Conservation of boundary decay and nonconvergent bounded gradients in degenerate diffusion problems

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Abstract

This paper investigates the boundary behavior of nonnegative classical solutions to the Dirichlet problem for

$$u_t = u^p \Delta u + g(u) \qquad \text{in } \Omega \times (0,T), \qquad p > 1,$$

and draws some consequences for the large time behavior of solutions. Here, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $g: [0,\infty) \to \mathbb{R}$ is locally Lipschitz continuous with $g(0) = 0$. The first goal is to study for which $\alpha \geq 1$ the implication

$$\begin{aligned} & u_0(x) \leq c_1(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad (c_1 > 0) \\ \Rightarrow \quad & u(x,t) \leq C(T')(\mathrm{dist}\,(x,\partial\Omega))^\alpha \quad \text{in } \Omega \times (0,T') \text{ for any } T' < T \qquad \text{(I)} \end{aligned}$$

is valid, and it turns out that this holds whenever either $p \geq 2$, or $p < 2$ and $\alpha \geq \frac{1}{p-1}$.
For $p \in (1,2]$ and $g \equiv 0$, this complements a previously known result, according to which the lower estimate $u_0(x) \geq c_0(\mathrm{dist}\,(x,\partial\Omega))^\alpha$ with some $\alpha < \frac{1}{p-1}$ and $c_0 > 0$ implies the existence of $T > 0$ and $C > 0$ such that $u(x,t) \geq C\mathrm{dist}\,(x,\partial\Omega)$ for all $x \in \Omega$ and $t \geq T$.
Using (I), we moreover show that whenever $p > 1$, there exist some values of $q \geq 1$ such that the particular equation $u_t = u^p u_{xx} + u^q$ possesses positive classical solutions which are nondecreasing w.r.t. to t and remain uniformly bounded in $C^1(\bar\Omega)$ for all times, but do not converge in $C^1(\bar\Omega)$ as $t \to \infty$.

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Introduction

We study nonnegative classical solutions of the Dirichlet problem

$$\begin{cases} u_t = u^p \Delta u + g(u) \qquad \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{0.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary of class C^3, where $p > 1$, $T > 0$, and $g: [0,\infty) \to \mathbb{R}$ is locally Lipschitz continuous in $[0,\infty)$ with $g(0) = 0$.
The first objective of this paper is to clarify under which conditions an initially given algebraic boundary decay is inherited by solutions in the following sense: Suppose that

$$u_0(x) \leq c_1(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \text{for all } x \in \Omega, \tag{0.2}$$

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with certain constants $\alpha \geq 1$ and $c_1 > 0$; does then

$$u(x,t) \leq C(T')(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \text{ and } 0 < t < T' < T \tag{0.3}$$

hold with some appropriately large constant $C(T')$?
As to the special case $g \equiv 0$, that is, for the unperturbed problem

$$\begin{cases} u_t = u^p \Delta u & \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{0.4}$$

it is known that for any $\alpha > 1$, (0.3) can hold only in presence of sufficiently strong degeneracies: Namely, in the non-degenerate case when $p = 0$, the well-known Hopf boundary point lemma states that each nontrivial nonnegative solution u has strictly negative normal derivative at $\partial\Omega$ for all $t > 0$; in particular, $u(\cdot,t)$ enters the cone

$$K := \Big\{\varphi : \ \Omega \to \mathbb{R} \ \Big| \ \exists\, c > 0 : \ \varphi(x) \geq c\mathrm{dist}\,(x,\partial\Omega) \text{ for all } x \in \Omega\Big\}$$

immediately, which is obviously incompatible with (0.3) when $\alpha > 1$. In [BP], a slightly weaker result was derived for the weakly degenerate case $p \in (0,1)$, in which (0.4) can be transformed into the porous medium equation $v_t = \Delta v^m$ via the substitution $u = av^m$ with $m = \frac{1}{1-p} > 1$ and $a = m^{\frac{1}{p}}$: It was proved there that every nonnegative (weak) solution $u \not\equiv 0$ of (0.4), though possibly lacking a normal derivative at $\partial\Omega$, again enters K after some finite time (which may or may not be positive, cf. [F] or [A], for instantce).
The results in [Win1] indicate that the picture in the case $p \geq 1$ becomes more involved in so far as, unlike the case $p \in [0,1)$, the boundary behavior of u_0 plays an important role: For instance, it was shown there that if $p > 2$ then (0.2) implies (0.3) for positive solutions of (0.4) whenever $\alpha > 1$, so that such solutions will never enter K. Concerning the intermediate regime $p \in [1,2)$, the decay exponent $\alpha_c := \frac{1}{p-1} > 1$ was detected to be critical with respect to the possibility of entering K in the following sense: If $u_0(x) \geq c_0(\mathrm{dist}\,(x,\partial\Omega))^\alpha$ holds throughout Ω with some $\alpha < \alpha_c$ then $u(\cdot,t) \in K$ for all sufficiently large t, provided that Ω is a ball and u_0 is radially symmetric; on the other hand, if u_0 satisfies (0.2) with some $\alpha \geq \alpha_c$ then for all $t > 0$ we have $u(\cdot,t) \notin K$. The precise boundary behavior of the latter type of solutions was left unstudied in [Win1]; for instance, it might be conceivable that a solution satisfying (0.2) behaves like $u(x,t) \sim (\mathrm{dist}\,(x,\partial\Omega))^{\alpha(t)}$ with some nonconstant $\alpha(t)$, where the known fact that $u(\cdot,t) \notin K$ would be compatible with any $\alpha(t) > 1$. However, the first main result of the present work implies that this cannot be the case. More precisely and more generally, in Theorem 1.8 we shall include the case $g \not\equiv 0$ and prove that

- if $p > 1$ and u_0 satisfies (0.2) with some $\alpha \geq 1$ such that $\alpha \geq \frac{1}{p-1}$ then every classical solution of (0.1) satisfies (0.3).

There is a number of examples where appropriate information on the boundary behavior could be turned into a rather precise knowledge on the time asymptotics of solutions to problems of type (0.4) with $p \geq 0$; various qualitative properties such as stabilization, convergence rates, asymptotic profiles, or localization of blow-up points were addressed using such methods in [Wie2], [L], [AP] and [FMcL1], for instance. Correspondingly, the second main goal of this work is to illustrate how the implication (0.2) $\Rightarrow$ (0.3) may affect the large time behavior of solutions to (0.4) in that it enables us to detect some rather strange solutions. In fact, using Theorem 1.8, we shall see that for any degeneracy parameter $p > 1$ there exist locally Lipschitz continuous sources $g(u)$ such that the one-dimensional version of (0.4) possesses trajectories which remain uniformly bounded in $C^1(\bar{\Omega})$ but are not precompact in $C^1(\bar{\Omega})$:

- If $n = 1$, $p > 1$ and $q \geq 1$ is such that $q \in (p-1, p+1)$ and $q \geq 3 - p$ then there exist $u_0 \in C^1(\bar{\Omega})$ such that the solution u of (0.4) with $g(u) = u^q$ is nondecreasing w.r.t. to t and satisfies $\sup_{t>0} \|u(\cdot,t)\|_{C^1(\bar{\Omega})} < \infty$ and $\lim_{t\to\infty} \|u(\cdot,t) - w\|_{C^0(\bar{\Omega})} = 0$ with some $0 \not\equiv w \in C^{1+\vartheta}(\bar{\Omega})$ ($\vartheta > 0$), but which has the property $\liminf_{t\to\infty} \|u(\cdot,t) - w\|_{C^1(\bar{\Omega})} > 0$ (Theorem 2.7).

A similar conclusion would clearly be impossible in any semilinear problem of the general form $u_t = \Delta u + g(x,t,u,\nabla u)$ with, say, g locally bounded in $\bar{\Omega} \times [0,\infty) \times \mathbb{R} \times \mathbb{R}^n$, because then standard parabolic theory ([LSU]) turns a supposedly given bound of the form $\sup_{t>0} \|u(\cdot,t)\|_{C^1(\bar{\Omega})} < \infty$ into an estimate $\sup_{t>0} \|u(\cdot,t)\|_{C^{1+\vartheta}(\bar{\Omega})} < \infty$ with some $\theta > 0$. But also when weak degeneracies are present, a phenomenon of the above type can be ruled out in a fairly general framework. This will be demonstrated in Lemma 2.8, which shall in fact indicate the parameter $p = 1$ to be critical in this respect: Namely, there we shall consider the case $p \in (0,1)$ in the corresponding Dirichlet problem for the one-dimensional porous-medium type equation $u_t = u^p u_{xx} + g(x,t,u,u_x)$, where g is bounded and locally Hölder continuous in $\bar{\Omega} \times [0,\infty) \times \mathbb{R}^2$. Our result will say that for any 'solution' u (cf. Section 2.2 to see what is meant here by 'solution') with the property that $\limsup_{t\to\infty} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} < \infty$ one can find a sequence of times $t_k \to \infty$ such that $(u(\cdot,t_k))_{k\in\mathbb{N}}$ is precompact in $C^1(\bar{\Omega})$, unless $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$.

Unexpected behavior of gradients in quasilinear and semilinear parabolic equations has attracted increasing interest through the last years. The phenomenon most frequently investigated in this respect is the *blow-up* of gradients of bounded solutions in either finite or infinite time, where both possibilities of boundary as well as interior derivative blow-up have been detected in appropriate examples (see [Sou], [SV], [AF] and the references therein, for example); however, the occurrence of bounded but nonconvergent gradients, as studied here, appears to be quite a rarely visible phenomenon.

Let us finally mention that as to the perturbed problem (0.1), little seems to be known about boundary behavior so far. The only results we are aware of in the literature concentrate on proving that (0.2) entails (0.3) for $\alpha = 1$. Since even classical solutions of the degenerate problem (0.1) are in general not in $C^1(\bar{\Omega})$ for $t \geq 0$ (because there may exist even *stationary* irregular solutions, cf. [Wie2]), this is not self-evident. Correspondingly, even for $\alpha = 1$ the proof of the implication (0.2) $\Rightarrow$ (0.3) in [FMcL2] and [Wie1] for the equation $u_t = u^p \Delta u + u^{p+1}$ with $p \geq 1$ required new (comparison-based) arguments.

1 Boundary behavior

The proof of the desired implication (0.2) $\Rightarrow$ (0.3) in Section 1.3 will rely on comparison of u with time-dependent barrier functions. Here, in Section 1.2 we concentrate on the unperturbed problem (0.4) first and then use a transformation to cover Lipschitz sources $g(u)$. For the delicate parameter regime $p \in (1,2]$, the main idea is to proceed in two steps: In the first one we show that (0.2) implies $u(x,t) \leq C(T')(\mathrm{dist}\,(x,\partial\Omega))^{\frac{2}{p}}$ instead of the stronger estimate (0.3).

This is prepared in Section 1.1 by constructing suitable barrier functions $\bar{v}$ defined near $\partial\Omega$ and satisfying the *nonlinear* inequality $\bar{v}_t \geq \bar{v}^p \Delta \bar{v}$. Secondly, we use this preliminary estimate to derive (0.3) by showing that u lies below an appropriate solution $\bar{u}$ of the *linear* parabolic inequality $\bar{u}_t \geq u^p \Delta \bar{u}$.

In order to fix terminology, we define that a *classical solution* of (0.4) is to be understood as a nonnegative function $u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T))$ that satisfies (0.4) in the pointwise sense. By a *nonnegative classical subsolution* we mean a nonnegative function u from the same regularity class that vanishes on $\partial\Omega$ and satisfies $u_t \leq u^p \Delta u$ in $\Omega \times (0,T)$.

1.1 A radially symmetric auxiliary problem in an annulus

In this section we consider a variant of problem (0.4) where the spatial domain is the annulus $\mathcal{A} := B_{R_1}(x_c) \setminus \bar{B}_R(x_c)$ with a given point $x_c \in \mathbb{R}^n$ and fixed radii satisfying $0 < R < R_1$. We intend to use solutions of such problems as (upper) barriers in order to estimate our solution u of (0.4) from above in a neighborhood of $\partial\Omega$. Therefore, we shall require these solutions to be zero on the inner part of $\partial\mathcal{A}$ and to attain a sufficiently large positive constant on its outer part. In conjunction with an adequate choice of the initial data, this suggests to study, for instance, the auxiliary problem

$$\begin{cases} v_t = v^p \Delta v \qquad \text{in } \mathcal{A} \times (0,\infty), \\ v|_{\partial B_R(x_c)} = 0, \\ v|_{\partial B_{R_1}(x_c)} = M, \\ v|_{t=0} = v_0, \end{cases} \tag{1.1}$$

where $M > 0$ and $v_0(x) := \frac{M}{(R_1-R)^\alpha} \cdot (|x-x_c| - R)^\alpha$ for $x \in \bar{\mathcal{A}}$.
Since v_0 is a convex radial function that is nondecreasing with respect to $r = |x - x_c|$, it is natural to suspect that a solution of (1.1) is also radially symmetric and nondecreasing with respect to r, and that furthermore $v_t \geq 0$. In order to construct a solution with these properties, we consider the approximate problems

$$\begin{cases} v_{\varepsilon t} = v_\varepsilon^p \Delta v_\varepsilon \qquad \text{in } \mathcal{A} \times (0,\infty), \\ v_\varepsilon|_{\partial\mathcal{A}} = v_0|_{\partial\mathcal{A}} + \varepsilon, \\ v_\varepsilon|_{t=0} = v_{0\varepsilon} \end{cases} \tag{1.2}$$

for $\varepsilon \in (0,1)$, where $v_{0\varepsilon}$ suitably approximates v_0. An appropriate selection of $v_{0\varepsilon}$ is provided by the following lemma. A similar statement, using a more involved construction, was given in [ACP, Lemma A].

Lemma 1.1 *For any $\alpha > 1$ and $M > 0$ there exists a family $(v_{0\varepsilon})_{\varepsilon\in(0,1)} \subset C^\infty(\bar{\mathcal{A}})$ of radially symmetric initial data $v_{0\varepsilon} = v_{0\varepsilon}(r)$, $r = |x - x_c|$, satisfying*

$$v_0 + \varepsilon \leq v_{0\varepsilon} \leq M + \varepsilon \qquad \text{in } \mathcal{A}, \tag{1.3}$$

$$v_{0\varepsilon r} \geq 0 \qquad \text{in } \mathcal{A}, \tag{1.4}$$

$$\Delta v_{0\varepsilon} \geq 0 \qquad \text{in } \mathcal{A}, \tag{1.5}$$

$$\Delta v_{0\varepsilon}|_{\partial\mathcal{A}} = 0 \tag{1.6}$$

as well as

$$v_{0\varepsilon} \to v_0 \qquad \text{monotonically in } \bar{\mathcal{A}} \text{ and in } C^1(\bar{\mathcal{A}}). \tag{1.7}$$

Proof. Let $(\chi_\varepsilon)_{\varepsilon\in(0,1)} \subset C_0^\infty(\mathcal{A})$ denote a nondecreasing family of radial functions $\chi_\varepsilon = \chi_\varepsilon(r)$ such that $0 \le \chi_\varepsilon \le 1$ in $\mathcal{A}$ and $\chi_\varepsilon \nearrow 1$ in $\mathcal{A}$ as $\varepsilon \searrow 0$. Then the boundary-value problem

$$\begin{cases} \Delta v_{0\varepsilon} = \chi_\varepsilon \cdot \Delta v_0 & \text{in } \mathcal{A}, \\ v_{0\varepsilon}|_{\partial\mathcal{A}} = v_0|_{\partial\mathcal{A}} + \varepsilon, \end{cases} \tag{1.8}$$

has a unique solution $v_{0\varepsilon} \in C^\infty(\bar{\mathcal{A}})$ that evidently is radially symmetric and fulfils (1.6). Moreover, (1.5) holds, because the particular choice of v_0 in combination with the hypothesis $\alpha > 1$ implies that $\Delta v_0 \ge 0$ in $\mathcal{A}$.
Next, since $\underline{z} := v_0 + \varepsilon$ satisfies

$$\Delta \underline{z} = \Delta v_0 \ge \chi_\varepsilon \Delta v_0 = \Delta v_{0\varepsilon} \qquad \text{in } \mathcal{A}$$

as well as $\underline{z} \le v_{0\varepsilon}$ on $\partial\mathcal{A}$, the elliptic maximum principle implies that the left inequality in (1.3) is valid. Similarly, the right inequality in (1.3) follows upon comparison of $v_{0\varepsilon}$ with the constant function $\bar{z} := M + \varepsilon$ which satisfies $\bar{z} \ge v_{0\varepsilon}$ on $\partial\mathcal{A}$ and

$$\Delta \bar{z} = 0 \le \Delta v_{0\varepsilon} \qquad \text{in } \mathcal{A}.$$

To see (1.4), we note that $v_{0\varepsilon}(R) = \varepsilon$ together with the left inequality in (1.3) yields $v_{0\varepsilon r}(R) \ge 0$. But (1.5) states that $r^{n-1} v_{0\varepsilon r}$ is nondecreasing with r, whence (1.4) follows.
Finally, since for $0 < \varepsilon < \varepsilon' < 1$ we have $v_{0\varepsilon} < v_{0\varepsilon'}$ on $\partial\mathcal{A}$ and

$$\Delta v_{0\varepsilon} = \chi_\varepsilon \cdot \Delta v_0 \ge \chi_{\varepsilon'} \cdot \Delta v_0 = \Delta v_{0\varepsilon'} \qquad \text{in } \mathcal{A},$$

elliptic comparison implies that $v_{0\varepsilon}$ decreases as $\varepsilon \searrow 0$. Thus, $v_{0\varepsilon} \searrow z$ in $\bar{\mathcal{A}}$ holds for some measurable $z \ge v_0$. Since $\Delta v_0(x) \le c(|x - x_c| - R)^{\alpha-2}$ for all $x \in \mathcal{A}$ with some $c > 0$, our assumption $\alpha > 1$ implies that $\|\Delta v_{0\varepsilon}\|_{L^q(\mathcal{A})}$ is uniformly bounded for some $q > 1$; in fact, this is true for any $q > 1$ satisfying $(2-\alpha)q < 1$. Therefore elliptic regularity theory yields uniform boundedness of $v_{0\varepsilon}$ in $W^{2,q}(\mathcal{A})$. In view of the radial symmetry, the compact embedding $W^{2,q}((R, R_1)) \hookrightarrow\hookrightarrow C^1([R, R_1])$ entails that the convergence $v_{0\varepsilon} \to z$ actually takes place in $C^1(\bar{\mathcal{A}})$. Therefore (1.8) shows that $z \in C^1(\bar{\mathcal{A}})$ is a weak solution of $\Delta z = \Delta v_0$ fulfilling $z = v_0$ on $\partial\mathcal{A}$ and hence, by uniqueness of solutions to this linear elliptic problem, must coincide with v_0. ////

Using the parabolic comparison principle, we can now assert the desired consequences for the corresponding solution v of (1.1).

Lemma 1.2 *Let $\alpha > 1$ and $M > 0$. Then the problem (1.1) has a global classical radially symmetric solution $v = v(r,t)$. This solution can be obtained as the limit $v = \lim_{\varepsilon\searrow 0} v_\varepsilon$ in $C^0_{loc}(\bar{\mathcal{A}} \times [0,\infty)) \cap C^{2,1}_{loc}(\mathcal{A} \times [0,\infty))$ of a decreasing family of positive solutions v_ε of (1.2) which satisfy*

$$v_{\varepsilon r} \ge 0 \qquad \text{in } \mathcal{A} \times (0,\infty) \tag{1.9}$$

and

$$v_{\varepsilon t} \ge 0 \qquad \text{in } \mathcal{A} \times (0,\infty). \tag{1.10}$$

Consequently, we also have $v_r \ge 0$ and $v_t \ge 0$ in $\mathcal{A} \times (0,\infty)$.

PROOF. We consider (1.2), where $v_{0\varepsilon}$ is as provided by Lemma 1.1. Since $\varepsilon \le v_{0\varepsilon} \le M+\varepsilon$ in $\mathcal{A}$ and $\underline{v} := \varepsilon$ and $\bar{v} := M+\varepsilon$ are a sub- and a supersolution of (1.2), respectively, it follows from standard arguments ([Wie2]) that (1.2) is actually non-degenerate and has a unique global classical solution v_ε that satisfies

$$\varepsilon \le v_\varepsilon \le M+\varepsilon \qquad \text{in } \mathcal{A}\times(0,\infty). \tag{1.11}$$

Due to (1.6), the first-order compatibility condition for (1.2) is fulfilled, so that v_ε in fact belongs to $C^{2,1}(\bar{\mathcal{A}}\times[0,\infty))\cap C^\infty(\mathcal{A}\times(0,\infty))$. Clearly, v_ε inherits the radial symmetry of $v_{0\varepsilon}$, and since differentiation of (1.2) with respect to r shows that $z(r,t) := v_{\varepsilon r}(r,t)$ satisfies the linear parabolic equation

$$z_t = v_\varepsilon^p z_{rr} + p v_\varepsilon^{p-1} v_{\varepsilon r} z_r + \tfrac{n-1}{r} v_\varepsilon^p z_r + \tfrac{n-1}{r} v_\varepsilon^p v_{\varepsilon r} z - \tfrac{n-1}{r^2} v_\varepsilon^p z \quad \text{in } (R,R_1)\times(0,\infty),$$

it follows from the maximum principle that z remains nonnegative if the same is true on the parabolic boundary of $(R,R_1)\times(0,\infty)$. Indeed, by (1.4) we have $z\ge 0$ at $t=0$, whereas (1.11) along with the boundary conditions in (1.2) imply that also $z\ge 0$ at $r=R$ and at $r=R_1$; thus, (1.9) has been proved.

Next, by the above regularity statement the function $w := v_{\varepsilon t}$ belongs to $C^0(\bar{\mathcal{A}}\times[0,\infty))\cap C^\infty(\mathcal{A}\times(0,\infty))$ and satisfies

$$w_t = v_\varepsilon^p \Delta w + p v_\varepsilon^{p-1}\Delta v_\varepsilon \cdot w \qquad \text{in } \mathcal{A}\times(0,\infty).$$

Since in view of $v_\varepsilon|_{\partial\mathcal{A}} = v_0|_{\partial\mathcal{A}} + \varepsilon$, we have $w|_{\partial\mathcal{A}} = 0$ and (1.5) yields $w\ge 0$ at $t=0$, the maximum principle guarantees that (1.10) holds.

Together with (1.3), this sharpens (1.11) according to $v_0+\varepsilon \le v_\varepsilon \le M+\varepsilon$ in $\mathcal{A}\times(0,\infty)$. Thus, parabolic Schauder estimates may be applied to (1.2) to yield a uniform bound for v_ε in $C_{loc}^{2+\theta,\frac{1+\theta}{2}}((\bar{\mathcal{A}}\setminus\partial B_R(x_c))\times[0,\infty))$ with some $\theta>0$. Since evidently $v_\varepsilon \searrow v$ as $\varepsilon\searrow 0$ with some nonnegative function v, the latter statement in conjunction with the Arzelà-Ascoli theorem ensures that actually $v_\varepsilon \to v$ in $C_{loc}^{2,1}((\bar{\mathcal{A}}\setminus\partial B_R(x_c))\times[0,\infty))$. Consequently, v solves $v_t = v^p\Delta v$ in $\mathcal{A}\times(0,\infty)$ and attains the desired values at $t=0$ and the outer part $\partial B_{R_1}(x_c)$ of the lateral boundary. The continuity of v at the corresponding inner part $\partial B_R(x_c)$ follows from the fact $v_\varepsilon\searrow v$ as $\varepsilon\searrow 0$; indeed, this monotone convergence implies that v is upper semicontinuous and hence, as a nonnegative function, continuous wherever v is zero. – which clearly is the case at $\partial B_R(x_c)$. Altogether, we see that v is continuous in $\bar{\mathcal{A}}\times[0,\infty)$, whence Dini's theorem ensures that the convergence $v_\varepsilon\to v$ is locally uniform in $\bar{\mathcal{A}}\times[0,\infty)$. ////

The following lemma will play a key role in the proof of (0.3). Here the assumption $\alpha \ge \frac{1}{p-1}$ is essentially used.

Lemma 1.3 *Let $p\in[1,2)$, $\alpha\ge\frac{1}{p-1}$ and $M>0$. Then for any $R_0\in(R,R_1)$ we have*

$$\int_{B_{R_0}(x_c)\setminus B_R(x_c)} v^{1-p}(x,t)\,dx = +\infty \qquad \textit{for all } t\ge 0. \tag{1.12}$$

PROOF. Since v is radially symmetric, we again may write $v=v(r,t)$, where $r=|x-x_c|$. We multiply (1.2) by $\frac{\varphi}{v_\varepsilon^p}$, where $\phi(r) := \frac{R_0-r}{r^{n-1}}$. Integrating by parts with respect to $r\in(R,R_0)$ gives

$$-\frac{1}{p-1}\frac{d}{dt}\int_R^{R_0} v_\varepsilon^{1-p}(r,t)\cdot(R_0-r)\,dr$$

$$
\begin{aligned}
&= \int_R^{R_0} (v_{\varepsilon rr} + \frac{n-1}{r} v_{\varepsilon r}) \cdot (R_0 - r) dr \\
&= \int_R^{R_0} v_{\varepsilon r} dr + v_{\varepsilon r} \cdot (R_0 - r)\Big|_R^{R_0} \\
&\qquad -(n-1) \int_R^{R_0} v_\varepsilon \cdot \Big(\frac{R_0 - r}{r}\Big)_r dr + (n-1) \cdot v_\varepsilon \cdot \frac{R_0 - r}{r}\Big|_R^{R_0} \\
&= v_\varepsilon(R_0, t) - v_\varepsilon(r, t) - v_{\varepsilon r}(R, t) \cdot (R_0 - R) \\
&\qquad +(n-1) \int_R^{R_0} \frac{v_\varepsilon(r,t)}{r^2} dr - (n-1) \cdot v_\varepsilon(R, t) \cdot \frac{R_0 - R}{R}
\end{aligned}
$$

for $t > 0$. Since v_ε increases with r by Lemma 1.2, we can estimate

$$
-\frac{1}{p-1} \frac{d}{dt} \int_R^{R_0} v_\varepsilon^{1-p}(r,t) \cdot (R_0 - r) dr \le (M + \varepsilon) \cdot \Big[1 + (n-1) \cdot \frac{R_0 - R}{R^2}\Big].
$$

From this it follows that

$$
\frac{d}{dt} \int_R^{R_0} v_\varepsilon^{1-p}(r,t) \cdot (R_0 - r) dr \ge -C \qquad \text{for all } t > 0
$$

with some $C > 0$, which upon integration implies that

$$
\int_R^{R_0} v_\varepsilon^{1-p}(r,t) \cdot (R_0 - r) dr \ge \int_R^{R_0} v_{0\varepsilon}^{1-p}(r) \cdot (R_0 - r) dr - Ct \qquad \text{for all } t > 0.
$$

Here we use that $v_\varepsilon(\cdot, t) \searrow v(;t)$ for $t > 0$ and $v_{0\varepsilon}(r) \searrow v_0(r) = \frac{M}{(R_1 - R)^\alpha}(r - R)^\alpha$ as $\varepsilon > 0$, and apply Beppo-Levi's theorem to obtain

$$
\int_R^{R_0} v^{1-p}(r,t) \cdot (R_0 - r) dr \ge \Big(\frac{M}{(R_1 - R)^\alpha}\Big)^{1-p} \int_R^{R_0} (r - R)^{(1-p)\alpha} \cdot (R_0 - r) dr - Ct
$$

for all $t > 0$. But since $\alpha \ge \frac{1}{p-1}$, the right-hand side equals $+\infty$, so that

$$
\int_R^{R_0} r^{n-1} v^{1-p}(r,t) dr \ge \frac{R^{n-1}}{R_0} \cdot \int_R^{R_0} v^{1-p}(r,t) \cdot (R_0 - r) dr = +\infty \qquad \text{for all } t > 0,
$$

as claimed. ////

We are now ready to complete the announced preliminary upper estimate for the solution v.

Lemma 1.4 *Let $p \in [1, 2)$, $\alpha \ge \frac{1}{p-1}$ and $M > 0$. Then for all $T > 0$ there exists $C = C(T, p, \alpha, M, R, R_1) > 0$ such that the solution $v = \lim_{\varepsilon \searrow 0} v_\varepsilon$ of (1.1) satisfies*

$$
v(x,t) \le C \cdot (|x - x_c| - R)^{\frac{2}{p}} \qquad \textit{for all } (x,t) \in \mathcal{A} \times (0, T). \tag{1.13}
$$

PROOF. Since v is radial, we may write $v(r, t)$ instead of $v(x, t)$, where $r = |x - x_c|$. Given $T > 0$, we fix a small positive number δ satisfying

$$
\delta \le (1 + pat)^{-\frac{1}{p}}, \tag{1.14}
$$

where $a := \frac{2(2+(n-2)p)}{p^2} > 0$. We claim that it is possible to find some $R_0 > R$ satisfying

$$R_0 \le \bar{R}_0 := \min \Big\{ R + \Big(\frac{\delta}{M}\Big)^{\frac{1}{\alpha - \frac{2}{p}}}, R_1 \Big\} \tag{1.15}$$

with the property that

$$v(R_0, T) \le \delta (R_0 - R)^{\frac{2}{p}}. \tag{1.16}$$

In fact, if such an R_0 did not exist, we would have $v(r,T) > \delta(r-R)^{\frac{2}{p}}$ for all $r \in (R_0, \bar{R}_0)$ and hence

$$\int_{B_{\bar{R}_0}(x_c)\setminus B_R(x_c)} v^{1-p}(x,t)dx \le \delta^{1-p} \cdot \int_R^{\bar{R}_0} r^{n-1} \cdot (r-R)^{\frac{2}{p}(1-p)} dr.$$

Since $\frac{2}{p}(1-p) > -1$ whenever $p < 2$, the integral on the right is finite, whence we would have a contradiction to Lemma 1.3.
We now let

$$\bar{v}_\eta(r,t) := y(t) \cdot (r - R + \eta)^{\frac{2}{p}}, \qquad r \in [R, R_0],\ t \in [0,T],$$

where $\eta \in (0,1)$ and y is the solution of the initial-value problem

$$\begin{cases} y'(t) = a y^{p+1}, & t \in (0,T), \\ y(0) = \delta, \end{cases} \tag{1.17}$$

that is,

$$y(t) = (\delta^{-p} - pat)^{-\frac{1}{p}}, \qquad t \in [0,T].$$

Observe that by (1.14),

$$y(t) \le (\delta^{-p} - paT)^{-\frac{1}{p}} \le 1 \qquad \text{for all } t \in [0,T]. \tag{1.18}$$

Using (1.17), we see that $\bar{v}_\eta$ satisfies

$$\begin{aligned}
\bar{v}_{\eta t} - \bar{v}_\eta^p \Delta \bar{v}_\eta &= y' \cdot (r-R+\eta)^{\frac{2}{p}} \\
&\quad - \Big[y \cdot (r-R+\eta)^{\frac{2}{p}} \Big]^p \cdot y \cdot \Big\{ \tfrac{2}{p}(\tfrac{2}{p} - 1)(r-R+\eta)^{\frac{2}{p}-2} \\
&\qquad\qquad + \tfrac{n-1}{r} \cdot \tfrac{2}{p} \cdot (r-R+\eta)^{\frac{2}{p}-1} \Big\} \\
&\ge y' \cdot (r-R+\eta)^{\frac{2}{p}} \\
&\quad - y^{p+1} \cdot (r-R+\eta)^2 \cdot \Big\{ \tfrac{2}{p}(\tfrac{2}{p}-1)(r-R+\eta)^{\frac{2}{p}-2} \\
&\qquad\qquad + \tfrac{2(n-1)}{p} (r-R+\eta)^{\frac{2}{p}-2} \Big\} \\
&= \Big[y' - \tfrac{2(2-p+(n-1)p}{p^2} y^{p+1} \Big] \cdot (r-R+\eta)^{\frac{2}{p}} \\
&= (y' - a y^{p+1}) \cdot (r-R+\eta)^{\frac{2}{p}} \\
&= 0 \qquad \text{for } r \in (R, R_0),\ t \in (0,T),
\end{aligned} \tag{1.19}$$

where we note that $r > r - R + \eta$ for $r \in (R, R_0)$ due to our restriction $\eta < R$. Moreover,

$$\bar{v}_\eta(R,t) > 0 = v(R,t) \qquad \text{for all } t \in (0,T) \tag{1.20}$$

and

$$\begin{aligned} v_\eta(R_0,t) &= y(t) \cdot (R_0 - R + \eta)^{\frac{2}{p}} \\ &> \delta \cdot (R_0 - R)^{\frac{2}{p}} \qquad \text{for all } t \in [0,T]. \end{aligned}$$

On the other hand, from Lemma 1.2 we know that v is nonincreasing in t, so that (1.16) yields

$$\begin{aligned} v(R_0,t) &\le v(R_0,T) \\ &\le \delta(R_0 - R)^{\frac{2}{p}} \qquad \text{for all } t \in [0,T] \end{aligned}$$

and thus

$$\bar{v}_\eta(R_0,t) > v(R_0,t) \qquad \text{for all } t \in [0,T]. \tag{1.21}$$

Finally, at $t = 0$ we have

$$\begin{aligned} \frac{v(r,0)}{\bar{v}_\eta(r,0)} &= \frac{M(r-R)^\alpha}{\delta(r-R+\eta)^{\frac{2}{p}}} \\ &< \frac{M(r-R)^\alpha}{\delta(r-R)^{\frac{2}{p}}} \\ &\le \tfrac{M}{\delta}(R_0 - R)^{\alpha - \frac{2}{p}} \\ &\le 1 \qquad \text{for } r \in [R, R_0] \end{aligned} \tag{1.22}$$

due to (1.15) and the fact that $\alpha \ge \frac{1}{p-1} > \frac{2}{p}$.
Since the inequalities (1.20)-(1.22) on the parabolic boundary of $(R, R_0) \times (0,T)$ are strict, we may invoke the comparison principle for the degenerate parabolic inequality (1.19) (see [Wie2] for an appropriate version) to conclude that $v \le \bar{v}_\eta$ holds for all $r \in [R, R_0],\ t \in [0,T]$ and $\eta \in (0,R)$. In particular, by (1.18), in the limit $\eta \to 0$ we find

$$\begin{aligned} v(r,t) &\le y(T) \cdot (r-R)^{\frac{2}{p}} \\ &\le (r-R)^{\frac{2}{p}} \qquad \text{for all } r \in [R, R_0] \text{ and } t \in [0,T], \end{aligned}$$

while for $r \in [R_0, R_1]$ we trivially have

$$\frac{v(r,t)}{(r-R)^{\frac{2}{p}}} \le \frac{M(R_1 - R)^\alpha}{(R_0 - R)^{\frac{2}{p}}}.$$

Accordingly, (1.13) follows if we set $C := \max\left\{1, \frac{M(R_1-R)^\alpha}{(R_0-R)^{\frac{2}{p}}}\right\}$. ////

1.2 Upper estimate for subsolutions of (0.4)

In this section we shall use the previously constructed radial functions in order to estimate arbitrary *sub*solutions of (0.4) from above. Admitting subsolutions here rather than restricting ourselves to solutions will enable us to easily cope with the perturbed problem (0.1) in Section 1.3.

We first assert that if $p \in (1,2)$ then (0.2) implies that each corresponding subsolution decays at least like $(\mathrm{dist}\,(x,\partial\Omega))^{\frac{2}{p}}$ near $\partial\Omega$ for $t>0$. Observe that since $\frac{2}{p} > \frac{1}{p-1}$, this is weaker than the desired conclusion (0.3).

Lemma 1.5 *Let $p \in (1,2)$, $T>0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^3. Suppose that u is a nonnegative classical subsolution of (0.4) with $u|_{t=0} = u_0$, where*

$$u_0(x) \le c_1 (\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \tag{1.23}$$

with certain constants $\alpha \ge \frac{1}{p-1}$ and $c_1 > 0$. Then there exists $c>0$ with

$$u(x,t) \le c \cdot (\mathrm{dist}\,(x,\partial\Omega))^{\frac{2}{p}} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T). \tag{1.24}$$

Proof. From the assumed smoothness of $\partial\Omega$ it follows that there exists $R>0$ such that for all $y \in \partial\Omega$ one can find $x_c(y) \in \mathbb{R}^n \setminus \Omega$ with $\bar{B}_R(x_c) \cap \bar{\Omega} = \{y\}$. Moreover, there is $d>0$ with the property that for any $x \in \Omega$ with $\mathrm{dist}\,(x,\partial\Omega) < d$ there exists exactly one point $y(x) \in \partial\Omega$ such that $\mathrm{dist}\,(x,\partial\Omega) = |x-y(x)|$ (cf. [Se], [AP]).

Now let $x_0 \in \Omega$ satisfy $\mathrm{dist}\,(x_0,\partial\Omega) < d$, and set $y := y(x_0)$ and $x_c := x_c(y)$. Then for all $x \in B_{R+d}(x_c) \cap \Omega$, the point $\bar{y}(x) := x_c + R \cdot \frac{x-x_c}{|x-x_c|}$ lies on $\partial B_R(x_c)$ and thus outside Ω, so that

$$\begin{aligned} \mathrm{dist}\,(x,\partial\Omega) &= \mathrm{dist}\,(x,\mathbb{R}^n \setminus \Omega) \\ &\le |x - \bar{y}(x)| \\ &= |x-x_c| - R \qquad \text{for all } x \in B_{R+d}(x_c) \cap \Omega. \end{aligned} \tag{1.25}$$

Let v_ε and $v := \lim_{\varepsilon \to 0} v_\varepsilon$ denote the solutions of (1.2) and (1.1), respectively, with $R_1 := R+d$ and $M := \max\{c_1 \cdot (R_1-R)^\alpha, \|u_0\|_{L^\infty(\Omega)}\}$, as constructed in Lemma 1.2). Then on $(\partial B_{R_1}(x_c) \cap \bar{\Omega}) \times [0,T]$, we have $v_\varepsilon > v \ge M \ge u$, because $u \le \|u_0\|_{L^\infty(\Omega)}$ by comparison. If $x \in \bar{B}_{R_1}(x_c) \cap \partial\Omega$, however, then trivially $v_\varepsilon(x,t) > 0 = u(x,t)$ for all $t \in [0,T]$. Furthermore, for $t=0$ and any $x \in \bar{B}_{R_1} \cap \bar{\Omega}$, by (1.25) we find

$$v_\varepsilon(x,0) > v(x,0) = \tfrac{M}{(R_1-R)^\alpha}(|x-x_c|-R)^\alpha \ge c_1(|x-x_c|-R)^\alpha \ge u_0(x),$$

whereby we have shown that the strict inequality $v_\varepsilon > u$ holds on the parabolic boundary of $\Omega_0 \times (0,T)$, where $\Omega_0 := B_{R_1}(x_c) \cap \Omega$. Therefore parabolic comparison yields $v_\varepsilon \ge u$ in $\Omega_0 \times (0,T)$. Thus, taking $\varepsilon \to 0$ and recalling Lemma 1.4, we see that

$$u(x,t) \le v(x,t) \le c \cdot (|x-x_c| - R)^{\frac{2}{p}} \qquad \text{for all } x \in \Omega_0 \text{ and } t \in (0,T)$$

holds with some constant c depending on T, p, α, M, R and R_1 only. Since $|x_0 - x_c| \le |x_0 - y(x_0)| + |y(x_0) - x_c| = \mathrm{dist}\,(x_0,\partial\Omega) + R$ in view of the definition of $y(x_0)$, this in particular entails

$$u(x_0,t) \le c \cdot (|x_0 - x_c| - R)^{\frac{2}{p}} \le c \cdot (\mathrm{dist}\,(x_0,\partial\Omega))^{\frac{2}{p}} \qquad \text{for all } t \in (0,T)$$

with some c independent of x_0. This immediately gives (1.24). ////

We next claim that the estimate gained in the last lemma in conjunction with (0.2) entails the desired result (0.3). In fact, this can be proved for any $p > 1$ by regarding the inequality $u_t \le u^p \Delta u$ as a *linear* parabolic inequality with a variable coefficient $u^p(x,t)$ satisfying an appropriate decay condition near $\partial\Omega$.

Lemma 1.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^3-domain, let $p > 1$, $T > 0$ and suppose that u is a nonnegative clasical subsolution of (0.4) with initial data $u|_{t=0} = u_0$ satisfying*

$$u_0(x) \le c_1(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \textit{for all } x \in \Omega \tag{1.26}$$

with some $c_1 > 0$ and $\alpha \ge 1$ such that $\alpha \ge \frac{1}{p-1}$. Assume furthermore that there exists $c_2 > 0$ such that

$$u(x,t) \le c_2(\mathrm{dist}\,(x,\partial\Omega))^{\frac{2}{p}} \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T). \tag{1.27}$$

Then

$$u(x,t) \le c_3(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T) \tag{1.28}$$

holds with a sufficiently large constant $c_3 > 0$.

Proof. We let $\varphi \in C^1(\bar\Omega) \cap C^2(\Omega)$ denote any function satisfying $\Delta\varphi \le 0$ in Ω and

$$c_\varphi \cdot \mathrm{dist}\,(x,\partial\Omega) \le \varphi(x) \le C_\varphi \cdot \mathrm{dist}\,(x,\partial\Omega) \qquad \text{for all } x \in \Omega$$

with positive constants c_φ and C_φ. (For instance, due to the smoothness of $\partial\Omega$ this is true if we take φ to be the solution of $-\Delta\varphi = 1$ in Ω with $\varphi|_{\partial\Omega} = 0$.)
We set

$$y(t) := y_0\, e^{\kappa t}, \qquad t \ge 0, \tag{1.29}$$

where $y_0 := \frac{c_1}{c_\varphi^\alpha} > 0$ and $\kappa := \alpha(\alpha-1) \cdot \frac{c_2^p}{c_\varphi^2} \cdot \|\nabla\varphi\|^2_{L^\infty(\Omega)} \ge 0$, and set

$$\bar u(x,t) := y(t) \cdot \varphi^\alpha(x), \qquad x \in \bar\Omega,\ t \in [0,T].$$

Then at $t = 0$ we have, by (1.26),

$$\begin{aligned}
\frac{u(x,0)}{\bar u(x,0)} &\le \frac{c_1(\mathrm{dist}\,(x,\partial\Omega))^\alpha}{y_0 \cdot \varphi\alpha(x)} \\
&< \frac{c_1}{y_0 \cdot c_\varphi^\alpha} \\
&= 1 \qquad \text{for all } x \in \Omega,
\end{aligned}$$

whereas evidently $u = 0 \le \bar u$ on $\partial\Omega \times (0,T)$. Furthermore, from the assumption (1.27) we obtain

$$\begin{aligned}
u^p(x,t) &\le c_2^p(\mathrm{dist}\,(x,\partial\Omega))^2 \\
&\le \frac{c_2^p}{c_\varphi^2} \cdot \varphi^2(x) \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T).
\end{aligned}$$

Using this, we obtain that $\bar{u}$ satisfies the linear parabolic inequality

$$\begin{aligned}
\bar{u}_t - u^p \Delta \bar{u} &= y' \cdot \varphi\alpha \\
&\quad - u^p \cdot y \cdot \Big[\alpha \varphi^{\alpha-1} \Delta\varphi + \alpha(\alpha-1)\varphi^{\alpha-2} |\nabla\varphi|^2 \Big] \\
&\geq y' \cdot \varphi^\alpha - \alpha(\alpha-1) u^p \varphi^{\alpha-2} |\nabla\varphi|^2 y \\
&\geq y' \cdot \varphi^\alpha - \alpha(\alpha-1) \cdot \frac{c_2^p}{c_\varphi^2} \cdot \varphi^\alpha |\nabla\varphi|^2 y \\
&\geq \Big\{ y' - \alpha(\alpha-1) \frac{c_2^p}{c_\varphi^2} \cdot \|\nabla\varphi\|^2_{L^\infty(\Omega)} \dot{y} \Big\} \cdot \varphi^\alpha \quad \text{in } \Omega \times (0,T).
\end{aligned}$$

Therefore the maximum principle implies

$$\begin{aligned}
u(x,t) &\leq \bar{u}(x,t) \\
&\leq y_0 \, e^{\kappa T} \varphi^\alpha(x) \\
&\leq y_0 \, e^{\kappa T} \cdot C_\varphi^\alpha \cdot (\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T),
\end{aligned}$$

which yields (1.28). ////

Combining the previous two lemmata with an additional argument for $p \geq 2$ (which essentially replaces Lemma 1.5 for such p), we obtain the desired conclusion for arbitrary classical subsolutions of (0.4).

Corollary 1.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^3. Suppose that $p > 1$ and $T > 0$, and that u is a nonnegative classical subsolution of (0.4) with $u|_{t=0} = u_0$, where*

$$u_0(x) \leq c_1 (\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \textit{for all } x \in \Omega \tag{1.30}$$

with some $c_1 > 0$ and some $\alpha \geq 1$ satisfying $\alpha \geq \frac{1}{p-1}$. Then there exists $C = C(T) > 0$ such that

$$u(x,t) \leq C (\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T). \tag{1.31}$$

PROOF. If $p \in (1,2)$, we apply Lemma 1.5 and then Lemma 1.6 to obtain (1.31) immediately. In the case $p \geq 2$, the claim results from Lemma 1.6 as soon as we have checked that the hypothesis (1.27) of this lemma is satisfied. To see this, as in the proof of Lemma 1.6 we once again pick any $\varphi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ with the properties $\Delta\varphi \leq 0$ in Ω and $c_\varphi \mathrm{dist}\,(x,\partial\Omega) \leq \varphi(x) \leq C_\varphi \mathrm{dist}\,(x,\partial\Omega)$ throughout Ω, where $0 < c_\varphi < C_\varphi$. Since (1.30) and our assumption on α implies that $u_0(x) \leq \tilde{c}_1 \mathrm{dist}\,(x,\partial\Omega)$ for some $\tilde{c}_1 \geq c_1$, we obtain $u_0 \leq a\varphi$ in Ω for $a := \frac{\tilde{c}_1}{c_\varphi}$. Therefore u does not exceed $\bar{u}(x,t) := a\varphi(x)$ on the parabolic boundary of $\Omega \times (0,T)$ for any $\eta > 0$. In conjunction with the linear parabolic inequality

$$\bar{u}_t - u^p \Delta \bar{u} = -u^p \cdot a \Delta\varphi \geq 0 \qquad \text{in } \Omega \times (0,T),$$

this yields $u \leq \bar{u}$ in $\Omega \times (0,T)$. Hence,

$$u(x,t) \leq a C_\varphi \mathrm{dist}\,(x,\partial\Omega) \leq c_2 (\mathrm{dist}\,(x,\partial\Omega))^{\frac{2}{p}} \qquad \text{for } x \in \Omega \text{ and } t \in (0,T)$$

holds with $c_2 := a C_\varphi \cdot \max_{x \in \Omega} (\mathrm{dist}\,(x,\partial\Omega))^{\frac{p-2}{p}}$, so that Lemma 1.6 may be applied to assert (1.28), that is, (1.31). ////

1.3 Boundary behavior: the main results

Using a simple transformation and the results obtained above, we can now easily prove our main statement on boundary decay of solutions to the full problem (0.1).

Theorem 1.8 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^3, and $p > 1$. Let u be a nonnegative classical solution of (0.1) in $\Omega \times (0, T)$, where*

$$u_0(x) \le c_1(\mathrm{dist}\,(x, \partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \tag{1.32}$$

with some $c_1 > 0$ and some $\alpha \ge 1$ satisfying $\alpha \ge \frac{1}{p-1}$. Then for all $T' < T$, there exists $C(T') > 0$ such that

$$u(x,t) \le C(T')(\mathrm{dist}\,(x, \partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T'). \tag{1.33}$$

Proof. As a classical solution, u is bounded for $t < T'$, say, $u \le M$ in $\Omega \times (0, T')$. Letting $L > 0$ denote a Lipschitz constant for g in the interval $[0, M]$, we obtain from $g(0) = 0$ that $g(s) \le Ls$ for all $s \in [0, M]$ and in particular $g(u) \le Lu$ in $\Omega \times (0, T')$. Now the substitution

$$u(x,t) = e^{Lt} \cdot z(x,s), \qquad x \in \Omega, \quad s := \tfrac{1}{pL}\,(e^{pLt} - 1) \in (0, S'),$$

where $S' := \frac{1}{pL}\,(e^{pLT'} - 1)$, transforms the PDE in (0.1) into the equation

$$\begin{aligned} e^{(p+1)Lt} z_s + L\,e^{Lt} z &= u_t \\ &= u^p \Delta u + g(u) \\ &\le u^p \Delta u + Lu \\ &= e^{(p+1)Lt} z^p \Delta z + L\,e^{Lt} z, \qquad x \in \Omega, s \in (0, S') \end{aligned}$$

which is equivalent to the inequality

$$z_s \le z^p \Delta z \qquad \text{in } \Omega \times (0, S').$$

Since $z(\cdot, 0) \equiv u_0$, Corollary 1.7 says that the assumption (1.32) in fact entails (1.33). ////

For the sake of completeness, let us briefly demonstrate that a corresponding *lower* bound for the boundary decay is preserved for any parameter $\alpha \ge 1$. The proof of the following lemma relies on none of the previous results.

Lemma 1.9 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^3$, and $p > 0$. Let u be a nonnegative classical solution of (0.1) in $\Omega \times (0, T)$, where*

$$u_0(x) \ge c_0(\mathrm{dist}\,(x, \partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \tag{1.34}$$

with some $c_0 > 0$ and $\alpha \ge 1$. Then for all $T' < T$ there exists $C(T') > 0$ such that

$$u(x,t) \ge C(T')(\mathrm{dist}\,(x, \partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T'). \tag{1.35}$$

Proof. Given $T' \in (0, T)$, we let $M := \|u\|_{L^\infty(\Omega \times (0,T'))}$ and $L := \|g'\|_{L^\infty((0,M))}$. Then $u_t \ge u^p \Delta u - Lu$ in $\Omega \times (0, T')$, since $g(0) = 0$. In order to construct an appropriate subsolution, we let Θ denote a positive Dirichlet eigenfunction of $-\Delta$ in Ω corresponding to the principal eigenvalue $\lambda_1 > 0$, and note that due to the smoothness of $\partial\Omega$ we have $c_\Theta \mathrm{dist}\,(x,\partial\Omega) \le \Theta(x) \le C_\Theta \mathrm{dist}\,(x,\partial\Omega)$ for all $x \in \Omega$ with positive constants c_Θ and C_Θ. We let

$$\underline{u}(x,t) := y(t)\cdot\Theta^\alpha(x), \qquad x\in\bar\Omega,\ t\in[0,T'],$$

where $y(t) := y_0\, e^{-\kappa t}$ with $y_0 := \frac{c_0}{C_\Theta^\alpha}$ and $\kappa := \lambda_1\alpha M^p + L$. Then it is easy to see that $\underline{u} \le u$ at $t=0$ and on $\partial\Omega$, and since $\underline{u}$ satisfies the linear parabolic inequality

$$\begin{aligned}
\underline{u}_t - u^p\Delta\underline{u} + L\underline{u} &= y'\Theta^\alpha - u^p y\cdot[\alpha\Theta^{\alpha-1}\Delta\Theta + \alpha(\alpha-1)\Theta^{\alpha-2}|\nabla\Theta|^2] + Ly\Theta^\alpha \\
&\le y'\Theta^\alpha + \lambda_1\alpha u^p y\Theta^\alpha + Ly\theta^\alpha \\
&= [y' + \lambda_1\alpha M^p y + Ly]\Theta^\alpha \\
&= 0 \qquad \text{in } \Omega\times(0,T'),
\end{aligned}$$

we conclude by the comparison principle that $\underline{u} \le u$ in $\Omega\times(0,T')$. Accordingly, (1.35) holds if we set $C(T') := y_0\, e^{-\kappa T'} c_\Theta^\alpha$. ////

2 Consequences for the large time asymptotics

2.1 Solutions with nonconvergent bounded gradients for $p > 1$

In this section we specialize on the problem

$$\begin{cases} u_t = u^p u_{xx} + u^q & \text{in } \Omega\times(0,\infty), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{2.1}$$

in the one-dimensional domain $\Omega = (0,L)$, where $p > 1$ and $q \ge 1$ is such that $q \in (p-1,p+1)$. The initial data u_0 are now assumed to be positive in Ω and to vanish on $\partial\Omega$.
The following preliminary result on global existence of a classical solution can be derived using standard methods (cf. [Win2] or [Wie2], for instance).

Lemma 2.1 *Let $p > 1$, $q \in [1,p+1)$ and $u_0 \in C^0(\bar\Omega)$ be positive in Ω with $u_0|_{\partial\Omega} = 0$. Then (2.1) has a unique global positive classical solution $u \in C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\Omega\times(0,\infty))$. Moreover, if $(u_{0k})_{k\in\mathbb{N}} \subset C^0(\bar\Omega)$ is a sequence of functions that are positive in Ω and satisfy $u_{0k} \to u_0$ in $C^0(\bar\Omega)$ then the approximating problems*

$$\begin{cases} u_{kt} = u_k^p u_{kxx} + u_k^q & \textit{in } \Omega\times(0,\infty), \\ u_k|_{\partial\Omega} = u_{0k}|_{\partial\Omega}, \\ u_k|_{t=0} = u_{0k}, \end{cases} \tag{2.2}$$

have global classical solutions u_k with

$$u_k \to u \qquad \textit{in } C^0_{loc}(\bar\Omega\times[0,\infty)) \cap C^{2,1}_{loc}(\Omega\times(0,\infty)).$$

Concerning the large time behavior of classical solutions, we recall the following result from [Win2]. It states that if $q < p+1$ then *all* classical solutions of (2.1) approach a uniquely determined continuous steady state as $t \to \infty$. If $q > p-1$ then this stationary solution even belongs to $C^{1+\frac{q+1-p}{2}}(\bar\Omega)$. For a detailed analysis of these steady states, we refer the reader to [Wie2].

Lemma 2.2 *Let $p > 1$ and $q \ge 1$ be such that $q \in (p-1, p+1)$, and assume that $u_0 \in C^0(\bar\Omega)$ is positive in Ω with $u_0|_{\partial\Omega} = 0$. Then the solution u of (2.1) satisfies*

$$u(\cdot,t) \to w \qquad \text{in } C^0(\bar\Omega) \qquad \text{as } t \to \infty,$$

where $w \in C^{1+\frac{q+1-p}{2}}(\bar\Omega) \cap C^2(\Omega)$ is the unique positive solution of the singular elliptic problem

$$\begin{cases} -w_{xx} = w^{q-p} & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases} \tag{2.3}$$

In our construction we shall consider a solution u approaching the above steady state monotonically from below. To this end, we first construct suitable initial data.

Lemma 2.3 *Let $p > 1$ and $q \in (p-1, p+1)$. Then for all $\alpha > 1$ there exist $c_0 > 0, c_1 > 0, \vartheta \in (0,1)$ and $u_0 \in C^{1+\vartheta}(\bar\Omega) \cap C^\infty(\Omega)$ such that*

$$c_0(\operatorname{dist}(x,\partial\Omega))^\alpha \le u_0(x) \le c_1(\operatorname{dist}(x,\partial\Omega))^\alpha \qquad \text{for all } x \in \Omega \tag{2.4}$$

and

$$u_{0xx} + u_0^{q-p} > 0 \qquad \text{in } \Omega. \tag{2.5}$$

Proof. According to [Wie2], due to the fact that $q - p \in (-1,1)$ the solution w of (2.3) belongs to $C^{1+\vartheta'}(\bar\Omega)$ for some $\vartheta' \in (0,1)$ and satisfies

$$\tilde c_0 \operatorname{dist}(x,\partial\Omega) \le w(x) \le \tilde c_1 \operatorname{dist}(x,\partial\Omega) \qquad \text{for all } x \in \Omega \tag{2.6}$$

with positive constants c_0 and c_1. Picking a small number $A > 0$ such that

$$\alpha A^{p+1-q} \|w\|_{L^\infty(\Omega)}^{(p+1-q)(\alpha-1)} < 1,$$

we set

$$u_0(x) := A w^\alpha(x), \qquad x \in \bar\Omega.$$

Then, since $\alpha > 1$, both u_0 and u_{0x} are Hölder continuous in $\bar\Omega$, and (2.4) immediately results from (2.6). Moreover, we have

$$\begin{aligned} u_{0xx} + u_0^{q-p} &= A\Big(\alpha w^{\alpha-1} w_{xx} + \alpha(\alpha-1) w^{\alpha-2} w_x^2\Big) + A^{q-p} w^{(q-p)\alpha} \\ &\ge -A\alpha w^{\alpha-1} \cdot w^{q-p} + A^{q-p} w^{(q-p)\alpha} \\ &= \Big(-\alpha A^{p+1-q} \cdot w^{(p+1-q)(\alpha-1)} + 1 \Big) \cdot A^{q-p} w^{(q-p)\alpha} \\ &> 0 \qquad \text{in } \Omega \end{aligned}$$

by definition of A, whereby (2.5) has been shown. ////

Next, we assert that such initial data in fact lead to solutions that are nondecreasing with time.

Lemma 2.4 *Let $p > 1$ and $q \in [1, p+1)$, and assume that $u_0 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ is positive in Ω and satisfies $u_0|_{\partial\Omega} = 0$ as well as $u_{0xx} + u_0^{q-p} > 0$ in Ω. Then the solution u of (2.1) satisfies $u_t \geq 0$ in $\Omega \times (0, \infty)$.*

Proof. We first claim that for all $\varepsilon \in (0,1)$ there exist $\delta_\varepsilon \in (0, 2\varepsilon]$ and $u_{0\varepsilon} \in C^\infty(\bar{\Omega})$ satisfying

$$\max\{\delta_\varepsilon, u_0 - \varepsilon\} \leq u_{0\varepsilon} \leq u_0 + 3\varepsilon \qquad \text{in } \Omega, \tag{2.7}$$

$$u_{0\varepsilon xx} + u_{0\varepsilon}^{q-p} \geq 0 \qquad \text{in } \Omega \qquad \text{and} \tag{2.8}$$

$$u_{0\varepsilon} \equiv \delta_\varepsilon \qquad \text{in a neighborhood of } \partial\Omega. \tag{2.9}$$

To this end, we begin with the case $q < p$ and let

$$\begin{aligned}
\Omega_\varepsilon &:= \{x \in \Omega \mid u_0(x) > \varepsilon\}, \\
\mu_\varepsilon &:= \min\Big\{1, \min\{u_{0xx}(x) + u_0^{q-p}(x) \mid x \in \bar{\Omega}_\varepsilon\}\Big\}, \\
\nu_\varepsilon &:= \min\Big\{\frac{\mu_\varepsilon}{2}, (2\varepsilon)^{q-p}\Big\} \qquad \text{and} \\
\delta_\varepsilon &:= \min\Big\{\frac{2\varepsilon}{3}, \frac{2\varepsilon}{3} \cdot \Big[(1 - \tfrac{1}{2}\mu_\varepsilon \cdot \varepsilon^{p-q})^{-\frac{1}{p-q}} - 1\Big]\Big\}.
\end{aligned}$$

Since $\bar{u}_{0\varepsilon} := (u_0 - \varepsilon)_+$ satisfies $\bar{u}_{0\varepsilon xx} \geq u_{0xx} \cdot \chi_{\Omega_\varepsilon}$ in the sense of distributions in Ω, we can apply the satndard mollifying procedure to $\bar{u}_{0\varepsilon}$ in order to obtain a nonnegative $\hat{u}_{0\varepsilon} \in C_0^\infty(\Omega)$ fulfilling $u_0 - \varepsilon - \frac{\delta_\varepsilon}{2} \leq \hat{u}_{0\varepsilon} \leq u_0 + \frac{\delta_\varepsilon}{2}$ in Ω and $\hat{u}_{0\varepsilon xx} \geq \bar{u}_{0\varepsilon xx} - \nu_\varepsilon \geq u_{0xx} \cdot \chi_{\Omega_\varepsilon} - \nu_\varepsilon$ in the distributional sense. Thus, the function $u_{0\varepsilon} := \hat{u}_{0\varepsilon} + \delta_\varepsilon$ evidently satisfies (2.9) and

$$\max\{\delta_\varepsilon, u_0 - \varepsilon + \tfrac{\delta_\varepsilon}{2}\} \leq u_{0\varepsilon} \leq u_0 + \tfrac{3}{2}\delta_\varepsilon \qquad \text{in } \Omega, \tag{2.10}$$

which implies (2.7), because $\frac{3}{2}\delta_\varepsilon \leq \varepsilon$. Moreover, if $x \in \Omega_\varepsilon$ then

$$\begin{aligned}
u_{0\varepsilon xx}(x) + u_{0\varepsilon}^{q-p}(x) &= \hat{u}_{0\varepsilon xx}(x) + u_{0\varepsilon}^{q-p}(x) \\
&\geq u_{0xx}(x) - \nu_\varepsilon + u_{0\varepsilon}^{q-p}(x) \\
&\geq \mu_\varepsilon - u_0^{q-p}(x) - \nu_\varepsilon + u_{0\varepsilon}^{q-p}(x) \\
&\geq \frac{\mu_\varepsilon}{2} - u_0^{q-p}(x) + \Big(u_0(x) + \frac{3}{2}\delta_\varepsilon\Big)^{q-p}.
\end{aligned}$$

Since $u_0(x) > \varepsilon$ for such x, by definition of δ_ε we have

$$u_0 q - p(x) - \Big(u_0(x) + \frac{3}{2}\delta_\varepsilon\Big)^{q-p} \leq \varepsilon^{q-p} - \Big(\varepsilon + \frac{3}{2}\delta_\varepsilon\Big)^{q-p} \leq \frac{\mu_\varepsilon}{2}$$

and hence $u_{0\varepsilon xx} + u_{0\varepsilon}^{q-p} \geq 0$ in Ω_ε. On the other hand, if $x \in \Omega \setminus \Omega_\varepsilon$ then (2.10) implies

$$\begin{aligned}
u_{0\varepsilon xx}(x) + u_{0\varepsilon}^{q-p}(x) &\geq -\nu_\varepsilon + u_{0\varepsilon}^{q-p}(x) \\
&\geq -\nu_\varepsilon + (2\varepsilon)^{q-p} \\
&\geq 0
\end{aligned} \tag{2.11}$$

in view of the definition of ν_ε, wich proves (2.8).
In the case $q \geq p$ the procedure is similar but less involved: We take $\Omega_\varepsilon, \mu_\varepsilon$ and ν_ε as before and let $\delta_\varepsilon := 2\varepsilon$ this time. Then the correspondingly constructed function $u_{0\varepsilon}$ again fulfills (2.9), (2.10) and (2.9), so that in particular for $x \in \Omega_\varepsilon$ we have $u_{0\varepsilon}(x) \geq u_0(x)$ and hence

$$\begin{aligned}
u_{0\varepsilon xx}(x) + u_{0\varepsilon}^{q-p}(x) &\geq \mu_\varepsilon - u_0^{q-p}(x) - \nu_\varepsilon + u_{0\varepsilon}^{q-p}(x) \\
&\geq 0.
\end{aligned}$$

If $x \in \Omega \setminus \Omega_\varepsilon$ then (2.11) remains valid, because $u_{0\varepsilon} \ge \delta_\varepsilon = 2\varepsilon$ by (2.10).

We now pick a sequence of numbers $1 > \varepsilon_k \searrow 0$ as $k \to \infty$ and let $u_{0k} := u_{0\varepsilon_k}$. Using these functions as approximations for u_0 in Lemma 2.1, we obtain a corresponding family of solutions u_k of (2.2) which even belong to $C^{2,1}(\bar\Omega \times [0,\infty))$, because (2.7) ensures that (2.2) is actually non-degenerate, whereas (2.9) implies that the compatibility condition of first order is fulfilled for (2.2). Thus, $z := u_{kt}$ is a classical solution of the linear parabolic equation

$$\begin{aligned} z_t &= u_{kt} \\ &= u_k^p z_{xx} + (p u_k^{p-1} u_{kt} + q u_k^{q-1}) z \qquad \text{in } \Omega \times (0,\infty) \end{aligned}$$

satisfying $z|_{\partial\Omega} = 0$ and $z|_{t=0} \ge 0$ in view of (2.8). Hence, from the maximum principle we gain that $u_{kt} \ge 0$ in $\Omega \times (0,\infty)$, which in the limit $k \to \infty$ yields the claim because of the pointwise convergence $u_{kt} \to u_t$ asserted by Lemma 2.1. ////

The fact that $u_t \ge 0$ will now be used to prove that $u(\cdot,t)$ remains uniformly bounded in $W^{1,\infty}(\Omega)$ for all times, provided that u_0 does not decay too fast near $\partial\Omega$.

Lemma 2.5 *Let $p > 1$ and $q \ge 1$ be such that $q \in (p-1, p+1)$. Let $u_0 \in C^0(\bar\Omega) \cap C^2(\Omega)$ be such that $u_{0xx} + u_0^{q-p} > 0$ in Ω. Moreover, assume that $c_0(\operatorname{dist}(x,\partial\Omega))^\alpha \le u_0(x) \le c_1(\operatorname{dist}(x,\partial\Omega))^\alpha$ holds for all $x \in \Omega$ with positive constants c_0, c_1 and $\alpha > 1$ fulfilling $\alpha \ge \frac{1}{p-1}$ and $(p-q)\alpha < 1$. Then there exists $C > 0$ such that the solution u of (2.1) satisfies*

$$|u_x| \le C \qquad \text{in } \Omega \times (0,\infty). \tag{2.12}$$

PROOF. We first claim that our assumptions guarantee that $u^{q-p}(x,t) \le \psi(x)$ holds for all $x \in \Omega, t > 0$ and some function $\psi \in L^1(\Omega)$. Indeed, if $q \ge p$ this directly results from the boundedness of u; in the case $q < p$, we use the fact that $u_t \ge 0$ by Lemma 2.4 to estimate

$$\begin{aligned} u^{q-p}(x,t) &\le u_0^{q-p}(x) \\ &\le c_0^{q-p}(\operatorname{dist}(x,\partial\Omega))^{-(p-q)\alpha} \end{aligned}$$

for all $x \in \Omega$ and $t > 0$. Since $(p-q)\alpha < 1$, the right-hand side belongs to $L^1(\Omega)$.
In order to prove that (2.12) holds with $C := \|\psi\|_{L^1(\Omega)}$, we take an arbitrary $(x,t) \in \Omega \times (0,\infty)$ and first consider the case $u_x(x,t) < 0$. Then, since $u(\cdot,t)$ is positive in Ω and vanishes on $\partial\Omega$, there exists $x_0 \in \Omega$ with $x_0 < x$ such that $u_x(x_0,t) = 0$. Now from Lemma 2.4 we know that $u_t \ge 0$ and thus $u_{xx} \ge -u^{q-p}$ in $\Omega \times (0,\infty)$, so that in particular

$$\begin{aligned} u_x(x,t) &= u_x(x_0,t) + \int_{x_0}^x u_{xx}(y,t)dy \\ &= \int_{x_0}^x u_{xx}(y,t)dy \\ &\ge -\int_{x_0}^x u^{q-p}(y,t)dy \\ &\ge -\|\psi\|_{L^1(\Omega)}. \end{aligned}$$

Combined with a similar reasoning in the case $u_x(x,t) > 0$, this proves (2.12). ////

The boundary behavior of u_x can now be controlled using our knowledge on the boundary behavior of u and an additional scaling argument similar to that demonstrated in [Wie1].

Lemma 2.6 *Let $p>1$ and $q\ge 1$ be such that $q\in(p-1,p+1)$ and $q\ge 3-p$, and let $u_0\in C^{1+\vartheta}(\bar\Omega)\cap C^2(\Omega)$ for some $\vartheta\in(0,1)$. Suppose that $u_{0xx}+u_0^{q-p}>0$ in Ω, and that*

$$c_0(\mathrm{dist}\,(x,\partial\Omega))^\alpha \le u_0(x)\le c_1(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \textit{for all } x\in\Omega \tag{2.13}$$

holds with positive constants c_0 and c_1 and some $\alpha>1$ satisfying $\alpha\ge\frac{1}{p-1}$ and $\alpha\le\frac{2}{p+1-q}$. Then the solution u of (2.1) satisfies

$$u(\cdot,t)\in C^1(\bar\Omega) \qquad \textit{for all } t>0 \tag{2.14}$$

and

$$u_x(\cdot,t)|_{\partial\Omega}=0 \qquad \textit{for all } t>0. \tag{2.15}$$

Remark. The additional restriction $q\ge 3-p$ is made to guarantee that there indeed exist some numbers α complying with the above assumptions.

Proof. Let $t_0>0$. In view of Theorem 1.8 and the monotonicity with respect to t asserted by Lemma 2.4, the assumption (2.13) implies that

$$c_0(\mathrm{dist}\,(x,\partial\Omega))^\alpha \le u(x,t)\le \tilde c_1(\mathrm{dist}\,(x,\partial\Omega))^\alpha \qquad \text{for } x\in\Omega \text{ and } t\in(0,2t_0)$$

holds with some $\tilde c_1\ge c_1$. Thus, fixing $x_0\in\Omega$ and writing $d:=\mathrm{dist}\,(x_0,\partial\Omega)$ and $s_0:=2d^{p\alpha-2}t_0$, we know that

$$v(y,s):=\frac{1}{d^\alpha}\cdot u(x_0+dy,d^{2-p\alpha}s), \qquad y\in(-1,1),\ s\in(0,s_0),$$

satisfies

$$\tfrac{c_0}{2^\alpha}\le v(y,s)\le (\tfrac{3}{2})^\alpha\tilde c_1 \qquad \text{for all } y\in(-\tfrac12,\tfrac12) \text{ and } s\in(0,s_0). \tag{2.16}$$

Moreover, a straightforward calculation shows that v satisfies

$$v_s=v^p v_{yy}+d^{2-(p+1-q)\alpha}v^q \qquad \text{in } (-1,1)\times(0,s_0).$$

Therefore, interior parabolic Schauder estimates in combination with the regularity of u_0, with (2.16) and the fact that $2-(p+1-q)\alpha>0$ provide some $\vartheta'\in(0,\vartheta)$ and a bound C for v in $C^{1+\vartheta',\frac{1+\vartheta'}{2}}([-\frac14,\frac14]\times[0,s_0])$ which does not depend on x_0. In particular, this means that $|u_x(x_0,t_0)|=d^{\alpha-1}|v_y(0,d^{p\alpha-2}t_0)|\le Cd^{\alpha-1}$. Since $\alpha>1$, this implies that $u_x(\cdot,t_0)$ belongs to $C^0(\bar\Omega)$ and vanishes on $\partial\Omega$. ////

Collecting the above facts, we easily obtain the following.

Theorem 2.7 *Suppose $p>1$ and $q\ge1$ is such that $q\in(p-1,p+1)$ and $q\ge 3-p$. Then there exist initial data $u_0\in C^1(\bar\Omega)$ such that the solution u of (2.1) is nondecreasing with respect to t and has the following properties: It satisfies $u(\cdot,t)\in C^1(\bar\Omega)$ for all $t\ge0$,*

$$\|u(\cdot,t)\|_{C^1(\bar\Omega)}\le C \qquad \textit{for all } t\ge 0$$

with some C independent of t, and

$$u(\cdot,t)\to w \qquad \textit{in } C^0(\bar\Omega) \quad \textit{as } t\to\infty,$$

where w denotes the solution of (2.3), but $u_x(\cdot,t)|_{\partial\Omega}\equiv 0$ for $t>0$. In particular, there exists $\delta>0$ such that

$$\|u(\cdot,t)-w\|_{C^1(\bar\Omega)}\ge\delta \qquad \textit{for all } t\ge 0.$$

PROOF. As mentioned above (cf. the remark following Lemma 2.6), since $q \geq 3-p$, we have $\frac{1}{p-1} \leq \frac{2}{p+1-q}$, and from the assumption $q > p-1$ we gain that $\frac{2}{p+1-q} > 1$. Therefore there exists $\alpha > 1$ such that $\alpha \geq \frac{1}{p-1}$ and $\alpha \leq \frac{2}{p+1-q}$. We now take u_0 as provided by Lemma 2.3 and immediately obtain from Lemma 2.4, Lemma 2.5 and Lemma 2.6 that the corresponding solution u has the desired properties. ////

2.2 Boundedness in $W^{1,\infty}(\Omega)$ implies precompactness in $C^1(\bar{\Omega})$ for $p < 1$

Let us finally illustrate that the observed phenomenon of boundedness in $C^1(\bar{\Omega})$ without precompactness in $C^1(\bar{\Omega})$ must in fact be due to the strong degeneracy in (2.1). In order to see that the above result is sharp in this direction, we shall detect the degeneracy measuring parameter p to be critical at $p = 1$ in this respect, even when very general nonlinear sources are taken into account.
To be more precise, let us consider the problem

$$\begin{cases} u_t = u^p u_{xx} + g(x,t,u,u_x) & \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{2.17}$$

where u_0 and g are nonnegative functions with $u_0 \in C^0(\bar{\Omega})$, $u_0|_{\partial\Omega} = 0$ and $g \in C^{\vartheta}_{loc}(\bar{\Omega} \times [0,\infty) \times [0,\infty) \times \mathbb{R})$ for some $\vartheta > 0$. Our goal is to show that if $p < 1$ then a result similar to Theorem 2.7 cannot be found; that is, we wish to prove that if $u(\cdot,t)$ remains bounded in $C^1(\bar{\Omega})$ (or, more generally, in $W^{1,\infty}(\Omega)$) then $u(\cdot,t_k)$ must converge in $C^1(\bar{\Omega})$ at least along some sequence of times $t_k \to \infty$.
Since our focus is on solutions which are (eventually) bounded in $W^{1,\infty}(\Omega)$, a natural assumption on g seems to be that g be bounded. And in fact, this rather mild requirement is already sufficient for our purpose. In order to circumvent any difficulties about existence and uniqueness within various concept of weak solutions of (2.17), we concentrate here on the 'viscosity limit' $u := \lim_{\varepsilon \searrow 0} u_\varepsilon$ of the family of solutions u_ε of the approximate problems

$$\begin{cases} u_{\varepsilon t} = u_\varepsilon^p u_{\varepsilon xx} + g(x,t,u_\varepsilon,u_{\varepsilon x}) & \text{in } \Omega \times (0,\infty), \\ u_\varepsilon|_{\partial\Omega} = \varepsilon, \\ u_\varepsilon|_{t=0} = u_0 + \varepsilon, \end{cases} \tag{2.18}$$

for $\varepsilon \in (0,1)$. Since u_ε evidently decreases when ε decreases, the limit u exists in the pointwise sense and defines a nonnegative upper semicontinuous function which in a large number of situations indeed is a solution (in an appropriate sense) of (2.17) (see [ACP], [A], [B], [Wie2] or [FPS], for instance).
Our main result concerning (2.17) reads as follows.

Lemma 2.8 *Suppose $p \in (0,1)$ and $g \in L^\infty(\Omega \times (0,\infty) \times (0,\infty) \times \mathbb{R})$ is nonnegative and locally Hölder continuous in $\bar{\Omega} \times [0,\infty) \times [0,\infty) \times \mathbb{R}$. Assume that $u := \lim_{\varepsilon \searrow 0} u_\varepsilon$ satisfies*

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq M \qquad \textit{for all } t \geq t_0 \tag{2.19}$$

with positive constants t_0 and M. Then eihter $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$, or there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset (t_0\infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$(u(\cdot,t_k))_{k\in\mathbb{N}} \quad \textit{is relatively compact in } C^1(\bar\Omega). \tag{2.20}$$

Remark. We require local Hölder continuity of g only in order to ensure that the approximate problems (2.18) are classically solvable.

Proof. If $u(\cdot,t)$ does not converge to zero as $t \to \infty$, then (2.19) and the Arzelà-Ascoli theorem provide a sequence $\tilde t_k \to \infty$ and a nonnegative nontrivial $w \in C^0(\bar\Omega)$ such that $u(\cdot,\tilde t_k) \geq w$ in Ω for all $k \in \mathbb{N}$. Since $u_\varepsilon \geq u$ and $g \geq 0$, the comparison principle ensures that for all $\varepsilon > 0$ we have

$$u_\varepsilon(x,\tilde t_k + s) \geq U_\varepsilon(x,s) \qquad \text{for all } x \in \Omega \text{ and } s \geq 0, \tag{2.21}$$

where U_ε denotes the positive classical solution of

$$\begin{cases} U_{\varepsilon s} = U_\varepsilon^p U_{\varepsilon xx} & \text{in } \Omega \times (0,\infty), \\ U_\varepsilon|_{\partial\Omega} = \varepsilon, \\ U_\varepsilon|_{s=0} = w + \varepsilon. \end{cases}$$

By [BP], the (weak) solution $U := \lim_{\varepsilon\searrow 0} U_\varepsilon$ of the corresponding porous medium equation satisfies

$$U(x,s) \geq c_U \cdot \operatorname{dist}(x,\partial\Omega) \qquad \text{for all } x \in \Omega \text{ and } s \in (s_0, s_0+1)$$

with some $c_U > 0$ and $s_0 > 0$. Hence, (2.21) entails

$$u_\varepsilon(x,t) \geq c_U \cdot \operatorname{dist}(x,\partial\Omega) \qquad \text{for all } x \in \Omega \text{ and } t \in (\hat t_k, \hat t_k + 1) \tag{2.22}$$

with $\hat t_k := \tilde t_k + s_0$.
We now multiply (2.18) by $-u_{\varepsilon xx}$ and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_{\varepsilon x}^2 + \int_\Omega u_\varepsilon^p u_{\varepsilon xx}^2 = -\int_\Omega g(x,t,u_\varepsilon,u_{\varepsilon x})u_{\varepsilon xx}$$

for $t > 0$, where we have used that $u_{\varepsilon t}$ vanishes on $\partial\Omega$ for $t > 0$. By Young's inequality,

$$\begin{aligned} -\int_\Omega g(x,t,u_\varepsilon,u_{\varepsilon x}) &\leq \frac{1}{2}\int_\Omega u_\varepsilon^p u_{\varepsilon xx}^2 + \frac{1}{2}\int_\Omega \frac{g^2(x,t,u_\varepsilon,u_{\varepsilon x})}{u_\varepsilon^p} \\ &\leq \frac{1}{2}\int_\Omega u_\varepsilon^p u_{\varepsilon xx}^2 + \frac{G^2}{2}\int_\Omega u_\varepsilon^{-p}, \end{aligned}$$

where G is an upper bound for the function g. By (2.22), however,

$$\int_\Omega u_\varepsilon^{-p}(x,t)dx \leq c_U^{-p}\int_\Omega (\operatorname{dist}(x,\partial\Omega))^{-p}dx \qquad \text{for all } t \in (\hat t_k, \hat t_k + 1),$$

and since $p < 1$ we thus have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_{\varepsilon x}^2 + \frac{1}{2}\int_\Omega u_{\varepsilon xx}^2 \leq c_1 \qquad \text{for all } t \in (\hat t_k, \hat t_k + 1) \tag{2.23}$$

with some $c_1 > 0$. Now since $u_\varepsilon|_{\partial\Omega} = \varepsilon$ and $u_\varepsilon \ge \varepsilon$ in Ω by comparison, another integration by parts yields

$$\begin{aligned}\int_\Omega u_{\varepsilon x}^2 &= \int_\Omega (u_\varepsilon - \varepsilon)_x \cdot u_{\varepsilon x} = -\int_\Omega (u_\varepsilon - \varepsilon)\cdot u_{\varepsilon xx} \\ &\le \Big(\int_\Omega (u_\varepsilon - \varepsilon)^p u_{\varepsilon xx}^2\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega (u_\varepsilon - \varepsilon)^{2-p}\Big)^{\frac{1}{2}} \qquad \text{for } t > 0. \end{aligned} \tag{2.24}$$

Here, by Hölder's and Poincaré's inequalities we have

$$\Big(\int_\Omega (u_\varepsilon - \varepsilon)^{2-p}\Big)^{\frac{1}{2}} \le |\Omega|^{\frac{p}{4}} \Big(\int_\Omega (u_\varepsilon - \varepsilon)^2\Big)^{\frac{2-p}{4}} \le |\Omega|^{\frac{p}{4}} \cdot (\tfrac{|\Omega|}{\pi})^{\frac{2-p}{2}} \Big(\int_\Omega u_{\varepsilon x}^2\Big)^{\frac{2-p}{4}},$$

so that from (2.24) we obtain

$$\int_\Omega u_{\varepsilon x}^2 \le c_2 \Big(\int_\Omega u_\varepsilon^p u_{\varepsilon xx}^2\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega u_{\varepsilon x}^2\Big)^{\frac{2-p}{4}} \qquad \text{for } t > 0,$$

that is,

$$\int_\Omega u_\varepsilon^p u_{\varepsilon xx}^2 \ge c_3 \Big(\int_\Omega u_{\varepsilon x}^2\Big)^{\frac{2+p}{2}} \qquad \text{for } t > 0$$

with positive constants c_2 and c_3. Inserted into (2.23), this shows that the function $y(t) := \int_\Omega u_{\varepsilon x}^2(x,t)dx$ satisfies

$$y'(t) \le 2c_1 - c_3 y^{\frac{2+p}{2}} \qquad \text{for all } t \in (\hat{t}_k, \hat{t}_k + 1).$$

Since $z(t) := A(t - \hat{t}_k)^{-\frac{2}{p}}$ satisfies

$$\begin{aligned} z' - 2c_1 + c_3 z^{\frac{2+p}{2}} &= (\tfrac{2A}{p} + c_3 A^{\frac{2+p}{2}})(t - \hat{t}_k)^{-\frac{2+p}{p}} - 2c_1 \\ &\ge 0 \qquad \text{for all } t \in (\hat{t}_k, \hat{t}_k + 1) \end{aligned}$$

whenever $A \ge \max\{(\frac{4}{pc_3})^{\frac{p}{2}}, (\frac{4c_1}{c_3})^{\frac{p}{2+p}}\}$, it results from an ODE comparison argument that $y \le z$ in $(\hat{t}_k, \hat{t}_k + 1)$, whence in particular

$$y(t) \le c_4 \qquad \text{for all } t \in (\hat{t}_k + \tfrac{1}{2}, \hat{t}_k + 1)$$

holds with some $c_4 > 0$. Now an integration of (2.23) yields

$$\int_{\hat{t}_k + \frac{1}{2}}^{\hat{t}_k + 1} \int_\Omega u_\varepsilon^p u_{\varepsilon xx}^2 \le c_1 + c_4,$$

which together with (2.21) entails

$$\int_{\hat{t}_k + \frac{1}{2}}^{\hat{t}_k + 1} \int_\Omega (\operatorname{dist}(x, \partial\Omega))^p u_{\varepsilon xx}^2 \le c_5$$

with some $c_5 > 0$. As a consequence, the convergence $u_\varepsilon \to u$ takes place in the weak topology of $L^2((\hat{t}_k + \frac{1}{2}, \hat{t}_k + 1); W^{2,2}(\Omega'))$ for all $\Omega' \subset\subset \Omega$, and

$$\int_{\hat{t}_k + \frac{1}{2}}^{\hat{t}_k + 1} \int_\Omega (\operatorname{dist}(x, \partial\Omega))^p u_{xx}^2 \le c_5.$$

In particular, there exists $t_k \in (\hat{t}_k + \frac{1}{2}, \hat{t}_k + 1)$ such that $u_{xx}(\cdot, t_k) \in W^{2,2}_{loc}(\Omega)$ and

$$\int_\Omega (\mathrm{dist}\,(x, \partial\Omega))^p u_{xx}^2(x, t_k) dx \le 2c_5.$$

At this time t_k, for all $x_1, x_2 \in \Omega$ with $x_1 < x_2$ we find

$$\begin{aligned} |u_x(x_2, t_k) - u_x(x_1, t_k)| &= \Big| \int_{x_1}^{x_2} u_{xx}(x, t_k) dx \Big| \\ &\le \Big(\int_\Omega (\mathrm{dist}\,(x, \partial\Omega))^p u_{xx}^2(x, t_k) dx \Big)^{\frac{1}{2}} \cdot \Big(\int_{x_1}^{x_2} (\mathrm{dist}\,(x, \partial\Omega))^{-p} dx \Big)^{\frac{1}{2}}. \end{aligned}$$

Since it can easily be checked that $\int_{x_1}^{x_2} (\mathrm{dist}\,(x, \partial\Omega))^{-p} dx \le c_6 |x_2 - x_1|^{1-p}$ with suitably large c_6, we thus obtain

$$|u_x(x_2, t_k) - u_x(x_1, t_k)| \le \sqrt{2c_5 c_6} \cdot |x_2 - x_1|^{\frac{1-p}{2}}$$

and thereby have proved that $u(\cdot, t_k)$ is bounded in $C^{1+\frac{1-p}{2}}(\bar{\Omega})$ by a constant independent of k. In view of the Arzelà-Ascoli theorem this immediately gives (2.20). ////

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