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Effects of heterogeneous environments on chemotactic aggregation

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1 Introduction

In microbiological processes, it is common for organisms to interact with their environment via positive chemotaxis, that is the tendency to move in the direction of the gradient of some signal substance. This behavior has been documented as early as 1881 for Bacterium termo and Spirillum tenue moving toward oxygen-producing plant cells [6].

In order to make such biological systems accessible to quantitative analysis and outline the governing factors of structural evolution, at the beginning of the 1970s Keller and Segel proposed a system resembling

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u) - \nabla \cdot (S(u,v)\nabla v), \\ v_t = \Delta v + u - v, \end{cases} \tag{1.1}$$

see [11] and [12], to describe the behavior of slime mold aggregation and Escherichia coli as outlined in [1], respectively. Herein, the bacteria concentration $u = u(x,t)$ is subject not only to chemotaxis toward a self-produced attractant with density $v = v(x,t)$ but also to diffusion in the form of Brownian motion. The expressions $D(u,v)$ and $S(u,v)$ represent the diffusive and chemotactic sensitivity, respectively.

This system and variations thereof have various applications [9], including pattern formation in bacterial colonies [31], tumor invasion processes [5] and embryogenesis [20].

In line with experimental observations, even the prototypical setting with $D(u,v) \equiv 1$ and $S(u,v) \equiv u$ considered in bounded domains with no-flux boundary conditions has been shown to exhibit aggregation phenomena already, in their most extreme form represented by finite-time blow-up. While solutions always exist globally in time and are bounded in the spatially one-dimensional case [18], for two-dimensional balls, a critical mass phenomenon arises: For all sufficiently regular and radially symmetric initial data u_0 with total mass $\int_\Omega u_0 < 8\pi$, solutions are global and bounded [17], whereas if $\int_\Omega u_0 > 8\pi$, finite-time blow-up is possible ([8], [15]). For balls as domains in higher dimensions, initial data with arbitrary total mass leading to blow-up can be constructed [25].

On account of the fact that in numerous biological applications, the chemo-attractant dissipates much faster than the microbes move, by [10] we may consider the parabolic-elliptic system given by

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u) - \nabla \cdot (S(u,v)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v + u - \mu, \qquad \mu = \frac{1}{|\Omega|}\int_\Omega u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ as a relevant limit case of (1.1). Concerning global boundedness and finite-time blow-up, the results are similar to the fully parabolic system (1.1) [16]. The matter of deducing local existence in both systems in absense of degeneracies, particularly for constant and linear sensitivities D and S, respectively, has for example been considered in [3]. Therein, heat semigroup theory is employed in order to obtain a mild solution which can be shown to have nice regularity properties in the interior ([13, III.12]) in turn. By means of a fixed point argument, such results have been extended to possibly degenerate cases for sufficiently regular $D > 0$ and $S \geq 0$ dependent on u ([29]). Related systems for nutrient taxis with $D(u,v) \equiv uv$ and $S(u,v) = \Psi(u)v$ with Ψ asymptotically growing quadratically at most as well as such with $D(u)$ and $S(x,u,v)$ supposed to be not too singular in certain ways have been investigated [28] and [30], respectively, and, utilizing approximations, have been shown to possess global weak solutions.
In this thesis we shall consider a spatially dependent diffusion sensitivity generalized by the prototype $D(x) \equiv |x|^\beta$, $x \in \Omega$, for $\beta > 0$. This leads to the problem

$$\begin{cases} u_t = \nabla \cdot (|x|^\beta \nabla u) - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, \quad \mu := \frac{1}{|\Omega|} \int\limits_\Omega u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

posed in $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 1$, $R > 0$, fulfilling

$$u_0 \in C^0_{rad}(\overline{\Omega}) := \left\{ \varphi \in C^0(\overline{\Omega}) \;\middle|\; \varphi \text{ is radially symmetric} \right\} \quad \text{is nonnegative with } u_0 \not\equiv 0. \tag{1.4}$$

We shall remark that in the contexts we consider, actually $\mu = \frac{1}{|\Omega|} \int\limits_\Omega u_0$ will be proven to hold, so that μ may be viewed as a constant.
This system can be interpreted as a prototype for describing biological applications where the motility of a cell or bacteria population is impaired near the origin. For instance, we find this to be the case when coagulation mechanisms are present. Their significance not just for structural healing but also in the context of immune responses for invertebrates has among others been established in [19] and [23]. In mammals, the coagulation system was long thought to be important exclusively for haemostasis. However, nowadays it is commonly recognized that coagulation contributes to the effective elimination of bacteria in those organisms as well [2]. In fact, besides restricting the motility of bacteria, coagulation triggers the release of bradykinin which interacts with macrophages to emit chemo-attractants supporting the immune response [7]. Furthermore, fibrinogen releases fibrinopeptides, chemo-attractants to aid clotting [22].

Thus in this example already, there are multiple chemotactic dynamics at play wherein heterogeneous environments roughly as described in (1.3) might occur.
The mathematical analysis of (1.3) however is accompanied by notable difficulties. Calculating

$$\nabla \cdot (|x|^\beta \nabla u) = |x|^\beta \Delta u + (\nabla |x|^\beta) \cdot \nabla u$$

reveals that for one, we are dealing with a spatial diffusion degeneracy which in Keller-Segel type systems appears to be without precedent in literature, and moreover, at least for $\beta < 1$ singular behavior of $(\nabla |x|^\beta) \cdot \nabla u$ at $x = 0$ is to be expected. This already indicates that at least generally, we should not assume to be able to obtain a classical solution of (1.3) in $\overline{\Omega} \times (0, T)$ for some $T > 0$; instead, we either have to resort to weak solution concepts or at least omit the spatial point $x = 0$. Our results feature the latter.

First we formulate a statement on local existence of classical solutions in $(\overline{\Omega} \setminus \{0\}) \times (0, T_0)$ satisfying mass conservation.

Theorem 1.1. *Let $n \geq 1$, $R > 0$, $\Omega = B_R(0) \subset \mathbb{R}^n$ and $\beta > 0$, and write $\Omega_0 := \overline{\Omega} \setminus \{0\}$. Then for $\theta \in (0,1)$ and $u_0 \in C^\theta(\overline{\Omega})$ complying with (1.4), there exists a radially symmetric classical solution (u,v) of (1.3) in the sense of Definition 3.1 fulfilling*

$$\begin{cases} u \in C^0(\Omega_0 \times [0, T_0)) \cap C^{2,1}(\Omega_0 \times (0, T_0)) \\ v \in C^{2,0}(\Omega_0 \times (0, T_0)) \end{cases} \tag{1.5}$$

for some $T_0 \in (0, \infty]$. This solution has the properties that u is nonnegative and satisfies the mass conservation property, that is

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 =: m \qquad \text{for all} \quad t \in (0, T_0). \tag{1.6}$$

The second theorem includes a result on local boundedness and uniqueness. In order to accomplish this, we need to presume much stronger conditions.

Theorem 1.2. *Suppose the conditions of Theorem 1.1 hold, and let (u, v) denote the classical solution to (1.3) established therein.*
Assume that additionally $u_0 \in C^{1+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ is radially decreasing and

has the properties that

$$u_0 = 0 \qquad \textit{and} \qquad \nabla u_0 \cdot \nu = 0 \qquad \textit{on} \quad \partial\Omega \tag{1.7}$$

as well as

$$|\nabla u_0(x)| \le C_0 |x|^{n-1+\theta} \tag{1.8}$$

for some $C_0 > 0$*, and either* $\beta \le 2 - n$ *or* $\beta \ge 2$*.*
Then for $T_0 > 0$ *as in Theorem 1.1 we have that*

$$u \in C^{1,0}(\Omega_0 \times [0, T_0)) \quad \textit{is radially decreasing.} \tag{1.9}$$

Moreover, for some $T^* \in (0, T_0]$ *and each* $T \in (0, T^*)$*, there exists* $C = C(T) > 0$ *such that*

$$u(x,t) \le C \qquad \textit{for all} \quad (x,t) \in \Omega_0 \times [0,T]. \tag{1.10}$$

If additionally $n \ge 2$*, there exists a unique solution* (u, v) *of (1.3) in* $\Omega_0 \times [0, T^*)$ *fulfilling*

$$\begin{cases} u \in C^0(\Omega_0 \times [0,T^*)) \cap C^{2,1}(\Omega_0 \times (0,T^*)), \\ v \in C^{2,0}(\Omega_0 \times (0,T^*)), \end{cases}$$

which has the properties that $\int_\Omega v(\cdot,t) = 0$ *for all* $t \in (0, T^*)$ *and*

$$0 \le u \in L^\infty(\Omega \times (0,T)) \quad \textit{and} \quad \nabla v \in L^\infty(\Omega \times (0,T); \mathbb{R}^n) \tag{1.11}$$

for all $T \in (0, T^*)$*.*

We close with a result ruling out global bounded solutions for initial mass distributions concentrated adequately close to the origin.

Theorem 1.3. *Let* $n \ge 2$*,* $R > 0$*,* $\Omega = B_R(0) \subset \mathbb{R}^n$ *as well as* $\beta > 0$*, and assume* u_0 *complies with (1.4).*
Then for $m := \int_\Omega u_0$ *and each* $m_0 \in (0, m]$*, there exists* $r_0 = r_0(m_0, m, R, \beta) > 0$ *such that if*

$$\int_{B_{r_0}(0)} u_0 \ge m_0, \tag{1.12}$$

there is no global classical solution (u,v) *of (1.3) fulfilling*

$$\begin{cases} u \in C^0(\Omega_0 \times [0,\infty)) \cap C^{2,1}(\Omega_0 \times (0,\infty)) \\ v \in C^{2,0}(\Omega_0 \times (0,\infty)) \end{cases} \tag{1.13}$$

such that for each $T \in (0,\infty)$

$$0 \le u \in L^\infty(\Omega \times (0,T)) \quad \textit{and} \quad \nabla v \in L^\infty(\Omega \times (0,T); \mathbb{R}^n). \tag{1.14}$$

Outline of arguments. The main idea is to transform the Keller-Segel type Neumann boundary value problem to a Dirichlet problem for which we are able to obtain a local solution, see subsections 3.1.1, 3.1.2 and 3.1.3, and then retransform in order to acquire a solution of (1.3) (subsection 3.1.5).
To that end, we first simplify (1.3) to (3.3) making use of the radial symmetry in Lemma 3.1. Our goal then is to reduce (3.3) to the scalar parabolic Dirichlet boundary problem (3.20) for the mass accumulation function w given by $w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t) d\rho$ as introduced by Jäger and Luckhaus to deduce blow-up in [10], which is done in Lemma 3.4. Since we do not assume solutions to be defined on a compact space, however, this is not as straightforward as usual; we shall need some subtle arguments to ensure that v_r is of desired form and that mass is conserved (Lemma 3.2 and 3.3).
By a fairly standard type of reasoning which has been sketched in [4] and [26, Lemma 3.2], in subsection 3.1.2 we then construct a solution w to an incomplete version of this problem, that is without regarding the boundary condition at $s = 0$. This is achieved utilizing solutions to auxiliary non-degenerate problems which are uniformly bounded and satisfy an ordering property, whereby monotone convergence and compactness arguments yield the aforementioned result.
In the following subsection, after collecting some simple properties of w (Lemma 3.11), by another application of the comparison principle Lemma 2.3, for small times we construct supersolutions for w which are linear in s, that is

$$w(s,t) \le y(t) \cdot s \qquad \text{for} \quad s \in (0, R^n] \quad \text{and} \quad t \in (0, T^*)$$

for some $T^* > 0$ and a function $y \in C^1([0,T^*))$. This entails that w indeed solves the complete Dirichlet problem (3.20) in $[0,R^n] \times [0,T^*)$ (Lemma 3.12). That is not sufficient to infer that w_s is bounded in $(0,R^n) \times (0,T^*)$ though. To that end, we shall establish a

monotonocity property of w_s; and since w_s increasing in s is not thus interesting with regard to the dichotomy between blow-up and global boundedness, we opt for concavity of w.
Due to the left boundary however, solutions to the auxiliary problem from subsection 3.1.2 generally cannot be concave. Thus we need to introduce another similar auxiliary problem for which we can indeed show that concavity is maintained over time under certain compatibility conditions on the initial data (Lemma 3.18). We can also repeat the arguments from subsection 3.1.2 for this new type of approximation, yet not Lemma 3.12. Therefore, we need to establish that the obtained limit functions from both auxiliary problems are actually the same. Under a restriction for β, this is done in Lemma 3.17. That in hand, we are finally in position to show boundedness of w_s for a short timespan and thus, by means of our comparison principle, also uniqueness in this time interval (Lemma 3.19, 3.13).
In subsection 3.1.5, we then retransform to obtain a local solution of (3.3) fulfilling the mass conservation property (Lemma 3.21, 3.22). It is remarkable that the boundary condition for w at R^n implies that $w_{ss}(R^n, t) = 0$ for all $t > 0$ as well (Lemma 3.20), thus entailing the boundary condition for u_r in (3.3). The results on uniqueness and local boundedness are transfered as well.
In subsection 3.2, we are concerned with ruling out global boundedness for sufficiently large $\beta > 0$ and properly concentrated initial data. The main idea here is to attach singular weights to the mass accumulation function w and thus construct a generalized moment functional. This functional is bounded, but shown to explode in finite time under the assumption that w solves (3.20) globally with w_s bounded locally time, implying that this cannot be the case.
The main theorems can then be obtained mainly by collecting previous results. One needs to be cautious how the conditions imposed on the initial data of the original system (1.3) translate to those in (3.20) though.

2 Preliminaries

We first gather some key statements. Despite the Schauder estimates being standard theory, not only for coherence of reading and notational convenience but also to clarify the intended version, we explicitly formulate them here.

Consider – in wider generality than needed in this thesis – the parabolic Dirichlet boundary value problem

$$\begin{cases} u_t = \nabla \cdot \big(a(x,t)\nabla u\big) + b(x,t)\cdot\nabla u + d(x,t)u + f(x,t), & x\in\Omega,\ t\in(0,T), \\ u = \Phi(x,t), & x\in\partial\Omega,\ t\in(0,T), \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

wherein $a, d, f \in C^0(\overline{\Omega}\times[0,T])$, $b \in C^0(\overline{\Omega}\times[0,T];\mathbb{R}^n)$, $u_0 \in L^\infty(\Omega)$, $\Phi \in C^0(\partial\Omega\times[0,T])$, $T>0$ and $\Omega\subset\mathbb{R}^n$ is a convex bounded domain with smooth boundary.

In the following passage, we formulate a collection of Schauder estimates. The proposed conditions are not optimal, yet easier to verify than those given in [14] and [13].
It seems worth mentioning that the estimates for Hölder regularity of u itself (see [13, Theorem V.1.1]) are instrumental in proving Lemma 3.5. We omit them here however, since in light of ensuring continuity up to $t = 0$ in the original system we shall utilize a stronger form of convergence than otherwise necessary in Lemma 3.9.
For a subtle line of argument, in Lemma 3.7 we also need the global version of

Lemma 2.1. *Let $T>0$ and $\theta\in(0,1)$.*
For any $M>0$ there exist $\eta=\eta(M,T,\theta)\in(0,1)$ and $C(M,T,\theta)>0$ such that for any choice of

$$\begin{aligned} &a\in C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T]), \quad d,f\in C^0(\overline{\Omega}\times[0,T]),\\ &b\in C^0(\overline{\Omega}\times[0,T];\mathbb{R}^n), \quad \Phi\in C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega\times[0,T]) \end{aligned}$$

as well as $u_0\in C^{1+\theta}(\overline{\Omega})$ with $u_0|_{\partial\Omega}=\Phi(\cdot,0)$ and

$$\max\Big\{ \|a\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])}, \|b\|_{L^\infty(\Omega\times(0,T))}, \|d\|_{L^\infty(\Omega\times(0,T))}, \|f\|_{L^\infty(\Omega\times(0,T))}, \|\Phi\|_{C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega\times[0,T])}\Big\} \le M \tag{2.2}$$

and

$$\|u_0\|_{C^{1+\theta}(\overline{\Omega})} \le M \tag{2.3}$$

as well as

$$a \ge \frac{1}{M} \qquad \text{in } \Omega \times (0,T), \tag{2.4}$$

the corresponding classical solution $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *of (2.1) lies in* $C^{1+\eta,\frac{1+\eta}{2}}(\overline{\Omega} \times [0,T])$ *and satisfies the inequality*

$$\|u\|_{C^{1+\eta,\frac{1+\eta}{2}}(\overline{\Omega}\times[0,T])} \le C(M,T,\theta).$$

ii) Let $\Omega' \subset \overline{\Omega}$ *be relatively open in* $\overline{\Omega}$*. For any open* $\Omega'' \subset \Omega'$ *with* $K = \overline{\Omega''} \subset \Omega'$ *and* $M > 0$ *there exist* $\eta = \eta(K,\Omega',M,T,\theta) \in (0,1)$ *and* $C(K,\Omega',M,T,\theta) > 0$ *such that if for*

$$\begin{aligned} a \in C^{\theta,\frac{\theta}{2}}(\Omega' \times [0,T]), &\quad d, f \in C^0(\Omega' \times [0,T]), \\ b \in C^0(\Omega' \times [0,T];\mathbb{R}^n), &\quad \Phi \in C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega \times [0,T]) \end{aligned} \tag{2.5}$$

and $u_0 \in L^\infty(\Omega) \cap C^{1+\theta}(\overline{\Omega'})$ *we have*

$$\max\Big\{ \|a\|_{C^{\theta,\frac{\theta}{2}}(\Omega'\times[0,T])}, \|b\|_{L^\infty(\Omega'\times(0,T))}, \|d\|_{L^\infty(\Omega'\times(0,T))}, \|f\|_{L^\infty(\Omega'\times(0,T))}, \|\Phi\|_{C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega\times[0,T])} \Big\} \le M \tag{2.6}$$

and

$$\|u_0\|_{C^{1+\theta}(\overline{\Omega'})} \le M \tag{2.7}$$

as well as

$$a \ge \frac{1}{M} \qquad \text{in } \Omega' \times (0,T), \tag{2.8}$$

and if additionally the classical solution $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *of (2.1) has the property*

$$\|u\|_{L^\infty(\Omega\times(0,T))} \le M \tag{2.9}$$

then $u \in C^{1+\eta,\frac{1+\eta}{2}}(K \times [0,T])$ *with*

$$\|u\|_{C^{1+\eta,\frac{1+\eta}{2}}(K\times[0,T])} \le C(K,\Omega',M,T,\theta).$$

PROOF. Confer [14, Theorem 1.1], [13, V.3,4].

We also note the following estimate, whereof a local version will be of use for us as well.

Lemma 2.2. *Let $T > 0$ and $\theta \in (0,1)$.*

i) For any $M > 0$ there exist $\eta = \eta(M,T,\theta) \in (0,1)$ and $C(M,T,\theta) > 0$ such that for any choice of

$$\begin{aligned} a \in C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega} \times [0,T]), \quad & d, f \in C^{\theta,\frac{\theta}{2}}(\overline{\Omega} \times [0,T]), \\ b \in C^{\theta,\frac{\theta}{2}}(\overline{\Omega} \times [0,T]; \mathbb{R}^n), \quad & \Phi \in C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega \times [0,T]) \end{aligned} \tag{2.10}$$

together with $u_0 \in C^{2+\theta}(\overline{\Omega})$ with $u_0|_{\partial\Omega} = \Phi(\cdot,0)$ and

$$\begin{aligned} \max\Big\{ & \|a\|_{C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega}\times[0,T])}, \|b\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])}, \|d\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])}, \\ & \|f\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])}, \|\Phi\|_{C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega\times[0,T])} \Big\} \le M \end{aligned} \tag{2.11}$$

and

$$\|u_0\|_{C^{2+\theta}(\overline{\Omega})} \le M \tag{2.12}$$

as well as

$$a \ge \frac{1}{M} \qquad \text{in } \Omega \times (0,T), \tag{2.13}$$

then the corresponding classical solution $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ of (2.1) lies in $C^{2+\eta,\frac{2+\eta}{2}}(\overline{\Omega} \times [0,T])$ and satisfies

$$\|u\|_{C^{2+\eta,1+\frac{\eta}{2}}(\overline{\Omega}\times[0,T])} \le C(M,T,\theta),$$

if additionally u_0 complies with the compatibility condition

$$\nabla \cdot \big(a(\cdot,0)\nabla u_0\big) + b(\cdot,0) \cdot \nabla u_0 + d(\cdot,0)u_0 + f(\cdot,0) = \Phi_t(\cdot,0) \qquad \text{on } \partial\Omega. \tag{2.14}$$

ii) Let $\Omega' \subset \overline{\Omega}$ be relatively open in $\overline{\Omega}$. For any open $\Omega'' \subset \Omega'$ with $K = \overline{\Omega''} \subset \Omega'$, $\tau \in (0,T)$ and $M > 0$ there exist $\eta = \eta(K,\Omega',M,\tau,T,\theta) \in (0,1)$ and $C(K,\Omega',M,\tau,T,\theta) > 0$

such that if for

$$a \in C^{1+\theta,\frac{1+\theta}{2}}(\Omega' \times [\tau,T]), \quad d,f \in C^{\theta,\frac{\theta}{2}}(\Omega' \times [\tau,T]), \quad b \in C^{\theta,\frac{\theta}{2}}(\Omega' \times [\tau,T];\mathbb{R}^n) \quad (2.15)$$

and $\Phi \in C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega \times [0,T])$ *we have*

$$\max\Big\{ \|a\|_{C^{1+\theta,\frac{1+\theta}{2}}(\Omega'\times[\tau,T])}, \|b\|_{C^{\theta,\frac{\theta}{2}}(\Omega'\times[\tau,T])}, \|d\|_{C^{\theta,\frac{\theta}{2}}(\Omega'\times[\tau,T])}, \|f\|_{C^{\theta,\frac{\theta}{2}}(\Omega'\times[\tau,T])}, \|\Phi\|_{C^{2+\theta,1+\frac{\theta}{2}}(\partial\Omega\times[0,T])}\Big\} \le M \quad (2.16)$$

as well as

$$a \ge \frac{1}{M} \qquad \text{in } \Omega' \times (0,T), \quad (2.17)$$

and if additionally the classical solution $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *of (2.1) has the property (2.9), then* $u \in C^{2+\eta,1+\frac{\eta}{2}}(K \times [\tau,T])$ *with*

$$\|u\|_{C^{2+\eta,1+\frac{\eta}{2}}(K\times[\tau,T])} \le C(K,\Omega',M,\tau,T,\theta)$$

holds true.

PROOF. See [13, Theorem IV.5.2, VII.5,6].

Another important tool in the analysis of the transformed systems is a comparison principle. Since we also need to be able to deal with diffusion degeneracy and considering that because of the expression ww_s our differential equation is of semilinear type, the standard comparison principles do not seem to apply. Also, for arguments as in [26], some information drawn from standard results regarding the original Keller-Segel system is missing. Therefore, we prove a comparison principle tailored to our specific cases:

Lemma 2.3. *Let* $T > 0$ *and* $l, L \ge 0$ *with* $l < L$. *Suppose that* $\underline{w}$ *and* $\overline{w}$ *belong to* $C^0([l,L] \times [0,T]) \cap C^{2,1}((l,L) \times (0,T])$, *and that additionally either*

$$\underline{w}_s \in L^\infty((l,L) \times (0,T)) \quad \text{or} \quad \overline{w}_s \in L^\infty((l,L) \times (0,T)).$$

Moreover, for $a, b, \gamma \ge 0$ *and* $\alpha, \delta, c, d \in \mathbb{R}$

$$\underline{w}_t \le as^\alpha \underline{w}_{ss} + bs^\gamma \underline{w}\underline{w}_s + cs^\delta \underline{w}_s + d\underline{w}_s \quad \text{and} \quad \overline{w}_t \ge as^\alpha \overline{w}_{ss} + bs^\gamma \overline{w}\overline{w}_s + cs^\delta \overline{w}_s + d\overline{w}_s \quad (2.18)$$

shall hold for all $(s,t) \in (l,L) \times (0,T)$, *as well as*

$$\underline{w}(s,0) \le \overline{w}(s,0) \qquad \textit{for all } s \in (l,L) \tag{2.19}$$

and

$$\underline{w}(l,t) \le \overline{w}(l,t) \quad \textit{and} \quad \underline{w}(L,t) \le \overline{w}(L,t) \qquad \textit{for all } t \in (0,T). \tag{2.20}$$

Then

$$\underline{w}(s,t) \le \overline{w}(s,t) \qquad \textit{for all } s \in [l,L] \quad \textit{and} \quad t \in [0,T). \tag{2.21}$$

PROOF. Since without loss of generality we may assume $\overline{w}_s \in L^\infty((l,L)\times(0,T))$, we can choose

$$\kappa := bL^\gamma \left\|\overline{w}_s\right\|_{L^\infty((l,L)\times(0,T))} + 1. \tag{2.22}$$

For arbitrary $\varepsilon > 0$, define $z \in C^0([l,L]\times[0,T]) \cap C^{2,1}((l,L)\times(0,T])$ via

$$z(s,t) := \overline{w}(s,t) - \underline{w}(s,t) + \varepsilon e^{\kappa t}.$$

Define

$$S := \{t \in [0,T] \mid z(s,\tilde{t}) > 0\ \forall (s,\tilde{t}) \in [l,L]\times[0,t]\} \ne \emptyset$$

due to (2.19), which along with $z \in C^0([l,L]\times[0,T])$ also guarantees $t_0 := \sup S > 0$. If now $t_0 < T$, then there exists $s_0 \in [l,L]$ such that $z(s_0,t_0) = 0$, and (2.20) asserts that actually $s_0 \in (l,L)$. Therefore, at (s_0,t_0) not only

$$z = \overline{w} - \underline{w} + \varepsilon e^{\kappa t_0} = 0 \quad \text{and} \quad z_t \le 0$$

but also

$$z_s = \overline{w}_s - \underline{w}_s = 0 \quad \text{and} \quad z_{ss} = \overline{w}_{ss} - \underline{w}_{ss} \ge 0$$

hold. Combined with (2.18) and (2.22), this yields

$$\begin{aligned}
0 \ge z_t(s_0,t_0) &= \overline{w}_t - \underline{w}_t + \kappa\varepsilon e^{\kappa t_0} \\
&\ge as^\alpha(\overline{w}_{ss} - \underline{w}_{ss}) + bs^\gamma(\overline{w}\overline{w}_s - \underline{w}\underline{w}_s) + (cs^\delta + d)(\overline{w}_s - \underline{w}_s) + \kappa\varepsilon e^{\kappa t_0} \\
&= as^\alpha(\overline{w}_{ss} - \underline{w}_{ss}) + bs^\gamma\overline{w}_s(\overline{w} - \underline{w}) + \kappa\varepsilon e^{\kappa t_0} \\
&\ge bs^\gamma\overline{w}_s(\overline{w} - \underline{w}) + \kappa\varepsilon e^{\kappa t_0} \\
&= bs^\gamma\overline{w}_s(-\varepsilon e^{\kappa t_0}) + \kappa\varepsilon e^{\kappa t_0} \\
&= (\kappa - bs^\gamma\overline{w}_s)\varepsilon e^{\kappa t_0} \\
&\ge (\kappa - bL^\gamma\overline{w}_s)\varepsilon e^{\kappa t_0}
\end{aligned}$$

$$
\begin{aligned}
&\geq \varepsilon e^{\kappa t_0} \\
&> 0,
\end{aligned}
$$

a contradiction.
Thus necessarily $t_0 = T$, which by taking $\varepsilon \searrow 0$ implies that indeed (2.21) holds true. $\square$

3 Spatial dependence of diffusion sensitivity

3.1 Existence of solutions

We shall establish the existence of sufficiently smooth solutions to (1.3).

In the scenario at hand, that represents a particular challenge. Not only is the term $|x|^\beta$ not differentiable at 0 for $0 < \beta \le 1$, but moreover the coefficient of the Laplacian of u vanishes at $x = 0$ for all $\beta > 0$, implying a diffusion degeneracy. Whereas examples of possible degeneracies depending on u or even on u and v have for instance been discussed in [29], [27] and [28], to the author's knowledge no case of a spatially dependent diffusion degeneracy in such systems with Neumann boundary conditions has yet been addressed in standard literature.

Our approach in principle relies on a strategy usually employed to detect blow-up in parabolic-elliptic Keller-Segel type chemotaxis systems, introduced by Jäger and Luckhaus in [10]. In the context of proving existence however, it is crucial to us that not only radial symmetry but also mass is conserved for adequately regular solutions.

3.1.1 Transformation to a scalar Dirichlet problem

Radial symmetry is preserved by solutions to the system (1.3). Utilizing this, we first rewrite (1.3) in radial coordinates.

Lemma 3.1. *Suppose that $n \ge 1$, $R > 0$, $\Omega = B_R(0) \subset \mathbb{R}^n$, $T > 0$, $\Omega_0 := \overline{\Omega} \setminus \{0\}$, and let u_0 comply with (1.4). Then $(u(x,t), v(x,t))$ with $(x,t) \in \Omega_0 \times [0,T)$ is such that*

$$\begin{cases} u \in C^0(\Omega_0 \times [0,T)) \cap C^{2,1}(\Omega_0 \times (0,T)), \\ v \in C^{2,0}(\Omega_0 \times (0,T)), \end{cases} \tag{3.1}$$

and solves (1.3) pointwise in $\Omega_0 \times [0,T)$ if and only if, by writing $r := |x|$, the pair of functions $(u,v) = (u(r,t), v(r,t))$ with

$$\begin{cases} u \in C^0((0,R] \times [0,T)) \cap C^{2,1}((0,R] \times (0,T)), \\ v \in C^{2,0}((0,R] \times (0,T)), \end{cases} \tag{3.2}$$

fulfills

$$
\begin{cases}
u_t \;=\; \frac{1}{r^{n-1}}(r^{n-1+\beta}u_r)_r - \frac{1}{r^{n-1}}(r^{n-1}uv_r)_r, & r \in (0,R),\ t>0,\\
0 \;=\; \frac{1}{r^{n-1}}(r^{n-1}v_r)_r - \mu + u, & r \in (0,R),\ t>0,\\
u_r = v_r = 0, & r = R,\ t>0,\\
u(r,0) = u_0(r), & r \in (0,R),
\end{cases}
\tag{3.3}
$$

pointwise in $(0,R] \times [0,T)$.

PROOF. The transformation of the initial and the boundary conditions are obvious. In order to verify the conversion of the differential equalities as well, note that with $r(x) := |x| = \sqrt{x_1^2 + ... + x_n^2}$ for $i \in \{1,...,n\}$ and $x \in \Omega \setminus \{0\}$,

$$
\frac{\partial r}{\partial x_i} = \frac{1}{2\sqrt{x_1^2 + ... + x_n^2}} \cdot 2x_i = \frac{x_i}{r(x)}. \tag{3.4}
$$

Due to its radial symmetry, u only depends on $|x| = r(x)$. Making use of this and (3.4), for sake of conciseness abbreviating $r = r(x)$, we compute

$$
\begin{aligned}
|x|^\beta \frac{\partial u}{\partial x_i} u(r) &= r^\beta \frac{\partial u(r)}{\partial r} \cdot \frac{\partial r}{\partial x_i}\\
&= r^\beta u_r(r) \cdot \frac{x_i}{r}\\
&= u_r(r) \cdot r^{\beta-1} x_i,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\nabla \cdot \Big(|x|^\beta \nabla u\Big) &= \sum_{i=1}^n \partial_{x_i}\Big(r^\beta \frac{\partial u}{\partial x_i} u(r)\Big)\\
&= \sum_{i=1}^n \partial_{x_i}\Big(u_r(r) \cdot r^{\beta-1} x_i\Big)\\
&= \sum_{i=1}^n \Big(u_r(r) \cdot r^{\beta-1} + u_{rr}(r) \cdot \frac{x_i}{r} \cdot r^{\beta-1} x_i + u_r(r) \cdot (\beta-1) \cdot r^{\beta-2} \frac{x_i}{r} x_i\Big)\\
&= \sum_{i=1}^n \Big(u_r(r) \cdot r^{\beta-1} + u_{rr}(r) \cdot r^{\beta-2} x_i^2 + (\beta-1) u_r(r) r^{\beta-3} x_i^2\Big)\\
&= n \cdot u_r(r) \cdot r^{\beta-1} + u_{rr}(r) \cdot r^{\beta-2}\Big(\sum_{i=1}^n x_i^2\Big) + (\beta-1) u_r(r) r^{\beta-3}\Big(\sum_{i=1}^n x_i^2\Big)\\
&= n \cdot u_r(r) \cdot r^{\beta-1} + u_{rr}(r) \cdot r^{\beta-2} r^2 + (\beta-1) u_r(r) r^{\beta-3} r^2
\end{aligned}
$$

$$
\begin{aligned}
&= n \cdot u_r(r) \cdot r^{\beta-1} + u_{rr}(r) \cdot r^{\beta} + (\beta - 1) u_r(r) r^{\beta-1} \\
&= u_{rr}(r) \cdot r^{\beta} + (n - 1 + \beta) \cdot u_r(r) \cdot r^{\beta-1} \\
&= \frac{1}{r^{n-1}} (r^{n-1+\beta} u_r(r))_r. \qquad (3.5)
\end{aligned}
$$

In an analogous fashion, one establishes that

$$
\nabla \cdot (u(r) \nabla v(r)) = \frac{1}{r^{n-1}} (r^{n-1} u(r) v_r(r))_r \qquad (3.6)
$$

and

$$
\Delta v(r) = \frac{1}{r^{n-1}} (r^{n-1} v_r(r))_r. \qquad (3.7)
$$

Combining (3.5) and (3.6) now indeed confirms the equivalence of the first differential equality in (1.3) and (3.3), respectively, whereas (3.7) implies that the same holds for the second one, thereby completing the proof. □

From here on further, we shall denote $r := |x|$ and without risk of confusion write $(u, v) = (u(r,t), v(r,t))$ in the context of (3.3).

Definition 3.1. *We call a pair of functions (u, v) satisfying (3.1) and solving (1.3) pointwise in $(\overline{\Omega} \setminus \{0\}) \times [0, T)$ a classical solution of (1.3) in $(\overline{\Omega} \setminus \{0\}) \times [0, T)$. Analogously, we name (u, v) with the property (3.2) solving (3.3) pointwise in $(0, R] \times [0, T)$ a classical solution of (3.3) in $(0, R] \times [0, T)$.*

In order to deal with these solutions defined on a non-compact space, we shall focus on bounded solutions.

Lemma 3.2. *Suppose that $n \geq 1$, $R > 0$, and let u_0 comply with (1.4). Let (u, v) with $u \in L^\infty((0, R) \times (0, T_0))$ be a classical solution of (3.3) in $(0, R] \times [0, T_0)$.*
If then

$$
v_r(r,t) = \frac{1}{r^{n-1}} \left(\frac{\mu r^n}{n} - \int_0^r \rho^{n-1} u(\rho, t) d\rho \right) \quad \text{for all } (r,t) \in (0, R] \times [0, T_0), \qquad (3.8)
$$

we have

$$
|v_r(r,t)| \leq Cr \qquad \text{for all } (r,t) \in (0, R] \times [0, T_0) \qquad (3.9)
$$

with $C := \frac{2}{n} \cdot \|u\|_{L^\infty((0,R)\times(0,T_0))}$, and hence in particular $v_r \in L^\infty((0, R) \times (0, T_0))$.

Moreover, for $n \geq 2$ the converse statement also holds true:
If $n \geq 2$ and $v_r \in L^\infty((0,R) \times (0,T_0))$, then necessarily (3.8).

PROOF. Observe that v_r defined as in (3.8) indeed complies with (3.3) since

$$(r^{n-1}v_r)_r = \left(\frac{\mu r^n}{n} - \int\limits_0^r \rho^{n-1}u(\rho,t)d\rho\right)_r = \mu r^{n-1} - r^{n-1}u$$

and thus

$$0 = \frac{1}{r^{n-1}}(r^{n-1}v_r)_r - \mu + u,$$

as well as

$$v_r(R,t) = \frac{1}{R^{n-1}}\left(\frac{\mu R^n}{n} - \int\limits_0^R \rho^{n-1}u(\rho,t)d\rho\right) = 0$$

due to

$$\mu = \frac{1}{|B_R(0)|}\int\limits_{B_R(0)} u = \frac{n}{\omega_n R^n}\omega_n \int_0^R \rho^{n-1}u(\rho,t)d\rho = \frac{n}{R^n}\int_0^R \rho^{n-1}u(\rho,t)d\rho.$$

As a consequence of the boundedness of u in $(0,R] \times [0,T_0)$, we now obtain that for $(r,t) \in (0,R] \times (0,T_0)$

$$\begin{aligned}
|v_r(r,t)| &= \frac{1}{r^{n-1}}\left|\frac{\mu r^n}{n} - \int\limits_0^r \rho^{n-1}u(\rho,t)d\rho\right| \\
&= \frac{1}{r^{n-1}}\left|\int\limits_0^r \rho^{n-1}(\mu - u(\rho,t))d\rho\right| \\
&\leq \frac{1}{r^{n-1}}\int\limits_0^r \rho^{n-1}\left\|\mu - u\right\|_{L^\infty((0,R)\times(0,T_0))} d\rho \\
&\leq \frac{1}{r^{n-1}}\frac{r^n}{n}\left\|\mu - u\right\|_{L^\infty((0,R)\times(0,T_0))} \\
&\leq \frac{r}{n}(\|\mu\|_{L^\infty((0,R)\times(0,T_0))} + \|u\|_{L^\infty((0,R)\times(0,T_0))}) \\
&\leq \frac{2}{n}\left\|u\right\|_{L^\infty((0,R)\times(0,T_0))} \cdot r \\
&= Cr
\end{aligned}$$

with $C := \frac{2}{n} \cdot \|u\|_{L^\infty((0,R)\times(0,T_0))}$ and therefore

$$\|v_r\|_{L^\infty((0,R)\times(0,T_0))} \le CR < \infty,$$

verifying the first part of the lemma.
If moreover $n \ge 2$, then the second equation in (3.3) yields

$$(r^{n-1}v_r)_r = r^{n-1}\mu + r^{n-1}u$$

and thus upon integration for $r \in (0, R]$ and $\delta \in (0, r)$

$$r^{n-1}v_r(r,t) - \delta^{n-1}v_r(\delta,t) = \frac{\mu r^n}{n} - \frac{\mu\delta^n}{n} - \int\limits_\delta^r \rho^{n-1}u(\rho,t)d\rho.$$

Since $n-1 > 0$ and $v_r \in L^\infty((0,R)\times(0,T_0))$, taking $\delta \searrow 0$ results in

$$r^{n-1}v_r(r,t) = \frac{\mu r^n}{n} - \int\limits_0^r \rho^{n-1}u(\rho,t)d\rho,$$

which upon dividing both sides by r^{n-1} gives rise to (3.8). □

Under a weak assumption on β, classical solutions to (3.3) conserve $\|u(\cdot,t)\|_{L^1((0,R))}$ for at least as long as u is bounded and (3.8) holds. These additional conditions are necessary since in contrast to usual settings our solution is not defined on a compact space.

Lemma 3.3. *Suppose that $n \ge 1$, $R > 0$, $\beta > 2 - n$, and u_0 fulfills (1.4). Let (u, v) be a classical solution of (3.3) in the sense of Definition 3.1, and assume that additionally*

$$0 \le u \in L^\infty((0,R)\times(0,T_0)) \qquad \text{for some } T_0 \in (0,T] \tag{3.10}$$

as well as (3.8) holds. Then the mass conservation property

$$\int_0^R \rho^{n-1}u(\rho,t)d\rho = \int_0^R \rho^{n-1}u_0(\rho)d\rho \tag{3.11}$$

is valid for all $t \in [0, T_0)$.

Proof. Let $(\zeta^{(\delta)})_{\delta\in(0,\frac{R}{2})}$ be a family of cutoff functions such that for all $\delta \in (0, \frac{R}{2})$ we have that $\zeta^{(\delta)} \in C^\infty([0,R])$ satisfies

$$\begin{cases} \zeta^{(\delta)}(r) = 0, & r \in [0, \frac{\delta}{2}], \\ 0 \le \zeta^{(\delta)}(r) \le 1, & r \in (\frac{\delta}{2}, \delta), \\ \zeta^{(\delta)}(r) = 1, & r \in [\delta, R], \end{cases} \tag{3.12}$$

as well as

$$0 \le \zeta_r^{(\delta)}(r) \le \frac{4}{\delta}, \qquad r \in \left(\frac{\delta}{2}, \delta\right), \tag{3.13}$$

and for some $C > 0$ independent of δ

$$|\zeta_{rr}^{(\delta)}(r)| \le \frac{C}{\delta^2}, \qquad r \in \left(\frac{\delta}{2}, \delta\right). \tag{3.14}$$

Since (3.12) guarantees that for all $\delta \in (0, \frac{R}{2})$ and $t \in (0, T_0)$ we have $\zeta^{(\delta)} u_t(\cdot, t) \in L^1((0,R))$, using (3.3) we may compute

$$\begin{aligned} \frac{d}{dt}\int_0^R r^{n-1}\zeta^{(\delta)} u dr &= \int_0^R r^{n-1}\zeta^{(\delta)} u_t dr \\ &= \int_0^R r^{n-1}\zeta^{(\delta)} \cdot \frac{1}{r^{n-1}}((r^{n-1+\beta}u_r)_r - (r^{n-1}uv_r)_r) dr \\ &= \int_0^R \zeta^{(\delta)} \cdot (r^{n-1+\beta}u_r - r^{n-1}uv_r)_r dr \\ &= -\int_0^R \zeta_r^{(\delta)} \cdot r^{n-1+\beta}u_r dr + \int_0^R \zeta_r^{(\delta)} \cdot r^{n-1}uv_r dr \\ &= \int_0^R (\zeta_r^{(\delta)} \cdot r^{n-1+\beta})_r u dr + \int_0^R \zeta_r^{(\delta)} \cdot r^{n-1}uv_r dr \end{aligned} \tag{3.15}$$

via partial integration. Herein, abbreviating $C_0 := \|u\|_{L^\infty((0,R)\times(0,T_0))}$, we further estimate for $t \in (0, T_0)$

$$\begin{aligned} \left|\int_0^R (\zeta_r^{(\delta)} \cdot r^{n-1+\beta})_r u dr\right| &\le (n-1+\beta)\int_{\frac{\delta}{2}}^{\delta} |r^{n-2+\beta}\zeta_r^{(\delta)} u| + \int_{\frac{\delta}{2}}^{\delta} |r^{n-1+\beta}\zeta_{rr}^{(\delta)} u| \\ &\le (n-1+\beta)C_0\int_{\frac{\delta}{2}}^{\delta} |r^{n-2+\beta}\zeta_r^{(\delta)}| + C_0\int_{\frac{\delta}{2}}^{\delta} |r^{n-1+\beta}\zeta_{rr}^{(\delta)}| \\ &\le (n-1+\beta)C_0\frac{4}{\delta}\int_{\frac{\delta}{2}}^{\delta} \delta^{n-2+\beta} + C_0 C \cdot \frac{1}{\delta^2}\int_{\frac{\delta}{2}}^{\delta} \delta^{n-1+\beta} \\ &= 2(n-1+\beta)C_0\delta^{n-2+\beta} + C_0 C \cdot \frac{1}{2}\delta^{n-2+\beta}, \end{aligned} \tag{3.16}$$

by $n-2+\beta>0$, also guaranteeing the right hand side converges towards 0 as we let $\delta \searrow 0$. Since due to (3.8), Lemma 3.2 ensures that for some $C_1>0$

$$|v_r(r,t)| \le C_1 r \qquad \text{for all } (r,t)\in(0,R]\times(0,T_0),$$

we deduce that

$$\begin{aligned}
\left|\int_0^R \zeta_r^{(\delta)}\cdot r^{n-1}uv_r dr\right| &\le C_0C_1\int_0^R \zeta_r^{(\delta)}\cdot r^n dr \\
&\le C_0C_1\frac{4}{\delta}\int_0^\delta r^n dr \\
&= C_0C_1\frac{4}{\delta}\frac{\delta^{n+1}}{n} \\
&= \frac{4C_0C_1}{n}\cdot\delta^n \\
&\overset{\delta\searrow 0}{\longrightarrow} 0,
\end{aligned} \tag{3.17}$$

since $n\ge 1$. For $T\in(0,T_0)$, integrating (3.15) over $(0,T)$ now yields

$$\begin{aligned}
\int_0^R r^{n-1}\zeta^{(\delta)}u(r,T)dr - \int_0^R r^{n-1}\zeta^{(\delta)}u_0(r)dr = &\int_0^T\int_0^R (\zeta_r^{(\delta)}\cdot r^{n-1+\beta})_r u dr dt \\
&+\int_0^T\int_0^R \zeta_r^{(\delta)}\cdot r^{n-1}uv_r dr dt.
\end{aligned}$$

By (3.16), (3.17) and monotone as well as dominated convergence, letting $\delta\searrow 0$ this results in

$$\int_0^R r^{n-1}u(r,T)dr - \int_0^R r^{n-1}u_0(r)dr = 0,$$

verifying (3.11). □

Of major importance to our further analysis is the transformation of (3.3) to a Dirichlet problem.

Lemma 3.4. *Suppose $n\ge 1$, $R>0$, $\beta>2-n$, and u_0 fulfills (1.4), and let (u,v) be a solution to (3.3) in the sense of Lemma 3.1 for which (3.8) and (3.10) hold for $T_0=T$. We introduce the mass accumulation function*

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)d\rho, \quad s=r^n\in[0,R^n], t\in[0,T). \tag{3.18}$$

Then $w \in C^0([0,R^n]\times[0,T))\cap C^{2,1}((0,R^n]\times(0,T))$, *and for all* $s \in (0,R^n)$ *and* $t \in (0,T)$, *its spatial derivatives are given by*

$$w_s(s,t) = \frac{1}{n}\cdot u(s^{\frac{1}{n}},t) \quad \text{and} \quad w_{ss}(s,t) = \frac{1}{n^2}\cdot s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}},t). \tag{3.19}$$

Furthermore, w *solves the Dirichlet problem*

$$\begin{cases} w_t = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} + nww_s - \mu s w_s, & s\in(0,R^n),\ t\in(0,T), \\ w(0,t)=0, \quad w(R^n,t)=\frac{m}{\omega_n}, & t\in(0,T), \\ w(s,0)=w_0(s), & s\in(0,R^n), \end{cases} \tag{3.20}$$

with $m := \int\limits_\Omega u_0$, $\mu = \frac{nm}{\omega_n R^n}$ *and*

$$w_0(s) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u_0(\rho)d\rho, \qquad s\in[0,R^n]. \tag{3.21}$$

PROOF. The proof is inspired by [24, p.264].
The derivatives with respect to s are obtained via straightforward calculation. Considering that (3.8) is equivalent to

$$r^{n-1}v_r(r,t) = \frac{\mu r^n}{n} - \int\limits_0^r \rho^{n-1}u(\rho,t)d\rho,$$

employing the first equation in (3.3) and recalling $s=r^n$, we may calculate

$$\begin{aligned} w_t(s,t) &= \int\limits_0^{s^{\frac{1}{n}}} \rho^{n-1}u_t(\rho,t)d\rho \\ &= \int\limits_0^{s^{\frac{1}{n}}} (\rho^{n-1+\beta}u_r(\rho,t))_r d\rho - \int\limits_0^{s^{\frac{1}{n}}} (\rho^{n-1}u(\rho,t)v_r(\rho,t))_r d\rho \\ &= s^{1-\frac{1}{n}+\frac{\beta}{n}}u_r(s^{\frac{1}{n}},t) - s^{\frac{n-1}{n}}u(s^{\frac{1}{n}},t)v_r(s^{\frac{1}{n}},t) \\ &= n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}}w_{ss}(s,t) - u(s^{\frac{1}{n}},t)s^{\frac{n-1}{n}}v_r(s^{\frac{1}{n}},t) \\ &= n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}}w_{ss}(s,t) - u(s^{\frac{1}{n}},t)\left(\frac{\mu s}{n} - \int\limits_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)d\rho\right) \end{aligned}$$

$$
\begin{aligned}
&= n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss}(s,t) - n w_s(s,t)\left(\frac{\mu s}{n} - w(s,t)\right) \\
&= n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss}(s,t) + n w(s,t) w_s(s,t) - \mu w_s(s,t) s
\end{aligned}
$$

for $(s,t) \in (0,R^n) \times (0,T)$, thus verifying the differential equality in (3.20).
Due to

$$
w(R^n,t) = \int_0^R \rho^{n-1} u(\rho,t) d\rho = \int_0^R \rho^{n-1} u_0(\rho) d\rho = \frac{1}{\omega_n} \int_\Omega u_0(x) dx = \frac{m}{\omega_n}
$$

for $t \in [0,T)$, the boundary condition at $s = R^n$ is a simple consequence of Lemma 3.3. The Lebesgue integrability of u induced by (3.10) on the other hand immediately confirms $w(0,t) = 0$ for all $t \in [0,T)$. $\square$

3.1.2 An auxiliary problem

By Lemma 3.4, under the mild assumption that $\beta > 2-n$, every bounded nonnegative classical solution of (3.3) satisfying (3.8) implies the existence of a classical solution to (3.20). Thus, in search of solutions to the former, it appears sensible to study the latter system.
Regarding similar systems emerging from the basic parabolic-elliptic Keller-Segel model in two dimensions, proofs for the existence of corresponding solutions have been sketched in [26] and [4]. Those serve as an orientation for our approach.

As (3.20) still contains a diffusion degeneracy, questions regarding its solvability are not covered by standard theory. Therefore, we resort to non-degenerate „approximating“ problems.

Here and in the following, we always let $m > 0$, which can be interpreted as the total mass in the original system as in Lemma 3.4.

Lemma 3.5. *Let $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and $w_0 \in C^1([0, R^n])$ be such that*

$$w_0(0) = 0, \qquad w_0(R^n) = \frac{m}{\omega_n} \quad \textit{as well as} \quad w_{0s} \geq 0. \tag{3.22}$$

For $\varepsilon \in (0, R^n)$, define $w_{0\varepsilon} \in C^1([\varepsilon, R^n])$ via

$$w_{0\varepsilon}(s) := w_0\Big(\frac{R^n(s-\varepsilon)}{R^n - \varepsilon}\Big), \qquad s \in [\varepsilon, R^n]. \tag{3.23}$$

Then the system

$$\begin{cases} w_{\varepsilon t} = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{\varepsilon ss} + n w_\varepsilon w_{\varepsilon s} - \mu s w_{\varepsilon s}, & s \in (\varepsilon, R^n),\ t > 0, \\ w_\varepsilon(\varepsilon, t) = 0, \quad w_\varepsilon(R^n, t) = \frac{m}{\omega_n}, & t > 0, \\ w_\varepsilon(s, 0) = w_{0\varepsilon}(s), & s \in (\varepsilon, R^n), \end{cases} \tag{3.24}$$

possesses a unique global classical solution $w_\varepsilon \in C^0([\varepsilon, R^n] \times [0, \infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0, \infty))$.

PROOF. By (3.22) and (3.23), it is ensured that

$$w_{0\varepsilon}(\varepsilon) = w_0(0) = 0 \quad \text{and} \quad w_{0\varepsilon}(R^n) = w_0(R^n) = \frac{m}{\omega_n}.$$

Thus, by standard theory (cf. [13, V.6]), (3.24) indeed admits a unique global classical solution as claimed. □

Further information regarding these solutions becomes accessible by means of Lemma 2.3, first enabling us to derive basic bounds for our approximating solutions w_ε.

Lemma 3.6. *Let $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and $w_0 \in C^1([0, R^n])$ be as in (3.22), and $w_{0\varepsilon} \in C^1([\varepsilon, R^n])$ shall be defined as in (3.23).*
Then the global classical solution $w_\varepsilon \in C^0([\varepsilon, R^n] \times [0, \infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0, \infty))$ of (3.24) can be estimated by

$$0 \leq w_\varepsilon(s,t) \leq \frac{m}{\omega_n} \qquad \text{for all } (s,t) \in [\varepsilon, R^n] \times [0, \infty). \tag{3.25}$$

PROOF. Let $T > 0$ be abritrary. With $a = n^2$, $\alpha = 2 - \frac{2}{n} + \frac{\beta}{n}$, $b = n$, $\gamma = 0$, $c = -\mu$, $d = 0$ and $\delta = 1$ we have

$$w_{\varepsilon t} = a s^\alpha w_{\varepsilon ss} + b s^\gamma w_\varepsilon w_{\varepsilon s} + c s^\delta w_{\varepsilon s} + d w_{\varepsilon s}$$

in $(\varepsilon, R^n) \times (0, T)$. Defining $\underline{w} \equiv 0 \in C^0([\varepsilon, R^n] \times [0, \infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0, \infty))$, we get that

$$\underline{w}_t = 0 = a s^\alpha \underline{w}_{ss} + b s^\gamma \underline{w}\,\underline{w}_s + c s^\delta \underline{w}_s + d \underline{w}_s$$

for all $(s,t) \in (\varepsilon, R^n) \times (0, T)$, as well as

$$\underline{w}(s, 0) = 0 \leq w_{0\varepsilon}(s) = w_\varepsilon(s, 0) \qquad \text{forall } s \in [\varepsilon, R^n]$$

due to $w_{0\varepsilon}(0) = 0$ and $w_{0\varepsilon s} = w_{0s}\left(\frac{R^n(s-\varepsilon)}{R^n-\varepsilon}\right) \cdot \frac{R^n}{R^n-\varepsilon} \geq 0$.
Moreover, at the spatial boundary

$$\underline{w}(\varepsilon, t) = 0 = w_\varepsilon(\varepsilon, t) \quad \text{and} \quad \underline{w}(R^n, t) = 0 \leq \frac{m}{\omega_n} = w_\varepsilon(R^n, t) \qquad \text{for all } t \in (0, T)$$

hold, and obviously $\underline{w}_s \equiv 0$ is bounded.
Therefore, we may invoke Lemma 2.3 to conclude that

$$0 = \underline{w}(s,t) \leq w_\varepsilon(s,t) \qquad \text{for all } s \in [\varepsilon, R^n] \quad \text{and} \quad t \in [0, T).$$

Since $T > 0$ was arbitrary, the inequality actually holds in $[\varepsilon, R^n] \times [0, \infty)$, as desired.
The upper bound can be confirmed analogously. □

In the former proof, we made critical use of the first spatial derivatives of the sub- respectively supersolution being bounded. In order to be able to compare solutions w_ε to (3.24) for different $\varepsilon \in (0, R^n)$ with each other however, we need to establish that $w_{\varepsilon s}$ is bounded locally in time. To that end, we require the initial data to be slightly more regular than imposed in Lemma 3.5.

Lemma 3.7. *Suppose the conditions of Lemma 3.5 are met, and that additionally there exists $\theta \in (0, 1)$ such that*

$$w_0 \in C^{1+\theta}([0, R^n]). \tag{3.26}$$

Then for any $T \in (0, \infty)$, the solution of (3.24) satisfies

$$w_\varepsilon \in C^{1+\eta, \frac{1+\eta}{2}}([\varepsilon, R^n] \times [0, T]) \tag{3.27}$$

for some $\eta = \eta(\theta, T, \varepsilon) \in (0, 1)$. In particular,

$$w_{\varepsilon s} \in L^\infty((\varepsilon, R^n) \times (0, T)) \qquad \text{for all} \quad T \in (0, \infty). \tag{3.28}$$

Proof. Let $T > 0$. With $\Omega := (\varepsilon, R^n) \subset \mathbb{R}$,

$$\Phi : \{\varepsilon, R^n\} \times [0, T] \to \mathbb{R}, \qquad \Phi(\varepsilon, t) = 0, \; \Phi(R^n, t) = \frac{m}{\omega_n}$$

as well as

$$a : [\varepsilon, R^n] \times [0, T] \to \mathbb{R}, \quad a(s, t) = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} \tag{3.29}$$

along with

$$b : [\varepsilon, R^n] \times [0, T] \to \mathbb{R}, \qquad b(s, t) = n w_\varepsilon(s, t) - \mu s - n^2 \left(2 - \frac{2}{n} + \frac{\beta}{n}\right) s^{1-\frac{2}{n}+\frac{\beta}{n}} \tag{3.30}$$

and $d \equiv f \equiv 0$, the system (3.24) can be rewritten in terms of (2.1).
Note that Lemma 3.5 ensures that $w_\varepsilon \in C^0([\varepsilon, R^n] \times [0, T])$, thus allowing us to regard (3.24) as a linear problem here and guaranteeing that b is in the desired regularity class. Therefore, (2.2) is fulfilled.
By (3.23) and (3.26), we have that

$$w_{0\varepsilon} \in C^{1+\theta}([\varepsilon, \mathbb{R}^n]),$$

securing (2.3), and as already outlined in the context of Lemma 3.5, $w_{0\varepsilon}$ is compatible at the

boundary.
Furthermore, there is no diffusion degeneracy because of

$$a(s,t) \geq n^2 \varepsilon^{2-\frac{2}{n}+\frac{\beta}{n}} > 0 \qquad \text{for all } (s,t) \in [\varepsilon, R^n] \times [0,T],$$

warranting that (2.4) must be true for some $M(\varepsilon) > 0$.
Hence we may employ Lemma 2.1 to infer that there exists $\eta \in (0,1)$ such that (3.27) holds.
This however implies that $w_{\varepsilon s} \in C^0([\varepsilon, R^n] \times [0,T])$, whereby we conclude (3.28). □
This puts us in position to prove

Lemma 3.8. *Let $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and $w_0 \in C^{1+\theta}([0,R^n])$ be as in (3.22).*
For $\varepsilon_i \in (0, R^n)$, $i \in \{1,2\}$, with $\varepsilon_2 \leq \varepsilon_1$ define $w_{0\varepsilon_i} \in C^{1+\theta}([\varepsilon, R^n])$ via

$$w_{0\varepsilon_i}(s) := w_0\Big(\frac{R^n(s-\varepsilon_i)}{R^n - \varepsilon_i}\Big), \qquad s \in [\varepsilon_i, R^n]. \tag{3.31}$$

Then for $i \in \{1,2\}$, the corresponding classical solutions $w_{\varepsilon_i} \in C^0([\varepsilon_i, R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon_i, R^n] \times (0,\infty))$ of

$$\begin{cases} w_{\varepsilon_i t} = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{\varepsilon_i ss} + n w_{\varepsilon_i} w_{\varepsilon_i s} - \mu s w_{\varepsilon_i s}, & s \in (\varepsilon_i, R^n),\ t > 0, \\ w_{\varepsilon_i}(\varepsilon_i, t) = 0, \quad w_{\varepsilon_i}(R^n, t) = \frac{m}{\omega_n}, & t > 0, \\ w_{\varepsilon_i}(s,0) = w_{0\varepsilon_i}(s), & s \in (\varepsilon_i, R^n), \end{cases} \tag{3.32}$$

satisfy the ordering property

$$w_{\varepsilon_1}(s,t) \leq w_{\varepsilon_2}(s,t) \qquad \textit{for all} \quad (s,t) \in [\varepsilon_1, R^n] \times [0,\infty). \tag{3.33}$$

PROOF. For each $T > 0$, we obviously have

$$w_{\varepsilon_1 t} \leq n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{\varepsilon_1 ss} + n w_{\varepsilon_1} w_{\varepsilon_1 s} - \mu s w_{\varepsilon_1 s} \quad \text{and} \quad w_{\varepsilon_2 t} \geq n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{\varepsilon_2 ss} + n w_{\varepsilon_2} w_{\varepsilon_2 s} - \mu s w_{\varepsilon_2 s}$$

in $(\varepsilon_1, R^n) \times (0,T)$, as outlined in the proof of Lemma 3.6 these inequalities being of type (2.18).
Furthermore, $\varepsilon_2 \leq \varepsilon_1$ implies that $\frac{R^n(s-\varepsilon_2)}{R^n-\varepsilon_2} \geq \frac{R^n(s-\varepsilon_1)}{R^n-\varepsilon_1}$ and thus combined with $w_{0s} \geq 0$

$$w_{0\varepsilon_2}(s) = w_0\Big(\frac{R^n(s-\varepsilon_2)}{R^n-\varepsilon_2}\Big) \geq w_0\Big(\frac{R^n(s-\varepsilon_1)}{R^n-\varepsilon_1}\Big) = w_{0\varepsilon_1}(s)$$

for all $s \in (\varepsilon_1, R^n)$. Moreover, due to (3.25)

$$w_{\varepsilon_2}(\varepsilon_1, t) \geq 0 = w_{\varepsilon_1}(\varepsilon_1, t) \qquad \text{for all } \ t \in (0, T),$$

whereas the inequality at the right boundary trivially holds with equality.
Lastly, since the conditions of Lemma 3.7 are met, $w_{\varepsilon_2 s}$ is bounded, and consequently we may invoke Lemma 2.3 to conclude (3.33). □

Now we can obtain a solution to – albeit yet an incomplete version of – the degenerate problem formulated in (3.20):

Lemma 3.9. *Let $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and suppose $w_0 \in C^{1+\theta}([0, R^n])$ is as in (3.22).*
Moreover, for arbitrary $s_0 \in (0, R^n)$ and $\varepsilon \in (0, s_0)$ we define $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon, R^n])$ via (3.23).
Denoting $w_\varepsilon \in C^0([\varepsilon, R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0,\infty))$ as the corresponding solution to (3.24), we have

$$w_\varepsilon \nearrow w_{s_0} \qquad \text{for } \varepsilon \searrow 0 \tag{3.34}$$

pointwise in $[s_0, R^n] \times [0, \infty)$ for some limit function $w_{s_0} \in C^{1,\frac{1}{2}}([s_0, R^n] \times [0,\infty)) \cap C^{2,1}([s_0, R^n] \times (0, \infty))$. Moreover,

$$w_\varepsilon \to w_{s_0} \quad \text{in } C^{1,\frac{1}{2}}_{loc}([s_0, R^n] \times [0,\infty)) \quad \text{and} \quad w_\varepsilon \to w_{s_0} \quad \text{in } C^{2,1}_{loc}([s_0, R^n] \times (0,\infty)). \tag{3.35}$$

Thus, letting $s_0 \searrow 0$, we acquire a function $w \in C^{1,\frac{1}{2}}((0, R^n] \times [0,\infty)) \cap C^{2,1}((0, R^n] \times (0,\infty))$ which solves the incomplete Dirichlet problem

$$\begin{cases} w_t = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} + nww_s - \mu s w_s, & s \in (0, R^n), \ t > 0, \\ w(R^n, t) = \frac{m}{\omega_n}, & t > 0, \\ w(s, 0) = w_0(s), & s \in (0, R^n). \end{cases} \tag{3.36}$$

Proof. By Lemma 3.8 and Lemma 3.6, $(w_\varepsilon)_{\varepsilon \in (0, s_0)}$ is monotonically increasing as $\varepsilon \searrow 0$ and bounded from above, whereby via monotone convergence we may indeed conclude (3.34). Now let $\varepsilon \in (0, \frac{s_0}{2})$ and $T \in (0, \infty)$. With designations as in the proof of Lemma 3.7 (see especially (3.29) and (3.30)) and $\Omega' := (\frac{s_0}{2}, R^n]$, $K := [s_0, R^n]$ in the context of Lemma 2.1 (ii) we can easily see that due to $\frac{s_0}{2} > 0$ the function a is in the desired regularity class, and Lemma 3.5 guarantees the same for b, so that (2.5) holds. Lemma 3.6 also ensures that $\|b\|_{L^\infty(\Omega \times (0,T))}$ does not depend on ε; therefore the constant M in (2.6) is determined solely by s_0. By the chain rule and

$$\left|\frac{R^n(s-\varepsilon)}{R^n-\varepsilon}-\frac{R^n(t-\varepsilon)}{R^n-\varepsilon}\right| = \frac{R^n}{R^n-\varepsilon}|s-t| \le \frac{R^n}{R^n-\frac{s_0}{2}}|s-t|$$

for $\varepsilon \in (0, \frac{s_0}{2})$ and $s, t \in [0, R^n]$, the quantity $\|w_{0\varepsilon}\|_{C^{1+\theta}(\Omega')}$ is also bounded independently of ε. Moreover,

$$a \ge n^2 s_0^{2-\frac{2}{n}+\frac{\beta}{n}} \qquad \text{in} \quad [s_0, R^n] \times [0, T]$$

and thus the lower bound in (2.8) does not depend on ε as well.

By means of Lemma 2.1, we infer the existence of $\eta \in (0,1)$ and $C_1 > 0$, dependent on s_0 and T but independent of ε, such that for all $\varepsilon \in (0, \frac{s_0}{2})$ the classical solution w_ε of (3.24) is in $C^{1+\eta,\frac{1+\eta}{2}}([s_0, R^n] \times [0, T])$ with

$$\|w_\varepsilon\|_{C^{1+\eta,\frac{1+\eta}{2}}([s_0,R^n]\times[0,T])} \le C_1. \tag{3.37}$$

Consequently, the Arzelà-Ascoli theorem yields

$$w_\varepsilon \to w_{s_0} \quad \text{in} \quad C^{1,\frac{1}{2}}([s_0, R^n] \times [0, T]). \tag{3.38}$$

Now let $\tau > 0$. By (3.37), in particular we established that there is a uniform bound for $\|w_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\Omega'\times[\tau,T])}$ with respect to ε, thus verifying that (2.15) and (2.16) hold with M independent of ε.

Just as above, (2.17) and (2.9) are valid with no dependence on ε as well, and therefore

$$\|w_\varepsilon\|_{C^{2+\eta,1+\frac{\eta}{2}}([s_0,R^n]\times[\tau,T])} \le C_2 \tag{3.39}$$

with $\eta \in (0,1)$ and $C_2 > 0$ depending on s_0, τ and T but not on ε. Hence, applying Arzelà-Ascoli's theorem results in

$$w_\varepsilon \to w_{s_0} \quad \text{in} \quad C^{2,1}([s_0, R^n] \times [\tau, T]), \tag{3.40}$$

together with (3.37) confirming (3.35).

Combining the first equation in (3.24) and (3.40), we easily see that $w_{s_0} \in C^{2,1}([s_0, R^n] \times (0, \infty))$ and

$$w_{s_0 t} = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{s_0 ss} + n w_{s_0} w_{s_0 s} - \mu s w_{s_0 s} \tag{3.41}$$

in $[s_0, R^n] \times [\tau, T]$, and letting $\tau \searrow 0$ and $T \nearrow \infty$, we certify that this differential equality actually holds in $[s_0, R^n] \times (0, \infty)$.

The boundary condition at $s = R^n$ is merely a result of $w_\varepsilon(R^n, t) = \frac{m}{\omega_n}$ and the pointwise convergence, whereas

$$w_{s_0}(s,0) = w_0(s)$$

for $s \in [s_0, R^n)$ and $w_{s_0} \in C^{1,\frac{1}{2}}([s_0, R^n] \times [0,\infty))$ readily follow from (3.38) and

$$w_{0\varepsilon} \to w_0 \qquad \text{in} \quad C^1([s_0, R^n]).$$

By letting $s_0 \searrow 0$, thus defining $w : (0, R^n] \times [0,\infty) \to \mathbb{R}$ via $w(s,t) := w_{\frac{s}{2}}(s,t)$, this results in (3.36). □

3.1.3 Properties of w

In the previous subsection, we constructed a global-in-time solution to the incomplete Dirichlet problem (3.36). In the following, we establish that in fact the function w from Lemma 3.9 solves (3.20) for some $T^* > 0$, and we gather some information about w_s, in light of it essentially representing u in our original problem (1.3).

First we verify that the first spatial derivatives of our approximating solutions w_ε are nonnegative.

Lemma 3.10. *Let $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\theta \in (0,1)$, $\beta > 0$ and $w_0 \in C^{1+\theta}([0, R^n])$ be as in (3.22) and $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon, R^n])$ be defined as in (3.23) for $\varepsilon \in (0, R^n)$.*
Then the global classical solution $w_\varepsilon \in C^{1,\frac{1}{2}}([\varepsilon, R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0,\infty))$ of (3.24) has the property that

$$w_{\varepsilon s}(s,t) \geq 0 \qquad \textit{for all} \quad (s,t) \in [\varepsilon, R^n] \times [0,\infty). \tag{3.42}$$

Proof. By standard theory, we actually have that

$$w_{\varepsilon s} \in C^0([\varepsilon, R^n] \times [0,\infty)) \cap C^{2,1}((\varepsilon, R^n) \times (0,\infty)). \tag{3.43}$$

On account of that and the first equation in (3.24), $z := w_{\varepsilon s}$ fulfills

$$z_t = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + n^2\left(2 - \frac{2}{n} + \frac{\beta}{n}\right) s^{1-\frac{2}{n}+\frac{\beta}{n}} z_s + n w_\varepsilon z_s + n z^2 - \mu s z_s - \mu z \tag{3.44}$$

in $(\varepsilon, R^n) \times (0,\infty)$. For fixed $\delta > 0$, define $\varphi \in C^{2,1}((\varepsilon, R^n) \times (0,\infty))$ via

$$\varphi(s,t) := z(s,t) + \delta e^t.$$

For $T > 0$ we set

$$S := \{t \in [0, T) \mid \varphi(s, \tilde{t}) > 0\, \forall (s, \tilde{t}) \in [\varepsilon, R^n] \times [0, t]\}.$$

Then S is not empty due to

$$\varphi(s, 0) = w_{0\varepsilon s}(s) + \delta \geq \delta > 0,$$

which along with $\varphi \in C^0([\varepsilon, R^n] \times [0, T))$ also guarantees $t_0 := \sup S > 0$.
If now $t_0 < T$, then there exists $s_0 \in [\varepsilon, R^n]$ such that $\varphi(s_0, t_0) = 0$.
However, as for all $t \geq 0$ we have

$$w_\varepsilon(\varepsilon, t) = 0 \qquad \text{and} \qquad w_\varepsilon(R^n, t) = \frac{m}{\omega_n}$$

and Lemma 3.6 ensures that $0 \leq w_\varepsilon \leq \frac{m}{\omega_n}$, we can conclude that $w_{\varepsilon s}(\varepsilon, t) \geq 0$ and $w_{\varepsilon s}(R^n, t) \geq 0$ and consequently, $s_0 \in (\varepsilon, R^n)$.
Therefore, at (s_0, t_0) not only

$$\varphi = z + \delta e^{t_0} = 0 \quad \text{and} \quad \varphi_t \leq 0$$

but also

$$\varphi_s = z_s = 0 \quad \text{and} \quad \varphi_{ss} = z_{ss} \geq 0$$

hold. Combined with (3.44), this yields

$$\begin{aligned}
0 &\geq \varphi_t(s_0, t_0) = z_t + \delta e^{t_0} \\
&= n^2 s_0^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + n^2\left(2 - \frac{2}{n} + \frac{\beta}{n}\right) s_0^{1-\frac{2}{n}+\frac{\beta}{n}} z_s + n w_\varepsilon z_s + n z^2 - \mu s_0 z_s - \mu z + \delta e^{t_0} \\
&= n^2 s_0^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + n z^2 + \mu \delta e^{t_0} + \delta e^{t_0} \\
&\geq (1 + \mu)\delta e^{t_0} \\
&> 0,
\end{aligned}$$

a contradiction.
Thus necessarily $t_0 = T$, which by taking $\delta \searrow 0$ implies that indeed (3.42) holds true, since $T > 0$ has been chosen arbitrarily. □

Now we can easily transfer some estimates for our approximating solutions to w, thereby also establishing that w can be extended to $s = 0$ in a natural way:

Lemma 3.11. *Let $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and suppose $w_0 \in C^{1+\theta}([0, R^n])$ is as in (3.22).*
Then $w \in C^{1,\frac{1}{2}}((0, R^n] \times [0,\infty)) \cap C^{2,1}((0, R^n] \times (0,\infty))$ constructed in Lemma 3.9 is bounded by

$$0 \leq w(s,t) \leq \frac{m}{\omega_n} \qquad \text{for all } (s,t) \in (0, R^n] \times [0,\infty) \tag{3.45}$$

and for its first spatial derivative

$$w_s(s,t) \geq 0 \qquad \text{for all} \quad (s,t) \in (0, R^n] \times [0,\infty) \tag{3.46}$$

holds. Thus, for fixed $t \geq 0$ we may continuously extend $w(\cdot,t)$ to $s = 0$ via

$$w(0,t) := \lim_{s \searrow 0} w(s,t) \geq 0. \tag{3.47}$$

Proof. The estimate (3.45) is a direct consequence of Lemma 3.6 and the pointwise convergence (3.34), whereas (3.46) results from (3.42) and the convergence in $C^{1,\frac{1}{2}}_{loc}((0, R^n] \times [0,\infty))$ established in (3.35).
Since thereby it is guaranteed that w is bounded from below by 0, and for any fixed $t \geq 0$ the function $w(s,t)$ monotonically decreases as s decreases,

$$\lim_{s \searrow 0} w(s,t) \geq 0$$

is well-defined and thus defines a continuous extension of $w(\cdot,t)$ to $s = 0$. □

Without risk of confusion, from here on we regard w as incorporating this extension.
Note that w is continuous at $(0,t)$ with respect to the spatial variable s for fixed time t, yet not necessarily continuous in spacetime. As results in related systems, such as [4, Theorem 3.1] or [21, Lemma 3.4], suggest, in this general framework this is probably not always the case globally in time. For small times however we can establish

Lemma 3.12. *Let $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and suppose $w_0 \in C^{1+\theta}([0, R^n])$ is as in (3.22).*
Then there is a $T^* > 0$ and $y : [0, T^*) \to \mathbb{R}$ such that with w as in Lemma 3.9

$$w(s,t) \leq y(t) \cdot s \qquad \text{for all} \quad (s,t) \in (0, R^n] \times [0, T^*). \tag{3.48}$$

In consequence, we obtain that

$$w \in C^0([0,R^n] \times [0,T^*)) \quad \textit{with} \quad w(0,t) = 0 \qquad \textit{for all} \quad t \in [0,T^*). \tag{3.49}$$

If additionally

$$w_{0s}(s) \le \frac{\mu}{n} \qquad \textit{for all} \quad s \in (0,R^n), \tag{3.50}$$

then $T^* = \infty$.

PROOF. Once more, we resort to a comparison argument utilizing the approximating solutions w_ε.
Define

$$y_0 := \max\left\{ \|w_{0s}\|_{L^\infty((0,R^n))}, \frac{\mu}{n} \right\}, \tag{3.51}$$

and let $y \in C^1([0,T^*))$ denote the solution to the ODE system

$$\begin{cases} y'(t) = ny^2(t) - \mu y(t), & t > 0, \\ y(0) = y_0, \end{cases} \tag{3.52}$$

extended up to its maximal time of existence $T^* := T^y_{max} > 0$.
Based thereupon, for $\varepsilon \in (0,R^n)$ and arbitrary $T \in (0,T^*)$, set

$$\overline{w}(s,t) := y(t) \cdot s, \qquad (s,t) \in [\varepsilon, R^n] \times [0,T]. \tag{3.53}$$

Then we have

$$\begin{aligned} \overline{w}_t - n^2 s^{2-\frac{2}{n}-\frac{\beta}{n}} \overline{w}_{ss} - n\overline{w}\overline{w}_s + \mu s \overline{w}_s &= y_t \cdot s - ny^2 \cdot s + \mu y \cdot s \\ &= (ny^2 - \mu y) \cdot s - (ny^2 - \mu y) \cdot s \\ &= 0 \end{aligned} \tag{3.54}$$

in $(\varepsilon, R^n) \times (0,T)$. Furthermore, by (3.22), (3.23) and (3.51) we deduce that for $s \in (\varepsilon, R^n)$

$$\begin{aligned} w_\varepsilon(s,0) = w_{0\varepsilon}(s) &\le w_0(s) \\ &= \underbrace{w_0(0)}_{=0} + \int_0^s w_{0s}(\rho)d\rho \\ &\le \|w_{0s}\|_{L^\infty((0,R^n))} \cdot s \\ &\le y_0 \cdot s \\ &= \overline{w}(s,0). \end{aligned} \tag{3.55}$$

By means of a simple ODE comparison argument, we can see that $y(t) \geq \frac{\mu}{n} \geq 0$ for all $t \geq 0$. Therefore, the required inequalities at the lateral boundary are easily confirmed via

$$w_\varepsilon(0,t) = 0 \leq y(t) \cdot \varepsilon = \overline{w}(\varepsilon, t) \tag{3.56}$$

and

$$w_\varepsilon(R^n,t) = \frac{m}{\omega_n} = \frac{\mu}{n} \cdot R^n \leq y(t) \cdot R^n = \overline{w}(R^n, t) \tag{3.57}$$

for $t \in (0,T)$.
Thus combining (3.54), (3.55), (3.56) and (3.57), an application of Lemma 2.3 yields

$$w_\varepsilon(s,t) \leq \overline{w}(s,t) = y(t) \cdot s \qquad \text{for all} \quad (s,t) \in [\varepsilon, R^n] \times [0,T],$$

and by taking $T \nearrow T^*$ this inequality holds in $[\varepsilon, R^n] \times [0, T^*)$. Since the right hand side is independent of ε and $\varepsilon \in (0, R^n)$ has been chosen arbitrarily, via the pointwise convergence (3.34) this results in (3.48).
Now for $t \in [0, T^*)$, let $(s_k, t_k) \subset (0, R^n] \times [0, T^*)$ be a sequence such that $(s_k, t_k) \to (0, t)$ for $k \to \infty$.
Then there exists $k_0 \in \mathbb{N}$ with the property that for all $k \geq k_0$ we have $t_k \leq \frac{t+T^*}{2}$. Utilizing this and (3.48), we get that for $k \geq k_0$

$$w(s_k, t_k) \leq y(t_k) \cdot s_k \leq \frac{t+T^*}{2} \cdot s_k \to 0$$

for $k \to \infty$, which together with the nonnegativity of w establishes its continuity in $\{0\} \times [0, T^*)$ and (3.49).

If (3.50) holds, then obviously $y \equiv \frac{\mu}{n}$ and therefore $T^* = \infty$. □

Note that Lemma 3.12 implies that w solves (3.20) in $[0, R^n] \times [0, T^*)$.
The case (3.50) though contains linear functions w_0 (in terms of the original system (1.3) constant initial data u_0) at most and thus is not very interesting.

The information on the boundary $s = 0$ on hand now allows for a statement on uniqueness of the function constructed in Lemma 3.9 as a solution to (3.20), as long as its spatial derivative is bounded.

Lemma 3.13. *Let $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and suppose $w_0 \in C^{1+\theta}([0, R^n])$ is as in (3.22).*

For the function w defined in Lemma 3.9, let $T > 0$ be such that $w \in C^0([0,R^n] \times [0,T))$ with

$$w(0,t) = 0 \qquad \textit{for all} \quad t \in (0,T), \tag{3.58}$$

and furthermore assume w_s is bounded in $(0,R^n) \times [0,T)$.
Then w is the unique solution of (3.20) in $C^0([0,R^n] \times [0,T)) \cap C^{2,1}((0,R^n] \times (0,T))$.

PROOF. By (3.36) and (3.58), w is a classical solution of (3.20) in $[0,R^n] \times [0,T)$. Combined with the boundedness of w_s, via Lemma 2.3 we may immediately infer uniqueness. □

3.1.4 Local boundedness of w_s

The linear supersolution of w established in Lemma 3.12 alone is not sufficient to deduce that the spatial derivative w_s is bounded in $(0,R^n) \times (0,T^*)$. In order to be able to draw this conclusion, we will ensure that w is concave. To that end, we introduce another auxiliary problem.

Lemma 3.14. *Let $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$, $\theta \in (0,1)$ and $w_0 \in C^{1+\theta}([0,R^n])$ be as in (3.22). For $\varepsilon \in (0,R^n)$, define $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon,R^n])$ via (3.23).*
Then the system

$$\begin{cases} \tilde{w}_{\varepsilon t} = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} \tilde{w}_{\varepsilon ss} + n\tilde{w}_\varepsilon \tilde{w}_{\varepsilon s} - \mu(s-\varepsilon)\tilde{w}_{\varepsilon s}, & s \in (\varepsilon,R^n),\ t > 0, \\ \tilde{w}_\varepsilon(\varepsilon,t) = 0, \quad \tilde{w}_\varepsilon(R^n,t) = \frac{m}{\omega_n}, & t > 0, \\ \tilde{w}_\varepsilon(s,0) = w_{0\varepsilon}(s), & s \in (\varepsilon,R^n), \end{cases} \tag{3.59}$$

possesses a unique global classical solution $\tilde{w}_\varepsilon \in C^{1,\frac{1}{2}}([\varepsilon,R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon,R^n] \times (0,\infty))$.
Furthermore, we have

$$0 \leq \tilde{w}_\varepsilon(s,t) \leq \frac{m}{\omega_n} \qquad \textit{for all } (s,t) \in [\varepsilon,R^n] \times [0,\infty), \tag{3.60}$$

and $\tilde{w}_{\varepsilon s} \in C^0([\varepsilon,R^n] \times [0,\infty))$ is nonnegative, i.e.

$$\tilde{w}_{\varepsilon s}(s,t) \geq 0 \qquad \textit{for all} \quad (s,t) \in [\varepsilon,R^n] \times [0,\infty). \tag{3.61}$$

PROOF. As in Lemma 3.5, by standard theory there is a unique solution $\tilde{w}_\varepsilon \in C^0([\varepsilon,R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon,R^n] \times (0,\infty))$ to (3.59). Due to $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon,R^n])$, by means of the

Schauder estimate Lemma 2.1 we may furthermore establish that even $\tilde{w}_\varepsilon \in C^{1,\frac{1}{2}}([\varepsilon, R^n] \times [0,\infty))$ and that therefore $\tilde{w}_{\varepsilon s} \in C^0([\varepsilon, R^n] \times [0,\infty))$, just as it has been illustrated in Lemma 3.7.
The lower and bound for $\tilde{w}_\varepsilon$ in (3.60) are direct consequences of the comparison principle Lemma 2.3, as was in Lemma 3.6, this time with $d = \mu\varepsilon$.
Lastly, (3.61) can be confirmed completely analogously to Lemma 3.10. □

Indeed, these solutions also converge as $\varepsilon \searrow 0$.

Lemma 3.15. *Let $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and suppose $w_0 \in C^{1+\theta}([0, R^n])$ is as in (3.22).*
Moreover, for $s_0 \in (0, R^n)$ and $\varepsilon \in (0, s_0)$ we define $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon, R^n])$ via (3.23).
Denoting $\tilde{w}_\varepsilon \in C^0([\varepsilon, R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0,\infty))$ as the corresponding solution to (3.59), for each $s_0 \in (0, R^n)$ there exists $\tilde{w}_{s_0} \in C^{1,\frac{1}{2}}([s_0, R^n] \times [0,\infty)) \cap C^{2,1}([s_0, R^n] \times (0,\infty))$ such that

$$\tilde{w}_\varepsilon \to \tilde{w}_{s_0} \quad \text{in } C^{1,\frac{1}{2}}_{loc}([s_0, R^n] \times [0,\infty)) \quad \text{for} \quad \varepsilon \searrow 0 \tag{3.62}$$

and

$$\tilde{w}_\varepsilon \to \tilde{w}_{s_0} \quad \text{in } C^{2,1}_{loc}([s_0, R^n] \times (0,\infty)) \quad \text{for} \quad \varepsilon \searrow 0. \tag{3.63}$$

By taking $s_0 \searrow 0$, we thus obtain a function $\tilde{w} \in C^{1,\frac{1}{2}}((0, R^n] \times [0,\infty)) \cap C^{2,1}((0, R^n] \times (0,\infty))$ with the properties (3.62) and (3.63).

Proof. Just as in Lemma 3.8, we can easily establish the ordering property

$$\tilde{w}_{\varepsilon'}(s,t) \leq \tilde{w}_\varepsilon(s,t) \qquad \text{for all} \quad (s,t) \in [s_0, R^n] \times [0,\infty)$$

for $0 < \varepsilon \leq \varepsilon' < s_0$, whereby combined with the boundedness (3.60), monotone convergence implies the existence of $\tilde{w}_{s_0} : [s_0, R^n] \times [0,\infty) \to \mathbb{R}$ such that

$$\tilde{w}_\varepsilon \nearrow \tilde{w}_{s_0}$$

pointwise.
Now let $\varepsilon \in (0, \frac{s_0}{2})$ and $T \in (0,\infty)$. Just as in Lemma 3.9, all prerequisites of Lemma 2.1 are satisfied. Therefore, we infer the existence of $\eta \in (0,1)$ and $C_1 > 0$, dependent on s_0 and T but independent of ε, such that for all $\varepsilon \in (0, \frac{s_0}{2})$ the classical solution $\tilde{w}_\varepsilon$ of (3.59) is in

$C^{1+\eta,\frac{1+\eta}{2}}([s_0,R^n]\times[0,T])$ with

$$\|\tilde{w}_\varepsilon\|_{C^{1+\eta,\frac{1+\eta}{2}}([s_0,R^n]\times[0,T])} \le C_1. \tag{3.64}$$

Consequently, the Arzelà-Ascoli theorem yields

$$\tilde{w}_\varepsilon \to \tilde{w}_{s_0} \quad \text{in} \quad C^{1,\frac{1}{2}}([s_0,R^n]\times[0,T]). \tag{3.65}$$

This confirms (3.62).
For $\tau > 0$, we similarly see that by Lemma 2.2 (ii)

$$\|\tilde{w}_\varepsilon\|_{C^{2+\eta,1+\frac{\eta}{2}}([s_0,R^n]\times[\tau,T])} \le C_2 \tag{3.66}$$

with $\eta \in (0,1)$ and $C_2 > 0$ depending on s_0, τ and T but not on ε and hence, again applying Arzelà-Ascoli's theorem results in

$$\tilde{w}_\varepsilon \to \tilde{w}_{s_0} \quad \text{in} \quad C^{2,1}([s_0,R^n]\times[\tau,T]), \tag{3.67}$$

warranting (3.63). □

We could furthermore establish that $\tilde{w}$ solves (3.36), but this is irrelevant for our means. At this point, we cannot conclude $\tilde{w} = w$ either way, since in order to employ our comparison theorem, we would need to establish boundedness of $\tilde{w}_s$ and w_s.

We can however at least establish an inequality between solution w_ε and $\tilde{w}_\varepsilon$ for (3.24) and (3.59), respectively, for fixed $\varepsilon \in (0, R^n)$, which will be vital in the Lemma afterwards.

Lemma 3.16. *Let $n \ge 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$, $\theta \in (0,1)$ and $w_0 \in C^{1+\theta}([0,R^n])$ be as in (3.22). For $\varepsilon \in (0,R^n)$, define $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon,R^n])$ via (3.23).*
Then the respective solutions w_ε and $\tilde{w}_\varepsilon$ of (3.24) and (3.59) satisfy the inequality

$$w_\varepsilon(s,t) \le \tilde{w}_\varepsilon(s,t) \qquad \textit{for all} \quad (s,t) \in [\varepsilon,R^n]\times[0,\infty). \tag{3.68}$$

Proof. For each $T > 0$, we have

$$w_{\varepsilon t} = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{\varepsilon ss} + n w_\varepsilon w_{\varepsilon s} - \mu s w_{\varepsilon s}$$

and, due to (3.61),

$$\begin{aligned}\tilde{w}_{\varepsilon t} &= n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} \tilde{w}_{\varepsilon ss} + n\tilde{w}_\varepsilon \tilde{w}_{\varepsilon s} - \mu s \tilde{w}_{\varepsilon s} + \mu\varepsilon \tilde{w}_{\varepsilon s} \\ &\geq n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} \tilde{w}_{\varepsilon ss} + n\tilde{w}_\varepsilon \tilde{w}_{\varepsilon s} - \mu s \tilde{w}_{\varepsilon s}\end{aligned}$$

in $(\varepsilon, R^n) \times (0,T)$. By definition, $w_\varepsilon = \tilde{w}_\varepsilon$ at the spatial boundary and at $t = 0$. Since moreover $w_{\varepsilon s}$ is bounded in $(\varepsilon, R^n) \times (0,T)$, our comparison principle Lemma 2.3 warrants that

$$w_\varepsilon(s,t) \leq \tilde{w}_\varepsilon(s,t) \qquad \text{for all} \quad (s,t) \in [\varepsilon, R^n] \times [0,T),$$

which letting $T \nearrow \infty$ verifies (3.68). □

We may now show that indeed $\tilde{w} = w$.

Lemma 3.17. *Let $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$, $\theta \in (0,1)$ and $w_0 \in C^{1+\theta}([0,R^n])$ be as in (3.22). For $\varepsilon \in (0,R^n)$, define $w_{0\varepsilon} \in C^{1+\theta}([\varepsilon,R^n])$ via (3.23).*
Then if either $\beta \leq 2-n$ or $\beta \geq 2$, for each $s_0 \in (0,R^n)$ and $\varepsilon \in (0,s_0)$ we have that for the respective solutions w_ε and $\tilde{w}_\varepsilon$ of (3.24) and (3.59)

$$\tilde{w}_\varepsilon - w_\varepsilon \to 0 \qquad \text{in} \quad C^0_{loc}([0,\infty), L^1((s_0,R^n))) \qquad \text{for} \quad \varepsilon \searrow 0. \tag{3.69}$$

Thereby, for $\tilde{w}$ and w as in Lemma 3.15 and Lemma 3.9, respectively,

$$\tilde{w}(s,t) = w(s,t) \qquad \text{for all} \quad (s,t) \in (0,R^n] \times [0,\infty). \tag{3.70}$$

PROOF. Note that by means of Lemma 3.16,

$$|\tilde{w}_\varepsilon(s,t) - w_\varepsilon(s,t)| = \tilde{w}_\varepsilon(s,t) - w_\varepsilon(s,t) \qquad \text{for all} \quad (s,t) \in [\varepsilon,R^n] \times [0,\infty).$$

Using (3.24) and (3.59), abbreviating $\gamma := 2 - \frac{2}{n} + \frac{\beta}{n}$ we calculate that for $t > 0$

$$\begin{aligned}\int_\varepsilon^{R^n} (\tilde{w}_\varepsilon(s,t) - w_\varepsilon(s,t))ds &= \int_0^t \int_\varepsilon^{R^n} (\tilde{w}_{\varepsilon t}(s,\tau) - w_{\varepsilon t}(s,\tau))dsd\tau \\ &= n^2 \int_0^t \int_\varepsilon^{R^n} s^\gamma (\tilde{w}_{\varepsilon ss} - w_{\varepsilon ss})dsd\tau \\ &+ n \int_0^t \int_\varepsilon^{R^n} (\tilde{w}_\varepsilon \tilde{w}_{\varepsilon s} - w_\varepsilon w_{\varepsilon s})dsd\tau \\ &- \mu \int_0^t \int_\varepsilon^{R^n} s(\tilde{w}_{\varepsilon s} - w_{\varepsilon s})dsd\tau + \mu\varepsilon \int_0^t \int_\varepsilon^{R^n} \tilde{w}_{\varepsilon s} dsd\tau.\end{aligned} \tag{3.71}$$

Herein for $\tau \in (0,t)$, due to the boundary conditions,

$$\begin{aligned} n\int_\varepsilon^{R^n} (\tilde{w}_\varepsilon \tilde{w}_{\varepsilon s} - w_\varepsilon w_{\varepsilon s})ds &= \frac{n}{2}\int_\varepsilon^{R^n} (\tilde{w}_\varepsilon^2 - w_\varepsilon^2)_s ds \\ &= \frac{n}{2}(\tilde{w}_\varepsilon^2(R^n,\tau) - w_\varepsilon^2(R^n,\tau) - \tilde{w}_\varepsilon^2(\varepsilon,\tau) + w_\varepsilon^2(\varepsilon,\tau)) \\ &= 0 \end{aligned} \tag{3.72}$$

as well as

$$-\mu\int_\varepsilon^{R^n} s(\tilde{w}_{\varepsilon s} - w_{\varepsilon s})ds = \mu\int_\varepsilon^{R^n} (\tilde{w}_\varepsilon - w_\varepsilon)ds \tag{3.73}$$

via partial integration, and

$$\mu\varepsilon\int_\varepsilon^{R^n} \tilde{w}_{\varepsilon s}ds = \mu\varepsilon\tilde{w}_\varepsilon(R^n,\tau) = \frac{\mu m}{\omega_n}\varepsilon. \tag{3.74}$$

In order to deal with the first summand, observe that due to (3.68) and $\tilde{w}_\varepsilon(R^n,\tau) = w_\varepsilon(R^n,\tau)$ necessarily

$$\tilde{w}_{\varepsilon s}(R^n,\tau) \le w_{\varepsilon s}(R^n,\tau) \qquad \text{for all} \quad \tau > 0,$$

and similarly

$$\tilde{w}_{\varepsilon s}(\varepsilon,\tau) \ge w_{\varepsilon s}(\varepsilon,\tau) \qquad \text{for all} \quad \tau > 0.$$

Therefore, integrating by parts yields

$$\begin{aligned} n^2\int_\varepsilon^{R^n} s^\gamma(\tilde{w}_{\varepsilon ss} - w_{\varepsilon ss})ds &= [n^2 s^\gamma(\tilde{w}_{\varepsilon s}(s,\tau) - w_{\varepsilon s}(s,\tau)]_{s=\varepsilon}^{s=R^n} - n^2\gamma\int_\varepsilon^{R^n} s^{\gamma-1}(\tilde{w}_{\varepsilon s} - w_{\varepsilon s})ds \\ &\le -n^2\gamma\int_\varepsilon^{R^n} s^{\gamma-1}(\tilde{w}_{\varepsilon s} - w_{\varepsilon s})ds \\ &= [-n^2\gamma s^{\gamma-1}(\tilde{w}_\varepsilon(s,\tau) - w_\varepsilon(s,\tau))]_{s=\varepsilon}^{s=R^n} \\ &\quad + n^2\gamma(\gamma-1)\int_\varepsilon^{R^n} s^{\gamma-2}(\tilde{w}_\varepsilon - w_\varepsilon)ds \\ &= n^2\gamma(\gamma-1)\int_\varepsilon^{R^n} s^{\gamma-2}(\tilde{w}_\varepsilon - w_\varepsilon)ds. \end{aligned} \tag{3.75}$$

If now $\beta \le 2-n$, then $\gamma - 1 = 2 - \frac{2}{n} + \frac{\beta}{n} - 1 \le 0$ and thus we could estimate this term against 0. This case is analogous to the other one which we are gonna follow.

If on the other hand $\beta \geq 2$, we have $\gamma - 2 = \frac{\beta}{n} - \frac{2}{n} \geq 0$ and thus

$$n^2\gamma(\gamma-1)\int_\varepsilon^{R^n} s^{\gamma-2}(\tilde{w}_\varepsilon - w_\varepsilon)ds \leq n^2\gamma(\gamma-1)R^{n(\gamma-2)}\int_\varepsilon^{R^n}(\tilde{w}_\varepsilon - w_\varepsilon)ds.$$

So defining $y \in C^1([0,\infty))$ via $y(t) := \int_\varepsilon^{R^n}(\tilde{w}_\varepsilon(s,t) - w_\varepsilon(s,t))ds$, by (3.71) – (3.75) we obtain that

$$y(t) \leq \frac{\mu m}{\omega_n}\varepsilon \cdot t + \int_0^t (\mu + n^2\gamma(\gamma-1)R^{n(\gamma-2)})y(\tau)d\tau$$

Hence by means of Grönwall's inequality, we may deduce that with $b := \mu + n^2\gamma(\gamma-1)R^{n(\gamma-2)}$

$$y(t) \leq \frac{\mu m}{\omega_n}\varepsilon t \cdot e^{bt}$$

and therefore, since $y(t) \geq 0$ for all $t \geq 0$, indeed

$$y \to 0 \quad \text{in} \quad C^0_{loc}([0,\infty)) \quad \text{for} \quad \varepsilon \searrow 0,$$

confirming (3.69).
Since however

$$\tilde{w}_\varepsilon \to \tilde{w}_{s_0} \quad \text{in } C^0_{loc}([s_0,R^n]\times[0,\infty)) \quad \text{for} \quad \varepsilon \searrow 0$$

and

$$w_\varepsilon \to w_{s_0} \quad \text{in } C^0_{loc}([s_0,R^n]\times[0,\infty)) \quad \text{for} \quad \varepsilon \searrow 0,$$

these limits also hold in $C^0_{loc}([0,\infty),L^1((s_0,R^n)))$. Thus however, due to (3.69), $\tilde{w}_{s_0} = w_{s_0}$ in $C^0([0,\infty),L^1((s_0,R^n)))$. In view of the fact that both $\tilde{w}_{s_0}$ and w_{s_0} lie in $C^0([s_0,R^n]\times[0,\infty))$, this equality is pointwise in $[s_0,R^n]\times[0,\infty)$.
Letting $s_0 \searrow 0$, this confirms (3.70). □

We impose stricter requirements on the initial data to ensure $w(\cdot,t)$ is concave.

Lemma 3.18. *Let $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\theta \in (0,1)$, $\beta > 0$ and $w_0 \in C^{2+\theta}([0,R^n])$ be as in (3.22) with*

$$w_{0ss}(s) \leq 0, \qquad s \in (0,R^n), \tag{3.76}$$

as well as

$$w_{0ss}(0) = 0, \quad w_{0s}(R^n) = 0 \quad \text{and} \quad w_{0ss}(R^n) = 0. \tag{3.77}$$

Let $\tilde{w}$ denote the function defined in Lemma 3.15. Then $\tilde{w} \in C^{2,1}((0,R^n]\times[0,\infty))$, and

$$\tilde{w}_{ss}(s,t) \leq 0 \qquad \text{for all} \quad (s,t) \in (0,R^n)\times[0,\infty). \tag{3.78}$$

PROOF. We shall proceed similarly as in Lemma 3.7 and Lemma 3.10.
For $\varepsilon \in (0, R^n)$, let $w_{0\varepsilon} \in C^{2+\theta}([\varepsilon, R^n])$ be defined as in (3.23), and let $\tilde{w}_\varepsilon \in C^{1,\frac{1}{2}}([\varepsilon, R^n] \times [0,\infty)) \cap C^{2,1}([\varepsilon, R^n] \times (0,\infty))$ be the solution of (3.59).
By standard theory, moreover

$$\tilde{w}_{\varepsilon ss} \in C^{2,1}((\varepsilon, R^n) \times (0,\infty)). \tag{3.79}$$

With designations as in the proof of Lemma 3.7 with slightly modified b (add $+\mu\varepsilon$ there), we may easily confirm that for any $T > 0$ we have $a \in C^{1+\frac{\theta}{2},\frac{1+\theta}{2}}((\varepsilon, R^n) \times [0,T])$ and, since $\tilde{w}_\varepsilon \in C^{1,\frac{1}{2}}([\varepsilon, R^n] \times [0,T])$, also $b \in C^{\theta,\frac{\theta}{2}}((\varepsilon, R^n) \times [0,T])$.
With that on hand, we may easily confirm (2.10) and the existence of $M = M(\varepsilon) > 0$ such that (2.11) and (2.13) hold for $\Omega := (\varepsilon, R^n)$. Furthermore, the regularity imposed on the initial data enables us to conclude (2.12).
Additionally, by (3.77) we may infer that

$$w_{0\varepsilon ss}(\varepsilon) = 0, \quad w_{0\varepsilon s}(R^n) = 0 \quad \text{and} \quad w_{0\varepsilon ss}(R^n) = 0$$

and thus

$$n^2 \varepsilon^{2-\frac{2}{n}+\frac{\beta}{n}} \tilde{w}_{0\varepsilon ss}(\varepsilon) + n\tilde{w}_{0\varepsilon}(\varepsilon)\tilde{w}_{0\varepsilon s}(\varepsilon) - \mu(\varepsilon - \varepsilon)\tilde{w}_{0\varepsilon s}(\varepsilon) = 0$$

as well as

$$n^2 R^{n(2-\frac{2}{n}+\frac{\beta}{n})} \tilde{w}_{0\varepsilon ss}(R^n) + n\tilde{w}_{0\varepsilon}(R^n)\tilde{w}_{0\varepsilon s}(R^n) - \mu(R^n - \varepsilon)\tilde{w}_{0\varepsilon s}(R^n) = 0,$$

asserting (2.14).
Therefore, by means of Lemma 2.2 (i) this entails that we can find $\eta = \eta(M,T,\theta) \in (0,1)$ and $C(M,T,\theta) > 0$ such that $\tilde{w}_\varepsilon \in C^{2+\eta,1+\frac{\eta}{2}}([\varepsilon, R^n] \times [0,T])$ with

$$\|\tilde{w}_\varepsilon\|_{C^{2+\eta,1+\frac{\eta}{2}}([\varepsilon,R^n]\times[0,T])} \leq C.$$

As in Lemma 3.9, for fixed $s_0 \in (0, R^n)$ we may deduce that $\tilde{w} \in C^{2,1}([s_0, R^n] \times [0,T])$ by Arzelá-Ascoli's theorem, since M can be be selected dependent on s_0 but independent of ε for an estimate of $\|\tilde{w}_\varepsilon\|_{C^{2+\eta,1+\frac{\eta}{2}}([s_0,R^n]\times[0,T])}$, by taking $s_0 \searrow 0$ and $T \nearrow \infty$ resulting in

$$\tilde{w} \in C^{2,1}((0, R^n] \times [0,\infty)).$$

Going back to $\tilde{w}_\varepsilon$, we get that in particular

$$\tilde{w}_\varepsilon \in C^{2,1}([\varepsilon, R^n] \times [0,\infty)), \tag{3.80}$$

since $T > 0$ was chosen arbitrarily. Now, on account of (3.79) and the first equation in (3.59), $z := \tilde{w}_{\varepsilon ss}$ fulfills

$$\begin{aligned} z_t = \; & n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + 2n^2 \left(2 - \tfrac{2}{n} + \tfrac{\beta}{n}\right) s^{1-\frac{2}{n}+\frac{\beta}{n}} z_s + 3n\tilde{w}_{\varepsilon s} z + n\tilde{w}_\varepsilon z_s \\ & -\mu s z_s + \mu\varepsilon z_s - 2\mu z + n^2 \left(2 - \tfrac{2}{n} + \tfrac{\beta}{n}\right)\left(1 - \tfrac{2}{n} + \tfrac{\beta}{n}\right) s^{-\frac{2}{n}+\frac{\beta}{n}} z \end{aligned} \tag{3.81}$$

in $(\varepsilon, R^n) \times (0,\infty)$. For fixed $\delta > 0$ and

$$\kappa := 3n\|\tilde{w}_{\varepsilon s}\|_{L^\infty((\varepsilon,R^n)\times(0,T))} + n^2(2+\beta)(1+\beta)\max\{\varepsilon^{-\frac{2}{n}+\frac{\beta}{n}}, R^{n(-\frac{2}{n}+\frac{\beta}{n})}\} + 1,$$

define $\varphi \in C^{2,1}((\varepsilon, R^n) \times (0,\infty))$ via

$$\varphi(s,t) := z(s,t) - \delta e^{\kappa t}.$$

For $T > 0$ we set

$$S := \{t \in [0,T) \mid \varphi(s,\tilde{t}) < 0 \; \forall (s,\tilde{t}) \in [\varepsilon, R^n] \times [0,t]\}.$$

Then S is not empty due to

$$\varphi(s,0) = w_{0\varepsilon ss}(s) - \delta \leq -\delta < 0. \tag{3.82}$$

Due to (3.80), we have $\varphi \in C^0([\varepsilon, R^n] \times [0,T])$, so that (3.82) moreover guarantees $t_0 := \sup S > 0$.
If now $t_0 < T$, then there exists $s_0 \in [\varepsilon, R^n]$ such that $\varphi(s_0,t_0) = 0$.
Since for all $t > 0$ however $\tilde{w}_\varepsilon(\varepsilon,t) = 0$ and thus also $\tilde{w}_{\varepsilon t}(\varepsilon,t) = 0$,

$$\tilde{w}_{\varepsilon ss}(\varepsilon,t) = \frac{1}{n^2\varepsilon^{2-\frac{2}{n}+\frac{\beta}{n}}}\left(\tilde{w}_{\varepsilon t}(\varepsilon,t) - n\tilde{w}_\varepsilon(\varepsilon,t)\tilde{w}_{\varepsilon s}(\varepsilon,t) + \mu(\varepsilon-\varepsilon)\tilde{w}_{\varepsilon s}(\varepsilon,t)\right) = 0$$

and due to $\tilde{w}_\varepsilon(R^n,t) = \frac{m}{\omega_n}$, $\tilde{w}_{\varepsilon t}(R^n,t) = 0$ and $\tilde{w}_{\varepsilon s}(R^n,t) \geq 0$

$$\tilde{w}_{\varepsilon ss}(R^n,t) = \frac{1}{n^2 R^{n(2-\frac{2}{n}+\frac{\beta}{n})}}\left(\tilde{w}_{\varepsilon t}(R^n,t) - n\tilde{w}_\varepsilon(R^n,t)\tilde{w}_{\varepsilon s}(R^n,t) + \mu(R^n-\varepsilon)\tilde{w}_{\varepsilon s}(R^n,t)\right)$$

$$
\begin{aligned}
&= \frac{1}{n^2 R^{n(2-\frac{2}{n}+\frac{\beta}{n})}}\bigg(\Big(\mu R^n - \frac{nm}{\omega_n}\Big)\tilde{w}_{\varepsilon s}(R^n,t) - \mu\varepsilon\tilde{w}_{\varepsilon s}(R^n,t)\bigg) \\
&= -\frac{1}{n^2 R^{n(2-\frac{2}{n}+\frac{\beta}{n})}}\mu\varepsilon\tilde{w}_{\varepsilon s}(R^n,t) \\
&\le 0
\end{aligned}
$$

for all $t > 0$, we can conclude that $s_0 \in (\varepsilon, R^n)$.
Therefore, at (s_0, t_0) not only

$$
\varphi = z - \delta e^{\kappa t_0} = 0 \quad \text{and} \quad \varphi_t \ge 0
$$

but also

$$
\varphi_s = z_s = 0 \quad \text{and} \quad \varphi_{ss} = z_{ss} \le 0
$$

hold. Combined with (3.81), this yields that at (s_0, t_0)

$$
\begin{aligned}
0 \;\le\;& \varphi_t \\
=\;& z_t - \delta\kappa e^{\kappa t_0} \\
=\;& n^2 s_0^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + 2n^2\Big(2-\frac{2}{n}+\frac{\beta}{n}\Big)s_0^{1-\frac{2}{n}+\frac{\beta}{n}} z_s + 3n\tilde{w}_{\varepsilon s} z + n\tilde{w}_{\varepsilon} z_s \\
& -\mu s_0 z_s + \mu\varepsilon z_s - 2\mu z + n^2\Big(2-\frac{2}{n}+\frac{\beta}{n}\Big)\Big(1-\frac{2}{n}+\frac{\beta}{n}\Big)s_0^{-\frac{2}{n}+\frac{\beta}{n}} z - \delta\kappa e^{\kappa t_0} \\
=\;& n^2 s_0^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + 3n\tilde{w}_{\varepsilon s} z - 2\mu z + n^2\Big(2-\frac{2}{n}+\frac{\beta}{n}\Big)\Big(1-\frac{2}{n}+\frac{\beta}{n}\Big)s_0^{-\frac{2}{n}+\frac{\beta}{n}} z - \delta\kappa e^{\kappa t_0} \\
=\;& n^2 s_0^{2-\frac{2}{n}+\frac{\beta}{n}} z_{ss} + \bigg(3n\tilde{w}_{\varepsilon s} - 2\mu + n^2\Big(2-\frac{2}{n}+\frac{\beta}{n}\Big)\Big(1-\frac{2}{n}+\frac{\beta}{n}\Big)s_0^{-\frac{2}{n}+\frac{\beta}{n}} - \kappa\bigg)\delta e^{\kappa t_0} \\
\le\;& \delta e^{\kappa t_0}\bigg(3n\tilde{w}_{\varepsilon s} - 2\mu + n^2\Big(2-\frac{2}{n}+\frac{\beta}{n}\Big)\Big(1-\frac{2}{n}+\frac{\beta}{n}\Big)s_0^{-\frac{2}{n}+\frac{\beta}{n}} - \kappa\bigg) \\
\le\;& \delta e^{\kappa t_0}\Big(3n\|\tilde{w}_{\varepsilon s}\|_{L^\infty((\varepsilon,R^n)\times(0,T))} + n^2(2+\beta)(1+\beta)s_0^{-\frac{2}{n}+\frac{\beta}{n}} - \kappa\Big) \\
\le\;& -\delta e^{\kappa t_0} \\
<\;& 0,
\end{aligned}
$$

a contradiction.
Thus necessarily $t_0 = T$, which by taking $\delta \searrow 0$ implies that

$$
\tilde{w}_{\varepsilon ss}(s,t) \le 0 \qquad \text{for all} \quad (s,t) \in (\varepsilon, R^n) \times [0,\infty), \tag{3.83}
$$

since $T > 0$ has been chosen arbitrarily.

By (3.83), (3.63) and the definition of $\tilde{w}$, we may now readily infer (3.78). □

Lemma 3.19. *Let* $n \geq 1$, $R > 0$, $\mu = \frac{nm}{\omega_n R^n}$, $\theta \in (0,1)$, $\beta > 0$ *and* $w_0 \in C^{2+\theta}([0, R^n])$ *be as in (3.22) with (3.76) as well as (3.77).*
Let w *denote the global solution to (3.36) from Lemma 3.9. If either*

$$\beta \leq 2 - n \qquad or \qquad \beta \geq 2, \tag{3.84}$$

then $w \in C^{2,1}((0, R^n] \times [0, \infty))$ *and*

$$w_{ss}(s,t) \leq 0 \qquad for\ all \quad (s,t) \in (0, R^n) \times [0, \infty). \tag{3.85}$$

Moreover, with $T^* > 0$ *as in Lemma 3.12 we have that for each* $T \in (0, T^*)$ *there exists* $C = C(T)$ *such that*

$$w_s(s,t) \leq C \qquad for\ all \quad (s,t) \in (0, R^n] \times [0, T]. \tag{3.86}$$

Proof. Since the conditions of Lemma 3.19 are met, the function $\tilde{w}$ defined in Lemma 3.15 lies in $C^{2,1}((0, R^n] \times [0, \infty))$ and fulfills

$$\tilde{w}_{ss}(s,t) \leq 0 \qquad \text{for all} \quad (s,t) \in (0, R^n) \times [0, \infty).$$

Due to the restriction (3.84) on β though, by Lemma 3.17 we have that $w = \tilde{w}$ and thus (3.85) holds.
Now let $0 < T < T^*$. Lemma 3.12 ensures that

$$w(s,t) \leq C \cdot s \qquad \text{for all} \quad (s,t) \in (0, R^n] \times [0, T] \tag{3.87}$$

for some $C = C(T) > 0$. Assume there exists $(s_0, t_0) \in (0, R^n] \times [0, T]$ with the property that $w_s(s_0, t_0) > C$. Then via (3.49) the fundamental theorem of calculus asserts

$$\begin{aligned} w(s_0, t_0) &= w(0, t_0) + \int_0^{s_0} w_s(s, t_0) ds \\ &= \int_0^{s_0} w_s(s, t_0) ds \\ &\geq \int_0^{s_0} w_s(s_0, t_0) ds \\ &> C s_0, \end{aligned}$$

as (3.85) implies that $w_s(\cdot, t_0)$ is monotonically decreasing. This however contradicts (3.87), whereby we infer that (3.86) must hold true. □

3.1.5 Retransformation to Keller-Segel type system

With a local-in-time solution of (3.20) on hand, we may now obtain a solution to our original problem (1.3), or rather to its counterpart (3.3) in radial coordinates.
As a preparation, we first denote

Lemma 3.20. *Let* $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ *and suppose* $w_0 \in C^{1+\theta}([0,R^n])$ *is as in (3.22).*
Then w *as in Lemma 3.9 satisfies*

$$w_{ss}(R^n,t) = 0 \qquad \forall t > 0. \tag{3.88}$$

Proof. Since $w \in C^{2,1}((0,R^n] \times (0,\infty))$ and

$$w_t = n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} + nww_s - \mu s w_s$$

in $(0,R^n) \times (0,\infty)$, for any $t > 0$ we necessarily have

$$w_t(R^n,t) = n^2 R^{n(2-\frac{2}{n}+\frac{\beta}{n})} w_{ss}(R^n,t) + nw(R^n,t)w_s(R^n,t) - \mu R^n w_s(R^n,t).$$

Due to $w(R^n,t) = \frac{m}{\omega_n} = \frac{\mu R^n}{n}$ for all $t > 0$ and thus also $w_t(R^n,t) = 0$ in $(0,\infty)$, this yields

$$0 = n^2 R^{n(2-\frac{2}{n}+\frac{\beta}{n})} w_{ss}(R^n,t) + w_s(R^n,t) \cdot \underbrace{\left(n\frac{\mu R^n}{n} - \mu R^n \right)}_{=0},$$

and therefore (3.88). □

Now we can establish

Lemma 3.21. *Let* $n \geq 1$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ *and suppose* $w_0 \in C^{1+\theta}([0,R^n])$ *is as in (3.22).*
Furthermore, we choose $T_0 > 0$ *maximally such that the function*

$$w \in C^{1,\frac{1}{2}}((0,R^n] \times [0,\infty)) \cap C^{2,1}((0,R^n] \times (0,\infty))$$

constructed in Lemma 3.9 with the extension as in Lemma 3.11 is in $C^0([0,R^n] \times [0,T_0))$ *with*

$$w(0,t) = 0 \qquad \text{for all} \quad t \in (0,T_0). \tag{3.89}$$

Then for $u_0 \in C^\theta([0,R])$ *defined via*

$$u_0(r) = n \cdot w_{0s}(r^n), \qquad r \in [0,R],$$

the pair of functions $u \in C^0((0,R] \times [0,T_0)) \cap C^{2,1}((0,R] \times (0,T_0))$ *given by*

$$u(r,t) = n \cdot w_s(r^n,t), \tag{3.90}$$

and $v \in C^{2,0}((0,R] \times (0,T_0))$ *fulfilling*

$$v_r(r,t) = \frac{1}{r^{n-1}}\left(\frac{\mu r^n}{n} - \int_0^r \rho^{n-1} u(\rho,t)d\rho\right), \quad (r,t) \in (0,R] \times (0,T_0), \tag{3.91}$$

for all $t \in (0,T_0)$ *solves*

$$\begin{cases} u_t = \frac{1}{r^{n-1}}(r^{n-1+\beta}u_r)_r - \frac{1}{r^{n-1}}(r^{n-1}uv_r)_r, & r \in (0,R),\ t>0, \\ 0 = \frac{1}{r^{n-1}}(r^{n-1}v_r)_r - \mu + u, & r \in (0,R),\ t>0, \\ u_r = v_r = 0, & r = R,\ t>0, \\ u(r,0) = u_0(r), & r \in (0,R), \end{cases} \tag{3.92}$$

classically in $(0,R] \times [0,T_0)$.
Moreover, with T^* *as in Lemma 3.12,*

$$T_0 \geq T^* > 0. \tag{3.93}$$

Proof. First we observe that v_r determined as in (3.91) is well-defined since because of (3.89), for $(r,t) \in (0,R] \times (0,T_0)$ we can ensure that

$$\begin{aligned} \int_0^r \rho^{n-1} u(\rho,t)d\rho &= n \int_0^r \rho^{n-1} w_s(\rho^n,t)d\rho \\ &= n \int_0^{r^n} s^{\frac{n-1}{n}} w_s(s,t) \cdot \frac{1}{n} s^{\frac{1}{n}-1} ds \\ &= \int_0^{r^n} w_s(s,t)ds \\ &= w(r^n,t) - w(0,t) \\ &= w(r^n,t) \end{aligned} \tag{3.94}$$

by substituting $s=\rho^n$. Now v_r indeed complies with the second equation in (3.92) since

$$(r^{n-1}v_r)_r = \left(\frac{\mu r^n}{n} - \int_0^r \rho^{n-1}u(\rho,t)d\rho\right)_r = \mu r^{n-1} - r^{n-1}u$$

and thus

$$0 = \frac{1}{r^{n-1}}(r^{n-1}v_r)_r - \mu + u$$

in $(0,R)\times(0,T_0)$. Employing (3.94), $\mu = \frac{nm}{\omega_n R^n}$ and $w(R^n,t) = \frac{m}{\omega_n}$, we confirm the boundary condition

$$v_r(R,t) = \frac{1}{R^{n-1}}\left(\frac{\mu R^n}{n} - \int_0^R \rho^{n-1}u(\rho,t)d\rho\right) = \frac{1}{R^{n-1}}\left(\frac{\mu R^n}{n} - \frac{m}{\omega_n}\right) = 0$$

is satisfied in $(0,T_0)$. Calculating

$$u_r(r,t) = n^2 w_{ss}(r^n,t)\cdot r^{n-1} \qquad \text{for} \quad (r,t)\in(0,R]\times(0,T_0) \tag{3.95}$$

and employing (3.88) also immediately yields

$$u_r(R,t) = 0 \qquad \text{for all} \quad (r,t)\in(0,R]\times(0,T_0),$$

verifying the boundary condition for u_r as well.
Note that via interior Schauder estimates and (3.43), $w_s \in C^{2,1}((0,R^n)\times(0,T_0))$. With (3.95) on hand, writing $s=r^n$ we may now also infer

$$\begin{aligned}
u_t(r,t) &= n w_{st}(s,t)\\
&= n\left(n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} + n w(s,t) w_s(s,t) - \mu s w_s(s,t)\right)_s\\
&= n\left(n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} - n w_s(s,t)\left(\frac{\mu s}{n} - w(s,t)\right)\right)_s\\
&= n\left(n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} - u(s^{\frac{1}{n}},t)\left(\frac{\mu s}{n} - \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)d\rho\right)\right)_s\\
&= n\left(n^2 s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss} - u(s^{\frac{1}{n}},t) s^{\frac{n-1}{n}} v_r(s^{\frac{1}{n}},t)\right)_s\\
&= n\left(s^{1-\frac{1}{n}+\frac{\beta}{n}} u_r(s^{\frac{1}{n}},t) - s^{\frac{n-1}{n}} u(s^{\frac{1}{n}},t) v_r(s^{\frac{1}{n}},t)\right)_s
\end{aligned}$$

$$= n\Bigg(\int\limits_0^{s^{\frac{1}{n}}}(\rho^{n-1+\beta}u_r(\rho,t))_r d\rho - \int\limits_0^{s^{\frac{1}{n}}}(\rho^{n-1}u(\rho,t)v_r(\rho,t))_r d\rho\Bigg)_s$$

$$= n\Bigg(\int\limits_0^{s^{\frac{1}{n}}}(\rho^{n-1+\beta}u_r(\rho,t))_r - (\rho^{n-1}u(\rho,t)v_r(\rho,t))_r d\rho\Bigg)_s$$

$$= r^{n-1}\Big((r^{n-1+\beta}u_r(r,t))_r - (r^{n-1}u(r,t)v_r(r,t))_r\Big)$$

in $(0,R)\times(0,T_0)$, thus verifying the first equation in (3.92). The initial condition transfers from (3.36) because $w \in C^{1,\frac{1}{2}}((0,R]\times[0,T_0))$.
Lastly, (3.49) entails (3.93). □

Let us now gather some additional properties of this solution (u,v).

Lemma 3.22. *Assume the conditions of Lemma 3.21 and let (u,v) denote the solution to (3.92) in $(0,R]\times[0,T_0)$ defined therein.*
Then u is nonnegative, and furthermore the total mass is conserved, that is

$$\int_0^R \rho^{n-1}u(\rho,t)d\rho = \int_0^R \rho^{n-1}u_0(\rho)d\rho \tag{3.96}$$

for all $t \in (0,T_0)$.

Proof.
The nonnegativity of u directy follows via its definition (3.90) and the nonnegativity of w_s ensured in Lemma 3.11.
On the other hand, (3.96) emerges from

$$\int_0^R \rho^{n-1}u(\rho,t)d\rho = w(R^n,t) = \frac{m}{\omega_n} = w_0(R^n) = \int_0^R \rho^{n-1}u_0(\rho)d\rho,$$

for $t \in (0,T_0)$, wherein w is as in Lemma 3.9. □

Combining multiple results of this section, we are moreover able to formulate a proposition regarding uniqueness of solutions to (3.92).

Lemma 3.23. *Let $n \geq 2$, $R > 0$, $\theta \in (0,1)$, $\mu = \frac{nm}{\omega_n R^n}$, $\beta > 0$ and $u_0 \in C^\theta([0,R])$.*

Then for $T > 0$ there is at most one solution (u, v) of (3.92) in $(0, R] \times [0, T)$ with

$$\begin{cases} u \in C^0((0, R] \times [0, T)) \cap C^{2,1}((0, R] \times (0, T)), \\ v \in C^{2,0}((0, R] \times (0, T)), \end{cases}$$

which has the properties that $\int_0^R v(r, t)dr = 0$ for all $t \in (0, T)$ and

$$0 \leq u \in L^\infty((0, R) \times (0, T)) \quad \text{and} \quad v_r \in L^\infty((0, R) \times (0, T)). \tag{3.97}$$

PROOF. Note that $n \geq 2$ implies that $\beta > 0 \geq 2 - n$. Therefore, if also (3.97) holds, the requirements of Lemma 3.2 and Lemma 3.4 are met and thus the existence of a solution of (3.92) implies the existence of a solution $w \in C^0([0, R^n] \times [0, T)) \cap C^{2,1}((0, R^n] \times (0, T))$ of (3.20) with $w_0 \in C^{1+\theta}([0, R^n])$, in which $w_s \in C^0([0, R^n] \times [0, T))$ is nonnegative and bounded.
Then Lemma 3.13 however warrants that w is the unique classical solution of (3.20). Via the fundamental theorem of calculus, we can also easily infer that then

$$w(s, t) = \underbrace{w(0, t)}_{=0} + \int_0^s w_s(\rho, t)d\rho \leq \|w_s\|_{L^\infty((0,R^n)\times(0,T))} \cdot s,$$

providing a pendant to (3.48), although we technically do not need it since we do not demand v_r to be extendable to $r = 0$.
Lemma 3.21 then guarantees the existence of a solution (u, v) of (3.92) in the sense specified above.
In conclusion, solutions of (3.92) satisfying the conditions of this lemma correspond with solutions of (3.20), thus transfering their uniqueness. □

We remark that actually $u_0 \in C^0([0, R])$ is sufficient for Lemma 3.23 to hold, since bounded $w_s \in C^0((0, R^n] \times [0, T))$ is sufficient for all relevant arguments in the subsections 3.1.3 and 3.1.5.

Sharpening the conditions in accordance with subsection 3.1.4 enables us to acquire additional properties, most prominently local-in-time boundedness of u in $(0, R] \times [0, T^*)$.

Lemma 3.24. *Suppose the conditions of Lemma 3.19 hold, and let (u, v) denote the solution of (3.92) constructed in Lemma 3.21.*

Then this solution has the additional properties that $u \in C^{1,0}((0,R] \times [0,T_0))$ *and*

$$u_r(r,t) \le 0 \qquad \textit{for all} \quad (r,t) \in (0,R] \times [0,T_0). \tag{3.98}$$

Moreover, for $T^* > 0$ *as in Lemma 3.12 and each* $T \in (0,T^*)$, *there exists* $C = C(T) > 0$ *such that*

$$u(r,t) \le C \qquad \textit{for all} \quad (r,t) \in (0,R] \times [0,T]. \tag{3.99}$$

PROOF. One only needs to adapt the results of Lemma 3.19.
The first spatial derivative of u is given by

$$u_r(r,t) = n^2 w_{ss}(r^n,t) \cdot r^{n-1} \qquad \text{for} \quad (r,t) \in (0,R] \times (0,T_0).$$

Therefore the claimed regularity follows from $w \in C^{2,1}((0,R^n] \times [0,T_0)$, whereas (3.98) is a consequence of (3.85).
Lastly, as $u(r,t) = n \cdot w_s(r^n,t)$ for $(r,t) \in (0,R] \times (0,T_0)$, (3.86) translates to (3.99). □

3.2 Ruling out global boundedness in (1.3) for sufficiently concentrated initial data

In usual settings of the Keller-Segel system and its variants, the occurence of blow-up at a finite time $T < \infty$ corresponds with the maximal time of existence T_{max} equaling T. Since our classical solution concept however does not require u to be defined continuously on a compact space, it is well possible that for a domain $\Omega \subset \mathbb{R}^n$

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$$

but $u \in C^0(\Omega \times [0,T_0)) \cap C^{2,1}(\Omega \times (0,T_0))$ for some $T_0 > T$. Moreover considering we have no extensibility criterion at hand, we restrict ourselves to ruling out the existence of global bounded solutions under certain circumstances.

As in [26, Lemma 3.3], we may establish that given a condition corresponding to the initial mass in the corresponding original system (1.3) being sufficiently concentrated, there is no global solution of (3.20) for which w_s is bounded locally in time in $(0,R^n) \times (0,\infty)$.

Lemma 3.25. *Let $n \geq 1$, $\beta > \max\{0, 2-n\}$, $R > 0$, $m_0 > 0$ and $m \geq m_0$. There exists $s_0 = s_0(m_0, m, R, \beta) \in (0, R^n)$ such that if $w_0 \in C^1([0,R^n])$, $\mu = \frac{nm}{\omega_n R^n}$, and furthermore*

$$w_0(s_0) \geq \frac{m_0}{\omega_n}, \tag{3.100}$$

then there is no global classical solution

$$w \in C^0([0,R^n] \times [0,\infty)) \cap C^{2,1}((0,R^n] \times (0,\infty))$$

of (3.20) with the property that for each $T > 0$

$$w_s \in L^\infty((0,R^n) \times (0,T)). \tag{3.101}$$

PROOF. Due to $\beta > 2 - n$, it is possible to fix $\gamma \in (0,1)$ with the property that

$$\gamma \leq 1 - \frac{2}{n} + \frac{\beta}{n}. \tag{3.102}$$

We abbreviate

$$c_1 := \frac{8\left(2 - \frac{2}{n} + \frac{\beta}{n} - \gamma\right)^2 n^3}{3 - \frac{4}{n} + \frac{2\beta}{n} - \gamma}, \qquad c_2 := \frac{2n}{(3-\gamma)\omega_n^2} \qquad \text{and}$$

$$c_3 := \frac{3}{4\omega_n} \cdot \Big(\frac{1}{1-\gamma} - \frac{1}{2-\gamma} \Big). \tag{3.103}$$

Observe that $\beta > 2-n$ implies that $2 - \frac{4}{n} + \frac{2\beta}{n} > 0$.
Therefore, given $R > 0$, $m_0 > 0$ and $m > 0$ we can fix $s_0 = s_0(m_0, m, R) \in (0, \frac{R^n}{2})$ such that $s_1 := 2s_0$ satisfies

$$s_1^{2-\frac{4}{n}+\frac{2\beta}{n}} \le \frac{(1-\gamma)c_3^2}{nc_1} m_0^2 \tag{3.104}$$

and

$$s_1^2 \le \frac{(1-\gamma)c_3^2}{nc_2} \cdot \frac{m_0^2 R^{2n}}{m^2}, \tag{3.105}$$

and henceforth we assume that $w \in C^0([0,R^n] \times [0,\infty)) \cap C^{2,1}((0,R^n] \times (0,\infty))$ is a global classical solution of (3.20) satisfying (3.101). For $\delta \in (0, \frac{s_1}{2})$, we then use (3.20) to compute

$$\begin{aligned}
&\frac{d}{dt} \int_\delta^{s_1} s^{-\gamma}(s_1 - s) w(s,t) ds \\
&= n^2 \int_\delta^{s_1} s^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-s) w_{ss}(s,t) ds + \frac{n}{2} \int_\delta^{s_1} s^{-\gamma}(s_1-s)(w^2)_s(s,t) ds \\
&\quad -\mu \int_\delta^{s_1} s^{1-\gamma}(s_1-s) w_s(s,t) ds \\
&= -n^2 \Big(2 - \frac{2}{n} + \frac{\beta}{n} - \gamma\Big) \int_\delta^{s_1} s^{1-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-s) w_s(s,t) ds + n^2 \int_\delta^{s_1} s^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma} w_s(s,t) ds \\
&\quad -n^2 \delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-\delta) w_s(\delta,t) + \frac{n}{2} \cdot \gamma \int_\delta^{s_1} s^{-\gamma-1}(s_1-s) w^2(s,t) ds \\
&\quad + \frac{n}{2} \int_\delta^{s_1} s^{-\gamma} w^2(s,t) ds - \frac{n}{2} \delta^{-\gamma}(s_1-\delta) w^2(\delta,t) \\
&\quad + \mu(1-\gamma) \int_\delta^{s_1} s^{-\gamma}(s_1-s) w(s,t) ds - \mu \int_\delta^{s_1} s^{1-\gamma} w(s,t) ds + \mu \delta^{1-\gamma}(s_1-\delta) w(\delta,t) \\
&= n^2 \Big(2 - \frac{2}{n} + \frac{\beta}{n} - \gamma\Big)\Big(1 - \frac{2}{n} + \frac{\beta}{n} - \gamma\Big) \int_\delta^{s_1} s^{-\frac{2}{n}+\frac{\beta}{n}-\gamma} w(s,t) ds \\
&\quad -2n^2 \Big(2 - \frac{2}{n} + \frac{\beta}{n} - \gamma\Big) \int_\delta^{s_1} s^{1-\frac{2}{n}+\frac{\beta}{n}-\gamma} w(s,t) ds + \frac{n}{2} \cdot \gamma \int_\delta^{s_1} s^{-\gamma-1}(s_1-s) w^2(s,t) ds
\end{aligned}$$

$$+\frac{n}{2}\int_\delta^{s_1} s^{-\gamma}w^2(s,t)ds + \mu(1-\gamma)\int_\delta^{s_1} s^{-\gamma}(s_1-s)w(s,t)ds - \mu\int_\delta^{s_1} s^{1-\gamma}w(s,t)ds$$
$$+n^2\left(2-\frac{2}{n}+\frac{\beta}{n}-\gamma\right)\delta^{1-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-\delta)w(\delta,t) - n^2\delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-\delta)w_s(\delta,t)$$
$$+n^2 s_1^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}w(s_1,t) - n^2\delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}w(\delta,t)$$
$$-\frac{n}{2}\delta^{-\gamma}(s_1-\delta)w^2(\delta,t) + \mu\delta^{1-\gamma}(s_1-\delta)w(\delta,t) \qquad \text{for all } t>0$$

with $\mu = \frac{nm}{\omega_n R^n}$. Note that due to (3.102), the first summand in the last equality is nonnegative. Neglecting some other nonnegative summands as well and integrating in time shows that

$$\begin{aligned}
\int_\delta^{s_1} s^{-\gamma}(s_1-s)w(s,t)ds \ \ \geq\ \ & \int_\delta^{s_1} s^{-\gamma}(s_1-s)w_0(s)ds \\
& -2n^2\left(2-\frac{2}{n}+\frac{\beta}{n}-\gamma\right)\int_0^t\int_\delta^{s_1} s^{1-\frac{2}{n}+\frac{\beta}{n}-\gamma}w(s,\tau)dsd\tau \\
& +\frac{n}{2}\int_0^t\int_\delta^{s_1} s^{-\gamma}w^2(s,\tau)dsd\tau - \mu\int_0^t\int_\delta^{s_1} s^{1-\gamma}w(s,\tau)dsd\tau \\
& -n^2\delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-\delta)\int_0^t w_s(\delta,\tau)d\tau \\
& -n^2\delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}\int_0^t w(\delta,\tau)d\tau \\
& -\frac{n}{2}\delta^{-\gamma}(s_1-\delta)\int_0^t w^2(\delta,\tau)d\tau \qquad \text{for all } t>0. \qquad (3.106)
\end{aligned}$$

Here since we assume $w(0,t)=0$ for all $t \geq 0$ and boundedness of w_s in $(0,R^n)\times(0,t)$, we may infer that $\sup_{(s,\tau)\in(0,R^n)\times(0,t)} \frac{w(s,\tau)}{s}$ is finite, and therefore

$$\begin{aligned}
& n^2\delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}(s_1-\delta)\int_0^t w_s(\delta,\tau)d\tau + n^2\delta^{2-\frac{2}{n}+\frac{\beta}{n}-\gamma}\int_0^t w(\delta,\tau)d\tau \\
& +\frac{n}{2}\delta^{-\gamma}(s_1-\delta)\int_0^t w^2(\delta,\tau)d\tau \to 0 \quad \text{as } \delta \searrow 0,
\end{aligned}$$

whence on several applications of the monotone convergence theorem we infer from (3.106) that $y(t) := \int_0^{s_1} s^{-\gamma}(s_1-s)w(s,t)ds$, $t \geq 0$, satisfies

$$\begin{aligned}
y(t) \ \ \geq\ \ & y(0) - 2n^2\left(2-\frac{2}{n}+\frac{\beta}{n}-\gamma\right)\int_0^t\int_0^{s_1} s^{1-\frac{2}{n}+\frac{\beta}{n}-\gamma}w(s,\tau)dsd\tau \\
& +\frac{n}{2}\int_0^t\int_0^{s_1} s^{-\gamma}w^2(s,\tau)dsd\tau - \mu\int_0^t\int_0^{s_1} s^{1-\gamma}w(s,\tau)dsd\tau \qquad \text{for all } t>0.
\end{aligned}$$

By Young's inequality,

$$\begin{aligned} & 2n^2\Big(2-\frac{2}{n}+\frac{\beta}{n}-\gamma\Big)\int_0^{s_1} s^{1-\frac{2}{n}+\frac{\beta}{n}-\gamma} w(s,\tau)ds \\ \le \ & \frac{n}{8}\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds + 8\Big(2-\frac{2}{n}+\frac{\beta}{n}-\gamma\Big)^2 n^3 \int_0^{s_1} s^{2-\frac{4}{n}+\frac{2\beta}{n}-\gamma}ds \\ = \ & \frac{n}{8}\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds + c_1 s_1^{3-\frac{4}{n}+\frac{2\beta}{n}-\gamma} \qquad \text{for all } \tau>0 \end{aligned}$$

and

$$\begin{aligned} \mu\int_0^{s_1} s^{1-\gamma}w(s,\tau)ds \ &\le \ \frac{n}{8}\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds + \frac{2\mu^2}{n}\int_0^{s_1} s^{2-\gamma}ds \\ &= \ \frac{n}{8}\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds + c_2\frac{m^2}{R^{2n}}s_1^{3-\gamma} \qquad \text{for all } \tau>0 \end{aligned}$$

as well as

$$\begin{aligned} y(\tau) \ &\le \ \Big\{\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds\Big\}^{\frac{1}{2}} \cdot \Big\{\int_0^{s_1} s^{-\gamma}(s_1-s)^2ds\Big\}^{\frac{1}{2}} \\ &\le \ \Big\{\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds\Big\}^{\frac{1}{2}} \cdot \Big\{s_1^2\int_0^{s_1} s^{-\gamma}ds\Big\}^{\frac{1}{2}} \\ &= \ \Big\{\int_0^{s_1} s^{-\gamma}w^2(s,\tau)ds\Big\}^{\frac{1}{2}} \cdot \Big\{\frac{1}{1-\gamma}s_1^{3-\gamma}\Big\}^{\frac{1}{2}} \qquad \text{for all } \tau>0 \end{aligned}$$

by the Cauchy-Schwarz inequality. This entails that

$$y(t) \ge y(0) + \frac{4(1-\gamma)}{n}s_1^{\gamma-3}\int_0^t y^2(\tau)d\tau - \Big\{c_1 s_1^{3-\frac{4}{n}+\frac{2\beta}{n}-\gamma} + c_2\frac{m^2}{R^{2n}}s_1^{3-\gamma}\Big\}\cdot t \qquad \text{for all } t>0. \tag{3.107}$$

Now since (3.100) along with our selections of s_0 and c_3 guarantees that

$$\begin{aligned} y(0) &\ge \frac{m_0}{\omega_n}\cdot\int_{\frac{s_1}{2}}^{s_1} s^{-\gamma}(s_1-s)ds \\ &= \frac{m_0}{\omega_n}\cdot\bigg(\frac{1}{1-\gamma}s_1\Big(s_1^{1-\gamma}-\Big(\frac{s_1}{2}\Big)^{1-\gamma}\Big) - \frac{1}{2-\gamma}\Big(s_1^{2-\gamma}-\Big(\frac{s_1}{2}\Big)^{2-\gamma}\Big)\bigg) \\ &= \frac{m_0}{\omega_n}\cdot\Big(\frac{3}{4(1-\gamma)}s_1^{2-\gamma} - \frac{3}{4(2-\gamma)}s_1^{2-\gamma}\Big) \\ &= c_3 m_0 s_1^{2-\gamma} \end{aligned}$$

and that hence, by (3.104) and (3.105),

$$\frac{c_1 s_1^{3-\frac{4}{n}+\frac{2\beta}{n}-\gamma} + c_2 \frac{m^2}{R^{2n}} s_1^{3-\gamma}}{\frac{2(1-\gamma)}{n} s_1^{\gamma-3} y^2(0)} \le \frac{nc_1}{2(1-\gamma)c_3^2 m_0^2} s_1^{2-\frac{4}{n}+\frac{2\beta}{n}} + \frac{nc_2 m^2}{2(1-\gamma)c_3^2 m_0^2 R^{2n}} s_1^2 \le \frac{1}{2} + \frac{1}{2} = 1,$$

it follows that there exists $T > 0$ such that the problem

$$\begin{cases} \underline{y}'(t) = \frac{4(1-\gamma)}{n} s_1^{\gamma-3} \underline{y}^2(t) - \left\{ c_1 s_1^{3-\frac{4}{n}+\frac{2\beta}{n}-\gamma} + c_2 \frac{m^2}{R^{2n}} s_1^{3-\gamma} \right\}, & t \in (0,T), \\ \underline{y}(0) = y(0), \end{cases}$$

admits a solution $\underline{y} \in C^1([0,T))$ fulfilling $\underline{y}(t) \nearrow +\infty$ as $t \nearrow T$. But an ODE comparison argument based on (3.107) ensures that $y(t) \ge \underline{y}(t)$ for all $t \in (0,T)$, which is incompatible with our hypothesis that w is a global classical solution of (3.20) for which the first spatial derivative w_s is bounded locally in time. □

3.3 Proof of main results

We shall now give proof to theorems 1.1 – 1.3.

The first two main theorems are obtained by utilizing the results from subsection 3.1.5 and transfering them to (1.3). Theorem 1.1 can essentially be deduced from Lemma 3.21 and Lemma 3.22.

PROOF of Theorem 1.1. Let $m := \int\limits_\Omega u_0$. We shall write $\tilde{u}_0 \in C^\theta([0,R])$, due to the radial symmetry of u_0 well-defined via

$$\tilde{u}_0(|x|) = u_0(x) \qquad \text{for} \quad x \in \overline{\Omega}. \tag{3.108}$$

Then $w_0 : [0, R^n] \to \mathbb{R}$ defined by

$$w_0(s) = \int_0^{s^{\frac{1}{n}}} \rho^{n-1} \tilde{u}_0 d\rho \qquad \text{for} \quad s \in [0, R^n] \tag{3.109}$$

is in $C^{1+\frac{\theta}{n}}([0,R^n])$ and satisfies

$$n \cdot w_{0s}(r^n) = \tilde{u}_0(r) \tag{3.110}$$

for $r \in [0,R]$ as well as

$$w_0(0) = 0, \qquad w_0(R^n) = \frac{m}{\omega_n} \quad \text{and} \quad w_{0s}(s) \geq 0, \quad s \in [0, R^n],$$

and thus with $\mu := \frac{nm}{\omega_n R^n}$, Lemma 3.21 guarantees the existence of $T_0 > 0$ and a pair of functions

$$\begin{cases} \tilde{u} \in C^0((0,R] \times [0,T_0)) \cap C^{2,1}((0,R] \times (0,T_0)), \\ \tilde{v} \in C^{2,0}((0,R] \times (0,T_0)), \end{cases} \tag{3.111}$$

such that $(\tilde{u}, \tilde{v})$ solves (3.92) classically. Moreover, Lemma 3.22 asserts that $\tilde{u}$ is nonnegative and fulfills

$$\int_0^R \rho^{n-1} \tilde{u}(\rho, t) d\rho = \int_0^R \rho^{n-1} \tilde{u}_0(\rho) d\rho \tag{3.112}$$

for all $t \in (0, T_0)$. Defining $u : \Omega_0 \times [0, T_0) \to \mathbb{R}$ via

$$u(x,t) = \tilde{u}(|x|, t) \qquad \text{for} \quad (x,t) \in \Omega_0 \times [0, T_0) \tag{3.113}$$

and $v : \Omega_0 \times (0, T_0) \to \mathbb{R}$ by

$$v(x,t) = \tilde{v}(|x|,t) \qquad \text{for} \quad (x,t) \in \Omega_0 \times (0,T_0), \tag{3.114}$$

this entails that (1.5) holds, u is nonnegative and by (3.112)

$$\begin{aligned}
\frac{1}{|\Omega|} \int\limits_\Omega u(\cdot,t) &= \frac{n}{\omega_n R^n} \omega_n \int_0^R \rho^{n-1} \tilde{u}(\rho,t) d\rho \\
&= \frac{n}{R^n} \int_0^R \rho^{n-1} \tilde{u}(\rho,t) d\rho \\
&= \frac{n}{R^n} \int_0^R \rho^{n-1} \tilde{u}_0(\rho) d\rho \\
&= \frac{n}{R^n} \frac{1}{\omega_n} \int\limits_\Omega u_0(x) dx \\
&= \frac{nm}{R^n \omega_n} \\
&= \mu
\end{aligned} \tag{3.115}$$

for $t \in (0, T_0)$. Therefore, $(\tilde{u}, \tilde{v})$ solves (3.3) classically in the sense of Definition 3.1, and hence by Lemma 3.1 we may infer that (u, v) is a classical solution of (1.3). Considering

$$|\Omega| = \frac{\omega_n R^n}{n},$$

(3.115) also contains (1.6). □

For the second theorem, the key lemmata are 3.24 and 3.23.

PROOF of Theorem 1.2. Define $\tilde{u}_0 \in C^{1+\theta}((0,R])$ the same way as in (3.108). Then w_0 as in (3.109) is in $C^{2+\frac{\theta}{n}}((0,R^n])$ and satisfies (3.110) as well as

$$\tilde{u}_{0r}(r) = n^2 w_{0ss}(r^n) \cdot r^{n-1} \qquad \text{for} \quad r \in (0,R].$$

Since u_0 is radially decreasing and therefore $\tilde{u}_{0r} \le 0$ in $(0,R]$, this implies

$$w_{0ss}(s) \le 0 \qquad \text{for all} \quad s \in (0,R^n]. \tag{3.116}$$

Moreover, (1.7) entails

$$w_{0s}(R^n) = 0 \qquad \text{and} \qquad w_{0ss}(R^n) = 0. \tag{3.117}$$

Furthermore, due to $|\nabla u_0(x)| = |\tilde{u}_{0r}(r)|$ with $r = |x|$ for $x \in \overline{\Omega}$, (1.8) results in

$$\begin{aligned} |w_{0ss}(r^n)| &= \left| \frac{1}{n^2} r^{1-n} \tilde{u}_{0r}(r) \right| \\ &\leq \frac{C_0}{n^2} r^\theta \\ &\xrightarrow{r \to 0} 0, \end{aligned}$$

warranting that $w_0 \in C^{2+\frac{\theta}{n}}([0, R^n])$ with

$$w_{0ss}(0) = 0. \tag{3.118}$$

Together with

$$\beta \leq 2 - n \qquad \text{or} \qquad \beta \geq 2,$$

(3.116), (3.117) and (3.118) certify that the conditions of Lemma 3.19 are fulfilled, and thus Lemma 3.24 warrants that $\tilde{u} \in C^{1,0}((0, R] \times [0, T_0))$ with

$$\tilde{u}_r(r, t) \leq 0 \qquad \text{for all} \quad (r, t) \in (0, R] \times [0, T_0), \tag{3.119}$$

and that moreover there exists $T^* \in (0, T_0]$ such that for $T \in (0, T^*)$

$$\tilde{u}(r, t) \leq C \qquad \text{for all} \quad (r, t) \in (0, R] \times [0, T] \tag{3.120}$$

with some $C = C(T) > 0$. This however implies that for the solution (u, v) of (1.3) from Theorem 1.1, (1.9) and (1.10) are valid.
By the definition of $\tilde{v}_r$ in (3.91) and (3.120), we may infer that furthermore

$$\begin{aligned} |\tilde{v}_r(r, t)| &= \left| \frac{1}{r^{n-1}} \left(\frac{\mu r^n}{n} - \int_0^r \rho^{n-1} \tilde{u}(\rho, t) d\rho \right) \right| \\ &\leq \frac{1}{r^{n-1}} \cdot \frac{\mu r^n}{n} + C(T) \frac{1}{r^{n-1}} \int_0^r \rho^{n-1} d\rho \\ &= \frac{\mu r}{n} + C(T) \frac{r}{n} \end{aligned}$$

for $T \in (0, T^*)$ and $(r,t) \in (0,R] \times (0,T)$, warranting

$$\tilde{v}_r \in L^\infty((0,R) \times (0,T)) \tag{3.121}$$

for all $T \in (0,T^*)$. This also entails that $\int_0^R \tilde{v}(r,t)dr$ is well-defined in $(0,T^*)$, and thus allows us to uniquely determine $\tilde{v} \in C^{2,0}((0,R] \times (0,T^*))$ by demanding

$$\int_0^R \tilde{v}(r,t)dr = 0.$$

If now additionally $n \geq 2$, then by (3.120), (3.121) and nonnegativity of $\tilde{u}$, Lemma 3.23 ensures that $(\tilde{u}, \tilde{v})$ is the unique solution of (3.92) in $(0,R] \times [0,T^*)$ satisfying these properties. By Lemma 3.1 and

$$|\nabla v(x,t)| = |\tilde{v}_r(|x|,t)| \quad \text{for} \quad (x,t) \in \overline{\Omega} \times (0,T^*)$$

though, this means that the corresponding pair of functions (u,v) gained via (3.113) and (3.114) is indeed the unique solution of (1.3) in $\Omega_0 \times [0,T^*)$ fulfilling

$$\begin{cases} u \in C^0(\Omega_0 \times [0,T^*)) \cap C^{2,1}(\Omega_0 \times (0,T^*)), \\ v \in C^{2,0}(\Omega_0 \times (0,T^*)), \end{cases}$$

which has the properties that $\int\limits_\Omega v(\cdot,t) = 0$ for all $t \in (0,T^*)$ and

$$0 \leq u \in L^\infty(\Omega \times (0,T)) \quad \text{and} \quad \nabla v \in L^\infty(\Omega \times (0,T); \mathbb{R}^n)$$

for all $T \in (0,T^*)$. □

The result on ruling out global boundedness is established by means of subsection 3.1.1 and Lemma 3.25.

PROOF of Theorem 1.3. Let $n \geq 2$, $R > 0$, $\Omega = B_R(0) \subset \mathbb{R}^n$, $\beta > 0$ and $u_0 \in C^0(\overline{\Omega})$ complying with (1.4). Then, with designations as in (3.108) and (3.109), $\tilde{u}_0 \in C^0([0,R])$. Furthermore, suppose u_0 is such that with $m = \int\limits_\Omega u_0$, $m_0 \in (0,m]$ and $r_0 = s_0^{\frac{1}{n}}$ for $s_0 = s_0(m_0,m,R,\beta) \in (0,R^n)$ as in Lemma 3.25, we have that (1.12) holds.
Assume there was a global classical solution to (1.3) satisfying (1.14) for all $T > 0$ and (1.13). Then Lemma 3.1 would guarantee that there exists a pair of functions $(\tilde{u}, \tilde{v})$ corresponding to

(u,v) via (3.113) and (3.114) which has the property that

$$\begin{cases} \tilde{u} \in C^0((0,R] \times [0,\infty)) \cap C^{2,1}((0,R] \times (0,\infty)), \\ \tilde{v} \in C^{2,0}((0,R] \times (0,\infty)), \end{cases}$$

and solves (3.3) for the initial condition $\tilde{u}(\cdot,0) = \tilde{u}_0$. Moreover, since $|\nabla v(x,t)| = |\tilde{v}_r(|x|,t)|$ for all $(x,t) \in \Omega_0 \times (0,\infty)$, (1.14) entails that for each $T > 0$

$$0 \le \tilde{u} \in L^\infty((0,R) \times (0,T)) \quad \text{and} \quad \tilde{v}_r \in L^\infty((0,R) \times (0,T)). \tag{3.122}$$

As $n \ge 2$, Lemma 3.2 now ensures that $\tilde{v}_r$ is as in (3.8), whereby in turn we may infer that the requirements of Lemma 3.4 are met, since $n \ge 2$ also implies that $\beta > 0 \ge 2 - n$.
Therefore, $w : [0,R^n] \times [0,\infty) \to \mathbb{R}$ defined via

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} \tilde{u}(\rho,t) d\rho, \quad s = r^n \in [0,R^n], \quad t \in [0,\infty),$$

is in $C^0([0,R^n] \times [0,T)) \cap C^{2,1}((0,R^n] \times (0,T))$ and solves (3.20) in $[0,R^n] \times [0,\infty)$ for $w_0 \in C^1([0,R^n])$ given by

$$w_0(s) = \int_0^{s^{\frac{1}{n}}} \rho^{n-1} \tilde{u}_0(\rho) d\rho, \qquad s \in [0,R^n].$$

Furthermore,

$$w_s(s,t) = \frac{1}{n} \cdot \tilde{u}(s^{\frac{1}{n}},t) \qquad \text{for all} \quad (s,t) \in (0,R^n) \times (0,\infty)$$

combined with (3.122) entails that for each $T > 0$

$$w_s \in L^\infty((0,R^n) \times (0,T)).$$

Thus a global classical solution of (3.20) exists, and its first spatial derivative is bounded locally in time. Since however for $s_0 \in (0,R^n)$ as above by (1.12),

$$\begin{aligned} w_0(s_0) &= \int_0^{s_0^{\frac{1}{n}}} \rho^{n-1} \tilde{u}_0(\rho) d\rho \\ &= \frac{1}{\omega_n} \int_{B_{s_0^{\frac{1}{n}}}(0)} u_0 \end{aligned}$$

$$= \frac{1}{\omega_n} \int_{B_{r_0}(0)} u_0$$
$$\geq \frac{m_0}{\omega_n},$$

this is inconsistent with Lemma 3.25, thus ruling out the existence of a global classical solution to (1.3) satisfying (1.14) and (1.13) under the given premises. □

4 Discussion of results

We shall further evaluate our results in the context of Keller-Segel systems and assess their supposed optimality with regards to their requirements. Also, we shall give some prospect of what, in addition to the established results, might be obtainable or would be desirable.

Theorem 1.1 should be nearly optimal. One might allow some range of negative values for β as well in dimensions $n > 2$, yet this was not an aim here. It seems however that for our technique, Hölder continuity of u_0 is indispensable. We do recognize that in Lemma 3.7, boundedness of $w_{\varepsilon s}$ might be achieved without additionally demanding $w_0 \in C^{1+\theta}([0, R^n])$, for example by means of [13, Theorem VI.3.2]. Then however, in Lemma 3.9 we would not receive continuity of w_s up to $t = 0$.

Establishing boundedness for w_s in some some time intervall has represented a particular challenge. In similar systems, this has been accomplished by drawing from results for Keller-Segel systems. Without such, an argumentation via concavity seems appropriate. This however requires compatibility conditions for w_0 which translate to relatively strong constraints on the initial data in (1.3). Although the restriction on β should not be optimal as well, it already seems like quite an achievement to have any result of this manner at all in view of the ad-hoc character of Lemma 3.17.

The established uniqueness class on the other hand appears to be very natural regarding that (1.3) places demands on ∇v but not v itself.

Theorem 1.3 can be interpreted as an indication of blow-up. However, yet lacking an extensibility criterion, we omit this term. The range of β in which this is possible for arbitrary initial mass though is not too surprising. That is due to the fact that there exists an interesting kind of duality to prototypical Keller-Segel systems becoming evident via the transformation to the scalar problem (3.20). The influence of β herein may be viewed as a distortion of the exponent in $s^{2-\frac{2}{n}+\frac{\beta}{n}} w_{ss}$, otherwise the sole decisive influence of the dimension n in regards to the occurence of blow-up. For $\beta = 0$, it is known that $n = 2$ and thus $2 - \frac{2}{2} = 1$ is critical. Therefore, it seems reasonable to suppose that $2 - \frac{2}{n} + \frac{\beta}{n} = 1$ which is equivalent to $\beta = 2 - n$ assumes this role. Of course, without a complementary result on global boundedness for $\beta < 2 - n$, this is not confirmed to be the case.

Moreover, it might be possible to establish that the constructed function w is the smallest nonnegative solution of (3.36). This would then further strengthen the result in Theorem 1.3 as well.

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