Kleine AG - Canonical models of Shimura Varieties

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Classically, a modular form is defined as a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying certain growth and periodicity conditions determined by an arithmetic subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$. To generalise such a construction, one first notices that the upper half plane can be realised as a quotient:

$$\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}),$$

and for an arithmetic subgroup Γ of $\mathrm{PGL}_2(\mathbb{Q})^+$, the manifold $\Gamma\backslash\mathcal{H}$ has the structure of a complex algebraic variety (and is often referred to as a modular curve), such that modular forms live in certain cohomology groups corresponding to this variety. A similar procedure can be followed for many other rational reductive groups. For instance, if we consider $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$, for a totally real field $F/\mathbb{Q}$, we can construct a Hilbert modular variety and recover Hilbert modular forms. And, if we consider GSp_{2g}, we can construct a Siegel modular variety and recover Siegel modular forms. In general, Shimura varieties are (families of) certain complex algebraic varieties whose analytifications are quotients of hermitian symmetric domains. The aforementioned Siegel modular varieties, Hilbert modular varieties and modular curves (more accurately, quotients of the form $\Gamma\backslash\mathcal{H}^{\pm}$) are examples of Shimura varieties (with additional level).

Although such varieties were studied by Shimura, it was Deligne who formalised the theory. Following Deligne's definition (reformulated in a more up to date language), a Shimura datum is a pair (G, X), where G is connected reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$, where $\mathbb{S}$ is the Deligne torus, along with three conditions, often denoted by SV1, SV2 and SV3 (which guarantees that X is a finite disjoint union of hermitian symmetric domains). For some 'sufficiently small' compact open subgroup $K \subseteq G(\mathbb{A}_f)$, a Shimura variety relative to (G, X) of level K is a complex variety, which we shall denote by ${}_KM_{\mathbb{C}}(G, X)$. This variety is defined such that its $\mathbb{C}$-points are of the form:

$${}_KM_{\mathbb{C}}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

This adelic interpretation of Shimura varieties defines a natural $G(\mathbb{A}_f)$-action on their cohomology groups. But, since they are defined over $\mathbb{C}$, one does not have any 'interesting' Galois action. Following works by Shimura and his students, Miyake and Shih, Deligne in [Del71] proved that Shimura varieties of Hodge type (which includes Shimura varieties of PEL type) have 'canonical' models over a number field $E(G, X)$, often referred to as the *reflex field* of the Shimura datum. Later, in [Del79], Deligne showed that a more general class of abelian type Shimura varieties have canonical models over their reflex field. The latter will be the main focus of this kleine AG (there is also an English translation of this paper, [Mil20] by Milne). Thanks to the work of Borovoi and Milne, the existence of canonical models was proved for general Shimura varieties in [Mil83].

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Notations

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are the integers, rationals, reals and complex numbers respectively.
- $\overline{\mathbb{Q}}$ is an algebraic closure of the rationals
- $\mathcal{H}$ and $\mathcal{H}^{\pm}$ are the upper half plane and the disjoint union of the upper and lower half planes respectively.
- $\mathcal{H}_g^{\pm}$ denotes the Siegel upper and lower half planes of real dimension $2g$.
- $\mathbb{S} = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the Deligne torus.
- $\mathbb{A}^f = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_p$ is the ring of finite adèles of $\mathbb{Q}$.
- G shall always be a connected reductive group over $\mathbb{Q}$ with centre Z.
- G^{ad} and G^{der} are the adjoint group and derived subgroup of G respectively.

Registration/Introduction: 10:00 - 10:15

Talk 1: Modular forms, elliptic curves and Hodge structures

Time slot: 10:15 - 11:15 (1 hour)
We start off by defining a modular form of weight k with respect to a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, as done in [DS06]. We shall then discuss why the complex upper half plane $\mathcal{H}$, is a moduli space of complex elliptic curves, and show that $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$. We then redefine a modular form of weight k and level n as a rule on elliptic curves (with additional structure) as done in [Kat73], and mention the connection between the two definitions. We shall also define rational and real Hodge structures, and discuss the equivalence between Hodge structures on real vector spaces V and morphisms $\mathbb{S} \to \mathrm{GL}(V)$ as done in §2 of [Mil05] (all vector spaces are finite dimensional). Lastly, we will show that $\mathcal{H}$ can be seen as a conjugacy class of Hodge structures $\mathbb{S} \to \mathrm{PGL}_{2,\mathbb{R}}$.

Talk 2: Shimura datum and Shimura varieties

Time slot: 11:30 - 12:30 (1 hour)
We shall define Shimura datum and Shimura variety as in [Mil05] (where he denotes the former by SV1-SV3) and [Del79] (where he denotes the former by 2.1.1.1-2.1.1.3). Then, we shall show that $(\mathrm{GL}_2, \mathcal{H}^{\pm})$ is an example of a Shimura datum, and mention why the Siegel Shimura datum $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ is a generalisation. Given a Shimura datum (G, X), we shall prove Proposition 2.1.10 in [Del79], which states that

$$M_{\mathbb{C}}(G,X) = \Big(\frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\Big)\backslash X \times (G(\mathbb{A}^f)/Z(\mathbb{Q})^-)$$

(please note that there is a typo in Milne's translation of this proposition), and discuss Corollary 2.1.11 in [Del79].

Lunch break: 12:30 - 14:00
Next Kleine AG discussion: 14:00 - 14:30

Talks 3: Variation of Hodge structures and types of Shimura varieties

Time slot: 14:30 - 15:30 (1 hour)
We will define variation of Hodge structures as in Definition 1.1.7 of [Del79], followed by Proposition 1.1.14 in [Del79], which shows its relationship with Shimura data. We will define what it means for a Shimura variety to be of Hodge type, as in Definition 7.1 of [Mil05], and discuss the associated the moduli problem, as proved in Theorem 7.4 in [Mil05] (if time permits, we will discuss Shimura varieties of PEL type as done in §8 of [Mil05] and mention Theorem 8.17 in the same article). We shall finally define what is means for a Shimura variety to be of abelian type, as done in Definition 9.2 of [Mil05], and mention Theorem 9.4, again in the same article.

Talks 4: Canonical models

Time slot: 15:45 - 16:45 (1 hour)
Given a Shimura datum (G, X), we shall first define the reflex fields $E(G, X)$ and $E(G, X^+)$, where $X^+ \subseteq X$ is a connected component, as done in section 2.2.1 of [Del79] (Deligne does mention the term reflex field, but defines it as *corps dual*, and uses this term later on - but reflex field is a lot more common nowadays). For a Shimura datum $(T, \{h\})$, where T is a torus over $\mathbb{Q}$, and any number field E containing $E(T, \{h\})$, we shall then define the reciprocity morphism

$$r_E(T, \{h\}) : \mathrm{Gal}(\overline{\mathbb{Q}}, E)^{\mathrm{ab}} \to T(\mathbb{A}^f)/T(\mathbb{Q})^-$$

as done in section 2.2.3 of [Del79]. Now, given a Shimura datum (G, X), and a compact open subgroup $K \subseteq G(\mathbb{A}^f)$, we shall define what it means for a point on ${}_K M_{\mathbb{C}}(G, X)$ (resp. $M_{\mathbb{C}}(G, X)$) to be special/CM (it is highly desirable that the speaker mentions the connection with CM abelian varieties). Then, for a number field E containing $E(G, X)$, we will define the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the set of special points of ${}_K M_{\mathbb{C}}(G, X)$ (resp. $M_{\mathbb{C}}(G, X)$) as done in 2.2.4 of [Del79]. Once we have defined all the necessary terms, we shall define and discuss canonical models and weakly canonical models of Shimura varieties as in 2.2.5 and 2.2.6 of [Del79]. Lastly, we shall discuss Deligne's criterion 2.3.1 from [Del79], which guarantees the existence of canonical models for Shimura varieties of Hodge type (if time permits, we will discuss a rough sketch of the proof).

Talk 5: Canonical models of Shimura varieties of abelian type

Time slot: 17:00 - 18:00 (1 hour)
Given a Shimura datum (G, X) and a connected component $X^+ \subseteq X$, we shall first define a (connected) canonical model $M^\circ(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ as done in 2.7.10 in [Del79]. We shall then discuss its functorial properties (i.e. 2.7.11 in [Del79]). We will then discuss Proposition 2.7.13 and 2.7.18, which relates the canonical models $M(G, X)$ and $M^\circ(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$, and Corollary 2.7.19 which guarantees that Shimura varieties of abelian type have unique canonical models, from [Del79]. A quick outline of the proof is also discussed in §14 of [Mil05]. Lastly, if time permits, we will discuss Theorem 2.7.20 from [Del79], which describes the types of reductive groups which admit canonical models as a consequence of 2.7.19 and criterion 2.3.1 (discussed in the previous talk).

References

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