# Does a 'volume-filling effect' always prevent chemotactic collapse? 

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#### Abstract

The parabolic-parabolic Keller-Segel system for chemotaxis phenomena, $$
\left\{\begin{array}{l} u_{t}=\nabla \cdot(\phi(u) \nabla u)-\nabla \cdot(\psi(u) \nabla v), \quad x \in \Omega, t>0, \\ v_{t}=\Delta v-v+u, \quad x \in \Omega, t>0, \end{array}\right.
$$ is considered under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^{n}$ with $n \geq 2$. It is proved that if $\frac{\psi(u)}{\phi(u)}$ grows faster than $u^{\frac{2}{n}}$ as $u \rightarrow \infty$ and some further technical conditions are fulfilled, then there exist solutions that blow up in either finite or infinite time. Here, the total mass $\int_{\Omega} u(x, t) d x$ may attain arbitrarily small positive values. In particular, in the framework of chemotaxis models incorporating a volume-filling effect in the sense of Painter and Hillen (Can. Appl. Math. Q. 10, 501-543 (2002)), the results indicate how strongly the cellular movement must be inhibited at large cell densities in order to rule out chemotactic collapse.


Key words: chemotaxis, global existence, boundedness, blow-up
AMS Classification: 92C17, 35K55, 35B35, 35B40

## Introduction

We consider the initial-boundary value problem for two strongly coupled parabolic equations,

$$
\left\{\begin{array}{l}
u_{t}=\nabla \cdot(\phi(u) \nabla u)-\nabla \cdot(\psi(u) \nabla v), \quad x \in \Omega, t>0  \tag{0.1}\\
v_{t}=\Delta v-v+u, \quad x \in \Omega, t>0, \\
\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0, \quad x \in \partial \Omega, t>0, \\
u(x, 0)=u_{0}(x), \quad v(x, 0)=v_{0}(x), \quad x \in \Omega
\end{array}\right.
$$

with nonnegative initial data $u_{0} \in C^{0}(\bar{\Omega})$ and $v_{0} \in C^{1}(\bar{\Omega})$ in a bounded domain $\Omega \subset \mathbb{R}^{n}$ with smooth boundary, where $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal $\nu$ on $\partial \Omega$. The functions $\phi$ and $\psi$ are assumed to belong to $C^{2}([0, \infty))$ and to satisfy $\phi>0$ on $[0, \infty), \psi(0)=0$ and, for simplicity, that $\psi>0$ on $(0, \infty)$.

PDE systems of this type were introduced by Keller and Segel ([KS]) as a first approach towards the modelling of the biological phenomenon of directed, or partially directed, movement of cells in response to a chemical signal. In typical examples, as a reaction to an internal or external stimulus, the cells within some population start to produce such a signal substance and, at the same time, begin to move towards regions of higher concentration of this substance. As is known from experimental observations, such a movement may eventually lead to an aggregation of cells around one point, or few points, in space. This phenomenon, also referred to as chemotaxis, is believed to play an essential role in numerous processes of self-organization at the microscopic level, not only in protozoal populations but also in higher developed organisms (cf. [HP2] for a recent survey).

In the framework of (0.1), one neglects all further physical and chemical circumstances other than the presence of the cells theirselves and the signal, denoted in their respective densities by $u(x, t)$ and $v(x, t)$. One mathematical challenge in this context consists of finding out under which conditions and, if, in which sense cellular aggregation can indeed be described by the model (0.1), as simple as it stands. Here, one possible and frequently used mathematical notion of 'aggregation' is that ( 0.1 ) possesses solutions for which the cell density $u$ becomes unbounded in space at some finite or infinite time, the so-called blow-up time. Since, formally, (0.1) preserves the total cellular mass in the sense that $\int_{\Omega} u(x, t) d x=\int_{\Omega} u_{0}(x) d x$ for all $t>0$, this concept appears to be meaningful, because it implies that near the blow-up time the mass of an unbounded solution should essentially concentrate near those points where $u$ becomes large. However, the analysis of such blow-up solutions brings about several technical difficulties; for instance, there are only few results available that deal with the asymptotic behavior of unbounded solutions near their blow-up time. (For literature on existence of blow-up solutions, we refer to [HV], [HWa] and [Wi], for instance, asymptotic properties are addressed in [HV] and [NSS]; cf. also the survey $[\mathrm{H}]$.) Moreover, from the point of view of modelling it is not completely clear how exploding cell densities are to be interpreted.

Accordingly, considerable effort has been made in developing models of Keller-Segel type that do not possess blowing up but exclusively bounded solutions ([HP2]). One such approach was pursued by Hillen and Painter ( $[\mathrm{PH}]$ ) and is based on a biased random walk analysis. Having as their main ingredient the assumption that the cells' movement is inhibited near points where the cells are densely packed, they derive a functional link between the self-diffusivity $\phi(u)$ and the chemotactic sensitivity $\psi(u)$ that, in a non-dimensionalized version, takes the form

$$
\begin{equation*}
\phi(u)=Q(u)-u Q^{\prime}(u), \quad \psi(u)=u Q(u), \quad u \geq 0 \tag{0.2}
\end{equation*}
$$

where $Q(u)$ denotes the density-dependent probability for a cell to find space somewhere in its current neighborhood. Since this probability is basically unknown, different choices for $Q$ are conceivable, each of these providing a certain version of (0.1) that incorporates this so-called volume-filling effect. In [HP1], the authors propose the choice

$$
\begin{equation*}
Q(u)=A(\bar{u}-u)_{+} \tag{0.3}
\end{equation*}
$$

with some $A>0$ and $\bar{u}>0$, presuming that there exists some critical cell density $\bar{u}$ beyond which no further movement is possible. In fact, this model admits global bounded solutions only ([HP1]), and 'describing aggregation' amounts to studying dynamical properties such as instability
of constant steady states or the existence of attractors ([Wr]).
It is the purpose of the present paper to relax the above 'decay' assumption that $Q(u)$ be identically zero for $u$ large enough, and investigate the question to which extent (0.1) is then still able to prevent a chemotactic collapse in the sense of blow-up. Specifically, we think of $Q(u)$ to be positive for all $u \in[0, \infty)$ and to satisfy $Q(u) \rightarrow 0$ as $u \rightarrow \infty$, including algebraic or exponential decay, for instance. Such generalizations were previously suggested in [CC] and seem to be adequate in view of the lack of any experimental hint about a reasonable value for $\bar{u}$ in ( 0.3 ), or about the particular behavior of $Q(u)$ near $u=\bar{u}$ in this particular model.

Our main results, actually not requiring that $\phi$ and $\psi$ be connected via (0.2), indicate that the asymptotic behavior of the quotient $\frac{\psi(u)}{\phi(u)}$ for large $u$ is crucial. The main purpose is to show that

- if $\Omega$ is a ball in $\mathbb{R}^{n}$ for some $n \geq 2$ and

$$
\begin{equation*}
\frac{\psi(u)}{\phi(u)} \text { grows faster than } u^{\frac{2}{n}} \text { as } u \rightarrow \infty \tag{0.4}
\end{equation*}
$$

in a certain sense then for any $m>0$, (0.1) possesses unbounded solutions having mass $\int_{\Omega} u(x, t) d x \equiv m$
by proving the claimed conclusion under various, technically inspired, specifications of (0.4); as will be proved in Corollary 4.2, precise conditions that are sufficient for the occurrence of blow-up are

- $n=2$ and $\frac{\psi(u)}{\phi(u)} \geq c_{0} u \ln u$ for some $c_{0}>0$ and sufficiently large $u$;
- $n \geq 3$ and $u^{-\alpha} \frac{\psi(u)}{\phi(u)} \rightarrow c_{0}>0$ as $u \rightarrow \infty$ with some $\alpha>\frac{2}{n}$;
- $n \geq 3$ and $\liminf _{u \rightarrow \infty} \frac{u\left(\frac{\psi}{\phi}\right)^{\prime}(u)}{\left(\frac{\psi}{\phi}\right)(u)}>\frac{2}{n}$.

To illustrate this, let us suppose that $\phi$ and $\psi$ are given by $(0.2)$ with $Q(u) \simeq u^{-\beta}$ for large $u$ for some $\beta>0$. Then $\frac{\psi(u)}{\phi(u)} \simeq \frac{1}{1+\beta} u$ and hence the above results imply that blow-up occurs for some data when $n \geq 3$. On the other hand, if $Q$ decays exponentially, $Q(u) \simeq e^{-\beta u}$ for some $\beta>0$, then $\frac{\psi(u)}{\phi(u)} \simeq \frac{1}{\beta}$, so that none of the above criteria is fulfilled, and accordingly no collapse is asserted. However, if $Q(u) \simeq e^{-\beta u^{\gamma}}$ with positive $\beta$ and $\gamma$ then $\frac{\psi(u)}{\phi(u)} \simeq \frac{1}{\beta \gamma} u^{1-\gamma}$; accordingly, we conclude that if $n \geq 3$ and $\gamma<\frac{n-2}{n}$ here then blow-up solutions exist.
The number $\frac{2}{n}$ in (0.4) cannot be diminished; this is implied by the outcome in [HWi], where it was proved that if $\phi \equiv 1$ and $\limsup _{u \rightarrow \infty} u^{-\alpha} \psi(u)<\infty$ for some $\alpha<\frac{2}{n}$ then all solutions remain bounded.
It is an interesting open question whether $\frac{2}{n}$ is indeed critical in respect of collapse prevention also when $\phi(u)$ decays to zero as $u \rightarrow \infty$. Proving this conjecture seems to be connected to overcoming some technical complications stemming from the degeneracy in the diffusion part (cf. [CC]). To the best of our knowledge, affirmative results are available only under stronger growth restrictions of $\frac{\psi}{\phi}$ ([C2]), or for elliptic-parabolic simplifications of (0.1), but then only under the additional hypothesis that $\phi$ decays at most at an algebraic rate ([CW], [DW]).

Let us mention that in space dimension $n=2$, our requirement $\frac{\psi(u)}{\phi(u)} \geq c_{0} u \ln u$ at first glance seems far away from being optimal, for even in the standard Keller-Segel model with $\phi \equiv 1$ and $\psi(u)=u$ collapse is known to occur. However, in this case the appearance of blow-up is coupled to a large total mass of cells ([HWa], [NSY]), whereas we assert blow-up for arbitrarily small mass.

Our strategy of proof is inspired by the arguments in [SeS] and in [HWa]. We first identify an appropriate Lyapunov functional $F$ for (0.1) in Section 1 and thereby prove that bounded solutions of (0.1) approach certain steady-state solutions in an appropriate sense. Restricting to the radially symmetric setting henceforth, we next show in Section 2 that if $\frac{\psi}{\phi}$ satisfies some growth hypotheses then $F$, when evaluated at such equilibria, is uniformly bounded from below. In Section 3, however, we prove that suitably fast growth of $\frac{\psi}{\phi}$ implies the existence of smooth initial data at which $F$ attains arbitrarily large negative values. For such initial data, using the Lyapunov property of $F$ we easily conclude in Section 4 that the corresponding solutions cannot remain bounded.

## 1 Linking $\omega$-limit sets and steady states via a Lyapunov functional

For our purpose, a highly favorable structural property of the considered form of the Keller-Segel system will be that

$$
F(u, v):=\int_{\Omega}\left(\frac{1}{2}|\nabla v|^{2}+\frac{1}{2} v^{2}-u v+G(u)\right)
$$

acts as a Lyapunov functional for $(0.1)$ in that $F(u(\cdot, t), v(\cdot, t))$ is nonincreasing along trajectories and thus plays a role similar to that of energy in physics. Here, for any $s_{0}>0$ one may define the positive function $G=G_{s_{0}}$ on $(0, \infty)$ by

$$
G(s):=\int_{s_{0}}^{s} \int_{s_{0}}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)} d \tau d \sigma, \quad s>0
$$

The use of a Lyapunov functional has proved to be a strong tool in a large number of models of Keller-Segel type, also in simplified versions where the second equation in (0.1) is stationary ( $[\mathrm{CC}]$ ); for an overview, we refer to the surveys $[\mathrm{H}]$ and $[\mathrm{HP} 2]$.
Lemma 1.1 Let $T \in(0, \infty]$ and suppose that $(u, v)$ is a classical solution of ( 0.1 ) in $\Omega \times(0, T)$ with initial data $\left(u_{0}, v_{0}\right)$ satisfying $\inf _{x \in \bar{\Omega}} u_{0}(x)>0$. Then

$$
\begin{equation*}
\int_{0}^{t} \int_{\Omega} v_{t}^{2}+\int_{0}^{t} \int_{\Omega} \psi(u) \cdot\left|\frac{\phi(u)}{\psi(u)} \nabla u-\nabla v\right|^{2}+F(u(\cdot, t), v(\cdot, t))=F\left(u_{0}, v_{0}\right) \quad \text { for all } t \in(0, T) \tag{1.1}
\end{equation*}
$$

Proof. We perform a straightforward extension of the respective computations for the classical Keller-Segel model where $\phi \equiv 1$ and $\psi(u)=u([G Z])$. By the strong maximum principle applied to the first equation in (0.1), $u$ inherits strict positivity from its initial data. Therefore $\frac{\phi(u)}{\psi(u)}$ and hence $G(u)$ and $G^{\prime}(u) \equiv \frac{d}{d u} G(u)$ are continuous in $\bar{\Omega} \times[0, T)$, so that the first equation in (0.1) yields

$$
\left.\int_{\Omega} G(u)\right|_{0} ^{t}=\int_{0}^{t} \int_{\Omega} G^{\prime}(u) \nabla \cdot(\phi(u) \nabla u-\psi(u) \nabla v)
$$

$$
\begin{aligned}
& =-\int_{0}^{t} \int_{\Omega} G^{\prime \prime}(u) \nabla u \cdot(\phi(u) \nabla u-\psi(u) \nabla v) \\
& =-\int_{0}^{t} \int_{\Omega} \frac{\phi^{2}(u)}{\psi(u)}|\nabla u|^{2}+\int_{0}^{t} \int_{\Omega} \phi(u) \nabla u \cdot \nabla v
\end{aligned}
$$

Since

$$
\frac{\phi^{2}(u)}{\psi(u)}|\nabla u|^{2}=\psi(u)\left|\frac{\phi(u)}{\psi(u)} \nabla u-\nabla v\right|^{2}-\psi(u)|\nabla v|^{2}+2 \phi(u) \nabla u \cdot \nabla v
$$

we obtain

$$
\begin{equation*}
\left.\int_{\Omega} G(u)\right|_{0} ^{t}=-\int_{0}^{t} \int_{\Omega} \psi(u)\left|\frac{\phi(u)}{\psi(u)} \nabla u-\nabla v\right|+\int_{0}^{t} \int_{\Omega} \psi(u)|\nabla v|^{2}-\int_{0}^{t} \int_{\Omega} \phi(u) \nabla u \cdot \nabla v \tag{1.2}
\end{equation*}
$$

Here, the last term can be rewritten using the first equation in (0.1) according to

$$
\begin{align*}
-\int_{0}^{t} \int_{\Omega} \phi(u) \nabla u \cdot \nabla v & =\int_{0}^{t} \int_{\Omega} \nabla \cdot(\phi(u) \nabla u) \cdot v \\
& =\int_{0}^{t} \int_{\Omega} u_{t} v+\int_{0}^{t} \int_{\Omega} \nabla \cdot(\psi(u) \nabla v) \cdot v \\
& =\int_{0}^{t} \int_{\Omega} u_{t} v-\int_{0}^{t} \int_{\Omega} \psi(u)|\nabla v|^{2} \tag{1.3}
\end{align*}
$$

where

$$
\begin{align*}
\int_{0}^{t} \int_{\Omega} u_{t} v & =\left.\int_{\Omega} u v\right|_{0} ^{t}-\int_{0}^{t} \int_{\Omega} u v_{t} \\
& =\left.\int_{\Omega} u v\right|_{0} ^{t}-\int_{0}^{t} \int_{\Omega}\left(v_{t}-\Delta v+v\right) \cdot v_{t} \\
& =\left.\int_{\Omega} u v\right|_{0} ^{t}-\int_{0}^{t} \int_{\Omega} v_{t}^{2}-\left.\frac{1}{2} \int_{\Omega}|\nabla v|^{2}\right|_{0} ^{t}-\left.\frac{1}{2} \int_{\Omega} v^{2}\right|_{0} ^{t} \tag{1.4}
\end{align*}
$$

Combining (1.2) and (1.3) with (1.4), after an obvious reorganization we end up with (1.1). ////
In what follows, an important role is played by the properties of solutions $\left(u_{\infty}, v_{\infty}\right)$ of

$$
\left\{\begin{array}{l}
-\Delta v_{\infty}+v_{\infty}=u_{\infty}, \quad x \in \Omega  \tag{1.5}\\
\phi\left(u_{\infty}\right) \nabla u_{\infty}=\psi\left(u_{\infty}\right) \nabla v_{\infty}, \quad x \in \Omega \\
\frac{\partial v_{\infty}}{\partial \nu}=0, \quad x \in \partial \Omega .
\end{array}\right.
$$

Evidently, such function pairs are stationary solutions of (0.1) in the classical sense.
The identity (1.1) will allow us to establish a connection between the set $\mathcal{E}_{m}$ of such equilibria that have mass $m>0$,

$$
\begin{equation*}
\mathcal{E}_{m}:=\left\{\left(u_{\infty}, v_{\infty}\right) \in\left(C^{2}(\bar{\Omega})\right)^{2} \mid\left(u_{\infty}, v_{\infty}\right) \text { solves (1.5) and } \int_{\Omega} u_{\infty}=m\right\} \tag{1.6}
\end{equation*}
$$

and the $\omega$-limit set of bounded solutions of ( 0.1 ) given by

$$
\omega(u, v):=\left\{\left(\bar{u}_{\infty}, \bar{v}_{\infty}\right) \in\left(C^{2}(\bar{\Omega})\right)^{2} \left\lvert\, \begin{array}{ll} 
& \exists t_{k} \rightarrow \infty \text { such that as } k \rightarrow \infty \\
& \left.\left(u\left(\cdot, t_{k}\right), v\left(\cdot, t_{k}\right)\right) \rightarrow\left(\bar{u}_{\infty}, \bar{v}_{\infty}\right) \text { in }\left(C^{2}(\bar{\Omega})\right)^{2}\right\} . \tag{1.7}
\end{array}\right.\right.
$$

Lemma 1.2 Suppose ( $u, v$ ) is a global bounded solution of (0.1) with $u_{0}>0$ in $\bar{\Omega}$ and $\int_{\Omega} u_{0}=m$. Then $\omega(u, v) \cap \mathcal{E}_{m} \neq \emptyset$.

Proof. Since $u$ is bounded, scalar parabolic regularity theory applied to the second, the first, the second and again the first equation in (0.1) ([LSU]) implies boundedness of $v$ in $C^{1+\beta, \frac{1+\beta}{2}}(\bar{\Omega} \times$ $[1, \infty)$ ), then of $u$ in $C^{\beta, \frac{\beta}{2}}(\bar{\Omega} \times[1, \infty))$, then of $v$ in $C^{2+\beta, 1+\frac{\beta}{2}}(\bar{\Omega} \times[1, \infty))$ and finally of $u$ in $C^{2+\beta, 1+\frac{\beta}{2}}(\bar{\Omega} \times[1, \infty))$ for some $\beta>0$. Moreover, the boundedness of $(u, v)$ entails that $F(u(\cdot, t), v(\cdot, t))$ is bounded from below for all times, so that Lemma 1.1 guarantees

$$
\int_{0}^{\infty} \int_{\Omega} v_{t}^{2}+\int_{0}^{\infty} \psi(u)\left|\frac{\phi(u)}{\psi(u)} \nabla u-\nabla v\right|^{2}<\infty
$$

Using this together with the Arzelà-Ascoli theorem, we can extract a sequence of times $t_{k} \rightarrow \infty$ such that

$$
\begin{align*}
& v_{t}\left(\cdot, t_{k}\right) \rightarrow 0 \quad \text { in } L^{2}(\Omega) \quad \text { and }  \tag{1.8}\\
& \psi\left(u\left(\cdot, t_{k}\right)\right)\left|\frac{\phi\left(u\left(\cdot, t_{k}\right)\right)}{\psi\left(u\left(\cdot, t_{k}\right)\right)} \nabla u\left(\cdot, t_{k}\right)-\nabla v\left(\cdot, t_{k}\right)\right|^{2} \rightarrow 0 \quad \text { a.e. in } \Omega \tag{1.9}
\end{align*}
$$

as well as

$$
\begin{array}{ll}
u\left(\cdot, t_{k}\right) \rightarrow u_{\infty} & \text { in } C^{2}(\bar{\Omega}) \\
v\left(\cdot, t_{k}\right) \rightarrow v_{\infty} & \text { in } C^{2}(\bar{\Omega}) \tag{1.11}
\end{array}
$$

hold with some nonnegative $u_{\infty}$ and $v_{\infty}$ belonging to $C^{2}(\bar{\Omega})$.
Clearly, from (1.11) we know that $\frac{\partial v_{\infty}}{\partial \nu}=0$ on $\partial \Omega$, whereas (1.8), (1.10) and (1.11) imply upon letting $t=t_{k} \rightarrow \infty$ in $v_{t}=\Delta v-v+u$ that $0=\Delta v_{\infty}-v_{\infty}+u_{\infty}$ in $\Omega$. The mass requirement in the definition of $\mathcal{E}_{m}$ is an immediate consequence of the obvious mass conservation property $\int_{\Omega} u(x, t) d x \equiv \int_{\Omega} u_{0}$.
In order to prove the second identity in (1.5), we first note that the nonnegativity of $u_{\infty}$ implies that $\nabla u_{\infty} \equiv 0$ holds in the set $\left\{u_{\infty}=0\right\}$ of zeros of $u_{\infty}$. Thus, $\phi\left(u_{\infty}\right) \nabla u_{\infty}=\psi\left(u_{\infty}\right) \nabla v_{\infty}=0$ in $\left\{u_{\infty}=0\right\}$, because $\psi(0)=0$. For fixed $x \in\left\{u_{\infty}>0\right\}$, however, we have $\liminf _{k \rightarrow \infty} \psi\left(u\left(x, t_{k}\right)\right)>$ 0 , whence from (1.9) we infer that

$$
\frac{\phi\left(u\left(\cdot, t_{k}\right)\right)}{\psi\left(u\left(\cdot, t_{k}\right)\right)} \nabla u\left(\cdot, t_{k}\right)-\nabla v\left(\cdot, t_{k}\right) \rightarrow 0 \quad \text { a.e. in }\left\{u_{\infty}>0\right\}
$$

and therefore

$$
\phi\left(u_{\infty}\right) \nabla u_{\infty}=\psi\left(u_{\infty}\right) \nabla v_{\infty} \quad \text { a.e. in }\left\{u_{\infty}>0\right\} .
$$

Since both sides of this equation are continuous in $\bar{\Omega}$, this proves that $\phi\left(u_{\infty}\right) \nabla u_{\infty} \equiv \psi\left(u_{\infty}\right) \nabla v_{\infty}$ also holds in the whole set $\left\{u_{\infty}>0\right\}$, as desired.

Remark. It can be shown that actually each $u_{\infty} \in \mathcal{E}_{m}$ must be positive throughout $\bar{\Omega}$. In fact, suppose that some component $\mathcal{C}$ of $\left\{u_{\infty}>0\right\}$ does not coincide with $\bar{\Omega}$. Then there exist $x_{0} \in \partial \mathcal{C}$ and a sequence of points $x_{j} \in \mathcal{C}$ such that $x_{j} \rightarrow x_{0}$. Writing $g(s):=\int_{s_{0}}^{s} \frac{\phi(\sigma)}{\psi(\sigma)} d \sigma$ for $s>0$, we know from (1.5) that $\nabla\left(g\left(u_{\infty}\right)-v_{\infty}\right) \equiv 0$ and hence $g\left(u_{\infty}\right)-v_{\infty} \equiv \Gamma$ in $\mathcal{C}$ with some $\Gamma \in \mathbb{R}$. Since $\phi(0)>0=\psi(0)$ and $\psi \in C^{1}([0, \infty))$, it is clear that $g(0)=-\infty$, so that taking $j \rightarrow \infty$ in $g\left(u_{\infty}\left(x_{j}\right)\right)-v_{\infty}\left(x_{j}\right)=\Gamma$ yields a contradiction. Accordingly, $\mathcal{C}=\bar{\Omega}$, which means that $u_{\infty}>0$ in $\bar{\Omega}$.

## 2 Lower bounds for steady-state energies

The goal of this section is to assert a lower bound for the values of $F(u, v)$ for all possible members $(u, v)$ of $\mathcal{E}_{m}$. According to slightly different technical approaches, we distinguish between the cases $n=2$ and $n \geq 3$.
As a first step, let us make a simple but useful observation that allows us to rewrite the energy $F(u, v)$ of a solution $(u, v)$ of (1.5) without the 'mixed' term $\int_{\Omega} u v$.
Lemma 2.1 If $(u, v)$ is a solution of (1.5) and $s_{0}>0$ then

$$
\begin{equation*}
F(u, v)=-\frac{1}{2} \int_{\Omega}|\nabla v|^{2}-\frac{1}{2} \int_{\Omega} v^{2}+\int_{\Omega} G_{s_{0}}(u) \tag{2.1}
\end{equation*}
$$

Proof. We multiply the first equation in (1.5) by $v$ to obtain $\int_{\Omega}|\nabla v|^{2}+\int_{\Omega} v^{2}=\int_{\Omega} u v$. Inserting this into the definition of $F$ immediately results in (2.1).
////
The following preliminary estimate for radial steady states will be a common ingredient for both cases $n=2$ and $n \geq 3$. It is inspired by the classical proof of Pohozaev's identity.

Lemma 2.2 Let $\Omega=B_{R}(0), s_{0}>0$ and

$$
\begin{equation*}
H(s):=\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \quad \text { for } s>0 \tag{2.2}
\end{equation*}
$$

Then for all nonnegative and nonincreasing $\zeta \in C^{\infty}([0, R])$ satisfying $\zeta^{\prime}(0)=0$ and $\zeta(R)=0$, the inequality

$$
\begin{align*}
\frac{n-2}{2} \int_{\Omega} \zeta(|x|)|\nabla v|^{2} & -\frac{1}{2} \int_{\Omega}|x| \zeta^{\prime}(|x|)|\nabla v|^{2} \\
& \leq \int_{\Omega}|x| \zeta(|x|) \cdot\left(v+s_{0}\right) \cdot|\nabla v|+n \int_{\left\{u>s_{0}\right\}} \zeta(|x|) H(u) \tag{2.3}
\end{align*}
$$

is valid for each radially symmetric solution $(u, v)$ of (1.5).
Proof. We multiply $\Delta v=v-u$ by $\zeta(|x|)(x \cdot \nabla v)$ and integrate over $\Omega$ to see that

$$
\begin{equation*}
\int_{\Omega} \zeta(|x|)(x \cdot \nabla v) \Delta v=\int_{\Omega} \zeta(|x|) v(x \cdot \nabla v)-\int_{\Omega} \zeta(|x|) u(x \cdot \nabla v) \tag{2.4}
\end{equation*}
$$

Since $\zeta(R)=0$, two integrations by parts on the left yield

$$
\begin{aligned}
\int_{\Omega} \zeta(|x|)(x \cdot \nabla v) \Delta v= & -\int_{\Omega} \zeta(|x|)|\nabla v|^{2} \\
& -\int_{\Omega} \frac{\zeta^{\prime}(|x|)}{|x|}(x \cdot \nabla v)^{2} \\
& -\frac{1}{2} \int_{\Omega} \zeta(|x|)\left(x \cdot \nabla|\nabla v|^{2}\right)
\end{aligned}
$$

and

$$
-\frac{1}{2} \int_{\Omega} \zeta(|x|)\left(x \cdot \nabla|\nabla v|^{2}\right)=\frac{n}{2} \int_{\Omega} \zeta(|x|)|\nabla v|^{2}+\frac{1}{2} \int_{\Omega}|x| \zeta^{\prime}(|x|)|\nabla v|^{2} .
$$

Thereupon, (2.4) turns into the identity

$$
\begin{aligned}
\frac{n-2}{2} \int_{\Omega} \zeta(|x|)|\nabla v|^{2} & -\int_{\Omega} \frac{\zeta^{\prime}(|x|)}{|x|}(x \cdot \nabla v)^{2}+\frac{1}{2} \int_{\Omega}|x| \zeta^{\prime}(|x|)|\nabla v|^{2} \\
& =\int_{\Omega} \zeta(|x|) v(x \cdot \nabla v)-\int_{\Omega} \zeta(|x|) u(x \cdot \nabla v)
\end{aligned}
$$

We now use the radial symmetry of $v$ which guarantees that $(x \cdot \nabla v)^{2}=|x|^{2}|\nabla v|^{2}$, so that

$$
\begin{equation*}
\frac{n-2}{2} \int_{\Omega} \zeta(|x|)|\nabla v|^{2}-\frac{1}{2} \int_{\Omega}|x| \zeta^{\prime}(|x|)|\nabla v|^{2}=\int_{\Omega} \zeta(|x|) v(x \cdot \nabla v)-\int_{\Omega} \zeta(|x|) u(x \cdot \nabla v) . \tag{2.5}
\end{equation*}
$$

In order to find an upper bound for the second term on the right, we recall the definition of $H$ and split the integral in question according to

$$
\begin{equation*}
-\int_{\Omega} \zeta(|x|) u(x \cdot \nabla v)=-\int_{\left\{u \leq s_{0}\right\}} \zeta(|x|) u(x \cdot \nabla v)-\int_{\left\{u>s_{0}\right\}} \zeta(|x|)(x \cdot \nabla H(u)) \tag{2.6}
\end{equation*}
$$

Here, integrating by parts we obtain

$$
\begin{align*}
-\int_{\left\{u>s_{0}\right\}} \zeta(|x|)(x \cdot \nabla H(u)) & =n \int_{\left\{u>s_{0}\right\}} \zeta(|x|) H(u)+\int_{\left\{u>s_{0}\right\}}|x| \zeta^{\prime}(|x|) H(u) \\
& \leq n \int_{\left\{u>s_{0}\right\}} \zeta(|x|) H(u) \tag{2.7}
\end{align*}
$$

because $H(x) \equiv 0$ on $\partial\left\{u>s_{0}\right\} \cap \Omega, \zeta(R)=0$ and $\zeta^{\prime} \leq 0$. Using the obvious estimates

$$
-\int_{\left\{u \leq s_{0}\right\}} \zeta(|x|) u(x \cdot \nabla v) \leq \int_{\Omega}|x| \zeta(|x|) s_{0}|\nabla v|
$$

and

$$
\int_{\Omega} \zeta(|x|) v(x \cdot \nabla v) \leq \int_{\Omega}|x| \zeta(|x|) v|\nabla v|
$$

we thus infer from (2.5)-(2.7) that (2.3) holds.

### 2.1 The case $n=2$

We now concentrate on the two-dimensional case first. In this, the first term on the left of (2.3) vanishes, so that taking advantage from this estimate means using the second term on its left appropriately. This is the main technical goal in the proof of the following statement.

Lemma 2.3 Assume that $\Omega=B_{R}(0) \subset \mathbb{R}^{2}$ for some $R>0$. If

$$
\begin{equation*}
\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \leq K \frac{s}{\ln s} \quad \text { for all } s \geq s_{0} \tag{2.8}
\end{equation*}
$$

holds with some $K>0$ and $s_{0}>1$, then for all $m>0$ there exists $C=C\left(R, n, K, s_{0}\right)>0$ such that

$$
\begin{equation*}
F(u, v) \geq-C \tag{2.9}
\end{equation*}
$$

is valid for all radial solutions $(u, v)$ of (1.5).
Proof. For $\eta \in(0,1)$, we let

$$
\zeta(r):=\ln \frac{R^{2}+\eta}{r^{2}+\eta}, \quad r \in[0, R] .
$$

Then $\zeta$ is smooth and nonnegative in $[0, R]$ with $\zeta(R)=0$ and $\zeta^{\prime}(r)=-\frac{2 r}{r^{2}+\eta}$, so that $\zeta^{\prime} \leq 0$ on $[0, R]$ and $\zeta^{\prime}(0)=0$. Thus, from (2.3) we obtain, again abbreviating $H(s)=\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma$ and using Young's inequality,

$$
\begin{aligned}
\int_{\Omega} \frac{|x|^{2}}{|x|^{2}+\eta} \cdot|\nabla v|^{2} \leq & \int_{\Omega}|x| \ln \left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) \cdot\left(v+s_{0}\right) \cdot|\nabla v|+2 \int_{\left\{u>s_{0}\right\}} \ln \left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) \cdot H(u) \\
\leq & \frac{1}{2} \int_{\Omega} \frac{|x|^{2}}{|x|^{2}+\eta} \cdot|\nabla v|^{2}+\int_{\Omega}\left(|x|^{2}+\eta\right) \cdot \ln ^{2}\left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) \cdot v^{2} \\
& +s_{0}^{2} \int_{\Omega}\left(|x|^{2}+\eta\right) \cdot \ln ^{2}\left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) \\
& +2 \int_{\left\{u>s_{0}\right\}} \ln \left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) \cdot H(u) .
\end{aligned}
$$

Since it can easily be checked that $\left(r^{2}+\eta\right) \cdot \ln ^{2} \frac{R^{2}+\eta}{r^{2}+\eta} \leq 4\left(R^{2}+\eta\right) e^{-2}$ for all $r \in[0, R]$, we thus find

$$
\begin{align*}
\frac{1}{2} \int_{\Omega} \frac{|x|^{2}}{|x|^{2}+\eta} \cdot|\nabla v|^{2} \leq & 4\left(R^{2}+1\right) e^{-2} \int_{\Omega} v^{2}+4\left(R^{2}+1\right) e^{-2} s_{0}^{2}|\Omega| \\
& +2 \int_{\left\{u>s_{0}\right\}} \ln \left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) \cdot H(u) \tag{2.10}
\end{align*}
$$

Now in Young's inequality in the form

$$
a b \leq \frac{1}{\delta e} e^{\delta a}+\frac{1}{\delta} b \ln b
$$

valid for all positive $a, b$ and $\delta$, we pick any $\delta \in(0,1)$ and thus estimate

$$
\begin{aligned}
2 \int_{\left\{u>s_{0}\right\}} \ln \left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right) H(u) & \leq \frac{2}{\delta e} \int_{\Omega}\left(\frac{R^{2}+\eta}{|x|^{2}+\eta}\right)^{\delta}+\frac{2}{\delta} \int_{\left\{u>s_{0}\right\}} H(u) \ln H(u) \\
& \leq \frac{2\left(R^{2}+1\right)^{\delta}}{\delta e} \int_{\Omega}|x|^{-2 \delta}+\frac{2}{\delta} \int_{\left\{u>s_{0}\right\}} H(u) \ln H(u)
\end{aligned}
$$

for all $\eta \in(0,1)$, where the first integral on the right is finite since $\delta<1$. According to (2.8), we have

$$
\begin{aligned}
H(s) \ln H(s) & \leq K \frac{s}{\ln s} \cdot \ln \frac{K s}{\ln s} \leq K \frac{s}{\ln s}\left(\ln s+\ln \frac{K}{\ln s_{0}}\right) \\
& \leq K\left(1+c_{1}\right) s \quad \text { for all } s>s_{0}
\end{aligned}
$$

with $c_{1}:=\max \left\{0, \frac{\ln \frac{K}{\ln s_{0}}}{\ln s_{0}}\right\}$. Since $\int_{\left\{u>s_{0}\right\}} u \leq \int_{\Omega} u=m,(2.10)$ therefore implies

$$
\frac{1}{2} \int_{\Omega} \frac{|x|^{2}}{|x|^{2}+\eta}|\nabla v|^{2} \leq c_{2} \int_{\Omega} v^{2}+c_{3}
$$

where
$c_{2}=4\left(R^{2}+1\right) e^{-2} \quad$ and $\quad c_{3}=4\left(R^{2}+1\right) e^{-2} s_{0}^{2}|\Omega|+\frac{2\left(R^{2}+1\right)^{\delta}}{\delta e} \cdot \int_{\Omega}|x|^{-2 \delta}+\frac{2 K\left(1+c_{1}\right) m}{\delta}$.
In the limit $\eta \rightarrow 0$, Fatou's lemma thus yields

$$
\frac{1}{2} \int_{\Omega}|\nabla v|^{2} \leq c_{2} \int_{\Omega} v^{2}+c_{3}
$$

Hence, by (2.1) and the nonnegativity of $G$,

$$
\begin{aligned}
F(u, v) & \geq \frac{1}{2} \int_{\Omega}|\nabla v|^{2}-\int_{\Omega}|\nabla v|^{2}-\frac{1}{2} \int_{\Omega} v^{2} \\
& \geq \frac{1}{2} \int_{\Omega}|\nabla v|^{2}-\left(2 c_{2}+\frac{1}{2}\right) \int_{\Omega} v^{2}-2 c_{3}
\end{aligned}
$$

Finally, from Ehrling's lemma we gain some $c_{4}>0$ such that

$$
\left(2 c_{2}+\frac{1}{2}\right) \int_{\Omega} w^{2} \leq \frac{1}{2} \int_{\Omega}|\nabla w|^{2}+c_{4}\left(\int_{\Omega} w\right)^{2} \quad \text { for all } w \in W^{1,2}(\Omega)
$$

and thereby conclude, recalling $\int_{\Omega} v=m$, that

$$
F(u, v) \geq-c_{4} m^{2}-2 c_{3}
$$

and finish the proof.

### 2.2 The case $n \geq 3$

In the three-dimensional situation, the first term on the left of (2.3) will be essentially responsible for the fact that a corresponding lower bound for all steady-state energies can be proved under a less restrictive growth restriction on $\frac{\psi}{\phi}$.
Lemma 2.4 Let $n \geq 3$ and $\Omega=B_{R}(0)$ with some $R>0$. Suppose that

$$
\begin{equation*}
\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \leq \frac{n-2-\varepsilon}{n} \int_{s_{0}}^{s} \int_{s_{0}}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)} d \tau d \sigma+K s \quad \text { for all } s \geq s_{0} \tag{2.11}
\end{equation*}
$$

holds for some $s_{0}>1$ and $\varepsilon \in(0,1)$. Then for each $m>0$ one can find $C=C\left(R, m, n, \varepsilon, s_{0}\right)>0$ with the property that

$$
\begin{equation*}
F(u, v) \geq-C \tag{2.12}
\end{equation*}
$$

is satisfied for all radial solutions $(u, v)$ of (1.5).
Proof. We fix a nondecreasing $\zeta_{0} \in C^{\infty}(\mathbb{R})$ such that $\zeta_{0} \equiv 0$ in $(-\infty, 1)$ and $\zeta_{0} \equiv 1$ on $(2, \infty)$, and let $\zeta(r) \equiv \zeta_{k}(r):=\zeta_{0}(k(R-r))$ for $r \in[0, R]$ and $k \in \mathbb{N}$ large satisfying $k>\frac{2}{R}$. Then Lemma 2.2 ensures that with $H(s)=\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma$, we have

$$
\begin{aligned}
\frac{n-2}{2} \int_{\Omega} \zeta_{k}(|x|)|\nabla v|^{2} & \leq \int_{\Omega}|x| \zeta_{k}(|x|) \cdot\left(v+s_{0}\right) \cdot|\nabla v|+n \int_{\left\{u>s_{0}\right\}} \zeta_{k}(|x|) H(u) \\
& \leq \int_{\Omega}|x|\left(v+s_{0}\right)|\nabla v|+n \int_{\left\{u>s_{0}\right\}} H(u)
\end{aligned}
$$

for all such $k$, whence by Fatou's lemma we gain

$$
\frac{n-2}{2} \int_{\Omega}|\nabla v|^{2} \leq \int_{\Omega}|x|\left(v+s_{0}\right)|\nabla v|+n \int_{\left\{u>s_{0}\right\}} H(u) .
$$

With $\varepsilon$ taken from (2.11), we use Young's inequality to estimate

$$
\begin{aligned}
\int_{\Omega}|x|\left(v+s_{0}\right)|\nabla v| & \leq \frac{\varepsilon}{4} \int_{\Omega}|\nabla v|^{2}+\frac{1}{\varepsilon} \int_{\Omega}|x|^{2}\left(v+s_{0}\right)^{2} \\
& \leq \frac{\varepsilon}{4} \int_{\Omega}|\nabla v|^{2}+\frac{2 R^{2}}{\varepsilon} \int_{\Omega} v^{2}+\frac{2 s_{0}^{2} R^{2}|\Omega|}{\varepsilon}
\end{aligned}
$$

so that

$$
\frac{n-2-\varepsilon}{2} \int_{\Omega}|\nabla v|^{2} \leq-\frac{\varepsilon}{4} \int_{\Omega}|\nabla v|^{2}+\frac{2 R^{2}}{\varepsilon} \int_{\Omega} v^{2}+\frac{2 s_{0}^{2} R^{2}|\Omega|}{\varepsilon}+n \int_{\left\{u>s_{0}\right\}} H(u) .
$$

Rearranging this, from (2.1) we infer that

$$
F(u, v) \geq \frac{\varepsilon}{4(n-2-\varepsilon)} \int_{\Omega}|\nabla v|^{2}-c_{1} \int_{\Omega} v^{2}-c_{2}-\frac{n}{n-2-\varepsilon} \int_{\left\{u>s_{0}\right\}} H(u)+\int_{\Omega} G(u)
$$

with

$$
c_{1}=\frac{2 R^{2}}{\varepsilon(n-2-\varepsilon)}+\frac{1}{2} \quad \text { and } \quad c_{2}=\frac{2 s_{0}^{2} R^{2}|\Omega|}{\varepsilon(n-2-\varepsilon)} .
$$

Now by Ehrling's lemma, we have

$$
c_{1} \int_{\Omega} v^{2} \leq \frac{\varepsilon}{4(n-2-\varepsilon)} \int_{\Omega}|\nabla v|^{2}+c_{3}\left(\int_{\Omega} v\right)^{2}
$$

for some $c_{3}>0$. In view of the mass identity $\int_{\Omega} v=m$, we thus find

$$
F(u, v) \geq-c_{3} m^{2}-c_{2}-\frac{n}{n-2-\varepsilon} \int_{\left\{u>s_{0}\right\}} H(u)+\int_{\Omega} G(u) .
$$

Since $G(s) \geq 0$ for $s \leq s_{0}$ and

$$
G(s)-\frac{n}{n-2-\varepsilon} H(s) \geq-\frac{n K}{n-2-\varepsilon} s \quad \text { for all } s \geq s_{0}
$$

by (2.11), we conclude that

$$
F(u, v) \geq-c_{3} m^{2}-c_{2}-\frac{n K m}{n-2-\varepsilon}
$$

because $\int_{\Omega} u=m$.

## 3 Initial data with large negative energy

We next assert that smooth initial data with arbitrarily large negative energies exist. The construction partly parallels that in [HWi, Lemma 5.2]. However, the growth assumptions on $\frac{\psi}{\phi}$ are weaker here, especially in the case $n=2$; moreover, we provide initial data in $C^{\infty}$ here.

Lemma 3.1 Let $n \geq 2, R>0$ and $\Omega=B_{R}(0)$, and suppose that there exist $k>0$ and $s_{0}>1$ such that

$$
\int_{s_{0}}^{s} \int_{s_{0}}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)} d \tau d \sigma \leq \begin{cases}k s(\ln s)^{\theta} & \text { if } n=2 \quad \text { with some } \theta \in(0,1),  \tag{3.1}\\ k s^{2-\alpha} & \text { if } n \geq 3 \quad \text { with some } \alpha>\frac{2}{n},\end{cases}
$$

holds for all $s \geq s_{0}$. Then for each $m>0$ and $C>0$ one can find $\left(u_{0}, v_{0}\right) \in\left(C^{\infty}(\bar{\Omega})\right)^{2}$ satisfying $\int_{\Omega} u_{0}=m$ and

$$
\begin{equation*}
F\left(u_{0}, v_{0}\right)<-C . \tag{3.2}
\end{equation*}
$$

Proof. First, in the case $n \geq 3$ we evidently may assume that (3.1) is valid with some $\alpha>\frac{2}{n}$ satisfying $\alpha<1$. We then pick positive numbers $\beta, \gamma$ and $\delta$ such that

$$
\begin{equation*}
\beta>n, \quad \gamma \in((1-\alpha) n, n-2) \quad \text { and } \quad \delta>\frac{n}{2} \tag{3.3}
\end{equation*}
$$

which is possible because $\alpha>\frac{2}{n}$. For small $\eta>0$, we define the smooth functions $u_{\eta}$ and $v_{\eta}$ by

$$
\begin{aligned}
& u_{\eta}(x):=a_{\eta} \cdot \eta^{\beta-n} \cdot\left(|x|^{2}+\eta^{2}\right)^{-\frac{\beta}{2}}, \quad \text { and } \\
& v_{\eta}(x):=\eta^{\delta-\gamma} \cdot\left(|x|^{2}+\eta^{2}\right)^{-\frac{\delta}{2}}
\end{aligned}
$$

for $x \in \bar{\Omega}$, where

$$
a_{\eta}:=\frac{\eta^{n-\beta} m}{\int_{\Omega}\left(|x|^{2}+\eta^{2}\right)^{-\frac{\beta}{2}} d x}
$$

Since upon the substitution $r=\eta s$ we see that

$$
\begin{equation*}
\eta^{\lambda-N} \int_{0}^{R} r^{N-1}\left(r^{2}+\eta^{2}\right)^{-\frac{\lambda}{2}} d r \rightarrow A(N, \lambda):=\int_{0}^{\infty} s^{N-1}\left(s^{2}+1\right)^{-\frac{\lambda}{2}} d s \quad \text { as } \eta \rightarrow 0 \tag{3.4}
\end{equation*}
$$

whenever $N>0$ and $\lambda>N$, it can easily be checked that $a_{\eta} \rightarrow a_{0}:=\frac{m}{\omega_{n} \cdot A(n, \beta)}$ as $\eta \rightarrow 0$, where $\omega_{n}$ denotes the $(n-1)$-dimensional surface area of the unit sphere in $\mathbb{R}^{n}$. In particular, $a_{\eta}$ is bounded from above and below by a positive constant, uniformly with respect to $\eta \in(0,1)$.
The choice of $a_{\eta}$ immediately implies that $\int_{\Omega} u_{\eta}=m$. Moreover, by straightforward computations using (3.4) we obtain

$$
\begin{aligned}
\eta^{-n+2 \gamma+2} \int_{\Omega}\left|\nabla v_{\eta}\right|^{2} & =\omega_{n} \delta^{2} \eta^{-n+2 \delta+2} \int_{0}^{R} r^{n+1}\left(r^{2}+\eta^{2}\right)^{-\delta-2} d r \\
& \rightarrow \omega_{n} \delta^{2} \cdot A(n+2,2 \delta+4) \quad \text { as } \eta \rightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
\eta^{-n+2 \gamma} \int_{\Omega} v_{\eta}^{2} & =\omega_{n} \eta^{-n+2 \delta} \int_{0}^{R} r^{n-1}\left(r^{2}+\eta^{2}\right)^{-\delta} d r \\
& \rightarrow \omega_{n} \cdot A(n, 2 \delta) \quad \text { as } \eta \rightarrow 0
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\eta^{\gamma} \int_{\Omega} u_{\eta} v_{\eta} & =\omega_{n} a_{\eta} \cdot \eta^{-n+\beta+\delta} \int_{0}^{R} r^{n-1}\left(r^{2}+\eta^{2}\right)^{-\frac{\beta+\delta}{2}} d r \\
& \rightarrow \omega_{n} a_{0} \cdot A(n, \beta+\delta) \quad \text { as } \eta \rightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
\eta^{(1-\alpha) n} \int_{\Omega} G\left(u_{\eta}\right) & \leq \omega_{n} k a_{\eta}^{2-\alpha} \cdot \eta^{(1-\alpha) n+(2-\alpha)(\beta-n)} \int_{0}^{R} r^{n-1}\left(r^{2}+\eta^{2}\right)^{-\frac{(2-\alpha) \beta}{2}} d r \\
& \rightarrow \omega_{n} k a_{0}^{2-\alpha} \cdot A(n,(2-\alpha) \beta) \quad \text { as } \eta \rightarrow 0 .
\end{aligned}
$$

Since $\gamma>0$ and

$$
\gamma>-n+2 \gamma+2, \quad \gamma>-n+2 \gamma \quad \text { and } \quad \gamma>(1-\alpha) n
$$

according to (3.3), it follows that $F\left(u_{\eta}, v_{\eta}\right) \rightarrow-\infty$ as $\eta \rightarrow 0$, whence (3.2) is true for all sufficiently small $\eta>0$.

In the case $n=2$ we define $u_{\eta}$ as above, but let

$$
v_{\eta}(x):=\left(\ln \frac{R}{\eta}\right)^{-\kappa} \cdot \ln \frac{R^{2}}{|x|^{2}+\eta^{2}}
$$

for $\eta \in\left(0, \frac{R}{2}\right)$ this time, where $\kappa \in(0,1)$ is small enough fulfilling $\kappa<1-\theta$. Then, again substituting $r=\eta s$, we find

$$
\begin{aligned}
\int_{\Omega}\left|\nabla v_{\eta}\right|^{2} & =8 \pi \cdot\left(\ln \frac{R}{\eta}\right)^{-2 \kappa} \cdot \int_{0}^{R} r^{3}\left(r^{2}+\eta^{2}\right)^{-2} d r \\
& =8 \pi \cdot\left(\ln \frac{R}{\eta}\right)^{-2 \kappa} \cdot \int_{0}^{\frac{R}{\eta}} s^{3}\left(s^{2}+1\right)^{-2} d s \\
& \leq 8 \pi \cdot\left(\ln \frac{R}{\eta}\right)^{-2 \kappa} \cdot\left(1+\ln \frac{R}{\eta}\right)
\end{aligned}
$$

whereas clearly

$$
\begin{aligned}
\int_{\Omega} v_{\eta}^{2} & =2 \pi \cdot\left(\ln \frac{R}{\eta}\right)^{-2 \kappa} \cdot \int_{0}^{R} r\left(\ln \frac{R^{2}}{r^{2}+\eta^{2}}\right)^{2} d r \\
& \leq 8 \pi \cdot(\ln 2)^{-2 \kappa} \cdot \int_{0}^{R} r\left(\ln \frac{R}{r}\right)^{2} d r
\end{aligned}
$$

for all $\eta \in\left(0, \frac{R}{2}\right)$. Moreover,

$$
\begin{aligned}
\left(\ln \frac{R}{\eta}\right)^{\kappa-1} \cdot \int_{\Omega} u_{\eta} v_{\eta}= & 2 \pi a_{\eta} \cdot \eta^{\beta-2} \cdot\left(\ln \frac{R}{\eta}\right)^{-1} \cdot \int_{0}^{R} r\left(r^{2}+\eta^{2}\right)^{-\frac{\beta}{2}} \cdot \ln \frac{R^{2}}{r^{2}+\eta^{2}} d r \\
= & 2 \pi a_{\eta} \cdot\left(\ln \frac{R}{\eta}\right)^{-1} \cdot \int_{0}^{\frac{R}{\eta}} s\left(s^{2}+1\right)^{-\frac{\beta}{2}} \cdot\left(2 \ln \frac{R}{\eta}-\ln \left(s^{2}+1\right)\right) d s \\
= & 4 \pi a_{\eta} \cdot \int_{0}^{\frac{R}{\eta}} s\left(s^{2}+1\right)^{-\frac{\beta}{2}} d s \\
& -2 \pi a_{\eta} \cdot\left(\ln \frac{R}{\eta}\right)^{-1} \cdot \int_{0}^{\frac{R}{\eta}} s\left(s^{2}+1\right)^{-\frac{\beta}{2}} \ln \left(s^{2}+1\right) d s \\
\rightarrow & 4 \pi a_{0} \cdot \int_{0}^{\infty} s\left(s^{2}+1\right)^{-\frac{\beta}{2}} d s \quad \text { as } \eta \rightarrow 0
\end{aligned}
$$

and, by (3.1),

$$
\begin{aligned}
\int_{\Omega} G\left(u_{\eta}\right) & \leq 2 \pi k a_{\eta} \cdot \eta^{\beta-2} \cdot \int_{0}^{R} r\left(r^{2}+\eta^{2}\right)^{-\frac{\beta}{2}} \cdot\left(\ln \left(a_{\eta} \eta^{\beta-2}\left(r^{2}+\eta^{2}\right)^{-\frac{\beta}{2}}\right)\right)^{\theta} d r \\
& =2 \pi k a_{\eta} \cdot \int_{0}^{\frac{R}{\eta}} s\left(s^{2}+1\right)^{-\frac{\beta}{2}} \cdot\left(\ln \left(a_{\eta} \eta^{-2}\left(s^{2}+1\right)^{-\frac{\beta}{2}}\right)\right)^{\theta} d s \\
& \leq 2 \pi k a_{\eta} \cdot\left(2 \ln \frac{\sqrt{a_{\eta}}}{\eta}\right)^{\theta} \cdot \int_{0}^{\infty} s\left(s^{2}+1\right)^{-\frac{\beta}{2}} d s
\end{aligned}
$$

It therefore follows that

$$
F\left(u_{\eta}, v_{\eta}\right) \leq-c_{2}\left(\ln \frac{R}{\eta}\right)^{1-\kappa}+c_{3}\left(1+\left(\ln \frac{R}{\eta}\right)^{1-2 \kappa}+\left(\ln \frac{\sqrt{a_{\eta}}}{\eta}\right)^{\theta}\right)
$$

for all $\eta \in\left(0, \frac{R}{2}\right)$ with positive constants $c_{2}$ and $c_{3}$. Since

$$
1-\kappa>0, \quad 1-\kappa>1-2 \kappa \quad \text { and } \quad 1-\kappa>\theta
$$

due to our choice of $\kappa$, we again infer that $F\left(u_{\eta}, v_{\eta}\right) \rightarrow-\infty$ as $\eta \rightarrow 0$ and conclude as before. ////

## 4 Blow-up

It is not the purpose of the present paper to develop a refined existence and uniqueness theory under optimal regularity assumptions on $\phi, \psi, \partial \Omega$ and the initial data. Since we intend to use $\left(u_{0}, v_{0}\right)$ as provided by Lemma 3.1 as initial data, the only element from existence theory that we need here is the fact that if $\Omega$ is a ball in $\mathbb{R}^{n}$ then for any positive $\left(u_{0}, v_{0}\right) \in\left(C^{\infty}(\bar{\Omega})\right)^{2}$, there exists a maximal existence time $T_{\max } \leq \infty$ such that (0.1) possesses at least one classical solution $(u, v)$ in $\Omega \times\left(0, T_{\max }\right)$, and that the alternative

$$
\begin{equation*}
\text { either } T_{\max }=\infty \quad \text { or } \quad \limsup _{t \nearrow T_{\max }}\|u(\cdot, t)\|_{L^{\infty}(\Omega)}=\infty \tag{4.1}
\end{equation*}
$$

holds. The existence of at least one such solution can be demonstrated by means of either Schauder's fixed point theorem or general theory of quasilinear parabolic systems ([A]; cf. also [C1], [C2] or [Wr] for corresponding procedures in closely related problems); the existence of a maximal existence time along with its property (4.1) can be deduced from standard extendibility arguments.

Theorem 4.1 Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n}$ be a ball, and suppose that there exist $s_{0}>1, \varepsilon \in(0,1)$, $K>0$ and $k>0$ such that

$$
\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \leq \begin{cases}K \frac{s}{\ln s} & \text { if } n=2  \tag{4.2}\\ \frac{n-2-\varepsilon}{n} \int_{s_{0}}^{s} \int_{s_{0}}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)} d \tau d \sigma+K s & \text { if } n \geq 3\end{cases}
$$

as well as

$$
\int_{s_{0}}^{s} \int_{s_{0}}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)} d \tau d \sigma \leq \begin{cases}k s(\ln s)^{\theta} & \text { with some } \theta \in(0,1) \text { if } n=2  \tag{4.3}\\ k s^{2-\alpha} & \text { with some } \alpha>\frac{2}{n} \text { if } n \geq 3\end{cases}
$$

hold for all $s \geq s_{0}$. Then for each $m>0$ there exist initial data $\left(u_{0}, v_{0}\right) \in\left(C^{\infty}(\bar{\Omega})\right)^{2}$ with $\int_{\Omega} u_{0}=m$ such that the corresponding solution $(u, v)$ blows up in either finite or infinite time.

Proof. Let $m>0$ be given. From Lemma 1.2 we know that each global bounded solution $(u, v)$ of (0.1) gives rise to a steady-state solution $\left(u_{\infty}, v_{\infty}\right)$ of (1.5) satisfying $F\left(u_{\infty}, v_{\infty}\right) \leq$ $F(u(\cdot, 0), v(\cdot, 0))$. By Lemma 2.4 and Lemma 2.3, this entails that for some $C>0$ we have $F(u(\cdot, 0), v(\cdot, 0)) \geq-C$ whenever $(u, v)$ is global and bounded. But Lemma 3.1 says that there exist smooth initial data $\left(u_{0}, v_{0}\right)$ with $\int_{\Omega} u_{0}=m$ but $F\left(u_{0}, v_{0}\right)<-C$. In view of (4.1), the corresponding solution of ( $u, v$ ) evidently must blow up.

Let us finally specify some conditions on $\frac{\psi}{\phi}$ that are sufficient to guarantee (4.2) and (4.3) but easier to verify.

Corollary 4.2 In each ball $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, for any $m>0$ there exist unbounded solutions of (0.1) having mass $\int_{\Omega} u(x, t) \equiv m$, provided that one of the following hypotheses is satisfied.
i) $n=2$ and there exist $c_{0}>0$ and $s_{0}>1$ such that

$$
\begin{equation*}
\frac{\psi(s)}{\phi(s)} \geq c_{0} s \ln s \quad \text { for all } s \geq s_{0} \tag{4.4}
\end{equation*}
$$

holds.
ii) $n \geq 3$ and for some $c_{0}>0$ and $s_{0}>1$, the lower estimate

$$
\begin{equation*}
\frac{\psi(s)}{\phi(s)} \geq c_{0} s \quad \text { for all } s \geq s_{0} \tag{4.5}
\end{equation*}
$$

is valid.
iii) $n \geq 3$ and for some $c_{0}>0$ and $\alpha>\frac{2}{n}$ we have

$$
\begin{equation*}
s^{-\alpha} \frac{\psi(s)}{\phi(s)} \rightarrow c_{0} \quad \text { as } s \rightarrow \infty \tag{4.6}
\end{equation*}
$$

iv) $n \geq 3$ and

$$
\begin{equation*}
\liminf _{s \rightarrow \infty} \frac{s\left(\frac{\psi}{\phi}\right)^{\prime}(s)}{\left(\frac{\psi}{\phi}\right)(s)}>\frac{2}{n} \tag{4.7}
\end{equation*}
$$

Proof. i) We may assume that $s_{0} \geq e^{2}$. From (4.4) we obtain $\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \leq \frac{1}{c_{0}} \int_{s_{0}}^{s} \frac{d \sigma}{\ln \sigma}$, and since

$$
\frac{d}{d s}\left(\int_{s_{0}}^{s} \frac{d \sigma}{\ln \sigma}-\frac{2 s}{\ln s}\right)=\frac{2-\ln s}{(\ln s)^{2}} \leq 0 \quad \text { for all } s \geq e^{2}
$$

(4.2) follows with $K:=\frac{2}{c_{0}}$ upon integrating this.

In order to show that (4.3) actually holds for all $\theta \in(0,1)$, we observe that because of $s_{0} \geq e^{2}$, we have

$$
\begin{aligned}
\frac{d^{2}}{d s^{2}}\left(s(\ln s)^{\theta}\right) & =\frac{\theta(\ln s)^{\theta}}{s \ln s} \cdot\left(1-\frac{1-\theta}{\ln s}\right) \\
& \geq \frac{\theta(\ln s)^{\theta}}{s}\left(1-\frac{1-\theta}{2}\right) \\
& \geq \frac{\theta \cdot 2^{\theta}}{s \ln s} \cdot \frac{1+\theta}{2} \quad \text { for all } s \geq s_{0}
\end{aligned}
$$

Integrating this twice, we easily derive (4.3) upon the choice $k:=\frac{2^{1-\theta}}{\theta(1+\theta) c_{0}}$.
ii) As, by (4.5),

$$
\int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \leq \frac{1}{c_{0}}\left(s-s_{0}\right) \leq \frac{1}{c_{0}} s \quad \text { for all } s \geq s_{0}
$$

and since the first term on the right of (4.2) is nonnegative, we may let $K:=\frac{1}{c_{0}}$ and see that (4.2) is satisfied, whereas (4.3) is trivial here.
iii) In view of ii) we only need to consider the case when $\alpha<1$, in which we fix $\kappa \in\left(1, \frac{1-\frac{2}{n}}{1-\alpha}\right)$ and let $c_{1}>0$ be such that $c_{1}<c_{0}<\kappa c_{1}$. Then (4.6) guarantees that for some $s_{0}>1$, we have

$$
\begin{equation*}
c_{1} s^{\alpha} \leq \frac{\psi(s)}{\phi(s)} \leq \kappa c_{1} s^{\alpha} \quad \text { for all } s \geq s_{0} \tag{4.8}
\end{equation*}
$$

We now pick $\varepsilon \in(0,1)$ such that $\varepsilon<n-2-n(1-\alpha) \kappa$ and use (4.8) to estimate

$$
\begin{aligned}
& \int_{s_{0}}^{s} \int_{s_{0}}^{\sigma} \frac{\phi(\tau)}{\psi(\tau)} d \tau d \sigma-\frac{n}{n-2-\varepsilon} \int_{s_{0}}^{s} \frac{\sigma \phi(\sigma)}{\psi(\sigma)} d \sigma \\
& \quad \geq \frac{1}{\kappa c_{1}(1-\alpha)(2-\alpha)}\left(s^{2-\alpha}-s_{0}^{2-\alpha}\right)-\frac{s_{0}^{1-\alpha}}{\kappa c_{1}(1-\alpha)}\left(s-s_{0}\right)-\frac{n}{(n-2-\varepsilon) c_{1}(2-\alpha)}\left(s^{2-\alpha}-s_{0}^{2-\alpha}\right) \\
& \quad \geq \frac{1}{c_{1}(2-\alpha)}\left(\frac{1}{\kappa(1-\alpha)}-\frac{n}{n-2-\varepsilon}\right) s^{2-\alpha}-\frac{s_{0}^{1-\alpha}}{\kappa c_{1}(1-\alpha)} s-\frac{s_{0}^{2-\alpha}}{\kappa c_{1}(1-\alpha)(2-\alpha)}
\end{aligned}
$$

for all $s \geq s_{0}$, where according to the choice of $\varepsilon$, the first term on the right is nonnegative. Hence it follows that (4.2) is true if we set $K:=\frac{(3-\alpha)(n-2-\varepsilon) s_{0}^{1-\alpha}}{\kappa c_{1}(1-\alpha)(2-\alpha) n}$, for instance, whereas (4.3) is obvious.
iv) We note that (4.7) implies the existence of $\varepsilon \in(0,1)$ and $s_{0}>1$ such that

$$
\begin{equation*}
\frac{d}{d s} \frac{s \phi(s)}{\psi(s)} \leq \frac{n-2-\varepsilon}{n} \cdot \frac{\phi(s)}{\psi(s)} \quad \text { for all } s \geq s_{0} \tag{4.9}
\end{equation*}
$$

From this, we immediately derive (4.2) with $K:=0$ upon two integrations. Rewriting (4.9) in the equivalent form

$$
\frac{\frac{d}{d s} \frac{\psi(s)}{\phi(s)}}{\frac{\psi(s)}{\phi(s)}} \geq \frac{2+\varepsilon}{n s} \quad \text { for all } s \geq s_{0}
$$

again by integration we also obtain (4.3) with $\alpha:=\frac{2+\varepsilon}{n} \in\left(\frac{2}{n}, 1\right)$ and $k:=\frac{s_{0}^{\alpha} \phi\left(s_{0}\right)}{(1-\alpha)(2-\alpha) \psi\left(s_{0}\right)}$. ////

## References

[A] Amann, H.: Dynamic Theory of Quasilinear Parabolic Systems. III.Global Existence. Math. Zeitschrift 202, 219-250 (1989)
[CC] Calvez, V., Carillo, J.A.: Volume effects in the Keller-Segel model: energy estimates preventing blow-up. J. Math. Pures Appl. 86, 155-175 (2006).
[C1] CieśLak, T.: Quasilinear nonuniformly parabolic system modelling chemotaxis. J. Math. Anal. Appl. 326 (2), 1410-1426 (2007)
[C2] Cieślak, T.: Global existence of solutions to a chemotaxis system with volume filling effect. Colloq. Math. 111 (1), 117-134 (2008)
[CM-R] Cieślak, T., Morales-Rodrigo, C.: Quasilinear non-uniformly parabolic-elliptic system modelling chemotaxis with volume filling effect. Existence and uniqueness of global-in-time solutions. Topol. Methods Nonlinear Anal. 29 (2), 361-381 (2007)
[CW] Cieślak, T., Winkler, M.: Finite-time blow-up in a quasilinear system of chemotaxis. Nonlinearity 21, 1057-1076 (2008)
[DW] Djie, K., Winkler, M.: Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect. Submitted
[GZ] Gajewski, H., Zacharias, K.: Global behavior of a reaction-diffusion system modelling chemotaxis. Math. Nachr. 195, 77-114 (1998)
[HV] Herrero, M. A., Velázquez, J. J. L.: A blow-up mechanism for a chemotaxis model. Ann. Scuola Normale Superiore 24, 663-683 (1997)
[HP1] Hillen, T., Painter, K.J.: Global existence for a parabolic chemotaxis model with prevention of overcrowding. Adv. Appl. Math. 26, 280-301 (2001)
[HP2] Hillen, T., Painter, K.: A users' guide to PDE models for chemotaxis. To appear in: J. Math. Biol.
[H] Horstmann, D.: From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I. Jahresberichte DMV 105 (3), 103-165 (2003)
[HWa] HWa Horstmann, D., Wang, G.: Blow-up in a chemotaxis model without symmetry assumptions. Eur. J. Appl. Math. 12, 159-177 (2001)
[HWi] Horstmann, D., Winkler, M.: Boundedness vs. blow-up in a chemotaxis system. J. Differential Equations 215 (1), 52-107 (2005)
[KS] Keller, E. F., Segel, L. A.: Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol. 26, 399-415 (1970)
[LSU] Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasi-linear Equations of Parabolic Type. AMS, Providence, 1968
[N] Nagai, T.: Global Existence and Blowup of Solutions to a Chemotaxis System. Nonlinear Anal. 47, 777-787 (2001)
[NSS] Nagai, T., Senba, T., Suzuki, T.: Chemotactic collapse in a parabolic system of mathematical biology. Hiroshima Math. J. 30, 463-497 (2000)
[NSY] Nagai, T., Senba, T., Yoshida, K.: Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. Funkc. Ekvacioj, Ser. Int. 40, 411-433 (1997)
[PH] Painter, K., Hillen, T.: Volume-filling and quorum-sensing in models for chemosensitive movement. Can. Appl. Math. Q. 10 (4), 501-543 (2002)
[SeS] Senba, T., Suzuki, T.: Parabolic system of chemotaxis: blowup in a finite and the infinite time. Methods Appl. Anal. 8, 349-367 (2001)
[Wi] Winkler, M.: Aggregation vs. global diffusive behavior in the higher-dimensional KellerSegel model. Submitted
[Wr] Wrzosek, D.: Long-time behaviour of solutions to a chemotaxis model with volume-filling effect. Proc. Roy. Soc. Edinburgh, 136A, 431-444 (2006)

