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Trigonometric convolution structures on $\mathbb{Z}$ derived from Jacobi polynomials

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Abstract

We introduce systems of trigonometric polynomials which are orthogonal on the unit circle and arise from Jacobi polynomials by a certain complexification. It is shown that the product formula of such a system, though containing negative linearization coefficients, leads to a Banach algebra of measures on $\mathbb{Z}$ in a canonical way.

Keywords: Jacobi polynomials; Trigonometric polynomials; Linearization of products; Convolution algebras

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1. Introduction

Let $P_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$, denote the Jacobi polynomials of order (α, β), $\alpha, \beta > -1$, and set

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{1}{\binom{n+\alpha}{n}} P_n^{(\alpha,\beta)}(x).$$

By a well-known result of Gasper [5], the coefficients $g^{(\alpha,\beta)}(n,m,k)$ in the product linearizations

$$R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} g^{(\alpha,\beta)}(n,m,k) R_k^{(\alpha,\beta)}(x)$$

are nonnegative if and only if $(\alpha, \beta) \in V$,

$$V = \{(\alpha, \beta) \in \mathbb{R}^2 \colon \alpha \geqslant \beta,\ a(a+5)(a+3)^2 \geqslant (a^2 - 7a - 24)\, b^2\},$$

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where $a = \alpha + \beta + 1$, $b = \alpha - \beta$. As a consequence, the system $R_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$ with $(\alpha,\beta) \in V$ induces the structure of a so-called polynomial hypergroup on $\mathbb{N}_0$, characterized by the convolution of point measures according to

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g^{(\alpha,\beta)}(n,m,k)\,\delta_k .$$

This convolution has a unique bilinear, norm-continuous extension to $M_b(\mathbb{N}_0)$, the set of bounded Borel measures on $\mathbb{N}_0$, making $M_b(\mathbb{N}_0)$ into a commutative Banach-*-algebra with involution $\mu^* = \overline{\mu}$, $\mu \in M_b(\mathbb{N}_0)$. For details see [6].

In this paper, we consider systems of trigonometric polynomials $\Psi_n^{(\alpha,\beta)}$, $n \in \mathbb{Z}$, where $\Psi_n^{(\alpha,\beta)}$ is a certain complexification of the Jacobi polynomial $R_n^{(\alpha,\beta)}$, generalizing Euler's formula

$$\mathrm{e}^{\mathrm{i}nt} = \cos nt + \mathrm{i}\sin nt.$$

We prove that within a large range of parameters (α,β), the product linearizations of such a "trigonometric Jacobi-system" lead to a convolution structure on $\mathbb{Z}$ which is not positivity-preserving, but still makes $M_b(\mathbb{Z})$ into a commutative Banach-*-algebra.

2. Trigonometric Jacobi-systems

For $\alpha, \beta > -1$, let $\nu^{(\alpha,\beta)}$ denote the orthogonalization measure of the Jacobi polynomials $R_n^{(\alpha,\beta)}$ on $[-1,1]$, that is, $\mathrm{d}\nu^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta\,\mathrm{d}x$. Further, set

$$h^{(\alpha,\beta)}(n) := \left(\int_{-1}^{1} R_n^{(\alpha,\beta)}(x)^2\,\mathrm{d}\nu^{(\alpha,\beta)}(x)\right)^{-1}, \quad n \in \mathbb{N}_0.$$

According to formula (4.3.3) in [8],

$$h^{(\alpha,\beta)}(n) = \binom{n+\alpha}{n}^2 \cdot \frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}} \cdot \frac{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}. \tag{2.1}$$

With $\nu^{(\alpha,\beta)}$ there corresponds a measure $\mu^{(\alpha,\beta)}$ on the unit circle $T := \{z \in \mathbb{C}: |z| = 1\}$ in a canonical way, namely

$$\mathrm{d}\mu^{(\alpha,\beta)}(\mathrm{e}^{\mathrm{i}t}) = \tfrac{1}{2}(1-\cos t)^{\alpha+1/2}(1+\cos t)^{\beta+1/2}\,\mathrm{d}t.$$

Note that $\nu^{(\alpha,\beta)}$ is just the image measure of $\mu^{(\alpha,\beta)}$ with respect to the mapping $\varphi : T \to [-1,1]$, $\varphi(\mathrm{e}^{\mathrm{i}t}) = \cos t$.

The trigonometric Jacobi-system $(\Psi_n^{(\alpha,\beta)})_{n\in\mathbb{Z}}$ is now defined as follows: For $\alpha, \beta > -1$ and $z = \mathrm{e}^{\mathrm{i}t} \in T$, we set

$$\Psi_0^{(\alpha,\beta)}(z) := 1,$$
$$\Psi_n^{(\alpha,\beta)}(z) := R_{|n|}^{(\alpha,\beta)}(\cos t) + \mathrm{i}\,\mathrm{sgn}\,(n)\, l_n^{(\alpha,\beta)}\, \sin t\, R_{|n|-1}^{(\alpha+1,\beta+1)}(\cos t) \quad \text{if } n \neq 0,$$

with

$$l_n^{(\alpha,\beta)} := \left(\frac{h^{(\alpha+1,\beta+1)}(|n|-1)}{h^{(\alpha,\beta)}(|n|)}\right)^{1/2} = \frac{\sqrt{|n|(|n|+\alpha+\beta+1)}}{2(\alpha+1)}. \tag{2.2}$$

Note that $\Psi_n^{(\alpha,\beta)}(z)$ is a trigonometric polynomial of degree $|n|$, and that

$$\Psi_{-n}^{(\alpha,\beta)}(z) = \Psi_n^{(\alpha,\beta)}(\bar{z}) = \overline{\Psi}_n^{(\alpha,\beta)}(z). \tag{2.3}$$

If (α, β) is a fixed pair of parameters, we shall for shortness often drop the superscript (α, β) and distinguish by a tilde those quantities having superscript $(\alpha+1, \beta+1)$. In particular, $\widetilde{\nu} := \nu^{(\alpha+1,\beta+1)}$, $\widetilde{R}_n := R_n^{(\alpha+1,\beta+1)}$.

In the special case $\alpha = \beta = -\frac{1}{2}$, we have

$$\Psi_n^{(-1/2,-1/2)}(z) = z^n \quad \text{for } n \in \mathbb{Z}.$$

So $(\Psi_n^{(-1/2,-1/2)})_{n\in\mathbb{Z}}$ is just the classical trigonometric basis of $L^2(T, \mathrm{d}t)$, whose product formula $z^n z^m = z^{n+m}$ corresponds to the usual addition on $\mathbb{Z}$. In the general case the following holds:

Theorem 1. (a) *The system* $(\Psi_n^{(\alpha,\beta)})_{n\in\mathbb{Z}}$ *is an orthogonal basis of* $L^2(T, \mu^{(\alpha,\beta)})$ *with*

$$\int_T \Psi_n^{(\alpha,\beta)} \overline{\Psi}_m^{(\alpha,\beta)} \,\mathrm{d}\mu^{(\alpha,\beta)} = \pi^{(\alpha,\beta)}(n)^{-1} \cdot \delta_{n,m},$$

where $\pi^{(\alpha,\beta)}(0) = h^{(\alpha,\beta)}(0)$ *and* $\pi^{(\alpha,\beta)}(n) = \frac{1}{2}h^{(\alpha,\beta)}(|n|)$ *for* $n \neq 0$.

(b) $\Psi_n^{(\alpha,\beta)}\Psi_m^{(\alpha,\beta)} = \sum_{k\in I(n,m)} h_{n,m,k}^{(\alpha,\beta)} \Psi_k^{(\alpha,\beta)}$ *for all* $n, m \in \mathbb{Z}$, *with unique real coefficients* $h_{n,m,k}^{(\alpha,\beta)}$ *and with*

$$I(n,m) = \{-|n|-|m|, \ldots, -||n|-|m||\} \cup \{||n|-|m||, \ldots, |n|+|m|\}.$$

Proof. (a) We use the announced abbreviations. As μ is symmetric, i.e.,

$$\int_T f(\bar{z})\,\mathrm{d}\mu(z) = \int_T f(z)\,\mathrm{d}\mu(z)$$

for every continuous function f on T, one obtains that

$$\int_T \Psi_n \overline{\Psi}_m \,\mathrm{d}\mu = \int_{-1}^{1} R_{|n|}R_{|m|}\,\mathrm{d}\nu + \mathrm{sgn}(nm)\, l_n l_m \int_{-1}^{1} \widetilde{R}_{|n|-1}\widetilde{R}_{|m|-1}\,\mathrm{d}\widetilde{\nu}$$
$$= \frac{1+\mathrm{sgn}(nm)}{h(|n|)} \cdot \delta_{|n|,|m|}.$$

This is the orthogonality of the Ψ_n, $n \in \mathbb{Z}$. Their completeness in $L^2(T, \mu)$ is clear, because the vector space $\mathscr{T}$ of trigonometric polynomials is dense in $L^2(T, \mu)$ with respect to $\|\cdot\|_{2,\mu}$ and $\{\Psi_k, k \in \mathbb{Z}\}$ is a maximal linearly independent subset of $\mathscr{T}$.

(b) In view of (a) we can write

$$\Psi_n\Psi_m = \sum_{|k|\leqslant|n|+|m|} h_{n,m,k}\Psi_k \quad \text{with } h_{n,m,k} = \pi(k)\int_T \Psi_n\Psi_m\overline{\Psi}_k\,\mathrm{d}\mu.$$

By (2.3) and the symmetry of μ it is clear that the $h_{n,m,k}$ are real. Finally, the identities

$$h_{n,m,k} = h_{n,-k,-m}\cdot\frac{\pi(k)}{\pi(m)} = h_{m,-k,-n}\cdot\frac{\pi(k)}{\pi(n)}$$

show that $h_{n,m,k} = 0$ if $|m| > |k|+|n|$ or $|n| > |k|+|m|$. □

3. The convolution structure on $\mathbb{Z}$ associated with a trigonometric Jacobi-system

Following the construction of polynomial hypergroups, we intend to derive a convolution structure on $\mathbb{Z}$ from the product linearizations of the $\Psi_n^{(\alpha,\beta)}$ by setting

$$\delta_n * \delta_m := \sum_{k\in I(n,m)} h_{n,m,k}^{(\alpha,\beta)}\,\delta_k\,, \quad n,m\in\mathbb{Z}. \tag{3.1}$$

However, contrary to the case of Jacobi polynomials with order $(\alpha,\beta)\in V$, definition (3.1) cannot lead to (nontrivial) hypergroup structures; the reason is that there always occur negative coefficients $h_{n,m,k}^{(\alpha,\beta)}$ unless $(\alpha,\beta)=(-\frac{1}{2},-\frac{1}{2})$, which is the group case. Indeed, for $\alpha,\beta>-1$, a short calculation yields that

$$2h_{1,1,-1}^{(\alpha,\beta)} = g^{(\alpha,\beta)}(1,1,1) - \frac{3}{a_0}(\widetilde{b}_0 - b_0), \tag{3.2}$$

where a_0, b_0, $\widetilde{b}_0$ are the constants in

$$R_1^{(\alpha,\beta)}(x) = \frac{1}{a_0}(x-b_0), \qquad R_1^{(\alpha+1,\beta+1)}(x) = \frac{1}{\widetilde{a}_0}(x-\widetilde{b}_0).$$

Employing explicit formulas for the coefficients in (3.2) (see e.g. [6, 3(a)]), one obtains

$$h_{1,1,-1}^{(\alpha,\beta)} = \frac{\beta-\alpha}{2(\alpha+1)(\alpha+\beta+4)}.$$

Thus $h_{1,1,-1}^{(\alpha,\beta)} < 0$ if $\alpha>\beta$, and for the case $\alpha=\beta>-\frac{1}{2}$, the assertion follows from the identity $h_{2,2,-2}^{(\alpha,\alpha)} = h_{1,1,-1}^{(\alpha,-1/2)}$ which will be proved in Lemma 3. All other parameters $(\alpha,\beta)\neq(-\frac{1}{2},-\frac{1}{2})$ do not belong to V, but for $(\alpha,\beta)\notin V$ there obviously occur negative coefficients $h_{n,m,k}^{(\alpha,\beta)}$.

Theorem 2. *If $\alpha\geqslant\beta\geqslant-\frac{1}{2}$, then there exists a constant $C=C(\alpha,\beta)$, such that*

$$\sum_{k\in I(n,m)} |h_{n,m,k}^{(\alpha,\beta)}| \leqslant C \quad \textit{for all } n,m\in\mathbb{Z}.$$

This theorem is the main result of our paper and will be proved in Section 4. It assures that within the cited range of parameters, the convolution of point measures as defined in (3.1) extends uniquely to a norm-continuous convolution on $M_b(\mathbb{Z})$, the space of bounded Borel measures on $\mathbb{Z}$:

Corollary. *Suppose $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and let the convolution $*$ of point measures on $\mathbb{Z}$ be defined according to* (3.1). *Then the following hold*:

(a) *$*$ extends uniquely to a bilinear convolution $* : M_b(\mathbb{Z}) \times M_b(\mathbb{Z}) \to M_b(\mathbb{Z})$ which is continuous with respect to the total variation norm $\|\cdot\|$.*

(b) *$(M_b(\mathbb{Z}), *)$ is a commutative Banach-*-algebra with unit δ_0, the involution $\sigma^*\{(n)\} := \sigma\{(-n)\}$ and the norm $\|\sigma\|' := \|L_\sigma\|$, where for $\sigma \in M_b(\mathbb{Z})$, L_σ is the multiplication operator on $M_b(\mathbb{Z})$ defined by $L_\sigma(\tau) := \sigma * \tau$.*

Proof. (a) The existence of a bilinear, $\|\cdot\|$-continuous extension of $*$ to $M_b(\mathbb{Z})$ is a consequence of Theorem 2; its uniqueness results from the $\|\cdot\|$-denseness of $M_e(\mathbb{Z}) = \{\sigma \in M_b(\mathbb{Z}): |\mathrm{supp}(\sigma)| < \infty\}$ in $M_b(\mathbb{Z})$.

(b) Commutativity and associativity of $*$ are clear by the definition of the convolution of point measures and the norm-continuity of $*$. Hence $(M_b(\mathbb{Z}), *)$ is a commutative normed algebra with unit δ_0, and $\|\cdot\|'$ is the canonical norm which makes it into a Banach algebra. It remains to show that * is an involution on this Banach algebra. First, we have to check that $(\sigma * \tau)^* = \sigma^* * \tau^*$ for all $\sigma, \tau \in M_b(\mathbb{Z})$. Again it is sufficient to consider point measures, and indeed, as the $h_{n,m,k}$ are real, we have

$$(\delta_n * \delta_m)^* = \sum_{k \in I(n,m)} h_{n,m,k}\delta_{-k} = \sum_{k \in I(n,m)} h_{-n,-m,k}\delta_k = \delta_n^* * \delta_m^*.$$

Moreover, if $\sigma \in M_b(\mathbb{Z})$, then

$$\|L_{\sigma^*}\| = \sup_{\|\tau\| \leqslant 1} \|\sigma^* * \tau\| = \sup_{\|\tau\| \leqslant 1} \|\sigma * \tau^*\| = \|L_\sigma\|.$$

Hence * is an isometry with respect to $\|\cdot\|'$. □

Remark. In fact, the convolution defined by (3.1) provides a special example of a class of convolution structures on $\mathbb{Z}$ associated with orthogonal trigonometric systems on the unit circle. These structures can be studied within the axiomatic frame of so-called signed hypergroups, which have recently been introduced in [7]. The investigation of this more general setting, in particular concerning the spectra of the involved Banach algebras, is the subject of a subsequent paper.

4. Proof of the main theorem

We stick close to the proof of the corresponding fact for Jacobi polynomials given in [2]. It was shown there that the linearization coefficients of the $R_n^{(\alpha,\beta)}$ satisfy

$$\sum_{k=|n-m|}^{n+m} |g^{(\alpha,\beta)}(n,m,k)| = \mathrm{O}(1)$$

uniformly in $n, m \in \mathbb{N}_0$, provided $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. The basic idea is an asymptotic reduction of integrals involving triple products of Jacobi polynomials to integrals which involve triple products of Bessel functions and can be evaluated explicitly.

We start with two auxiliary results; the first one is a straightforward analogue to Gegenbauer's formula for ultraspherical polynomials:

Lemma 3. *Let* $\alpha > -1$. *Then*:

(i) $\Psi_n^{(\alpha,-1/2)}(z^2) = \Psi_{2n}^{(\alpha,\alpha)}(z)$ *for all* $z \in T$ *and* $n \in \mathbb{Z}$.

(ii) $h_{2n,2m,k}^{(\alpha,\alpha)} = \begin{cases} h_{n,m,l}^{(\alpha,-1/2)} & \text{if } k = 2l, \\ 0 & \text{else.} \end{cases}$

Proof. (i) follows immediately from the identities (3.13) and (3.14) in [1] between the Jacobi polynomials $R_n^{(\alpha,-1/2)}$ and the ultraspherical polynomials $R_n^{(\alpha,\alpha)}$.

(ii) results from (i) by comparing the linearizations of $\Psi_n^{(\alpha,-1/2)}\Psi_m^{(\alpha,-1/2)}$ and of $\Psi_{2n}^{(\alpha,\alpha)}\Psi_{2m}^{(\alpha,\alpha)}$. $\square$

Lemma 4. *Let* $\delta > 0$ *be a fixed constant. For* $n, m, k \in \mathbb{N}_0$ *set* $N = n + \delta, M = m + \delta, K = k + \delta$ *and let*

$$\Delta(n,m,k) := \tfrac{1}{4}\sqrt{(M+N+K)(M+N-K)(K+M-N)(K+N-M)}$$

denote the area of the triangle with sides N, M, K. *Then for* $\alpha > -\frac{1}{2}$, *the estimate*

$$\sum_{k=|n-m|}^{n+m} k \cdot \Delta(n,m,k)^{2\alpha-1} = \mathrm{O}(n^{2\alpha}m^{2\alpha})$$

holds uniformly in $n, m \in \mathbb{N}$.

Proof. We may assume that $n \geqslant m$. Furthermore, in case $\alpha \geqslant \frac{1}{2}$ the statement is obvious, because $\Delta(n,m,k) \leqslant \frac{1}{2}nm$ and $2\alpha - 1 \geqslant 0$.

For the proof in the remaining case $|\alpha| < \frac{1}{2}$, it is convenient to set $l := k - n + m$ and $L := l + \delta = K - N + M$. With these notations,

$$\Delta(n,m,k)^2 = \left(N + \frac{L}{2}\right)\left(M - \frac{L}{2}\right)\frac{L}{2}\left(N - M + \frac{L}{2}\right) =: \delta(n,m,l),$$

as well as

$$S(n,m) := (nm)^{-2\alpha}\sum_{k=n-m}^{n+m} k \cdot \Delta(n,m,k)^{2\alpha-1}$$

$$= (nm)^{-2\alpha}\sum_{l=0}^{2m}\frac{l}{2}\cdot\delta(n,m,l)^{\alpha-1/2} + (nm)^{-2\alpha}\sum_{l=0}^{2m}\left(\frac{l}{2} + n - m\right)\cdot\delta(n,m,l)^{\alpha-1/2}.$$

Hence the following estimation holds:

$$S(n,m) = O\left((NM)^{-2\alpha}\sum_{l=0}^{2m}\frac{L}{2}\left(N\left(M-\frac{L}{2}\right)\left(\frac{L}{2}\right)^2\right)^{\alpha-1/2}\right)$$

$$+O\left((NM)^{-2\alpha}\sum_{l=0}^{2m}\left(\frac{L}{2}+N-M\right)\left(N\left(M-\frac{L}{2}\right)\frac{L}{2}\left(N-M+\frac{L}{2}\right)\right)^{\alpha-1/2}\right)$$

$$= O\left((NM)^{-\alpha-1/2}\sum_{l=0}^{2m}\left(\frac{L}{2}\right)^{2\alpha}\left(1-\frac{L}{2M}\right)^{\alpha-1/2}\right)$$

$$+O\left(M^{-\alpha-1/2}\sum_{l=0}^{2m}\left(\frac{L}{2}\right)^{\alpha-1/2}\left(1-\frac{L}{2M}\right)^{\alpha-1/2}\right)$$

$$= O\left(\sum_{l=0}^{2m}\frac{1}{2M}\left(\frac{L}{2M}\right)^{2\alpha}\left(1-\frac{L}{2M}\right)^{\alpha-1/2}\right) + O\left(\sum_{l=0}^{2m}\frac{1}{2M}\left(\frac{L}{2M}\right)^{\alpha-1/2}\left(1-\frac{L}{2M}\right)^{\alpha-1/2}\right).$$

Both sums in this last expression may be interpreted as Riemannian sums for certain beta-integrals, namely

$$\lim_{m\to\infty}\sum_{l=0}^{2m}\frac{1}{2M}\left(\frac{L}{2M}\right)^{2\alpha}\left(1-\frac{L}{2M}\right)^{\alpha-1/2} = \int_0^1 x^{2\alpha}(1-x)^{\alpha-1/2}\,dx < \infty;$$

$$\lim_{m\to\infty}\sum_{l=0}^{2m}\frac{1}{2M}\left(\frac{L}{2M}\right)^{\alpha-1/2}\left(1-\frac{L}{2M}\right)^{\alpha-1/2} = \int_0^1 x^{\alpha-1/2}(1-x)^{\alpha-1/2}\,dx < \infty.$$

This implies that $S(n,m) = O(1)$ uniformly in $n, m \in \mathbb{N}$. □

Proof of Theorem 2. We start with a series of reductions:

1. The assertion is clearly true for the "Chebyshev-system" ($\Psi_n^{(-1/2,-1/2)}$, $n \in \mathbb{Z}$), so it can be assumed without loss of generality that $\alpha > -\frac{1}{2}$. In view of Lemma 3, we can further assume that $\beta > -\frac{1}{2}$ as well.

2. It suffices to consider the case $|n| \geqslant |m| \geqslant 1$. Moreover, summations over $I(n,m)$ can be replaced by summations over $I(n,m)\setminus\{0\}$; this is because $(\alpha,\beta) \in V$ and hence

$$|h_{n,-n,0}^{(\alpha,\beta)}| = 4g^{(\alpha,\beta)}(n,n,0) = 4\,\frac{h^{(\alpha,\beta)}(0)}{h^{(\alpha,\beta)}(n)} \leqslant 4 \quad (n \in \mathbb{N}).$$

3. Suppose $|n|, |m|, |k| \geqslant 1$. Leaving again the superscripts, we can write

$$h_{n,m,k} = \frac{h(|k|)}{2}\left(\int_{-1}^1 R_{|n|}R_{|m|}R_{|k|}\,d\nu \pm l_m l_k \int_{-1}^1 R_{|n|}\,\widetilde{R}_{|m|-1}\,\widetilde{R}_{|k|-1}\,d\widetilde{\nu}\right.$$

$$\left.\pm l_n l_k \int_{-1}^1 R_{|m|}\,\widetilde{R}_{|n|-1}\,\widetilde{R}_{|k|-1}\,d\widetilde{\nu} \pm l_n l_m \int_{-1}^1 R_{|k|}\,\widetilde{R}_{|n|-1}\,\widetilde{R}_{|m|-1}\,d\widetilde{\nu}\right),$$

with $\pm$-distribution depending on the signs of n, m, k. For abbreviation, set

$$I(n,m,k) := \int_{-1}^1 R_{|n|}\,\widetilde{R}_{|m|-1}\,\widetilde{R}_{|k|-1}\,d\widetilde{\nu}.$$

With this notation,

$$|h_{n,m,k}| \leqslant \tfrac{1}{2}\, g(|n|,|m|,|k|) + \frac{h(|k|)}{2}\left(l_m l_k |I(n,m,k)| + l_n l_k |I(m,n,k)| + l_n l_m |I(k,n,m)|\right).$$

For a further reduction, the following relations between Jacobi polynomials of order (α,β) and $(\alpha+1,\beta+1)$ are employed; see formulas (4.10.1) and (4.21.7) in [8]:

$$R_n'(x) = 2(\alpha+1)\, l_n^2\, \widetilde{R}_{n-1}(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\, \widetilde{R}_{n-1}(x)\right) = -2(\alpha+1)(1-x)^{\alpha}(1+x)^{\beta} R_n(x).$$

Integration by parts thus yields

$$\begin{aligned} I(m,n,k) &= \frac{-1}{2(\alpha+1)l_n^2}\int_{-1}^{1} R_{|n|}(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(R_{|m|}(x)\,\widetilde{R}_{|k|-1}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right)\mathrm{d}x \\ &= -\frac{l_m^2}{l_n^2}\int_{-1}^{1} R_{|n|}\,\widetilde{R}_{|m|-1}\,\widetilde{R}_{|k|-1}\,\mathrm{d}\widetilde{\nu} + \frac{1}{l_n^2}\int_{-1}^{1} R_{|n|}R_{|m|}R_{|k|}\,\mathrm{d}\nu. \end{aligned}$$

Hence

$$|I(m,n,k)| \leqslant \frac{l_m^2}{l_n^2}\,|I(n,m,k)| + \frac{1}{l_n^2\, h(|k|)}\, g(|n|,|m|,|k|).$$

The same estimation, with k and m interchanged, holds for $|I(k,n,m)|$. Recalling that $l_m \leqslant l_n$ and $l_k \leqslant 2l_n$ for $|m| \leqslant |n|$ and $k \in I(n,m)$, we obtain

$$|h_{n,m,k}| \leqslant 2\, g(|n|,|m|,|k|) + 2h(|k|)\, l_m l_k |I(n,m,k)|.$$

The $g(n,m,k)$ being nonnegative, it remains to show that, for $n \geqslant m$,

$$\sum_{k=n-m,\, k\neq 0}^{n+m} h(k) l_m l_k \left|\int_{-1}^{1} R_n\, \widetilde{R}_{m-1}\, \widetilde{R}_{k-1}\, \mathrm{d}\widetilde{\nu}\right| = \mathrm{O}(1). \tag{4.1}$$

4. For the proof of (4.1), we essentially follow the exposition of Askey and Wainger [2]. First, we turn over to the classical normalization of the Jacobi polynomials,

$$P_n^{(\alpha,\beta)} = \binom{n+\alpha}{n} R_n^{(\alpha,\beta)}, \quad n \in \mathbb{N}_0.$$

According to the asymptotic relations

$$\binom{n+\alpha}{n} = \mathrm{O}(n^{\alpha}), \quad l_n = \mathrm{O}(n), \quad h(n) = \mathrm{O}(n^{2\alpha+1})$$

(the latter two are easily obtained from the explicit formulas (2.1) and (2.2)), it is sufficient to estimate

$$\sum_{k=n-m,\, k\neq 0}^{n+m} k^{2\alpha+1}(knm)^{-\alpha}$$
$$\times\left|\int_0^{\pi} P_n^{(\alpha,\beta)}(\cos t)\, P_{m-1}^{(\alpha+1,\beta+1)}(\cos t)\, P_{k-1}^{(\alpha+1,\beta+1)}(\cos t)\left(\sin\frac{t}{2}\right)^{2\alpha+3}\left(\cos\frac{t}{2}\right)^{2\beta+3}\mathrm{d}t\right|.$$

Moreover, the symmetry relation $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ (see [8, (4.1.3)]) reduces our assertion to the proof of

$$\sum_{k=n-m,\, k\neq 0}^{n+m} k^{\alpha+1}$$
$$\times\left|\int_0^{\pi/2} P_n^{(\alpha,\beta)}(\cos t)\, P_{m-1}^{(\alpha+1,\beta+1)}(\cos t)\, P_{k-1}^{(\alpha+1,\beta+1)}(\cos t)\left(\sin\frac{t}{2}\right)^{2\alpha+3}\left(\cos\frac{t}{2}\right)^{2\beta+3}\mathrm{d}t\right| = \mathrm{O}(n^\alpha m^\alpha) \tag{4.2}$$

for arbitrary $\alpha, \beta > -\frac{1}{2}$ (not necessarily $\alpha \geqslant \beta$).

The tool for replacing the Jacobi polynomials in (4.2) by terms involving Bessel functions is the following version of Hilb's formula, see [2]:

$$\left(\sin\frac{t}{2}\right)^{\alpha+1/2}\left(\cos\frac{t}{2}\right)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos t) = \frac{N^{-\alpha}\Gamma(n+\alpha+1)}{n!}\sqrt{\frac{t}{2}}\, J_\alpha(Nt) + R^{(\alpha,\beta)}(n,t)$$
$$\left(\alpha, \beta > -1,\ n\in\mathbb{N},\ N := n + \frac{\alpha+\beta+1}{2}\right).$$

Here J_α is the Bessel function of order α, and for $0 < t \leqslant 3\pi/4$, the error term $R^{(\alpha,\beta)}(n,t)$ is given by

$$R^{(\alpha,\beta)}(n,t) = \begin{cases}\mathrm{O}(tn^{-3/2}) & \text{for all } t, \qquad (4.3)\\ \mathrm{O}(t^{\alpha+5/2}n^\alpha) & \text{if } nt \leqslant C, \qquad (4.4)\end{cases}$$

with an arbitrary fixed constant $C > 0$.

Furthermore, the following estimates of Bessel functions will be involved:

$$|J_\alpha(x)| \leqslant Ax^\alpha \quad \text{for } 0 < x \leqslant 1, \tag{4.5}$$

$$|J_\alpha(x)| \leqslant Ax^{-1/2} \quad \text{for } x > 0,\ \alpha \geqslant -\tfrac{1}{2}. \tag{4.6}$$

We set $N := n + \delta$, $M := m + \delta$, $K := k + \delta$, with $\delta = (\alpha+\beta+1)/2$. Then according to Hilb's formula above, we can write

$$\left(\sin\frac{t}{2}\right)^{2\alpha+3}\left(\cos\frac{t}{2}\right)^{2\beta+3} P_n^{(\alpha,\beta)}(\cos t)\, P_{m-1}^{(\alpha+1,\beta+1)}(\cos t)\, P_{k-1}^{(\alpha+1,\beta+1)}(\cos t)$$
$$= C_{n,m,k}\left(\sin\frac{t}{2}\right)^{-\alpha-1/2}\left(\cos\frac{t}{2}\right)^{-\beta-1/2}\left(\frac{t}{2}\right)^{3/2} J_\alpha(Nt)\, J_{\alpha+1}(Mt)\, J_{\alpha+1}(Kt) + S_{n,m,k}(t),$$

where

$$C_{n,m,k} = \frac{N^{-\alpha}\Gamma(n+\alpha+1)}{n!} \cdot \frac{M^{-\alpha-1}\Gamma(m+\alpha+1)}{(m-1)!} \cdot \frac{K^{-\alpha-1}\Gamma(k+\alpha+1)}{(k-1)!}$$

and the error term $S_{n,m,k}(t)$ can be estimated in a first step as follows:

$$\begin{aligned}|S_{n,m,k}(t)| \leqslant At^{-\alpha-1/2}\Big(& t\,|J_\alpha(Nt)J_{\alpha+1}(Kt)\,R^{(\alpha+1,\beta+1)}(m-1,t)| \\ &+ t\,|J_\alpha(Nt)J_{\alpha+1}(Mt)\,R^{(\alpha+1,\beta+1)}(k-1,t)| + t\,|J_{\alpha+1}(Kt)J_{\alpha+1}(Mt)\,R^{(\alpha,\beta)}(n,t)| \\ &+ \sqrt{t}\,|J_\alpha(Nt)\,R^{(\alpha+1,\beta+1)}(m-1,t)\,R^{(\alpha+1,\beta+1)}(k-1,t)| \\ &+ \sqrt{t}\,|J_{\alpha+1}(Mt)\,R^{(\alpha,\beta)}(n,t)\,R^{(\alpha+1,\beta+1)}(k-1,t)| \\ &+ \sqrt{t}\,|J_{\alpha+1}(Kt)\,R^{(\alpha,\beta)}(n,t)\,R^{(\alpha+1,\beta+1)}(m-1,t)| \\ &+ |R^{(\alpha,\beta)}(n,t)\,R^{(\alpha+1,\beta+1)}(k-1,t)\,R^{(\alpha+1,\beta+1)}(m-1,t)|\Big),\end{aligned}$$

with a constant A independent of n, m, k and $t \in [0, \pi/2]$. Once it is shown that

$$I := \sum_{k=n-m,\,k\neq 0}^{n+m} k^{\alpha+1} \int_0^{\pi/2} |S_{n,m,k}(t)|\,\mathrm{d}t = \mathrm{O}(n^\alpha m^\alpha), \tag{4.7}$$

it remains to estimate the sum

$$\sum_{k=n-m}^{n+m} k^{\alpha+1} \left| \int_0^{\pi/2} t^{3/2} \left(\sin\frac{t}{2}\right)^{-\alpha-1/2} \left(\cos\frac{t}{2}\right)^{-\beta-1/2} J_\alpha(Nt)J_{\alpha+1}(Mt)J_{\alpha+1}(Kt)\,\mathrm{d}t \right|. \tag{4.8}$$

We are going to carry out the proof of (4.7) in one step. For this, we set $I := I_1 + I_2$, where in I_1 the range of integration is $[0, 1/m]$ and in I_2 the range of integration is $[1/m, \pi/2]$. In I_1, we use the estimates (4.4) and (4.5) respectively for the terms containing m, as well as (4.3) and (4.6) respectively for the terms containing n and k. It follows that

$$\begin{aligned} I_1 &= \mathrm{O}\Bigg(\sum_{k=n-m,\,k\neq 0}^{n+m} (km)^{\alpha+1} \int_0^{1/m} (n^{-1/2}k^{-1/2}t^3 + n^{-1/2}k^{-3/2}t^2 + k^{-1/2}n^{-3/2}t^2 \\ &\qquad + n^{-1/2}k^{-3/2}t^4 + n^{-3/2}k^{-3/2}t^3 + n^{-3/2}k^{-1/2}t^4 + n^{-3/2}k^{-3/2}t^5)\,\mathrm{d}t \Bigg) \\ &= \mathrm{O}\Bigg(\sum_{k=n-m,\,k\neq 0}^{n+m} (km)^{\alpha+1} \cdot (nk)^{-1/2} m^{-2} \int_0^{1/m} 1 \cdot \mathrm{d}t \Bigg) = \mathrm{O}(n^\alpha m^{\alpha-1}) = \mathrm{O}(n^\alpha m^\alpha). \end{aligned}$$

In I_2, the estimates (4.3) and (4.6) lead to

$$\begin{aligned} I_2 &= \mathrm{O}\Bigg(\sum_{k=n-m,\,k\neq 0}^{n+m} k^{\alpha+1} \int_{1/m}^{\pi/2} \Bigg(\left(\frac{1}{n} + \frac{1}{m} + \frac{1}{k}\right)(nmk)^{-1/2}\,t^{-\alpha+1/2} \\ &\qquad + \left(\frac{1}{nm} + \frac{1}{km} + \frac{1}{kn}\right)(nmk)^{-1/2}\,t^{-\alpha+3/2} + (nmk)^{-3/2}\,t^{-\alpha+5/2}\Bigg)\,\mathrm{d}t \Bigg) \\ &= \mathrm{O}\left(n^\alpha m^{-1/2} \int_{1/m}^{\pi/2} t^{-\alpha+1/2}\mathrm{d}t \right) = \mathrm{O}(n^\alpha m^{-1/2}(C + m^{\alpha-3/2} + \delta_{\alpha,3/2} \ln m)) = \mathrm{O}(n^\alpha m^\alpha). \end{aligned}$$

Thus (4.7) is proved. The next step is a further reduction of the sum (4.8). This runs exactly in the same way as it was carried out in [2] for the case of three Bessel functions of the same order; we shall therefore restrict ourselves to a sketch of the important steps.

First, the term $(\sin(t/2))^{-\alpha-1/2}(\cos(t/2))^{-\beta-1/2}$ has to be replaced by $t^{-\alpha-1/2}$. For this, write $(\sin(t/2))^{-\alpha-1/2}(\cos(t/2))^{-\beta-1/2} = (t/2)^{-\alpha-1/2}G(t)$, where $G : [0,\pi/2] \to \mathbb{R}$ is continuous with $1 - G(t) = \mathrm{O}(t^2)$. As in [2, p. 29f], it is then checked that

$$\sum_{k=n-m}^{n+m} k^{\alpha+1} \left| \int_0^{\pi/2} t^{1-\alpha}(1-G(t))J_\alpha(Nt)J_{\alpha+1}(Mt)J_{\alpha+1}(Kt)\,\mathrm{d}t \right| = \mathrm{O}(n^\alpha m^\alpha).$$

It remains to estimate $S(0,\pi/2)$, where for $\sigma,\tau \in \mathbb{R}_+ \cup \infty$, $S(\sigma,\tau)$ is defined by

$$S(\sigma,\tau) = \sum_{k=n-m}^{n+m} k^{\alpha+1} \left| \int_\sigma^\tau t^{1-\alpha}J_\alpha(Nt)J_{\alpha+1}(Mt)J_{\alpha+1}(Kt)\,\mathrm{d}t \right|.$$

This in turn is achieved by splitting up the range of integration: we show that

(a) $S(\pi/2,\infty) = \mathrm{O}(n^\alpha m^\alpha)$,

(b) $S(0,\infty) = \mathrm{O}(n^\alpha m^\alpha)$.

The proof of (a) is the same as in [2, p. 30f]; hereby the condition $\alpha > -\frac{1}{2}$ is of decisive importance. The proof of (b), however, affords some additional argumentation: For $\alpha > -\frac{1}{2}$ the integral

$$J(n,m,k) := \int_0^\infty t^{1-\alpha}J_\alpha(Nt)J_{\alpha+1}(Mt)J_{\alpha+1}(Kt)\,\mathrm{d}t$$

is a generalization of the Weber–Schafheitlin integral, whose explicit evaluation is known [4, 8.11.34]. As $n-m \leqslant k \leqslant n+m$ and therefore $|N-M| < K < N+M$, it is given by

$$J(n,m,k) = \frac{(MK)^{\alpha-1}}{\sqrt{2\pi}\,N^\alpha}(1-\sigma^2)^{\alpha/2-1/4}P_{\alpha+1/2}^{1/2-\alpha}(\sigma),$$

where $\sigma = (1/(2MK))(M^2+K^2-N^2)$ and $P_{\alpha+1/2}^{1/2-\alpha}$ is the associated Legendre function of order $\frac{1}{2}-\alpha$ and degree $\alpha+\frac{1}{2}$. According to formulas 3.8.(17) and 3.6.1.(14) in [3],

$$P_{\alpha+1/2}^{1/2-\alpha}(\sigma) = P_{\alpha+1/2}^{-(\alpha+1/2)+1}(\sigma) = \frac{2^{-\alpha+1/2}}{\Gamma\left(\alpha+\frac{1}{2}\right)}\,\sigma\cdot(1-\sigma^2)^{\alpha/2-1/4}.$$

With $\Delta(n,m,k)$ defined as in Lemma 4, we can write

$$(MK)^2(1-\sigma^2) = 4\,\Delta(n,m,k)^2.$$

It follows that

$$J(n,m,k) = \frac{2^{\alpha-1}}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\cdot\frac{\sigma}{(NMK)^\alpha}\,\Delta(n,m,k)^{2\alpha-1}.$$

As $|\sigma| < 1$, Lemma 4 now assures that (b) is satisfied, and thus the proof of the theorem is finished. □

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