# Special values of $L$-functions
# Abstracts

**Baskar Balasubramanyam** (IISER Pune)

*Title:* $p$-adic adjoint $L$-functions for Hilbert modular forms.

*Abstract:* Let $F$ be a totally real field. Let $\pi$ be a cuspidal cohomological automorphic representation for $\mathrm{GL}_2/F$. Let $L(s, \mathrm{Ad}^0, \pi)$ denote the adjoint $L$-function associated to $\pi$. The special values of this $L$-function and its relation to congruence primes have been studied by Hida, Ghate and Dimitrov. Let $p$ be an integer prime. In this talk, I will discuss the construction of a $p$-adic adjoint $L$-function in neighbourhoods of very decent points of the Hilbert eigenvariety. As a consequence, we relate the ramification locus of this eigenvariety to the zero set of the $p$-adic $L$-functions. This was first established by Kim when $F = \mathbb{Q}$. We follow Bellaiche's description of Kim's method, generalizing it to arbitrary totally real number fields. This is joint work with John Bergdall and Matteo Longo.

**Denis Benois** (Université de Bordeaux)

*Title:* Iwasawa theory for critical modular forms

*Abstract:* An elliptic eigenform $f$ of weight $k$ is said to be $\theta$-critical if it belongs to the image of the $\theta$-operator $\left(\frac{d}{dq}\right)^{k-1}$. The Iwasawa theory of non-$\theta$-critical forms has been extensively studied. In this talk, we are interested in the $\theta$-critical case. Using the geometry of the eigencurve and the theory of overconvergent modular symbols, Bellaïche constructed a two-variable $p$-adic $L$-function in an infinitesimal neighborhood of $f$. We give an 'étale' construction of this $p$-adic $L$-function. We also define some Selmer complex which can be seen as the algebraic counterpart of Bellaïche's $p$-adic $L$-function and discuss the Iwasawa Main Conjecture in this context. This is a joint work with K. Büyükboduk.

**Siegfried Böcherer** (Universität Mannheim)

*Title:* On denominators of some L-values, when twisted by characters

*Abstract:* Can we expect 'more' denominators, when we twist $L$-values by arbitrary Dirichlet characters? We look at this question for $L$-values arising in some way from Siegel Eisenstein series, i. e. for standard $L$-functions attached to Siegel modular forms and also for triple product $L$-functions. The first case is due to the author and H. Katsurada and the second case is treated in an ongoing Mannheim PhD thesis by T. Keller.

**Shih-Yu Chen** (Kyoto University)

*Title:* Algebraicity of ratios of special values of Rankin-Selberg $L$-functions.

*Abstract:* For consecutive critical values of Rankin–Selberg $L$-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n'}$, we have the celebrated result of Harder and Raghuram on the algebraicity of the ratios (in terms of relative periods when $n$ or $n'$ is odd). As a different aspect of ratios of critical values, we consider ratios of product of different Rankin–Selberg $L$-functions at a fixed critical point. In this talk, we introduce our result on the algebraicity of the ratios under some parity and regularity conditions. The parity condition can be removed subject to certain archimedean non-vanishing hypothesis. As applications, we prove new cases of Blasius' and Deligne's conjectures on critical values of tensor product $L$-functions and symmetric power $L$-functions of modular forms.

**Laurent Clozel** (Université Paris-Sud 11)

*Title:* Invariance, by conjugation of the coefficients, of the central zero of certain Rankin $L$-functions. (Joint work with A. Kret).

*Abstract:* Deligne has conjectured that certain 'critical' special values of the $L$-functions of motives are given by periods. A consequence is that their vanishing is invariant by conjugation of the coefficients of the motive.

Given the conjectural correspondence between motives over a number field F and 'algebraic' representations of $\mathrm{GL}(N, \mathbb{A}_F)$, this implies the corresponding properties of the algebraic automorphic, cuspidal $L$-functions of this group. If $N$ is even, this applies to the central value at $s = 1/2$.

Many results of this kind have been proved by Moeglin, Grobner, Raghuram and others, sometimes as consequences of explicit formulas for the special values. In this lecture I will describe a proof of this invariance for Rankin $L$-functions associated to a pair of suitably regular algebraic, self-dual cuspidal representations of $\mathrm{GL}(m)$ and $\mathrm{GL}(n)$ when $m$, $n$ have opposite parities and $F$ is totally real.

**Masaaki Furusawa** (Osaka Metropolitan University)

*Title:* On a certain Ichino-Ikeda type formula and the generalized Böcherer conjecture.

*Abstract:* We discuss the Ichino-Ikeda type formula for the Bessel periods in the case of $(\mathrm{SO}(5), \mathrm{SO}(2))$. As a corollary of our formula, we obtain an explicit formula relating certain weighted averages of Fourier coefficients of holomorphic Siegel cusp forms which are Hecke eigenforms to central special values of $L$-functions, thanks to computations of local integrals by Dickson, Pitale, Saha and Schmidt. The formula is regarded as a natural generalization of Böcherer's conjecture to the nontrivial torus character case. This is a joint work with Kazuki Morimoto at Kobe University.

**Günter Harder** (Universität Bonn)

*Title:* Eisenstein cohomology and special values: An outlook.

*Abstract:* Various authors have proved algebraicity (rationality) results for special values of $L$-functions, these $L$-functions are attached to certain automorphic forms (or perhaps better to eigenclasses in the cohomology of arithmetic groups). The guiding principle is the rational structure on the transcendentally defined Eisenstein cohomology. But these authors look at special situations and specific subspaces in the cohomology.

In my talk I will explain my view of Eisenstein cohomology and will discuss the goal of this concept. Furthermore I will explain the possible implications for special values that can be derived from an understanding of the Eisenstein cohomology.

**Ming-Lun Hsieh** (National Taiwan University)

*Title:* $p$-adic $L$-functions for $\mathrm{U}(2,1) \times \mathrm{U}(1,1)$ and the Ichino–Ikeda conjecture.

*Abstract:* In this talk, I will talk about the $p$-adic interpolation of Gross–Prasad periods in the setting of $\mathrm{U}(2,1) \times \mathrm{U}(1,1)$. Thanks to the Ichino–Ikeda conjecture established by R. Beuzart-Plessis–Y. Liu–W. Zhang–X. Zhu in the stable case and R. Beuzart-Plessis–Chaudouard–Zydor in the endoscopic case, this leads to a construction of the five variable $p$-adic $L$-functions associated with Hida families for $\mathrm{U}(3) \times \mathrm{U}(2)$. This is a joint work in progress with M. Harris and S. Yamana.

**Chi-Yun Hsu** (Université de Lille)

*Title:* Eigenvariety for partially classical Hilbert modular forms.

*Abstract:* It is often useful to regard modular forms as in the larger space of $p$-adic overconvergent modular forms, so that $p$-adic analytic techniques can be used to study them. The geometric interpretation of this is an eigenvariety, which is a rigid analytic space parametrizing finite-slope overconvergent Hecke eigenforms. For Hilbert modular forms, Andreatta–Iovita–Pilloni constructed $p$-adic families of modular sheaves as well as the eigenvariety. Moreover, for Hilbert modular forms, it makes sense to consider an intermediate notion — the partially classical overconvergent forms. I will talk about the construction of the eigenvariety in this scenario following the approach of Andreatta–Iovita–Pilloni. As an application, it can be proved that the Galois representation associated to a partially classical Hilbert Hecke eigenform is partially de Rham.

**Debanjana Kundu** (Fields Institute)

*Title:* $p \neq q$ Iwasawa Theory.

*Abstract:* Let $K$ be an imaginary quadratic field of class number 1 such that both $p$ and $q$ split in $K$. We show that under appropriate hypotheses, the $p$-part of the ideal class groups is bounded over finite subextensions of an anticyclotomic $\mathbb{Z}_q$-extension of $K$. This is joint work with Antonio Lei.

**Kenichi Namikawa** (Tokyo Denki University)

*Title:* An integrality of critical values of Rankin-Selberg $L$-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$.

*Abstract:* By using the generalized modular symbol method due to Kazhdan–Mazur–Schmidt, Raghuram proved the algebraicity of ratios of period integrals of Rankin-Selberg $L$-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ to Whittaker periods. In this talk, we consider a refinement of the generalized modular symbol method and we prove the algebraicity of critical values of Rankin–Selberg $L$-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ if the base fields are totally imaginary. Key ingredients of the proof are a description of local systems on $\mathrm{GL}(n)$ by Gel'fand–Tsetlin basis and an explicit formula for the archimedean local zeta integrals. Furthermore, we also propose a formulation for $p$-integral properties of these critical values. This is a joint work with Takashi Hara (Tsuda university) and Tadashi Miyazaki (Kitasato university).

**Peter Schneider** (Universität Münster)

*Title:* The Mellin transform over Robba rings.

*Abstract:* After briefly recalling the classical as well as the $p$-adic Mellin transform which is a basic tool to construct $L$-functions I will emphazise that in the $p$-adic case its background is a statement about certain distribution algebras. Then I will present joint work with O. Venjakob in which we generalize this formalism to Robba rings and $(\phi, \Gamma)$-modules. The main result is a structural

statement about the $\psi$-operator on such a module. Moreover, this leads to a basic relation between multiplicative and additive Robba rings which is the starting point of all reciprocity laws in this area.

**Michael Spiess** (Universität Bielefeld)

*Title:* Adelic Eisenstein Classes and special values of partial $\zeta$-functions of totally real fields.

*Abstract:* I will introduce an adelic variant of the Eisenstein classes of Beilinson, Kings and Levin and present applications to partial zeta values of totally real fields. This is joint work with Alexandros Galanakis.

**Joachim Schwermer** (Universität Wien)

*Title:* On residues of Eisenstein series and related classes in the cohomology of arithmetic groups.

*Abstract:* The cohomology of an arithmetic congruence subgroup of a connected reductive algebraic group defined over a number field is strongly related to the theory of automorphic forms. We are interested in the part of the cohomology which can be represented by square-integrable residues of Eisenstein series. Their existence depends on a subtle combination of geometric conditions (coming from cohomological reasons) and arithmetic conditions in terms of analytic properties of automorphic $L$-functions (coming from the study of poles of Eisenstein series). We discuss some cases in which an eventual construction can be carried through.

**Jacques Tilouine** (Université Paris 13)

*Title:* Results and Conjectures on integral period relations.

*Abstract:* In a joint work with E. Urban we established integral period relations for quadratic base changes of a classical cusp form. In view of results by Balasubramanyam–Raghuram and Ichino, we can formulate integral period relation conjectures for other Langlands functorialities. For the symmetric square, our conjecture is related to a remark by Kazhdan–Mazur and Claus Schmidt on periods for linear groups. The potential proof of one divisibility predicted by each conjectural integral period relation relies on the construction of an integral 'period linear form' vanishing on non-base changes, which has to be constructed in each case.

**Christopher Williams** (University of Warwick)

*Title:* $p$-adic $L$-functions for GL(3).

*Abstract:* Let $\pi$ be a $p$-ordinary cohomological cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$. Coates and Perrin-Riou conjectured that the (algebraic parts of) special values of its $L$-function satisfy beautiful $p$-adic congruences, captured in the existence of a $p$-adic $L$-function. For $n = 1, 2$ constructions of such $p$-adic $L$-functions predate the conjecture by decades, but for $n > 2$ our understanding is quite incomplete: in all previous constructions, $\pi$ is assumed to be (at least) a functorial transfer via a proper subgroup of $\mathrm{GL}(n)$ (e.g. for symmetric squares from GL(2), Rankin–Selberg transfers, $\pi$ admitting Shalika models, etc).
In this talk, I will describe joint work with David Loeffler, where we construct a $p$-adic $L$-function when $n = 3$, without any transfer/self-duality assumptions. Our method uses a 'Betti Euler system', where we construct a tower of integral Eisenstein classes in the Betti cohomology for GL(3). If time permits, I'll discuss an application to the (automorphic realisation of) Deligne's algebraicity conjecture in this case, strengthening a theorem of Mahnkopf (by proving the expected compatibility of periods at infinity).