FIFTY YEARS OF TOPOLOGICAL ALGEBRA

Dedicated to Karl H. Hofmann
On the Occasion Of
His 75th Birthday

Jimmie D. Lawson
Louisiana State University

Abstract. Over the past 50 years Karl Hofmann has made major, wide-ranging contributions across the spectrum in topological algebra:

compact semigroups, transformation groups, rings and sheaves, continuous lattices and domain theory, C^*-algebras, Lie semigroups, loops, divisible groups and semigroups, the exponential function, compact groups, and proLie groups.

The Fifties

TOPOLOGICAL SEMIGROUPS EMERGE

A *topological semigroup* consists of a semigroup S endowed with a Hausdorff topology such that the multiplication function

$$(s,t) \mapsto st\colon S \times S \to S$$

is (jointly) continuous. The pioneering work in the study of compact (topological) semigroups was carried out by A. D. Wallace and his students and co-workers at Tulane during the fifties. The accent in those early days (reflecting Wallace's mathematical interests) was on semigroup connections with algebraic topology and the topology of continua.

Semigroups appear more naturally in a physical universe than in a geometric one...[They] might be regarded as exemplars of irreversible actions.
A. D. Wallace

TOPOLOGICAL GEOMETRY

Meanwhile, in Germany, a young mathematician by the name of Karl H. Hofmann was seeking to apply topological algebra to topological geometry. There are natural ways to assign coordinate fields and more general algebraic structures to geometric objects such as projective planes. Hofmann's early explorations into topological geometry led to the consideration and characterization of locally compact semigroups that were the union of a group and a zero element, or more generally a group and a compact boundary.

Topological Geometry At Work

HOFMANN UNDERGOES TRANSATLANTIC TRANSLATION

HOFMANN'S semigroup investigations touched on work of MOSTERT and SHIELDS, two Tulane mathematicians, who studied semigroups on manifolds.

The Tulane connection ripened through further contact with MOSTERT and WALLACE, eventually leading to HOFMANN'S moving to New Orleans and spending 20 years on the Tulane faculty until he returned to Germany in the early 80's.

Street Scene
New Orleans

ONE-PARAMETER SEMIGROUPS

The additive semigroup of nonnegative reals, its one-point compactification

$$([0, \infty], +) \cong ([1, 0], \cdot),$$

and the corresponding one-parameter semigroups (continuous homomorphic images) are basic to the theory of topological semigroups (just as one-parameter groups are in Lie and topological group theory).

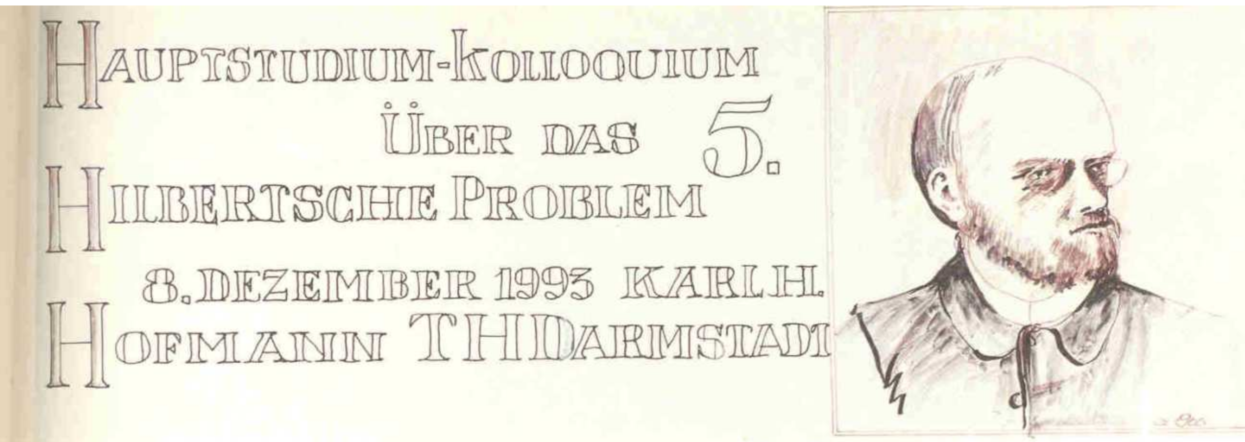
The One-Parameter Semigroup Theorem. (Mostert and Shields, 1957) Let $S \neq S^{-1}$ be a compact connected semigroup with identity 1 isolated in the set of idempotents. Then there exists a one-parameter semigroup starting at 1 and immediately leaving the group of units $H(1)$. If the only other idempotent is 0, then the image runs all the way to 0.

Paul Mostert

VARIATIONS ON HILBERT'S FIFTH

In 1959 HOFMANN independently made heavily overlapping discoveries on the existence of one-parameter semigroups. This work in some sense extended work of YAMABE, GLEASON, HILLE and others on one-parameter groups and semigroups, work that had arisen in connection with HILBERT'S Fifth Problem.

HOFMANN went on to consider versions of HILBERT'S Fifth Problem for loops and later for semigroups.



THEORY OF THREADS SEWN UP

In the early theory a *thread* denoted a topological semigroup with 1 defined on an interval. Besides the basic example $([0,\infty],+) \cong ([1,0],\cdot)$, there is also the "calculator semigroup" constructed from $[0,\infty]$ with $[M,\infty]$ identified to a single point called "overflow":

$$\text{for } 0 \le a, b < M,\ \ a \oplus b = \begin{cases} a+b, & \text{if } a+b < M, \\ \text{overflow}, & \text{otherwise.} \end{cases}$$

A. H. Clifford, a pioneer algebraic semigrouper and Tulane colleague, made major contributions to the following

Theorem. Any metric $\{1,0\}$-thread (with endpoints 1 and 0) arises by deleting countably many pairwise disjoint open intervals of $[0,1]$, pasting a multiplicative copy of one of the above two semigroups into each gap, and multiplying by taking the minimum in all other cases.

A. H. Clifford

The Sixties

NEW WAYS OF "WINDING DOWN"

Let ϕ: $\mathbb{R}^+ \to G$ be a continuous homomorphism (a one-parameter semigroup) onto a dense subset of a topological group G. Then one can create a compact semigroup S consisting of a one-parameter semigroup winding densely on the group G, which is the minimal ideal of S as follows:

1. Form the product semigroup $T := [0, \infty] \times G$.

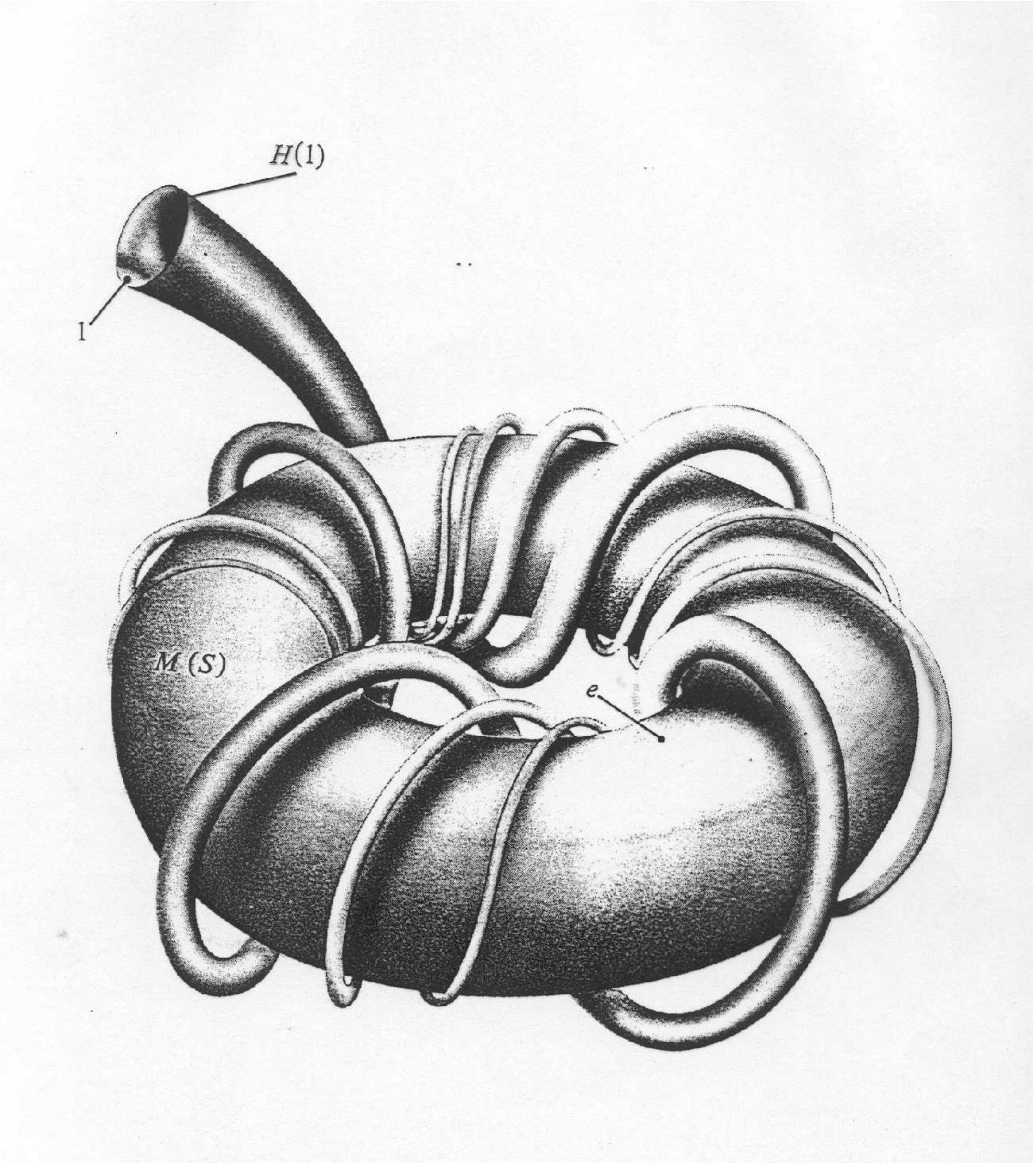
2. Define a one-parameter semigroup $\sigma : \mathbb{R}^+ \to T$ by $\sigma(t) = \big(t, \phi(t)\big)$.

3. Form the semigroup

$$S := \sigma(\mathbb{R}^+) \cup (\{\infty\} \times G) = \overline{\sigma(\mathbb{R}^+)}.$$

T $\qquad$ $\sigma(\mathbb{R}^+)$ $\qquad$ S

More elaborate winding semigroups may be manufactured by replacing $[0, \infty]$ by $K \times [0, \infty]$, where K is a compact group.



A PATH THROUGH THE WILDERNESS

The notion of a compact semigroup proved too diverse for any comprehensive theory to emerge. But a remarkable achievement of the sixties was a structure theorem by Hofmann and Mostert for *irreducible* (that is, minimal) compact connected subsemigroups stretching from one end of the semigroup, the identity 1, to the other end, the minimal ideal M. After their work, such subsemigroups provided a known path through an uncharted jungle.

Theorem. Let T be an irreducible compact connected subsemigroup stretching from 1 to M of a compact semigroup S. Then

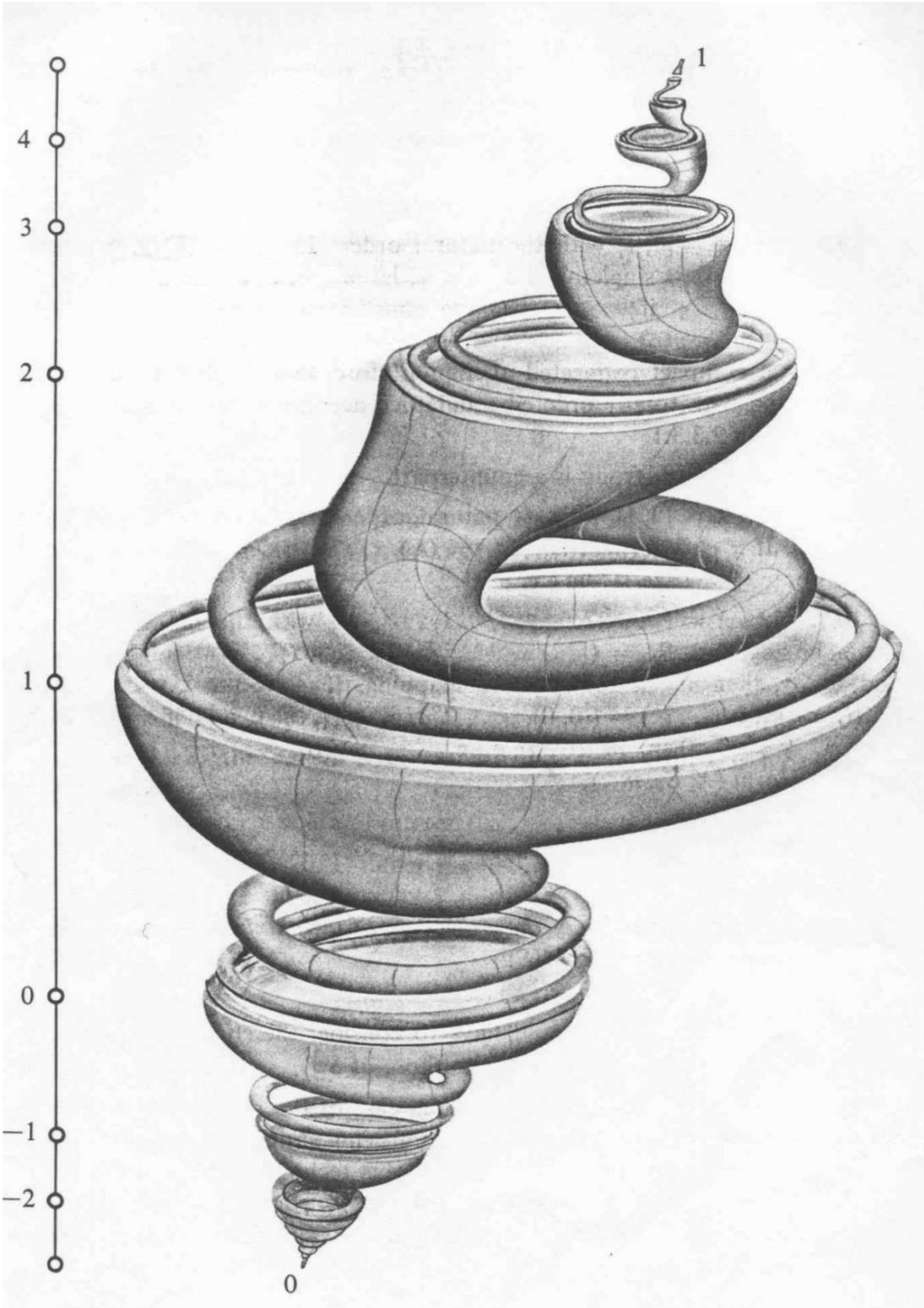
1. T is abelian;

2. T meets the group of units only in $\{1\}$;

3. $T/\mathcal{H}$ is a $\{0,1\}$-thread, where $\mathcal{H}$ denotes the congruence relation of mutual divisibilty;

4. T can be constructed by an appropriate, concretely given "chaining" procedure of winding semigroups that generalizes the construction of $\{0,1\}$-threads.

An irreducible semigroup T together with $T/\mathcal{H}$, a thread with idempotents all integers together with $\pm\infty$.



WHAT NEEDS TO BE FIXED?

The proof of the irreducibilty theorem (the centerpiece of *Elements of Compact Semigroups*, 1966) required extensive semigroup machinery plus a major new fixed-point theorem from the theory of compact connected transformation groups giving sufficient conditions for a compact, connected group action on a compact acyclic space to have an acyclic set of fixed-points. HOFMANN and MOSTERT were assisted with the latter by insights of ARMAND BOREL, who visited at Tulane during the period they were working on this theorem and the book.

One big problem remained (and remains) open:
The Centralizer Conjecture. Can the irreducible semigroup T be chosen in the centralizer of the (necessarily compact) group of units $H(1)$?

HOFMANN ENCOUNTERS UPS AND DAUNS

In 1968 Hofmann published with Tulane colleague John Dauns an AMS Memoir on representations of rings by sections of sheaves and bundles over the spectrum. Here the celebrated Dauns-Hofmann Theorem appeared, so dubbed by Dixmier, who was visiting Tulane at the time.

The Dauns-Hofmann Theorem. Let A be a C^*-algebra, $x \in A$, and f a bounded continuous scalar function on $\mathrm{Prim}(A)$, the space of primitive ideals endowed with the Jacobson topology. Then there exists a unique $fx \in A$ such that

$$fx \equiv f(P)x \bmod P \quad \text{for all } P \in \mathrm{Prim}(A).$$

The Seventies

A SEARCH FOR ORDER

The lack of a comprehensive general theory led topological semigroupers to focus on important special classes. A hundred miles up the Mississippi from Tulane, a young researcher at Louisiana State University was busily studying compact *semilattices*, commutative semigroups in which every element is idempotent. These carry a natural (partial) order

$$s \leq t \Leftrightarrow st = s;$$

this order provides an alternative characterization of semilattices as partially ordered sets in which any two elements have a greatest lower bound (their semigroup product). Compactness forces "lower completeness," every nonempty subset has a greatest lower bound.

Investigations by LAWSON uncovered a tractible class of compact semilattices, namely those which had a basis of neighborhoods at each point which were subsemilattices. Standard examples, such as the space of non-empty compact subsets of a compact space equipped with the Vietoris topology and the operation of union, possessed this property.

Meanwhile,....

SCOTT ORDERS COMPUTER SCIENCE

DANA SCOTT at Oxford realized that many structures and operations of computer science have interesting order-theoretic interpretation. He discovered a class of lower complete semilattices, later called *Scott domains*, in which each element was the directed supremum of elements "compactly" below it. These turned out to be admirably suited for modeling the lambda-calculus, denotational semantics of programming languages, and a variety of basic concepts in theoretical computer science. For his ground breaking work he was awarded the Turing Prize.

TOPOLOGICAL ALGEBRA VISITS "SCOTT"LAND

In a 1976 paper of major consequence, Hofmann and Stralka (implicitly) proved:

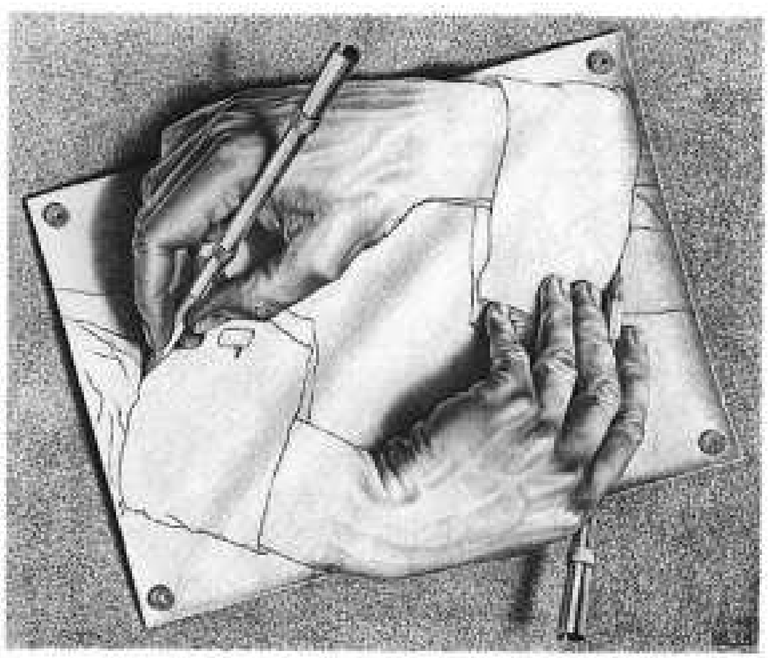
Theorem. Given a compact topological semilattice S with a neighborhood basis of subsemilattices, then as a partially ordered set S is a Scott domain. Conversely, a Scott domain admits a unique topology (which may be defined directly from the order) making it into a compact semilattice (with a neighborhood basis of subsemilattices).

This remarkable convergence from diverse investigations to the same mathematical structure gave impetus to a far-reaching joint collaboration that resulted in the development of a mature theory of continuous lattices, semilattices, and domains, which was set down in *A Compendium of Continuous Lattices*, 1980, with no less then six authors, Gierz, Keimel, Hofmann, Lawson, Mislove, and Scott, three from Darmstadt.

A STRANGE, NEW WORLD

In "Scott"land the topological algebraists found themselves in a strange, new world. In Scott's topology, spaces were no longer Hausdorff, only T_0. But even here significant work was done, such as the HOFMANN-MISLOVE Theorem, which characterized compact sets in terms of Scott-open filters in the open-set lattice.

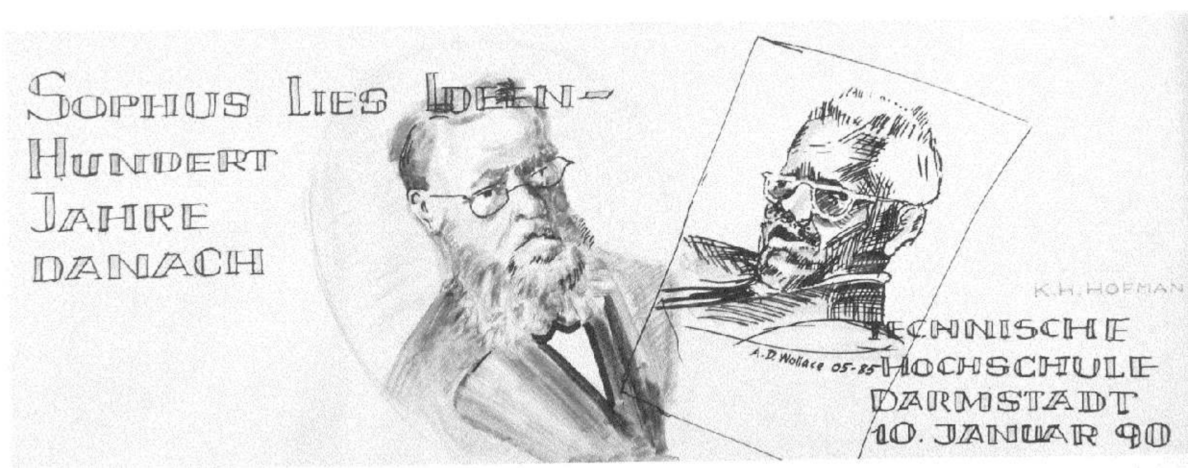
Spaces could satisfy strange "domain equations," such as being homeomorphic to their own function spaces, so that one could apply an element resp. function to itself.



TRANSITION

As the next decade rolled around, like Dorothy in Oz, or Alice in Wonderland, it was time to click the magic red slippers together, get a little recovery time, and then return to a more normal world.

Coming Up: Lie Meets Wallace, Lie Theory and Semigroups



The Eighties

IT'S AN (INFINITESIMALLY) SMALL WORLD AFTER ALL

One reason for the huge success of the classical Lie theory of groups is the capability that often exists to pull back nonlinear problems at the group level to problems at the Lie algebra (or infinitesimal) level which can be attacked by the tools of linear algebra. Indeed one typically tries to reinterpret the problem at the Cartan subalgebra level and use the elementary geometry of roots and their transformations (via the finite Weyl group).

A significant theme in the work of HOFMANN has been the effort to extend this program to larger classes of algebraic structures. In particular, he, HILGERT and LAWSON (together with several of HOFMANN'S talented graduate students) pioneered an extensive Lie theory of semigroups in the eighties, in which one studied those those subsemigroups of Lie groups that were "infinitesimally generated," i.e., could be recovered from their (sub)-tangent sets at the identity in the Lie algebra.

Actually (as gradually emerged) such a program had been launched in the fifties by Charles Loewner, who is most famous for his work on the Bieberbach conjecture, and re-emerged independently in the eighties in the work of G. Ol'shanskiĭ (with an eye toward applications in representation theory) and in the work of V. Jurjevic and I. Kupka in geometric control theory.

Theorem Given a closed (local) subsemigroup S of a Lie group G, there is a naturally associated tangent object in the Lie algebra of G which is a special type of closed convex cone called a Lie wedge (the appropriate analog of the Lie subalgebra corresponding to an analytic subgroup). Conversely given any Lie wedge, there is a corresponding local semigroup in the Lie group infinitesimally generated by the Lie wedge.

Unlike the Lie group case, there is not a global variant of this theorem, i.e., no third fundamental theorem of Lie. A major problem in the theory has been the effort to understand those Lie wedges for which there exists a corresponding (global) semigroup.

A POSITIVE OUTLOOK

The Lie theory of semigroups has introduced a significant new aspect into the Lie group theory, namely an appropriate notion of "positivity." At the group level the semigroups may be thought of as the set of positive elements, and the tangent cone is the set of infinitesimal positive elements. There is an associated notion of ordered homogeneous spaces. The development of the theory has necessitated new tools such as convex analysis, the theory of causal structures (drawn from physics), and methods of geometric control theory (where one has notions of evolution in positive time) in addition to the classical methods of Lie theory. Applications were found in various areas such as representation theory and harmonic analysis.

The new theory found ample documentation in two comprehensive works: *Lie Groups, Convex Cones, and Semigroups* by HILGERT, HOFMANN, and LAWSON (1989) and *Lie Semigroups and Their Applications* by HILGERT and NEEB (1993).

The Nineties

HOFMANN LOOPS BACK

Loops, nonassociative groups, sometimes turn up as the coordinatizing algebraic structures in topological geometry and had been considered by HOFMANN in the the late 50s and early 60s. He returned to this subject in the early 90s with KARL STRAMBACH. HOFMANN'S work in this area has been far-ranging, including analogies with topological and Lie group theory, local and global theory, and an examination of the corresponding version of Hilbert's fifth problem.

HOFMANN and STRAMBACH showed that the tangent space $L(G)$ at the identity e of a (local) Lie loop carries two operations, a binary commutator bracket and a trilinear associator bracket linked by what they called the Akivis identity. This tangent algebra $L(G)$ then forms an Akivis algebra, and if G is a Lie group, the Akivis identity reduces to the classical Jacobi identity. These considerations were tied to differential geometry by associating with a Lie loop G two left canonical connections, which coincide if G is a Lie group.

FINDING ONE'S ROOTS

A subsemigroup of a Lie group is called *divisible* if given $g \in S$ and $n \in \mathbb{N}$, there exists an n-th root $h \in S$ (i.e., $h^n = g$). Repeated random taking of roots can display rather unsystematic behavior, as even examples in the complex plane can demonstrate, and trying to take them in an orderly way in a Lie group proved something of a nightmare. However in a lengthy detailed study (yet another AMS Memoir) Hofmann and Wolfgang Ruppert showed that divisibility in Lie semigroups implies

(i) local divisibility (and in the local setting root extraction is uniquely determined),
(ii) all elements lie on one-parameter semigroups and hence in the image of the exponential mapping,
(iii) these semigroups can essentially be classified from their Lie tangent wedges, called semialgebras, and
(iv) the only simple Lie algebra containing a generating semialgebra is $\mathfrak{sl}(2, \mathbb{R})$.

HOFMANN REPORTS EXPONENTIAL RESEARCH GROWTH

Many considerations involved in the divisibility problem required detailed information about the exponential function (dubbed "Hofmann's favorite function" by students) from a Lie algebra to a Lie group. This research has naturally led to important new general insights and results concerning the exponential function of a Lie group. In particular, one now has an almost complete understanding of those Lie groups for which the exponential function is surjective or has dense image.

HOFMANN REGROUPS*

In the 60s Hofmann published an *Introduction to Compact Groups I,II* in the Tulane Notes and later began a 30-year research collaboration on the topic with Sid Morris around 1977. This project moved to the front burner of the stove in the 90s, resulting in the 858 page tome *The Structure of Compact Groups* (a far from compact book). SCG appeared in 1998 and in a revised version in 2006.

The aim of SCG was to present the structure of compact groups without unnecessary assumptions. The approach was not that of analysis (Hofmann had lectured and written on "analytic groups without analysis"), but, not suprisingly, through the development of a Lie theory for compact groups. The power of this general Lie theoretic approach is then demonstrated by obtaining in a systematic way not only the known structure results of compact groups, but also new and extended results along the way.

* Thanks to Sid Morris for generous help with this section

THE BIG FOUR

The postscript to Chapter 9 of SCG recaps four principal structure theorems for compact groups that appear in that chapter.

(I) **The Levi-Mal'cev Theorem**. A connected compact group is nearly a direct product of its connected center and the commutator subgroup.

(II) **The Maximal Pro-torus Theorem**. A connected compact group has maximal pro-tori which are all conjugate, and the center is their intersection. Each maximal pro-torus is a maximal abelian subgroup.

(III) **The Borel-Scheerer-Hofmann Theorem**. The commutator of a connected compact group is (topologically and algebraically) a semidirect factor.

(IV) **The Dong Hoon Lee Supplement Theorem**. A compact group G contains a totally disconnected compact subgroup D such that $G = G_0D$ and $G_0 \cap D$ is normal in G and central in G_0.

THE LITTLEST OUTLAW

Lie theoretic arguments now and then employ inductive arguments on Lie algebra dimension by eliminating a "smallest criminal" or "littlest outlaw." Here is HOFMANN in hot pursuit of the "littlest outlaw" (or is it the infamous "Mexican bandito"?).

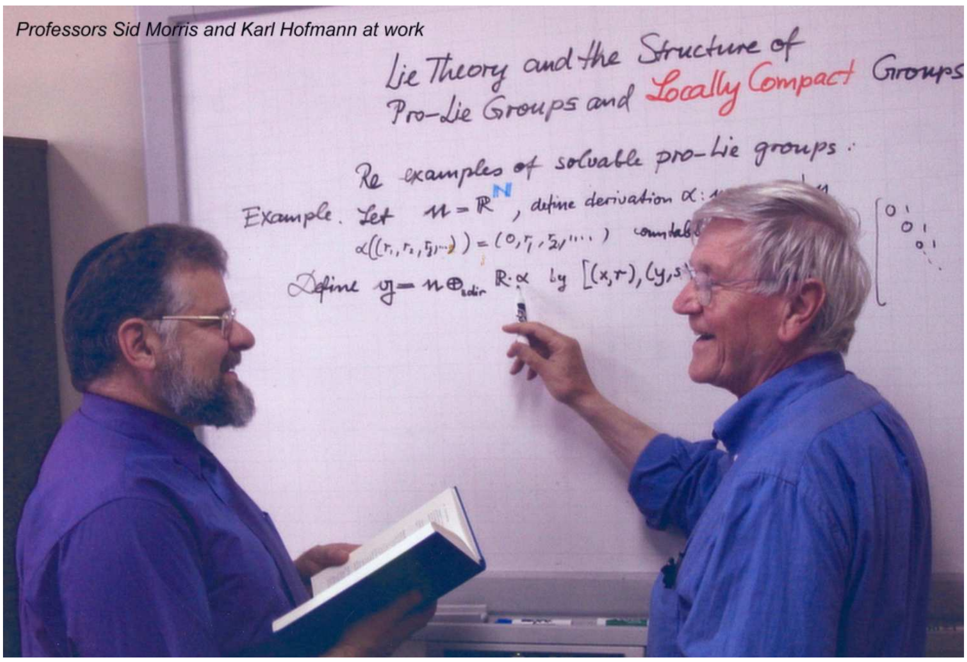
Sheriff Hofmann

WANTED: DEAD OR ALIVE

The Mexican Bandit

BEYOND LOCALLY COMPACT GROUPS

Combining the work of Iwasawa and Yamabe from the first half of the 20th century, one can reduce the structure theory of connected locally compact groups to that of compact groups and Lie groups. But is there a well-behaved category that contains all these objects, and is there a systematic way to study and obtain their structure? In their book *The Lie Theory of Connected Pro-Lie Groups*, 2007, Hofmann and Morris propose the category of *pro-Lie groups*.



Professors Sid Morris and Karl Hofmann at work

The Hofmann-Morris Company

PRO-LIE GROUPS

A *pro-Lie group* can be defined in any of the following equivalent ways:

(1) A projective limit of finite dimensional Lie groups.

(2) A complete group G for which every identity neighborhood contains a normal subgroup N such that G/N is a Lie group.

(3) A group that is (isomorphic to) a closed subgroup of a product of finite dimensional Lie groups.

Favorable Properties: The category of pro-Lie groups is closed under all limits, particularly all products, in the category of topological groups and under passing to closed subgroups. It is not closed under quotient groups, but these have completions that are again pro-Lie.

THE LIE THEORY OF PRO-LIE GROUPS

Every pro-Lie group has a Lie algebra and an exponential function whose image generates (algebraically) a dense subgroup of the identity component. The Lie algebra is an analogue of a pro-Lie group, called a *pro-Lie algebra.* Basic familiar properties such as existence of a radical and a Levi-Mal'cev theorem carry over.

The Lie algebra functor preserves limits and quotients, and its left adjoint provides a functorial version of Lie's Third Theorem.

One uses Lie theoretic methods to derive structure theorems for pro-Lie groups.

Sample Theorem. Each connected pro-Lie group contains maximal compact connected subgroups , which are all conjugate under inner automorphisms.

FOOTNOTES

(1) HOFMANN'S sketches of A. D. WALLACE, PAUL MOSTERT, and A. H. CLIFFORD appear in "An interview with Karl Hofmann on the occasion of his seventieth birthday," Semigroup Forum 65(3), 317-328. Supplementary information to that presented here can be found there.

(2) HOFMANN'S drawings of a semigroup consisting of a wind on a torus and of an irreducible semigroup consisting of chained winding semigroups appear in K. H. HOFMANN and P. MOSTERT, *Elements of Compact Semigroups*, Charles Merrill, 1966.

(3) The posters on topological planes, Hilbert's Fifth Problem, topologies on lattices, and Lie and Wallace were selections from reproductions of HOFMANN'S weekly colloquium posters for the Mathematics Department at the Technical University, Darmstadt. These appeared in a selection of these posters, *Poster Cartoons 1983-1998*, published in 1998 by the TU Darmstadt University Press.

(4) HOFMANN'S New Orleans street scence appeared as: Cover of the AMS Notices, October 2006, an issue publicizing the national meeting of the AMS in New Orleans in January, 2007.

(5) Thanks to SID MORRIS for pictures of Sheriff Hofmann and the Hofmann-Morris Company and KARL HOFMANN for a scan of his drawing of the Mexican Bandit (Karl's original (1986) is property of AL STRALKA).