# Nontrivial ordered $\omega$-limit sets in a linear degenerate parabolic equation 

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#### Abstract

The paper deals with the initial-boundary value problem for $$
u_{t}=a(x)\left(\Delta u+\lambda_{1} u\right)
$$ with zero Dirichlet data in a smoothly bounded domain $\Omega \subset \mathbb{R}^{n}, n \geq 1$. Here $a$ is positive in $\Omega$ and Hölder continuous in $\bar{\Omega}$, and $\lambda_{1}>0$ denotes the principal eigenvalue of $-\Delta$ in $\Omega$ with Dirichlet data. It is shown that if $\int_{\Omega} \frac{\operatorname{dist}(x, \partial \Omega))^{2}}{a(x)} d x=\infty$ then there exist initial data in $W^{1, \infty}(\Omega)$ such that the solution of $(\star)$ has an ordered $\omega$-limit set homeomorphic to a compact real interval. Under this condition, also unbounded $\omega$-limit sets occur. Conversely, if $\frac{\operatorname{dist}(x, \partial \Omega))^{2}}{a(x)}$ is integrable then any solution emanating from initial data in $W^{1, \infty}(\Omega)$ converge to some stationary solution of $(\star)$ as time approaches infinity.


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## 1 Introduction

We consider the problem

$$
\begin{align*}
& u_{t}=a(x)\left(\Delta u+\lambda_{1} u\right) \quad \text { in } \Omega \times(0, \infty), \\
& \left.u\right|_{\partial \Omega}=0,  \tag{1.1}\\
& \left.u\right|_{t=0}=u_{0}
\end{align*}
$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^{n}, n \geq 1$, where $a$ is a Hölder continuous function on $\bar{\Omega}$ that is positive in $\Omega$. The initial data $u_{0}$ are supposed to be continuous in $\bar{\Omega}$ and to vanish on $\partial \Omega$, and $\lambda_{1}>0$ denotes the smallest eigenvalue of the Laplacian in $\Omega$ subject to Dirichlet boundary conditions.
It appears to be a common feature of large classes of nonlinear parabolic equations that bounded solutions stabilize. This means that bounded solutions converge to some timeindependent function in a suitable topology as $t \rightarrow \infty$, where the most reasonable candidates for such limit functions are the stationary (i.e. time-independent) solutions of the parabolic problem. For instance, as to the semilinear generalization of (1.1) given by

$$
\begin{equation*}
u_{t}=a(x)(\Delta u+g(x, u)) \tag{1.2}
\end{equation*}
$$

it is well known that if $a \equiv 1$ then each bounded solution $u$ approaches its $\omega$-limit set

$$
\left\{w \in L^{2}(\Omega) \mid u\left(\cdot, t_{k}\right) \rightarrow w \text { in } L^{2}(\Omega) \text { for some } t_{k} \rightarrow \infty\right\}
$$

as $t \rightarrow \infty$. Moreover, $\omega\left(u_{0}\right)$ is a compact and connected subset of the set $\mathcal{E}$ of solutions $w \in W_{0}^{1,2}(\Omega)$ of $\Delta w+g(x, w)=0$. Particularly, if $\mathcal{E}$ is known to be a singleton or merely a discrete set then this entails that $u$ converges, that is, it stabilizes, to some $w \in \mathcal{E}$ as $t \rightarrow \infty$. But the latter conclusion is also true in more complicated situations: If, for example, $\mathcal{E}$ is arbitrarily large but ordered then it is a consequence of the Hopf boundary point lemma that all bounded solutions stabilize (cf. [Li]). Also, it is known that bounded solutions converge to a steady state if $n=1([\mathrm{Ze}],[\mathrm{Ma}])$, if $g$ is analytic (see [Si], [Je]), or if $\Omega$ is a ball, $g=g(u)$ and $u$ is nonnegative ([HP]).
In any event, it seems that stabilization is at least a generic phenomenon in the sense that it occurs for all $u_{0}$ from an open dense subset of $W^{1, \infty}(\Omega)$ ([Li], [ST]), or, allowing arbitrary $u_{0}$, for 'almost every' $g$ and 'almost every' $\Omega$; for this and a more detailed discussion including further references, consult [BP], [PS], [PR] and [Po2].
Most of these results can easily be extended to the case of a smooth function $a \not \equiv 1$ which is bounded below by a positive constant on $\bar{\Omega}$. Concerning more general equations (including quasilinear and degenerate types or higher order equations), also most results in the literature concentrate on proving stabilization (see $[\mathrm{Ar}]$ and the references therein, $[\mathrm{LP}],[\mathrm{Wi}]$, [Je], and ???).
Accordingly, only few nonconvergent bounded solutions of problems related to (1.2) have been found so far. Even in the case that $g$ is allowed to depend on $\nabla u$ explicitly (which in general destroys a certain energy-dissipating property of (1.2), cf. the consideration around (2.11) below), it is not trivial to detect them; examples for this and more complex types of behavior on some domains in $\mathbb{R}^{n}, n \geq 2$, are given in [Po1] (see also the references therein). As to (1.2), on arbitrary domains in $\mathbb{R}^{n}, n \geq 2$, Poláčik and Simondon constructed nonlinearities $g(x, u) \in C^{\infty}$ such that (1.2) with $a \equiv 1$ has bounded nonstabilizing solutions ([PS]; see also $[\mathrm{PR}])$. An example for $g$ independent of $x$ is given by Poláčik and Yanagida in [PY], where various oscillating solutions are constructed for the Cauchy problem associated with

$$
\begin{equation*}
u_{t}=\Delta u+u^{p} \quad \text { in } \mathbb{R}^{n} \times(0, \infty), \quad n \geq 11, \quad p \geq \frac{(n-2)^{2}-4 n+8 \sqrt{n-1}}{(n-2)(n-10)} \tag{1.3}
\end{equation*}
$$

It is the main purpose of the present work to demonstrate that if $a$ vanishes on some part of the boundary then it may occur that (1.1) has some global solutions with nonstabilizing large time behavior. To make this more precise, we observe that according to the choice of $\lambda_{1}$, the set $\mathcal{E}$ of equilibria of (1.1) is precisely the one-dimensional eigenspace $\{\gamma \Theta \mid \gamma \in \mathbb{R}\}$ associated with the principal Laplacian eigenfunction $\Theta \geq 0$. Then our main results state that if

$$
\begin{equation*}
\int_{\Omega} \frac{(\operatorname{dist}(x, \partial \Omega))^{2}}{a(x)} d x=\infty \tag{1.4}
\end{equation*}
$$

then

- for any $-\infty<\alpha<\beta<\infty$, there exists $u_{0} \in W^{1, \infty}(\Omega)$ with $\alpha \Theta \leq u_{0} \leq \beta \Theta$, and such that the solution of (1.1) is global and bounded and its $\omega$-limit set is given by $\omega\left(u_{0}\right)=\{\gamma \Theta \mid \gamma \in[\alpha, \beta]\}$ (Theorem 4.3);

Under assumption (1.4), even unbounded $\omega$-limit sets are possible, just as in the case of problem (1.3). Namely,

- given any $\alpha \in \mathbb{R}$ and $p<\infty$, one can find initial data $u_{0} \in W^{1, p}(\Omega)$ with the property that the solution of (1.1) is global and unbounded but has a nontrivial $\omega$-limit set $\omega\left(u_{0}\right)=\{\gamma \Theta \mid \gamma \in[\alpha, \infty)\}$ (Corollary 4.4);
- for each $p<\infty$ there exists a global unbounded solution of (1.1), emanating from some $u_{0} \in W^{1, p}(\Omega)$, which is such that

$$
\begin{aligned}
& \max _{x \in \bar{\Omega}} u\left(x, t_{k}^{+}\right) \rightarrow \infty \quad \text { and } \\
& \min _{x \in \bar{\Omega}} u\left(x, t_{k}^{-}\right) \rightarrow-\infty
\end{aligned}
$$

along suitable sequences $t_{k}^{ \pm} \rightarrow \infty$ (Corollary 4.5).
In fact, (1.4) is sharp in respect of stabilization in the sense that

- if $\int_{\Omega} \frac{(\text { dist }(x, \partial \Omega))^{2}}{a(x)} d x<\infty$ then any solution emanating from initial data in $W^{1, \infty}(\Omega)$ converges to some multiple of $\Theta$ in $L^{2}(\Omega)$ as $t \rightarrow \infty$ (Lemma 3.1).

In other words, a degeneracy of (1.1), even if occuring only at one single point on the boundary, may destroy the mentioned convergence results, regardless of the choice of $\Omega$ and the space dimension $n$. This particularly means - let us emphasize this - that neither the restriction to space dimension one, nor to analytic $a$, nor to positive solutions can enforce stabilization, despite of the fact that (1.1) is a linear problem, and even though $\mathcal{E}$ is strictly ordered due to the positivity of $\Theta$.

## 2 Well-posedness of the problem and a basic property of $\omega$-limit sets

Throughout the rest of the paper we assume that

$$
\begin{equation*}
a \in C^{\mu}(\bar{\Omega}) \text { for some } \mu>0 \quad \text { and } \quad a(x)>0 \text { for all } x \in \Omega \tag{2.1}
\end{equation*}
$$

as well as

$$
\begin{equation*}
u_{0} \in C^{0}(\bar{\Omega}) \quad \text { and }\left.\quad u_{0}\right|_{\partial \Omega}=0 \tag{2.2}
\end{equation*}
$$

Moreover, for later reference let us normalize $\Theta$ in any convenient way, for instance by the requirement $\|\Theta\|_{L^{\infty}(\Omega)}=1$. Then there exist positive constants $\theta_{ \pm}$such that

$$
\begin{equation*}
\theta_{-} \operatorname{dist}(x, \partial \Omega) \leq \Theta(x) \leq \theta_{+} \operatorname{dist}(x, \partial \Omega) \tag{2.3}
\end{equation*}
$$

holds for all $x \in \Omega$.
In both the construction and the further examination of solutions of (1.1) it will be useful to
consider suitable regularizations of (1.1) that remove the possibly existing degeneracy. For definiteness, let us utilize

$$
\begin{align*}
u_{\varepsilon t} & =a_{\varepsilon}(x)\left(\Delta u_{\varepsilon}+\lambda_{1} u_{\varepsilon}\right) \quad \text { in } \Omega \times(0, \infty), \\
\left.u_{\varepsilon}\right|_{\partial \Omega} & =0, \\
\left.u_{\varepsilon}\right|_{t=0} & =u_{0} \tag{2.4}
\end{align*}
$$

for this purpose, where $a_{\varepsilon}(x):=a(x)+\varepsilon$. By classical linear parabolic Schauder theory ([LSU]), these problems have unique global solutions $u_{\varepsilon}$ for each $\varepsilon>0$. Taking $\varepsilon \rightarrow 0$, one can see that (1.1) is well-posed in the following sense.

Lemma 2.1 If a and $u_{0}$ satisfy (2.1) and (2.2) then (1.1) possesses a unique global classical solution $u \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\Omega \times(0, \infty))$. This solution can be obtained as the limit in $C_{l o c}^{0}(\bar{\Omega} \times[0, \infty)) \cap C_{l o c}^{2,1}(\Omega \times(0, \infty))$ of the solutions $u_{\varepsilon}$ of (2.4) as $\varepsilon \rightarrow 0$.
Morevoer, for all $T>0$ and $\delta>0$ there exists $\nu>0$ such that for all $\tilde{u}_{0}$ satisfying (2.2),

$$
\begin{equation*}
\left\|\tilde{u}_{0}-u_{0}\right\|_{L^{\infty}(\Omega)} \leq \nu \quad \text { implies } \quad\|\tilde{u}(\cdot, t)-u(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \delta \quad \forall t \in[0, T] \tag{2.5}
\end{equation*}
$$

when $\tilde{u}$ denotes the solution of (1.1) with initial data $\tilde{u}_{0}$.
Proof. The proof uses a series of rather well-known arguments and thus in some places we confine ourselves to an outline.
First of all, since $a$ is positive in $\Omega$, the classical maximum principle ensures uniqueness of solutions in the indicated class.
Next, it follows upon comparison with spatially homogeneous functions that

$$
\begin{equation*}
\left|u_{\varepsilon}(x, t)\right| \leq\left\|u_{0}\right\|_{L^{\infty}(\Omega)} \cdot e^{\lambda_{1}\left\|a_{\varepsilon}\right\|_{L^{\infty}(\Omega)} t} \quad \text { in } \Omega \times(0, \infty) \tag{2.6}
\end{equation*}
$$

Hence, parabolic Schauder theory in combination with the Arzelà-Ascoli theorem implies that $u_{\varepsilon} \rightarrow u$ in $C^{2,1}(K \times[\tau, T])$ holds for all $0<\tau<T<\infty$ and any compact $K \subset \subset \Omega$ along some sequence $\varepsilon=\varepsilon_{j} \rightarrow 0$, which entails that $u$ solves $u_{t}=a(x)\left(\Delta u+\lambda_{1} u\right)$ in $\Omega \times(0, \infty)$ classically.
In order to obtain that the convergence $u_{\varepsilon} \rightarrow u$ actually takes place in $C_{l o c}^{0}(\bar{\Omega} \times[0, \infty))$, it is now sufficient to prove that for each $T>0$ and $\mu>0$ there exist $K \subset \subset \Omega$ and $\tau>0$ such that for all $\varepsilon \in(0,1)$ we have

$$
\begin{equation*}
\left|u_{\varepsilon}(x, t)\right| \leq \mu \quad \text { in }(\Omega \backslash K) \times(0, T) \tag{2.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|u_{\varepsilon}(x, t)-u_{0}(x)\right| \leq \mu \quad \text { in } K \times(0, \tau) \tag{2.8}
\end{equation*}
$$

To see (2.7), we fix $\eta \in(0,1)$ small such that $2 \eta e^{\gamma T} \leq \mu$ holds with $\gamma:=\lambda_{1} \cdot \sup _{\varepsilon \in(0,1)}\left\|a_{\varepsilon}\right\|_{L^{\infty}(\Omega)}$. Then, since $\left.u_{0}\right|_{\partial \Omega}=0$, there exists $c_{\eta}>0$ such that $u_{0}(x) \leq \eta+c_{\eta} \Theta(x)$ in $\Omega$. As $v(x, t):=$ $e^{\gamma t} \cdot\left(\eta+c_{\eta} \Theta(x)\right)$ satisfies

$$
\begin{aligned}
v_{t}-a_{\varepsilon}(x)\left(\Delta v+\lambda_{1} v\right) & =\gamma e^{\gamma t}\left(\eta+c_{\eta} \Theta\right)-a_{\varepsilon}(x) e^{\gamma t}\left(-c_{\eta} \lambda_{1} \Theta+\lambda_{1} \eta+c_{\eta} \lambda_{1} \Theta\right) \\
& \geq \gamma e^{\gamma t}-\lambda_{1} \eta a_{\varepsilon}(x) e^{\gamma t} \\
& \geq 0 \quad \text { in } \Omega \times(0, T),
\end{aligned}
$$

it follows from the maximum principle that $\left|u_{\varepsilon}\right| \leq v$ in $\Omega \times(0, T)$. Particularly, defining $K$ to be the set where $c_{\eta} \Theta(x) \geq \eta$, we end up with (2.7).
The proof of (2.8) now easily follows upon comparing $u_{\varepsilon}$, for fixed $x_{0} \in K$, from above and below with the barrier functions

$$
v_{ \pm}(x, t):=u_{0}\left(x_{0}\right) \pm\left(\frac{\mu}{2}+c_{1}\left|x-x_{0}\right|^{2}+c_{2} t\right) \quad \text { in } \Omega \times(0, \tau),
$$

where $c_{1}$ and $c_{2}$ are large and $\tau>0$ is small, all these numbers depending on $\mu$ and $K$ only. As a consequence, $u$ is a classical solution of (1.1) and hence, by uniqueness, $u_{\varepsilon} \rightarrow u$ holds along the entire net $\varepsilon \rightarrow 0$.
Finally, the continuous dependence property (2.5) can be proved along the same lines upon performing minor modifications to (2.6)-(2.8).

By adapting the standard Lyapunov procedure to the present case, we obtain that the $\omega$ limit set of any solution, either bounded or unbounded, can only consist of steady states of (1.1).

Lemma 2.2 Assume that, besides (2.2), $u_{0}$ belongs to $W^{1,2}(\Omega)$, and that there exists $t_{k} \rightarrow$ $\infty$ such that

$$
\begin{equation*}
\left\|u\left(\cdot, t_{k}\right)\right\|_{L^{2}(\Omega)} \leq M \quad \forall k \in \mathbb{N} \tag{2.9}
\end{equation*}
$$

holds with some $M>0$. Then $\left(u\left(\cdot, t_{k}\right)\right)_{k \in \mathbb{N}}$ is precompact with respect to the strong topology in $L^{2}(\Omega)$ and

$$
\begin{equation*}
\bigcap_{k_{0} \in \mathbb{N}} \overline{\left(u\left(\cdot, t_{k}\right)\right)_{k \geq k_{0}}} \subset\{\gamma \theta \mid \gamma \in \mathbb{R}\}, \tag{2.10}
\end{equation*}
$$

where the closure is taken in $L^{2}(\Omega)$.
Proof. We multiply (2.4) by $\frac{u_{s t}}{a_{\varepsilon}}$ to see upon integrating by parts that

$$
\begin{equation*}
\int_{0}^{t_{k}} \int_{\Omega} \frac{u_{\varepsilon t}^{2}}{a_{\varepsilon}(x)}+\frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\left(\cdot, t_{k}\right)\right|^{2}-\frac{\lambda_{1}}{2} \int_{\Omega} u_{\varepsilon}^{2}\left(\cdot, t_{k}\right)=\frac{1}{2} \int_{\Omega}\left|\nabla u_{0}\right|^{2}-\frac{\lambda_{1}}{2} \int_{\Omega} u_{0}^{2} \tag{2.11}
\end{equation*}
$$

Due to (2.9) and the assumption that $u_{0} \in W^{1,2}(\Omega)$, this gives

$$
\int_{0}^{t_{k}} \int_{\Omega} \frac{u_{\varepsilon t}^{2}}{a_{\varepsilon}(x)}+\frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\left(\cdot, t_{k}\right)\right|^{2} \leq \frac{1}{2} \int_{\Omega}\left|\nabla u_{0}\right|^{2}+\frac{\lambda_{1} M^{2}}{2}=: C
$$

for all $k \in \mathbb{N}$. Thus, since $a_{\varepsilon} \leq\|a\|_{L^{\infty}(\Omega)}+\varepsilon$,

$$
\begin{equation*}
\int_{0}^{\infty} \int_{\Omega} u_{t}^{2} \leq C_{0}\|a\|_{L^{\infty}(\Omega)} \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u\left(\cdot, t_{k}\right)\right|^{2} \leq C_{0} \quad \text { for all } k \in \mathbb{N} \tag{2.13}
\end{equation*}
$$

by Fatou's lemma, for instance. Now (2.13) ensures the claimed $L^{2}(\Omega)$-precompactness, whereas (2.10) will follow from an argumentation similar to that presented in [Ar] for the
porous medium equation (cf. also [Wi]): From (2.12) we particularly gain

$$
\begin{aligned}
\int_{0}^{1} \int_{\Omega}\left|u\left(x, t_{k}+\tau\right)-u\left(x, t_{k}\right)\right|^{2} d x d \tau & =\int_{0}^{1} \int_{\Omega}\left|\int_{t_{k}}^{t_{k}+\tau} u_{t}(x, s) d s\right|^{2} d x d \tau \\
& \leq \int_{0}^{1} \tau \int_{\Omega} \int_{t_{k}}^{t_{k}+1} u_{t}^{2}(x, s) d s d x d \tau \\
& \leq \frac{1}{2} \int_{t_{k}}^{\infty} \int_{\Omega} u_{t}^{2} \rightarrow 0 \quad \text { as } k \rightarrow \infty
\end{aligned}
$$

Hence, if $w$ is any element of the set on the left-hand side of (2.10), that is, if $u\left(\cdot, t_{k_{j}}\right) \rightarrow w$ in $L^{2}(\Omega)$ for some subsequence $t_{k_{j}} \rightarrow \infty$, we also have

$$
\begin{equation*}
\int_{0}^{1} \int_{\Omega}\left|u\left(x, t_{k_{j}}+\tau\right)-w(x)\right|^{2} d x d \tau \rightarrow 0 \quad \text { as } j \rightarrow \infty \tag{2.14}
\end{equation*}
$$

Let us fix $\zeta \in C_{0}^{\infty}((0,1))$ with $\int_{0}^{1} \zeta(\tau) d \tau=1$ and let $\varphi \in C_{0}^{\infty}(\Omega)$ be arbitrary. Then multiplying (1.1) by $\frac{\zeta\left(t-t_{k}\right) \varphi(x)}{a(x)}$ and integrating yields

$$
\begin{aligned}
0= & \int_{t_{k_{j}}}^{t_{k_{j}}+1} \int_{\Omega} \zeta^{\prime}\left(t-t_{k_{j}}\right) u(x, t) \frac{\varphi(x)}{a(x)} d x d t+\int_{t_{k_{j}}}^{t_{k_{j}}+1} \int_{\Omega} \zeta\left(t-t_{k_{j}}\right) u(x, t) \Delta \varphi(x) d x d t \\
& +\lambda_{1} \int_{t_{k_{j}}}^{t_{k_{j}}+1} \int_{\Omega} \zeta\left(t-t_{k_{j}}\right) u(x, t) \varphi(x) d x d t \\
= & \int_{0}^{1} \int_{\Omega} \zeta^{\prime}(\tau) u\left(x, t_{k_{j}}+\tau\right) \frac{\varphi(x)}{a(x)} d x d \tau+\int_{0}^{1} \int_{\Omega} \zeta(\tau) u\left(x, t_{k_{j}}+\tau\right) \Delta \varphi(x) d x d \tau \\
& +\lambda_{1} \int_{0}^{1} \int_{\Omega} \zeta(\tau) u\left(x, t_{k_{j}}+\tau\right) \varphi(x) d x d \tau .
\end{aligned}
$$

According to (2.14) we may take $j \rightarrow \infty$ here to achieve

$$
\begin{aligned}
0 & \int_{0}^{1} \int_{\Omega} \zeta^{\prime}(\tau) w(x) \frac{\varphi(x)}{a(x)} d x d \tau+\int_{0}^{1} \int_{\Omega} \zeta(\tau) w(x) \Delta \varphi(x) d x d \tau \\
& +\lambda_{1} \int_{0}^{1} \int_{\Omega} \zeta(\tau) w(x) \varphi(x) d x d \tau \\
= & \int_{\Omega} w\left(\Delta \varphi+\lambda_{1} \varphi\right)
\end{aligned}
$$

in view of the properties of $\zeta$. Therefore $w$, belonging to $W_{0}^{1,2}(\Omega)$ due to (2.13), is a weak solution of $-\Delta w=\lambda_{1} w$ in $\Omega$. By simplicity of $\lambda_{1}, w$ must coincide with some multiple of $\Theta$.

## 3 The case $\int_{\Omega} \frac{(\text { dist }(x, \partial \Omega))^{2}}{a(x)} d x<\infty$

Let us first make sure that if the degeneracy is sufficiently weak then all solutions emanating from Lipschitz continuous initial data stabilize.

Lemma 3.1 Suppose that $\int_{\Omega} \frac{(\text { dist }(x, \partial \Omega))^{2}}{a(x)} d x<\infty$. Then for any $u_{0} \in W^{1, \infty}(\Omega)$ vanishing on $\partial \Omega$, the solution $u$ of (1.1) satisfies

$$
\begin{equation*}
u(\cdot, t) \rightarrow \alpha \Theta \quad \text { in } L^{2}(\Omega) \quad \text { as } t \rightarrow \infty \tag{3.1}
\end{equation*}
$$

where $\alpha$ is given by

$$
\begin{equation*}
\alpha=\frac{\int_{\Omega} \frac{\Theta(x) u_{0}(x)}{a(x)} d x}{\int_{\Omega} \frac{\Theta^{2}(x)}{a(x)}} \tag{3.2}
\end{equation*}
$$

Remark. Due to (2.3), the assumption $\int_{\Omega} \frac{(\operatorname{dist}(x, \partial \Omega))^{2}}{a(x)} d x<\infty$ is equivalent to the requirement $\int_{\Omega} \frac{\Theta^{2}(x)}{a(x)} d x<\infty$.

Proof. Since $u_{0} \in W^{1, \infty}(\Omega)$ and $\left.u_{0}\right|_{\partial \Omega}=0$, there exists $c>0$ such that $\left|u_{0}(x)\right| \leq$ $c \operatorname{dist}(x, \partial \Omega)$, so that, by (2.3),

$$
\begin{equation*}
\left|u_{0}(x)\right| \leq \frac{c}{\theta_{-}} \Theta(x) \quad \text { for all } x \in \Omega \tag{3.3}
\end{equation*}
$$

Thus the comparison principle implies $\left|u_{\varepsilon}(x, t)\right| \leq \frac{c}{\theta_{-}} \Theta(x)$ in $\Omega \times(0, \infty)$, whence $u$ is bounded. Therefore Lemma 2.2 says that its $\omega$-limit set is not empty and consists of multiples of $\Theta$ only. To see that this set actually is a singleton, we multiply (2.4) by $\frac{\Theta(x)}{a_{\varepsilon}(x)}$ and integrate to obtain

$$
\begin{aligned}
\int_{\Omega} \frac{\Theta(x)}{a_{\varepsilon}(x)} u_{\varepsilon}(x, t) d x-\int_{\Omega} \frac{\Theta(x)}{a_{\varepsilon}(x)} u_{0}(x) d x & =\int_{0}^{t} \int_{\Omega}\left(\Delta u_{\varepsilon}+\lambda_{1} u_{\varepsilon}\right) \cdot \Theta \\
& =\int_{0}^{t} \int_{\Omega} u_{\varepsilon} \cdot\left(\Delta \Theta+\lambda_{1} \Theta\right) \\
& =0
\end{aligned}
$$

From (3.3) we gain the uniform majorization

$$
\left|\frac{\Theta(x)}{a_{\varepsilon}(x)} u_{\varepsilon}(x, t)\right| \leq \frac{c}{\theta_{-}} \frac{\Theta^{2}(x)}{a(x)} \quad \text { for all } t \geq 0 \text { and } \varepsilon>0
$$

Therefore our assumption $\int_{\Omega} \frac{\Theta^{2}(x)}{a(x)} d x<\infty$ together with the dominated convergence theorem implies that in the limit $\varepsilon \rightarrow 0$

$$
\int_{\Omega} \frac{\Theta(x)}{a(x)} u(x, t) d x=\int_{\Omega} \frac{\Theta(x)}{a(x)} u_{0}(x) d x \quad \text { for all } t>0
$$

Now suppose $\alpha \in \mathbb{R}$ is such that $u\left(\cdot, t_{k}\right) \rightarrow \alpha \Theta$ in $L^{2}(\Omega)$ along some sequence $t_{k} \rightarrow \infty$. Passing to a subsequence, we may assume that this convergence is also pointwise a.e. in $\Omega$ and hence repeating the above argument in taking $t_{k} \rightarrow \infty$ yields

$$
\int_{\Omega} \frac{\Theta(x)}{a(x)} \cdot \alpha \Theta(x) d x=\int_{\Omega} \frac{\Theta(x)}{a(x)} u_{0}(x) d x .
$$

This means that $\alpha$ is uniquely determined by (3.2).

## 4 The case $\int_{\Omega} \frac{(\text { dist }(x, \partial \Omega))^{2}}{a(x)} d x=\infty$

A key ingredient in the proof of our main results is the following lemma. It states that under codition (1.4), any signed perturbation from a steady state that initially is compactly supported will not change the large time behavior, no matter how large this perturbation is at $t=0$. A similar global attractivity property of steady states is shared by the equilibria of 1.3 ; in [PY] this discovery also is the starting point for the construction of oscillating solutions.
Lemma 4.1 Suppose that $\int_{\Omega} \frac{(\operatorname{dist}(x, \partial \Omega))^{2}}{a(x)} d x=\infty$, and assume that

$$
u_{0}=\alpha \Theta+\varphi
$$

with some $\alpha \in \mathbb{R}$ and $\varphi \in W^{1,2}(\Omega) \cap C^{0}(\bar{\Omega})$ such that

$$
\begin{aligned}
& \varphi \text { has compact support in } \Omega \quad \text { and } \\
& \text { either } \varphi \geq 0 \text { in } \Omega \quad \text { or } \quad \varphi \leq 0 \text { in } \Omega .
\end{aligned}
$$

Then the solution $u$ of (1.1) fulfils

$$
\begin{equation*}
u(\cdot, t) \rightarrow \alpha \Theta \quad \text { in } L^{2}(\Omega) \quad \text { as } t \rightarrow \infty \tag{4.1}
\end{equation*}
$$

Proof. Since $\tilde{u}_{ \pm}:= \pm(u-\alpha \Theta)$ also solves (1.1) by linearity, it is sufficient to consider the case $\alpha=0$ and $\varphi \geq 0$ only.
Starting with the same strategy as in the proof of Lemma 3.1, we first use $\frac{\Theta(x)}{a_{\varepsilon}(x)}$ as a test function in (2.4) to gain the conservation property

$$
\int_{\Omega} \frac{\Theta(x)}{a_{\varepsilon}(x)} u_{\varepsilon}(x, t) d x=\int_{\Omega} \frac{\Theta(x)}{a_{\varepsilon}(x)} u_{0}(x) d x \quad \text { for all } t>0
$$

Since the right-hand side tends to the finite number $\int_{\Omega} \frac{\Theta(x)}{a(x)} \varphi(x) d x$ as $\varepsilon \rightarrow 0$, it follows from Fatou's lemma that

$$
\begin{equation*}
\int_{\Omega} \frac{\Theta(x)}{a(x)} u(x, t) d x \quad \text { remains uniformly bounded for all } t>0 \tag{4.2}
\end{equation*}
$$

Next, we multiply (2.4) by $\frac{u_{\varepsilon}(x, t)}{a_{\varepsilon}(x)}$ and use the Poincaré inequality to obtain the dissipation property

$$
\begin{aligned}
\frac{1}{2} \int_{\Omega} \frac{u_{\varepsilon}^{2}(x, t)}{a_{\varepsilon}(x)} d x-\frac{1}{2} \int_{\Omega} \frac{\varphi^{2}(x)}{a_{\varepsilon}(x)} d x & =-\int_{0}^{t} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+\lambda_{1} \int_{0}^{t} \int_{\Omega} u_{\varepsilon}^{2} \\
& \leq 0 \quad \text { for all } t>0
\end{aligned}
$$

Again by Fatou's lemma, we deduce from this that

$$
\int_{\Omega} u^{2}(x, t) d x \leq\|a\|_{L^{\infty}(\Omega)} \int_{\Omega} \frac{u^{2}(x, t)}{a(x)} d x \leq\|a\|_{L^{\infty}(\Omega)} \int_{\Omega} \frac{\varphi^{2}(x)}{a(x)} d x \quad \text { for all } t>0 .
$$

This allows us to apply Lemma 2.2 to conclude that the whole semi-orbit $\{u(\cdot, t) \mid t>0\}$ is precompact in $L^{2}(\Omega)$, and that each member of the $\omega$-limit set is a multiple of $\Theta$. But it is impossible that $u\left(\cdot, t_{k}\right) \rightarrow \beta \Theta$ in $L^{2}(\Omega)$ occurs for some $\beta>0$ and $t_{k} \rightarrow \infty$ : In this case, namely, for a subsequence we would have $u\left(\cdot, t_{k_{j}}\right) \rightarrow \beta \Theta$ also a.e. in $\Omega$ and hence, once more due to Fatou's lemma, could conclude from (4.2) that

$$
\int_{\Omega} \frac{\Theta(x)}{a(x)} \cdot \beta \Theta(x) d x<\infty
$$

In light of (2.3), this would contradict the hypothesis $\int_{\Omega} \frac{(\operatorname{dist}(x, \partial \Omega))^{2}}{a(x)} d x=\infty$, whence in fact (4.1) must hold.

We now follow an idea originally used in [PY] to construct nonstabilizing solutions of (1.3) and repeatedly apply the last lemma with suitable $\varphi$ to obtain initial distributions leading to large $\omega$-limit sets. The construction is somewhat more involved than in [PY], because we treat the bounded and the unbounded case at the same time.
Lemma 4.2 Assume that $\int_{\Omega} \frac{\left(\operatorname{dist}(x, \partial \Omega)^{2}\right.}{a(x)} d x=\infty$. Let $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be a nondecreasing and $\left(\beta_{k}\right)_{k \in \mathbb{N}}$ be an increasing sequence of real numbers such that $\alpha_{1}<\beta_{1}$. Then for all $p \in[2, \infty)$ there exists $u_{0} \in W^{1, p}(\Omega) \cap C^{0}(\bar{\Omega})$ vanishing on $\partial \Omega$ and a sequence of times $t_{k} \rightarrow \infty$ such that the solution $u$ of (1.1) satisfies

$$
\left\{\begin{align*}
\left\|u\left(\cdot, t_{k}\right)-\alpha_{k} \Theta\right\|_{L^{2}(\Omega)} \leq \frac{1}{k} & \text { if } k \text { is odd } \quad \text { and }  \tag{4.3}\\
\left\|u\left(\cdot, t_{k}\right)-\beta_{k} \Theta\right\|_{L^{2}(\Omega)} \leq \frac{1}{k} & \text { if } k \text { is even } .
\end{align*}\right.
$$

Moreover, $u_{0}$ can be chosen in such a way that

$$
\begin{equation*}
\alpha \Theta \leq u_{0} \leq \beta \Theta \quad \text { in } \Omega \tag{4.4}
\end{equation*}
$$

holds with $\alpha:=\lim _{k \rightarrow \infty} \alpha_{k} \geq-\infty$ and $\beta:=\lim _{k \rightarrow \infty} \beta_{k} \leq+\infty$, and if both $\alpha_{k} \equiv \alpha$ and $\beta_{k} \equiv \beta$ for all $k$ then it is possible to achieve $u_{0} \in W^{1, \infty}(\Omega)$.
Proof. The idea is to define $u_{0}$ as the limit of an inductively defined seuqnece of initial data $u_{0 k}$ lying between $\alpha_{k} \Theta$ and $\beta_{k} \Theta$. Here, in a small neighborhood of $\partial \Omega, u_{0 k}$ shall coincide with $\alpha_{k} \Theta$ if $k$ is odd, and with $\beta_{k} \Theta$ if $k$ is even, so that Lemma 4.1 will yield that the corresponding solutions $u_{k}$ of (1.1) become close to $\alpha_{k} \Theta$ and $\beta_{k} \Theta$, respectively, at some large but finite $t_{k}$. The continuous dependence property asserted in Lemma 2.1 then enables us to slightly modify $u_{0 k}$ near $\partial \Omega$ without a significant effect at $t_{k}$.
To make this more precise, we fix a cut-off function $\zeta \in C^{\infty}([0, \infty))$ such that $\zeta(s)=1$ for $s \leq \frac{1}{2}, \zeta(s)=0$ for $s \geq 1$ and $-4 \leq \zeta^{\prime}(s) \leq 0$ for $s \geq 0$. For each $\delta>0$, set $\zeta_{\delta}(x):=\zeta\left(\frac{\operatorname{dist}(x, \partial \Omega)}{\delta}\right)$ for $x \in \bar{\Omega}$. Evidently, $\zeta_{\delta}(x)=1$ if dist $(x, \partial \Omega) \leq \frac{\delta}{2}$ and $\zeta_{\delta}(x)=0$ if $\operatorname{dist}(x, \partial \Omega) \geq \delta$. Moreover, since $x \mapsto \operatorname{dist}(x, \partial \Omega)$ has Lipschitz constant 1 , we have

$$
\begin{equation*}
\left|\nabla \zeta_{\delta}(x)\right| \leq \frac{4}{\delta} \cdot \chi_{\left\{\frac{\delta}{2} \leq \operatorname{dist}(x, \partial \Omega) \leq \delta\right\}}(x) \quad \text { for a.e. } x \in \Omega \tag{4.5}
\end{equation*}
$$

To simplify notation, from now on we may assume - expanding the sequneces if necessary that

$$
\begin{equation*}
\alpha_{k}=\alpha_{k-1} \text { whenever } k \text { is even and } \beta_{k}=\beta_{k-1} \text { for all odd } k \tag{4.6}
\end{equation*}
$$

The rest of the proof will be organized in three steps.
Step 1: Initiation of the recursion.
We let

$$
\begin{equation*}
u_{01}(x):=\alpha_{1} \Theta(x), \quad x \in \bar{\Omega} \tag{4.7}
\end{equation*}
$$

and fix an rabitrary $t_{1}>1$. Then, since $\alpha_{1} \Theta$ is a stationary solution of (1.1), Lemma 2.1 states that there exists $\nu_{1}>0$ such that for $\tilde{u}_{0}$ satisfying (2.2),

$$
\left\|\tilde{u}_{0}-u_{0}\right\|_{L^{\infty}(\Omega)} \leq \nu_{1} \quad \text { implies } \quad\left\|\tilde{u}\left(\cdot, t_{1}\right)-\alpha_{1} \Theta\right\|_{L^{2}(\Omega)} \leq 1
$$

when $\tilde{u}$ denotes the solution of (1.1) with $\left.\tilde{u}\right|_{t=0}=\tilde{u}_{0}$. Observe that if we choose $\delta_{1}>\operatorname{diam} \Omega$ then $\zeta_{\delta_{1}} \equiv 1$ and (4.7) can trivially be rewritten in the form

$$
u_{01}=\beta_{1} \Theta-\left(\beta_{1}-\alpha_{1}\right) \zeta_{\delta_{1}} \Theta
$$

Generalizing, let us assume that for some $k \geq 2$ and all $j \in\{1, \ldots, k-1\}$ we have already defined $u_{0 j}, \delta_{j}, \nu_{j}$ and $t_{j}$ with the follwing properties:

$$
\begin{align*}
& \delta_{j}<\frac{\delta_{j-1}}{2}, \quad 2 \leq j<k  \tag{4.8}\\
& \nu_{j}<\frac{\nu_{j-1}}{2^{j-1}}, \quad 2 \leq j<k  \tag{4.9}\\
& t_{j}>j, \quad 1 \leq j<k  \tag{4.10}\\
& u_{0 j}=\beta_{1} \Theta+\sum_{i=1}^{j}(-1)^{i}\left(\beta_{i}-\alpha_{i}\right) \zeta_{\delta_{i}} \Theta, \quad 1 \leq j<k,  \tag{4.11}\\
& \left\|u_{0 j}-u_{0, j-1}\right\|_{L^{\infty}(\Omega)} \leq \frac{\nu_{j-1}}{2}, \quad 2 \leq j<k  \tag{4.12}\\
& \left\|\nabla u_{0 j}-\nabla u_{0, j-1}\right\|_{L^{p}(\Omega)} \leq \frac{1}{2^{j}}, \quad 2 \leq j<k \tag{4.13}
\end{align*}
$$

and, for $1 \leq j<k$,

$$
\left\|\tilde{u}_{0}-u_{0 j}\right\|_{L^{\infty}(\Omega)} \leq \nu_{j} \quad \text { implies } \quad \begin{cases}\left\|\tilde{u}\left(\cdot, t_{j}\right)-\alpha_{j} \Theta\right\|_{L^{2}(\Omega)} \leq \frac{1}{j} & \text { if } j \text { is odd }  \tag{4.14}\\ \left\|\tilde{u}\left(\cdot, t_{j}\right)-\beta_{j} \Theta\right\|_{L^{2}(\Omega)} \leq \frac{1}{j} & \text { if } j \text { is even }\end{cases}
$$

where $\tilde{u}_{0}$ satisfies (2.2) and $\tilde{u}$ is the corresponding solution of (1.1). Observe that (4.8) and (4.11) imply that

$$
u_{0 j} \equiv \begin{cases}\beta_{j} \Theta & \text { on } \operatorname{supp} \zeta_{\delta_{j}} \text { if } j<k \text { and } j \text { is even } \\ \alpha_{j} \Theta & \text { on } \operatorname{supp} \zeta_{\delta_{j}} \text { if } j<k \text { and } j \text { is odd }\end{cases}
$$

so that (4.11) can be written in the recursive form

$$
u_{0 j}=\left\{\begin{align*}
u_{0, j-1}+\zeta_{\delta_{j}}\left(\beta_{j} \Theta-\alpha_{j} \Theta\right) & \equiv u_{0, j-1}+\zeta_{\delta_{j}}\left(\beta_{j} \Theta-u_{0, j-1}\right)  \tag{4.15}\\
& \equiv\left(1-\zeta_{\delta_{j}}\right) u_{0, j-1}+\zeta_{\delta_{j}} \cdot \beta_{j} \Theta \quad \text { if } j<k \text { is even, } \\
u_{0, j-1}-\zeta_{\delta_{j}}\left(\beta_{j} \Theta-\alpha_{j} \Theta\right) & \equiv u_{0, j-1}+\zeta_{\delta_{j}}\left(\alpha_{j} \Theta-u_{0, j-1}\right) \\
& \equiv\left(1-\zeta_{\delta_{j}}\right) u_{0, j-1}+\zeta_{\delta_{j}} \cdot \alpha_{j} \Theta \quad \text { if } j<k \text { is odd. }
\end{align*}\right.
$$

From this and (4.6) it can easily be seen by induction that

$$
\begin{equation*}
\alpha_{j} \Theta \leq u_{0 j} \leq \beta_{j} \Theta, \quad 1 \leq j<k \tag{4.16}
\end{equation*}
$$

Step 2: Continuation of the recursion - the inductive step.
$\overline{\text { We proceed to define }} u_{0 k}, \delta_{k}, \nu_{k}$ and $t_{k}$ such that (4.8)-(4.14) continue to hold up to $j=k$. To initiate this next step, we fix $\delta_{k}<\frac{\delta_{k-1}}{2}$ such that

$$
\begin{equation*}
\left(\beta_{k}-\alpha_{k}\right) \cdot \theta_{+} \delta_{k} \leq \frac{\nu_{k-1}}{2} \tag{4.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(4 \theta_{+}+\|\nabla \Theta\|_{L^{\infty}(\Omega)}\right) \cdot\left(\beta_{k}-\alpha_{k}\right) \cdot\left|\Omega \backslash \Omega_{\delta_{k}}\right|^{\frac{1}{p}} \leq \frac{1}{2^{k}} \tag{4.18}
\end{equation*}
$$

hold, where $\theta_{+}$is the constant from (2.3) and $\Omega_{\delta}:=\{x \in \Omega \mid \operatorname{dist}(x, \partial \Omega)>\delta\}$. Now we define through (4.11) for $j=k$, that is, we set

$$
u_{0 k}:=\beta_{1} \Theta+\sum_{i=1}^{k}(-1)^{i}\left(\beta_{i}-\alpha_{i}\right) \zeta_{\delta_{i}} \Theta .
$$

Then (4.15) and (4.16), now also valid for $j=k$, say that

$$
u_{0 k}= \begin{cases}\beta_{k} \Theta-\left(1-\zeta_{\delta_{k}}\right)\left(\beta_{k} \Theta-u_{0, k-1}\right)=: \beta_{k} \Theta-\varphi_{k} & \text { if } k \text { is even } \\ \alpha_{k} \Theta-\left(1-\zeta_{\delta_{k}}\right)\left(\alpha_{k} \Theta-u_{0, k-1}\right)=: \alpha_{k} \Theta+\varphi_{k} & \text { if } k \text { is odd }\end{cases}
$$

and that $\varphi_{k}$ is nonnegative. Therefore Lemma 4.1 asserts that if, for instance, $k$ is even then the solution $u_{k}$ of (1.1) with $\left.u_{k}\right|_{t=0}=u_{0 k}$ converges to $\beta_{k} \Theta$ as $t \rightarrow \infty$ and particularly satisfies

$$
\begin{equation*}
\left\|u_{k}\left(\cdot, t_{k}\right)-\beta_{k} \Theta\right\|_{L^{2}(\Omega)} \leq \frac{1}{2 k} \tag{4.19}
\end{equation*}
$$

for some $t_{k}>k$. Hence, according to Lemma 2.1, if we pick $\nu_{k}<\frac{\nu_{k-1}}{2^{k-1}}$ small enough then for any $\tilde{u}_{0}$ compatible with (2.2),

$$
\left\|\tilde{u}_{0}-u_{0 k}\right\|_{L^{\infty}(\Omega)} \leq \nu_{k} \quad \text { implies } \quad\left\|\tilde{u}\left(\cdot, t_{k}\right)-u_{k}\left(\cdot, t_{k}\right)\right\|_{L^{2}(\Omega)} \leq \frac{1}{2 k}
$$

In conjunction with (4.19) and a similar reasoning for odd $k$ this shows that (4.14) continues to hold for $j=k$.
To check (4.12) up to $j=k$, we use (4.15), (2.3) and (4.17) in estimating

$$
\begin{aligned}
\left\|u_{0 k}-u_{0, k-1}\right\|_{L^{\infty}(\Omega)} & =\left\|\zeta_{\delta_{k}}\left(\beta_{k} \Theta-\alpha_{k} \Theta\right)\right\|_{L^{\infty}(\Omega)} \\
& \left.\leq\left(\beta_{k}-\alpha_{k}\right)\|\Theta\|_{L^{\infty}\left(\Omega \backslash \Omega_{\delta_{k}}\right.}\right) \\
& \leq\left(\beta_{k}-\alpha_{k}\right) \cdot \theta_{+} \delta_{k} \\
& \leq \frac{\nu_{k-1}}{2} .
\end{aligned}
$$

Finally, from (4.15), (2.3), (4.5) and (4.18) we gain

$$
\begin{aligned}
\left\|\nabla u_{0 k}-\nabla u_{0, k-1}\right\|_{L^{p}(\Omega)} \leq & \left(\beta_{k}-\alpha_{k}\right)\left\|\Theta \nabla \zeta_{\delta_{k}}\right\|_{L^{p}\left(\Omega \backslash \Omega_{\delta_{k}}\right)}+\left(\beta_{k}-\alpha_{k}\right)\left\|\zeta_{\delta_{k}} \nabla \Theta\right\|_{L^{p}\left(\Omega \backslash \Omega_{\delta_{k}}\right)} \\
\leq & \left(\beta_{k}-\alpha_{k}\right) \cdot \theta_{+} \delta_{k} \cdot \frac{4}{\delta_{k}} \cdot\left|\Omega \backslash \Omega_{\delta_{k}}\right|^{\frac{1}{p}} \\
& +\left(\beta_{k}-\alpha_{k}\right) \cdot\|\nabla \Theta\|_{L^{\infty}(\Omega)} \cdot\left|\Omega \backslash \Omega_{\delta_{k}}\right|^{\frac{1}{p}} \\
\leq & \frac{1}{2^{k}}
\end{aligned}
$$

which asserts (4.13) up to $j=k$.
Step 3. Limit process.
$\overline{\text { As a result of (4.12) and (4.9), for } 1 \leq k<k^{\prime}<\infty \text { we have }}$

$$
\begin{aligned}
\left\|u_{0 k}-u_{0 k^{\prime}}\right\|_{L^{\infty}(\Omega)} & \leq \sum_{i=k+1}^{k^{\prime}}\left\|u_{0 i}-u_{0, i-1}\right\|_{L^{\infty}(\Omega)} \\
& \leq \sum_{i=k+1}^{k^{\prime}} \frac{\nu_{i-1}}{2} \\
& =\frac{1}{2}\left(\nu_{k}+\sum_{i=k+2}^{k^{\prime}} \nu_{i-1}\right) \\
& \leq \frac{1}{2}\left(\nu_{k}+\sum_{i=k+2}^{k^{\prime}} \frac{\nu_{i-2}}{2^{2-2}}\right) \\
& \leq \frac{\nu_{k}}{2}\left(1+\sum_{i=k+2}^{k^{\prime}} \frac{1}{2^{2-2}}\right) \\
& \leq \frac{\nu_{k}}{2}\left(1+\sum_{l=1}^{\infty} \frac{1}{2^{c}}\right) \\
& =\nu_{k}
\end{aligned}
$$

which proves that the $u_{0 k}$ form a Cauchy sequence in $C^{0}(\bar{\Omega})$. Taking now $k^{\prime} \rightarrow \infty$ here shows that for its limit $u_{0}$ we have $\left\|u_{0}-u_{0 k}\right\|_{L^{\infty}(\Omega)} \leq \nu_{k}$ for all $k \in \mathbb{N}$. Hence, by (4.14), the solution $u$ of (1.1) emanating from $u_{0}$ satisfies (4.3).
Moreover, (4.16) proves (4.4), and the inclusion $u_{0} \in W^{1, p}(\Omega)$ is a consequence of (4.13), the triangle inequality in $L^{p}(\Omega)$ and the finiteness of $\sum_{k=1}^{\infty} \frac{1}{2^{k}}$.
Finally, if $\alpha_{k} \equiv \alpha$ and $\beta_{k} \equiv \beta$ then from (4.11) we gain

$$
\begin{align*}
\left\|\nabla u_{0 k}\right\|_{L^{\infty}(\Omega)} \leq & \beta\|\nabla \Theta\|_{L^{\infty}(\Omega)}+(\beta-\alpha) \sum_{i=1}^{k}\left\|\nabla \zeta_{\delta_{i}} \cdot \Theta\right\|_{L^{\infty}(\Omega)} \\
& +(\beta-\alpha)\left\|\sum_{i=1}^{k}(-1)^{i} \zeta \zeta_{\delta_{i}}\right\|_{L^{\infty}(\Omega)} \cdot\|\nabla \Theta\|_{L^{\infty}(\Omega)} . \tag{4.20}
\end{align*}
$$

The first and the third term on the right are bounded independently of $k$, because $\zeta_{\delta_{i}}$ is nonincreasing with $i$ and thus $0 \geq \sum_{i=1}^{k}(-1)^{i} \zeta_{\delta_{i}} \geq-1$. As to the second term, we use that
by construction the gradients $\nabla \zeta_{\delta_{i}}$ have mutually disjoint supports; that is, recalling (4.5), (4.8) and (2.3) we find

$$
\begin{aligned}
\sum_{i=1}^{k}\left\|\nabla \zeta_{\delta_{i}} \cdot \Theta\right\|_{L^{\infty}(\Omega)} & \leq \sum_{i=1}^{k} \frac{4}{\delta_{i}} \cdot \theta_{+} \delta_{i} \cdot\left|\Omega_{\frac{\delta_{i}}{2}} \backslash \Omega_{\delta_{i}}\right| \\
& \leq 4 \theta_{+} \cdot|\Omega|
\end{aligned}
$$

Accordingly, (4.20) shows that in this case we have $u_{0} \in W^{1, \infty}(\Omega)$.
Now the main results actually are corollaries of this lemma.
Theorem 4.3 Suppose $\alpha$ and $\beta$ are real numbers such that $\alpha<\beta$. Then there exists $u_{0} \in W^{1, \infty}(\Omega)$ vanishing on $\partial \Omega$ such that the $\omega$-limit set of the solution (1.1) is given by

$$
\omega\left(u_{0}\right)=\{\gamma \Theta \mid \gamma \in[\alpha, \beta]\} .
$$

Proof. Applying Lemma 4.2 to $\alpha_{k} \equiv \alpha$ and $\beta_{k} \equiv \beta$, we obtain $u_{0} \in W^{1, \infty}(\Omega)$ with

$$
\begin{equation*}
\alpha \Theta \leq u_{0} \leq \beta \Theta \quad \text { in } \Omega \tag{4.21}
\end{equation*}
$$

and such that $\omega\left(u_{0}\right)$ contains $\alpha \Theta$ and $\beta \Theta$. Since the solution $u$ of (1.1) is continuous, $\omega\left(u_{0}\right)$ is connected and thus $\{\gamma \Theta \mid \gamma \in[\alpha, \beta]\} \subset \omega\left(u_{0}\right)$. By (4.21), also the opposite inclusion holds.

Corollary 4.4 Let $\alpha \in \mathbb{R}$. Then for all $p \in[2, \infty)$ there exists $u_{0} \in W^{1, p}(\Omega) \cap C^{0}(\bar{\Omega})$ with $\left.u_{0}\right|_{\partial \Omega}=0$ such that the solution of (1.1) is unbounded and

$$
\omega\left(u_{0}\right)=\{\gamma \Theta \mid \gamma \in[\alpha, \infty)\} .
$$

Proof. We set $\alpha_{k} \equiv \alpha$ and $\beta_{k}:=\alpha+k$ in Lemma 4.2 to obtain a function $u_{0}$ from the indicated class, fulfilling

$$
\begin{equation*}
u_{0} \geq \alpha \Theta \quad \text { in } \Omega \tag{4.22}
\end{equation*}
$$

and such that $\omega\left(u_{0}\right)$ contains $\alpha \Theta$. Since, by (4.3),
$\left\|u\left(\cdot, t_{k}\right)-\alpha \Theta\right\|_{L^{2}(\Omega)} \leq \frac{1}{k} \quad$ if $k$ is odd, and
$\left\|u\left(\cdot, t_{k}\right)-\alpha \Theta\right\|_{L^{2}(\Omega)} \geq k\|\Theta\|_{L^{2}(\Omega)}-\left\|\left(u\left(\cdot, t_{k}\right)-\alpha \Theta\right)-k \Theta\right\|_{L^{2}(\Omega)} \geq k\|\Theta\|_{L^{2}(\Omega)}-1$ if $k$ is even,
we infer from the continuity of the solution $u$ of (1.1) that for each $\beta>\alpha$ and $m \in \mathbb{N}$ large enough one can find $\tilde{t}_{m} \in\left(t_{2 m}, t_{2 m 1}\right)$ such that

$$
\begin{equation*}
\left\|u\left(\cdot, \tilde{t}_{m}\right)-\alpha \Theta\right\|_{L^{2}(\Omega)}=\|(\beta-\alpha) \Theta\|_{L^{2}(\Omega)} . \tag{4.23}
\end{equation*}
$$

Thus, Lemma 2.2 (which is still applicable since $u_{0} \in W^{1,2}(\Omega)$ ) entails that

$$
\bigcap_{m_{0} \in \mathbb{N}} \overline{\left(u\left(\cdot, \tilde{t}_{m}\right)\right)_{m \geq m_{0}}} \subset\{\gamma \Theta \mid \gamma \in \mathbb{R}\}
$$

By (4.23), however, the only conceivable accumulation point of $u\left(\cdot, \tilde{t}_{m}\right)-\alpha \Theta$ is $(\beta-\alpha) \Theta$, from which we conclude

$$
u\left(\cdot, \tilde{t}_{m}\right) \rightarrow \beta \Theta \quad \text { as } m \rightarrow \infty
$$

We thus have proved $\{\gamma \Theta \mid \gamma \in[\alpha, \infty)\} \subset \omega\left(u_{0}\right)$. In view of (4.22) and, again, Lemma 2.2, this completes the proof.

Corollary 4.5 For any $p<\infty$ one can find $u_{0} \in W^{1, p}(\Omega) \cap C^{0}(\bar{\Omega})$, vanishing on $\partial \Omega$, such that the solution $u$ of (1.1) satisfies

$$
\begin{aligned}
& \max _{x \in \bar{\Omega}} u\left(x, t_{k}^{+}\right) \rightarrow \infty \quad \text { and } \\
& \min _{x \in \bar{\Omega}} u\left(x, t_{k}^{-}\right) \rightarrow-\infty
\end{aligned}
$$

as $k \rightarrow \infty$ for suitable sequences $t_{k}^{ \pm} \rightarrow \infty$.
Proof. The claim immediately follows upon an application of Lemma 4.2 to $\alpha_{k}:=-k$ and $\beta_{k}:=k$.

Remark. Unfortunately, it does not result from Corollary 4.5 that the solution constructed there has its $\omega$-limit set equal to the entire steady state set $\{\gamma \Theta \mid \gamma \in \mathbb{R}\}$, and we have to leave open here wheter the latter can be achieved for appropriate initial data. In fact, this seems to be hardly achievable by our construction, which, invoking Lemma 4.1 in each recursion step, essentially relied on the fact that the compactly supported difference of $u_{0 k}$ to some multiple of $\Theta$ had one sign throughout $\Omega$.

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