidean scalar product $\langle\,.\,,\,.\,\rangle$, and let R be the reduced root system of G. A G-invariant function $k : R \to \mathbb{C}$ is called a multiplicity function on R. In the present context, we shall be concerned with root system $B_q = \{\pm e_i, 1 \leq i \leq q\} \cup \{\pm e_i \pm e_j, 1 \leq i < j \leq q\}$. Each multiplicity on B_q is of the form $k = (k_1, k_2)$ where k_1 is the value on the roots $\pm e_i$ and k_2 is the value on the roots $\pm e_i \pm e_j$.

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For a fixed multiplicity k, the associated (rational) Dunkl operators are given by

$$T_\xi(k) = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{\langle \alpha, \, . \, \rangle} (1 - \sigma_\alpha), \quad \xi \in \mathbb{R}^q.$$

Here R_+ is a positive subsystem of R, σ_α denotes the reflection in the hyperplane perpendicular to α and the action of G is extended to functions on $\mathbb{R}^q$ in the usual way. The $T_\xi(k)$ commute (Ref. 3) and therefore generate a commutative algebra of differential-reflection operators on $\mathbb{R}^q$. For $k \geq 0$ and spectral parameter $\eta \in \mathbb{C}^q$, consider the so-called Bessel system

$$p(T(k))f \, = \, p(\eta)f \quad \forall \, p \in \mathcal{P}^G; \quad f(0) = 1.$$

$\mathcal{P}^G$ denotes the subalgebra of G-invariant polynomials in $\mathcal{P}$, and $p(T(k))$ is the Dunkl operator associated with the polynomial $p(x) = p(x_1, \ldots, x_n)$ which is obtained by replacing x_i by $T_{e_i}(k)$. When restricted to G-invariant functions, $p(T(k))$ as a differential operator. As proven in Ref. 12, the Bessel system has a unique analytic G-invariant solution $\xi \mapsto J_k(\xi, \eta)$ which is called the Bessel function associated with R. In rank one, one obtains the one-variable Bessel functions $J_k(\xi, \eta) = j_{k-1/2}(i\xi\eta)$. In the general case, J_k satisfies $J_k(\xi, \eta) = J_k(\eta, \xi)$ and is G-invariant in both arguments. It is known that the algebra of invariant differential operators of a flat symmetric space is (essentially) given by the operators $p(T(k))$ where R and k are determined by the underlying root space data. The spherical functions are then given by the corresponding Bessel functions $J_k(\, . \, , \eta)$.

Let Ξ denote the closed Weyl chamber associated with R_+. The Bessel function J_k gives rise to an integral transform on Ξ which is a symmetrized version of the Dunkl transform (see Ref. 7 for a detailed study) and generalizes the usual Hankel transform to higher rank. Let w_k denote the weight function

$$w_k(\xi) = \prod_{\alpha \in R_+} |\langle \alpha, \xi \rangle|^{2k_\alpha}$$

on Ξ. Then the symmetric Dunkl transform on $L^1(\Xi, w_k)$ is defined by

$$\widehat{f}^k(\eta) = \int_\Xi f(\xi) J_k(-i\xi, \eta) w_k(\xi) d\xi, \quad \eta \in \Xi.$$

This transform has many properties in common with the usual Fourier transform. In particular, there are Plancherel and Paley-Wiener theorems available. However, there is so far no bounded generalized translation matching the Dunkl transform in general. In Ref. 13, interpolations of the

orbit hypergroup convolutions associated with $(U_p \times U_q) \ltimes M_{p,q}$ are obtained, which build three continuous series of hypergroups with Bessel functions of type B_q as characters. The results, which will be described in the next section, are based on the following representation of the B_q-Dunkl-type Bessel function as a hypergeometric function $_0F_1$ of two arguments:

Proposition 7.1. *[Refs. 1,13] Let $k = (k_1, k_2) \geq 0$ and $k_2 > 0$. Let J_k^B denote the Dunkl-type Bessel function of type B_q and with multiplicity k. For $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{C}^q$ put $\xi^2 = (\xi_1^2, \ldots, \xi_q^2)$. Then for all $\xi, \eta \in \mathbb{C}^q$,*

$$J_k^B(\xi, \eta) = {_0F_1^\alpha}\big(\mu; \frac{\xi^2}{2}, \frac{\eta^2}{2}\big) \quad \text{with} \ \ \alpha = \frac{1}{k_2}, \ \mu = k_1 + (q-1)k_2 + \frac{1}{2}.$$

8. Hypergroups on the Weyl chamber

In Section 5 we saw that the cone Π_q carries a continuously parametrized family of commutative hypergroup structures $*_\mu$ with $\mu > \varrho - 1$, as well as additional orbit hypergroup structures for $\mu = pd/2$, $p \geq q$ an integer. Let

$$\mathcal{M}_q := \big\{ \frac{pd}{2}, \ p = q, q+1, \ldots \big\} \cup (\rho - 1, \infty).$$

Under the action of U_q on Π_q by conjugation $(v, r) \mapsto vrv^{-1}$, each convolution $*_\mu$ with $\mu \in \mathcal{M}$ induces a commutative hypergroup convolution $\circ_\mu$ on Ξ_q which is obtained by the technique of orbital hypergroup morphisms (see Ref. 8). For this it is important that the mapping $r \mapsto vrv^{-1}$ is an automorphism of X_μ. As before, we identify $\xi \in \Xi_q$ with the associated diagonal matrix from Π_q and denote by σ the canonical mapping $r \mapsto \sigma(r)$ on Π_q. The following is proven in Ref. 13:

Theorem 8.1.

(1) *For fixed $d \in \{1, 2, 4\}$ and each $\mu \in \mathcal{M}_q$ the chamber Ξ_q carries a commutative hypergroup structure $Y_\mu = (\Xi_q, \circ_\mu)$ with convolution*

$$(\delta_\xi \circ_\mu \delta_\eta)(f) = \int_{U_q} (f \circ \sigma)(\xi *_\mu v\eta v^{-1}) dv.$$

The neutral element is 0 and the involution is given by the identity mapping.

(2) *A Haar measure on Y_μ is given by*

$$\widetilde{\omega}_\mu = d_\mu h_\mu(\xi) d\xi \quad \text{with} \quad h_\mu(\xi) = \prod_{i=1}^q \xi_i^{2\gamma+1} \prod_{i<j} (\xi_i^2 - \xi_j^2)^d$$

as in Lemma 3.1.

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(3) *The dual space of Y_μ is parametrized by Ξ_q and consists of the functions*

$$\psi_\xi^\mu(\eta) = \int_{U_q} \varphi_\xi^\mu(v\eta v^{-1})dv = J_k^B(\xi, i\eta), \quad \xi \in \Xi_q$$

where the multiplicity k is given by $k = k(\mu, d) = \left(\mu - \frac{d}{2}(q-1) - \frac{1}{2}, \frac{d}{2}\right)$.

In particular, the Bessel function J_k^B with $k = k(\mu, d)$ satisfies the positive product formula

$$J_k^B(\xi, z)J_k^B(\eta, z) = \int_{\Xi_q} J_k^B(\zeta, z) \, d(\delta_\xi \circ_\mu \delta_\eta)(\zeta) \quad \forall \xi, \eta \in \Xi_q,\, z \in \mathbb{C}^q.$$

Parts (1), (2) and the first identity of (3) are proven by hypergroup techniques as indicated above. For the second identity of (3), one needs the product formula

$$\frac{Z_\lambda(r)Z_\lambda(s)}{Z_\lambda(I)} = \int_{U_q} Z_\lambda(\sqrt{r}usu^{-1}\sqrt{r})du \quad \forall r, s \in \Pi_q$$

see Ref. 4, Cor. XI.3.2. Together with Proposition 7.1 this shows that

$$\Psi_\xi^\mu(\eta) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda^{2/d}|\lambda|!} \frac{Z_\lambda\left(\frac{\xi^2}{2}\right)Z_\lambda\left(\frac{\eta^2}{2}\right)}{Z_\lambda(I)} = {}_0F_1^{2/d}\left(\mu; \frac{\xi^2}{2}, -\frac{\eta^2}{2}\right) = J_k^B(\xi, i\eta)$$

with the stated value of k.

The hypergroups Y_μ are again self-dual via the homeomorphism $Y_\mu \to \widehat{Y}_\mu$, $\xi \mapsto \psi_\xi^\mu$. Under this identification, the Plancherel measure of Y_μ coincides with the Haar measure $\widetilde{\omega}_\mu$.

In the geometric cases $\mu = pd/2$, the convolution of the hypergroup Y_μ coincides with the convolution of the Gelfand pair $(U_p \times U_q) \ltimes M_{p,q}/(U_p \times U_q)$, and the support of the probability measure $\delta_\xi \circ_\mu \delta_\eta$ on Ξ_q describes the set of possible singular spectra of sums $x + y$ with matrices $x, y \in M_{p,q}$ having given singular spectra ξ and η.

Acknowledgments

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