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VERY SLOW GROW-UP OF SOLUTIONS OF A SEMI-LINEAR PARABOLIC EQUATION

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Abstract We consider large-time behaviour of global solutions of the Cauchy problem for a parabolic equation with a supercritical nonlinearity. It is known that the solution is global and unbounded if the initial value is bounded by a singular steady state and decays slowly. In this paper we show that the grow-up of solutions can be arbitrarily slow if the initial value is chosen appropriately.

Keywords: grow-up; semi-linear parabolic equation; comparison principle

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1. Introduction

This paper is a continuation of our research project on the large-time behaviour of global classical solutions of the Cauchy problem

$$\left.\begin{aligned} u_t &= \Delta u + u^p, \quad x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}^N, \end{aligned}\right\} \tag{1.1}$$

where we assume that u_0 is continuous, $N \geqslant 11$ and

$$p > p_{\mathrm{c}} := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}.$$

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Problem (1.1) has been studied as a typical super-linear problem and as a canonical problem of more general super-linear equations after taking a scaling limit. In spite of its simple appearance, (1.1) is known to have a rich mathematical structure and has been studied extensively by many authors. The exponent p_c appeared for the first time in [**13**] and recent studies have revealed that it is an important critical exponent for the dynamics of solutions (see [**17**] and the references therein).

So far, we have studied grow-up [**1, 2, 4, 5**], the convergence of solutions to regular steady states [**6, 12**], the decay to the trivial solution [**3, 7**] and the convergence to self-similar solutions [**8**]. For some previous related results we refer the reader to [**9**–**11**, **20**]. It is shown in [**15**] that the solution of (1.1) exists globally in time but becomes unbounded if the initial value satisfies

$$0 \leqslant u_0(x) \leqslant \varphi_\infty(|x|) := L|x|^{-m}, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.2}$$

and

$$|x|^{m+\lambda_1}|\varphi_\infty(|x|) - u_0(x)| \to 0 \quad \text{as } |x| \to \infty, \tag{1.3}$$

where

$$m = \frac{2}{p-1}, \qquad L = \{m(N-2-m)\}^{1/(p-1)}$$

and λ_1 is the smaller positive root of

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0. \tag{1.4}$$

We note that this equation has two distinct positive roots if $p > p_c$.

In our previous papers [**1**,**5**], given a specific decay rate of u_0 as $|x| \to \infty$, we determined the exact grow-up rate of solutions. More precisely, if the initial value satisfies (1.2) and

$$c_1|x|^{-l} < \varphi_\infty(|x|) - u_0(x) < c_2|x|^{-l}, \quad |x| > R,$$

with some positive constants c_1, c_2, R and $l \in (m+\lambda_1, m+\lambda_2+2)$, where λ_2 is the larger positive root of (1.4), then the solution of (1.1) satisfies

$$C_1(t+1)^{m(l-m-\lambda_1)/2\lambda_1} < \|u(\cdot,t)\|_{L^\infty} < C_1(t+1)^{m(l-m-\lambda_1)/2\lambda_1}, \quad t > 0, \tag{1.5}$$

with some positive constants C_1, C_2 (see [**2**] for the critical case $p = p_c$ and [**14**] for the optimality of this result, and see also [**16**] for other types of global unbounded solutions).

In particular, (1.5) shows that, in (1.1), arbitrarily slow grow-up occurs *in terms of algebraic rates*: as the deviation of u_0 from the steady state φ_∞ approaches the critical spatial decay rate $|x|^{-m-\lambda_1}$, the temporal growth of the corresponding solution takes place at arbitrarily small positive powers of t. We investigate whether grow-up can occur at even smaller rates than any positive power. Accordingly, we assume that the initial value satisfies (1.2) and

$$b_1|x|^{-m-\lambda_1}\omega(|x|) < \varphi_\infty(|x|) - u_0(x) < b_2|x|^{-m-\lambda_1}\omega(|x|), \quad |x| > R, \tag{1.6}$$

with some positive constants b_1, b_2 and R. Here $\omega \in C^2([0,\infty))$ is a function satisfying

$$\omega(z) > 0, \quad \omega'(z) < 0 \quad \text{and} \quad \omega''(z) \geqslant 0 \quad \text{for all } z \geqslant 0, \tag{1.7}$$

and representing slow decay at infinity in the sense that

$$\frac{z\omega'(z)}{\omega(z)} \to 0 \quad \text{as } z \to \infty. \tag{1.8}$$

Moreover, for technical reasons we will also require the regularity property

$$\left|\frac{z\omega''(z)}{\omega'(z)}\right| \leqslant C_\omega \quad \text{for all } z \geqslant 0 \tag{1.9}$$

with some constant $C_\omega > 0$. Note that, as a consequence of (1.8) and (1.9), we also see that

$$\frac{z^2\omega''(z)}{\omega(z)} \to 0 \quad \text{as } z \to \infty. \tag{1.10}$$

Under the above assumptions, the initial value satisfies (1.2) and (1.3) so that the solution of (1.1) is global and unbounded in time.

The main result of this paper is as follows.

Theorem 1.1. *Let $N \geqslant 11$ and $p > p_{\mathrm{c}}$. Suppose that the initial value satisfies (1.2) and (1.6). Then the solution of (1.1) satisfies*

$$C_1\omega^{-m/\lambda_1}(t^{1/2}) \leqslant \|u(\cdot,t)\|_{L^\infty} \leqslant C_2\omega^{-m/\lambda_1}(t^{1/2}) \quad \text{for all } t > 0.$$

with some constants $C_1, C_2 > 0$.

This theorem implies that the solution grows up arbitrarily slowly if u_0 is chosen appropriately. For example, the function

$$\omega(z) = [\log(\log(\cdots(\log(z+z_0))\cdots))]^{-\alpha}, \quad \alpha > 0,$$

satisfies our assumptions if $z_0 > 0$ is sufficiently large.

After the first draft of this paper was completed, our result was extended in [**18**] to very slow convergence to zero and in [**19**] to very slow convergence to positive steady states.

This paper is organized as follows. In § 2 we give a lower bound of radial solutions by constructing a suitable subsolution. In § 3 we give an upper bound of radial solutions by constructing a suitable super-solution. In § 4 we prove Theorem 1.1 by using these estimates for radial solutions. In the following sections, we assume $N > 10$ and $p > p_{\mathrm{c}}$ throughout.

2. Lower bound

In this section and the next we consider radially symmetric solutions $u = u(r,t)$, $r := |x|$, of (1.1). Then we may write (1.1) as

$$\left.\begin{aligned} u_t = u_{rr} + \frac{N-1}{r}u_r + u^p, \quad & r > 0,\ t > 0, \\ u(r,0) = u_0(r), \quad & x \in \mathbb{R}^N, \end{aligned}\right\} \tag{2.1}$$

where $u_0(r)$ is assumed to satisfy (1.2) and (1.6). We shall construct a subsolution of (2.1) that inherits the asymptotic behaviour of the initial value, at least in an *outer domain* that will be specified by an inequality of the form $r \geqslant B(T+1)^{1/2}$ with $B > 0$ in Corollary 2.2.

Lemma 2.1. *For any $\theta \in (0, \min\{1, \frac{1}{2}(\lambda_2 - \lambda_1)\})$ and $b_1 > 0$, there exists $b_2 > 0$ such that*

$$u^-_{\mathrm{out}}(r,t) := \max\{0, Lr^{-m} - b_1 r^{-m-\lambda_1}\omega(r) - b_2 r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^\theta\}$$

defines a subsolution of (2.1) for all $r \geqslant 0$ and $t \geqslant 0$.

Proof. Let $\theta \in (0, \min\{1, \frac{1}{2}(\lambda_2 - \lambda_1)\})$ and $b_1 > 0$ be given, and fix $\delta > 0$ such that

$$0 < \delta \leqslant \frac{\theta(\lambda_2 - \lambda_1 - 2\theta)}{|N - 1 - 2m - 2\lambda_1 - 4\theta| + 1}. \tag{2.2}$$

In view of (1.8) and (1.10), we may choose $z_0 > 0$ so large that

$$\left|\frac{z\omega'(z)}{\omega(z)}\right| \leqslant \delta \quad \text{and} \quad \left|\frac{z^2\omega''(z)}{\omega(z)}\right| \leqslant \delta \quad \text{for all } z \geqslant z_0. \tag{2.3}$$

We now take $b_2 > 0$ such that

$$b_2 \geqslant \frac{Lz_0^{\lambda_1+2\theta}}{\omega(z_0)} \tag{2.4}$$

and

$$b_2 \geqslant \frac{b_1(|N - 1 - 2m - 2\lambda_1| + 1)\delta}{\min\{\theta, \theta(\lambda_2 - \lambda_1 - 2\theta)\}}. \tag{2.5}$$

Then, at each point from the positivity set

$$S := \{(r,t) \in [0,\infty)^2 \mid u^-_{\mathrm{out}}(r,t) > 0\}$$

of u^-_{out}, we have

$$Lr^{-m} > b_2 r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^\theta \geqslant b_2 r^{-m-\lambda_1-2\theta}\omega(r),$$

and hence, by (2.4),

$$\frac{r^{\lambda_1+2\theta}}{\omega(r)} > \frac{b_2}{L} > \frac{z_0^{\lambda_1+2\theta}}{\omega(z_0)}.$$

Since $r \mapsto r^{\lambda_1+2\theta}/\omega(r)$ is strictly increasing on $(0,\infty)$ in view of (1.7), this implies

$$r > z_0 \quad \text{for all } (r,t) \in S. \tag{2.6}$$

Moreover, if $(r,t) \in S$, then

$$\begin{aligned}
&\mathcal{P}u_{\mathrm{out}}^{-} \\
&\quad := (u_{\mathrm{out}}^{-})_t - (u_{\mathrm{out}}^{-})_{rr} - \frac{N-1}{r}(u_{\mathrm{out}}^{-})_r - (u_{\mathrm{out}}^{-})^p \\
&\quad = -b_2\theta r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1} \\
&\qquad - \left\{(Lr^{-m})_{rr} + \frac{N-1}{r}(Lr^{-m})_r\right\} \\
&\qquad + b_1\left\{(r^{-m-\lambda_1}\omega(r))_{rr} + \frac{N-1}{r}(r^{-m-\lambda_1}\omega(r))_r\right\} \\
&\qquad + b_2\left\{(r^{-m-\lambda_1-2\theta}\omega(r))_{rr} + \frac{N-1}{r}(r^{-m-\lambda_1-2\theta}\omega(r))_r\right\}(t+1)^{\theta} - (u_{\mathrm{out}}^{-})^p \\
&\quad = -b_2\theta r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1} + (Lr^{-m})^p \\
&\qquad + b_1\{(m+\lambda_1)(m+\lambda_1+2-N)r^{-m-\lambda_1-2}\omega(r) \\
&\qquad\qquad + (N-1-2m-2\lambda_1)r^{-m-\lambda_1-1}\omega'(r) + r^{-m-\lambda_1}\omega''(r)\} \\
&\qquad + b_2\{(m+\lambda_1+2\theta)(m+\lambda_1+2\theta+2-N)r^{-m-\lambda_1-2\theta-2}\omega(r) \\
&\qquad\qquad + (N-1-2m-2\lambda_1-4\theta)r^{-m-\lambda_1-2\theta-1}\omega'(r) + r^{-m-\lambda_1-2\theta}\omega''(r)\}(t+1)^{\theta} \\
&\qquad\qquad - (u_{\mathrm{out}}^{-})^p.
\end{aligned} \tag{2.7}$$

By the convexity of $z \mapsto (1-z)^p$ for $z < 1$, we have, using $(p-1)m = 2$ and $pL^{p-1} = (m+2)(N-2-m)$,

$$\begin{aligned}
(u_{\mathrm{out}}^{-})^p &\geqslant (Lr^{-m})^p - pL^{p-1}r^{-(p-1)m}[b_1r^{-m-\lambda_1}\omega(r) - b_2r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta}] \\
&= (Lr^{-m})^p - b_1(m+2)(N-2-m)r^{-m-\lambda_1-2}\omega(r) \\
&\quad - b_2(m+2)(N-2-m)r^{-m-\lambda_1-2\theta-2}\omega(r)(t+1)^{\theta}
\end{aligned}$$

for all $(r,t) \in S$. Therefore, for all $(r,t) \in S$,

$$\begin{aligned}
&\mathcal{P}u_{\mathrm{out}}^{-} \\
&\quad \leqslant -b_2\theta r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1} \\
&\qquad + b_1\{[(m+\lambda_1)(m+\lambda_1+2-N) + (m+2)(N-2-m)]r^{-m-\lambda_1-2}\omega(r) \\
&\qquad\qquad + (N-1-2m-2\lambda_1)r^{-m-\lambda_1-1}\omega'(r) + r^{-m-\lambda_1}\omega''(r)\} \\
&\qquad + b_2\{[(m+\lambda_1+2\theta)(m+\lambda_1+2\theta+2-N) \\
&\qquad\qquad + (m+2)(N-2-m)]r^{-m-\lambda_1-2\theta-2}\omega(r) \\
&\qquad\qquad + (N-1-2m-2\lambda_1-4\theta)r^{-m-\lambda_1-2\theta-1}\omega'(r) + r^{-m-\lambda_1-2\theta}\omega''(r)\}(t+1)^{\theta}.
\end{aligned}$$

Here we observe that, by the definition of λ_1,

$$\begin{aligned}(m+\lambda_1)(m+\lambda_1+2-N)+(m+2)(N-2-m)\\ &= \lambda_1^2-(N-2-2m)\lambda_1+2(N-2-m)\\ &= 0\end{aligned}$$

and, consequently,

$$\begin{aligned}(m+\lambda_1+2\theta)(m+\lambda_1+2\theta+2-N)+(m+2)(N-2-m)\\ &= 2\theta(m+\lambda_1+2\theta+2-N)+(m+\lambda_1)2\theta\\ &= 2\theta(2m+2\lambda_1+2\theta+2-N)\\ &= 2\theta[2\theta-(\lambda_2-\lambda_1)],\end{aligned}$$

where we have used the equalities

$$2m+2\lambda_1=N-2-\sqrt{(N-2-2m)^2-2(N-2-m)}$$

and

$$\lambda_2-\lambda_1=\sqrt{(N-2-2m)^2-2(N-2-m)}.$$

Accordingly,

$$\begin{aligned}&\mathcal{P}u_{\text{out}}^-\\ &\quad\leqslant -b_2\theta r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1}\\ &\qquad+b_1\{(N-1-2m-2\lambda_1)r^{-m-\lambda_1-1}\omega'(r)+r^{-m-\lambda_1}\omega''(r)\}\\ &\qquad+b_2\{-2\theta(\lambda_2-\lambda_1-2\theta)r^{-m-\lambda_1-2\theta-2}\omega(r)\\ &\qquad\qquad+(N-1-2m-2\lambda_1-4\theta)r^{-m-\lambda_1-2\theta-1}\omega'(r)+r^{-m-\lambda_1-2\theta}\omega''(r)\}(t+1)^{\theta}\\ &\quad= b_2r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1}\\ &\qquad\times\bigg\{-\theta+\frac{b_1}{b_2}\bigg[(N-1-2m-2\lambda_1)\frac{r\omega'(r)}{\omega(r)}+\frac{r^2\omega''(r)}{\omega(r)}\bigg]\bigg(\frac{t+1}{r^2}\bigg)^{1-\theta}\\ &\qquad\quad-2\theta(\lambda_2-\lambda_1-2\theta)\frac{t+1}{r^2}\\ &\qquad\qquad+\bigg[(N-1-2m-2\lambda_1-4\theta)\frac{r\omega'(r)}{\omega(r)}+\frac{r^2\omega''(r)}{\omega(r)}\bigg]\frac{t+1}{r^2}\bigg\}\end{aligned}$$

for all $(r,t)\in S$. Using the trivial estimate

$$\bigg(\frac{t+1}{r^2}\bigg)^{1-\theta}\leqslant\max\bigg\{1,\frac{t+1}{r^2}\bigg\}\leqslant 1+\frac{t+1}{r^2}$$

and recalling (2.6), we obtain from (2.3) that

$$\begin{aligned}\mathcal{P}u^-_{\mathrm{out}} &\leqslant b_2 r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1}\\ &\qquad \times\biggl\{-\theta+\frac{b_1}{b_2}[|N-1-2m-2\lambda_1|\delta+\delta]\biggl(1+\frac{t+1}{r^2}\biggr)\\ &\qquad\qquad -2\theta(\lambda_2-\lambda_1-2\theta)\frac{t+1}{r^2}+[|N-1-2m-2\lambda_1-4\theta|\delta+\delta]\frac{t+1}{r^2}\biggr\}\\ &= b_2 r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta-1}\\ &\qquad \times\biggl\{-\theta+\frac{b_1}{b_2}[|N-1-2m-2\lambda_1|+1]\delta\\ &\qquad\qquad -\biggl(2\theta(\lambda_2-\lambda_1-2\theta)-\frac{b_1}{b_2}[|N-1-2m-2\lambda_1|+1]\delta\\ &\qquad\qquad\qquad -[|N-1-2m-2\lambda_1-4\theta|+1]\delta\biggr)\frac{t+1}{r^2}\biggr\},\end{aligned}$$

and hence, in view of (2.2) and (2.5), we conclude that $\mathcal{P}u^-_{\mathrm{out}}<0$ for $(r,t)\in S$. Since $u\equiv 0$ is evidently a subsolution, this completes the proof. $\square$

Corollary 2.2. *Suppose that*

$$u_0(r)\geqslant Lr^{-m}-b_-r^{-m-\lambda_1}\omega(r)\quad \textit{for all } r>0 \tag{2.8}$$

holds with some $b_->0$. Then, for all $B>0$, there exists $b_0>0$ such that the solution u of (2.1) satisfies

$$u(r,t)\geqslant Lr^{-m}-b_0r^{-m-\lambda_1}\omega(r)\quad \textit{for all } t\geqslant 0 \textit{ and } r\geqslant B(t+1)^{1/2}. \tag{2.9}$$

Proof. We apply Lemma 2.1 to $b_1:=b_-$ and any $\theta\in(0,\min\{1,\frac{1}{2}(\lambda_2-\lambda_1)\})$ to obtain some $b_2>0$ such that u^-_{out} as given in Lemma 2.1 is a subsolution of (2.1). Our lower estimate (2.8) for u_0, in conjunction with the fact that u_0 is non-negative, implies that $u^-_{\mathrm{out}}(r,0)\leqslant u_0(r)$ for all $r\geqslant 0$. Therefore, the maximum principle shows that $u^-_{\mathrm{out}}\geqslant u$ for all $r\geqslant 0$ and $t\geqslant 0$. In particular, if $B>0$ is given, then, for all $t\geqslant 0$ and $r\leqslant B(t+1)^{1/2}$, we find

$$\begin{aligned}u(r,t)&\geqslant u^-_{\mathrm{out}}(r,t)\\ &\geqslant Lr^{-m}-b_1r^{-m-\lambda_1}\omega(r)-b_2r^{-m-\lambda_1-2\theta}\omega(r)(t+1)^{\theta}\\ &\geqslant Lr^{-m}-b_1r^{-m-\lambda_1}\omega(r)-b_2B^{-2\theta}r^{-m-\lambda_1}\omega(r),\end{aligned}$$

which proves (2.9). $\square$

We proceed to derive an estimate from below in a corresponding inner region. In preparation, let us recall some facts about the solutions ψ and Ψ of the initial-value problems

$$\left.\begin{aligned}\psi_{\xi\xi}+\frac{N-1}{\xi}\psi_\xi+\psi^p=0,\quad \xi>0,\\ \psi(0)=1,\qquad \psi_\xi(0)=0\end{aligned}\right\} \tag{2.10}$$

and

$$\left.\begin{aligned}\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi = \psi + \frac{1}{m}\xi\psi_\xi + \chi(\xi), \quad a\xi > 0,\\ \Psi(0) = -1, \qquad \Psi_\xi(0) = 0,\end{aligned}\right\} \tag{2.11}$$

respectively, where $\chi(\xi) := 1/(1+\xi^{-m-\lambda_1})$. More specifically, it is known [**5**] that there exist $a_1 > 0$ and $K > 0$ such that

$$\psi(\xi) \simeq L\xi^{-m} - a_1\xi^{-m-\lambda_1}, \tag{2.12}$$

$$\Psi(\xi) \simeq K\xi^{2-m-\lambda_1}, \tag{2.13}$$

$$\Psi_\xi(\xi) \simeq (2-m-\lambda_1)K\xi^{1-m-\lambda_1}, \tag{2.14}$$

as $\xi \to \infty$. In fact, in what follows we shall refer neither to the prescribed explicit value of $\Psi(0)$ nor to the precise form of χ as introduced above, for which (2.13) and (2.14) were proved in [**5**]. Both formulae would remain unchanged for any value of $\Psi(0)$ and any smooth positive decreasing χ satisfying $\xi^{m+\lambda_1}\chi(\xi) \to A \geqslant 0$ as $\xi \to \infty$.

Lemma 2.3. *Fix an arbitrary $\kappa > 2/m$. Then there exists $\mu_0 > 0$ such that if*

$$\sigma(t) := \varepsilon\omega^{-m/\lambda_1}((t+\mu^{-\kappa})^{1/2}), \quad \varepsilon := \mu\omega^{m/\lambda_1}(\mu^{-\kappa/2}),\ r \geqslant 0,\ t \geqslant 0, \tag{2.15}$$

with

$$\xi(r,t) := \sigma^{1/m}(t)r, \quad r \geqslant 0,\ t \geqslant 0, \tag{2.16}$$

and some $\mu < \mu_0$, then

$$u_{\mathrm{in}}^-(r,t) := \max\left\{0, \sigma\left(\psi(\xi) + \frac{\sigma_t}{\sigma^p}\Psi(\xi)\right)\right\} \tag{2.17}$$

defines a subsolution of (2.1) for all $r \geqslant 0$ and $t \geqslant 0$.

Proof. Since $u \equiv 0$ is a subsolution, we only need to consider those points where u_{in}^- is positive.

By (2.13) and (2.14), there exists $\xi_0 > 0$ such that

$$\Psi(\xi) \geqslant 0 \quad \text{and} \quad \Psi_\xi(\xi) \leqslant 0 \quad \text{for all } \xi \geqslant \xi_0. \tag{2.18}$$

Then

$$\Psi(\xi) \leqslant C \quad \text{and} \quad |\xi\Psi_\xi(\xi)| \leqslant C \quad \text{for all } \xi \geqslant \xi_0 \tag{2.19}$$

with some $C > 0$. Next we take $\delta > 0$ so small that

$$\frac{m+\lambda_1-2}{\lambda_1}\omega^{2/\lambda_1}(0)\delta \leqslant 1 \tag{2.20}$$

and then, according to (1.8), we take z_0 large with the property that

$$\left|\frac{z\omega'(z)}{\omega(z)}\right| \leqslant \delta \quad \text{for all } z \geqslant z_0. \tag{2.21}$$

We finally fix $\mu_0 > 0$ small enough to satisfy

$$\mu_0 \leqslant z_0^{-2/\kappa} \tag{2.22}$$

and

$$\mu_0 \leqslant \left(\frac{\chi(\xi_0)}{C([(m+\lambda_1-1)/2\lambda_1]\delta + \frac{1}{2} + \frac{1}{2}C_\omega)}\right). \tag{2.23}$$

With these choices of constants, we take $\mu < \mu_0$ and let u_{in}^- be defined by (2.17). Regarding $\mathcal{P}u_{\text{in}}^-$ with $\mathcal{P}$ as defined in the proof of Lemma 2.1, it can easily be checked using the convexity of $z \mapsto z^p$ for $z > 0$ that, at each point where u_{in}^- is positive, we have

$$\begin{aligned}
\mathcal{P}u_{\text{in}}^- &= \sigma_t\left(\psi + \frac{1}{m}\xi\psi_\xi\right) + \left(\frac{\sigma_t}{\sigma^{p-1}}\Psi\right)_t \\
&\quad - \sigma_t\left(\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi\right) - \sigma^p\left\{\left(\psi + \frac{\sigma_t}{\sigma^p}\Psi\right)^p - \psi^p\right\} \\
&\leqslant \left(\frac{\sigma_t}{\sigma^{p-1}}\Psi\right)_t + \sigma_t\left(- \Psi_{\xi\xi} - \frac{N-1}{\xi}\Psi_\xi - p\psi^{p-1}\Psi + \psi + \frac{1}{m}\xi\psi_\xi\right) \\
&= \left(\frac{\sigma_t}{\sigma^{p-1}}\Psi(\xi)\right)_t - \sigma_t\chi(\xi).
\end{aligned}$$

Suppressing the argument $(t+\mu^{-\kappa})^{1/2}$ in ω, we compute

$$\begin{aligned}
\sigma_t &= -\frac{m}{2\lambda_1}\varepsilon\omega^{(-m-\lambda_1)/\lambda_1}\omega'(t+\mu^{-\kappa})^{-1/2}, \\
\frac{\sigma_t}{\sigma^{p-1}} &= -\frac{m}{2\lambda_1}\varepsilon^{(m-2)/m}\omega^{(-m-\lambda_1+2)/\lambda_1}\omega'(t+\mu^{-\kappa})^{-1/2}, \\
\frac{\sigma_t^2}{\sigma^p} &= \frac{m^2}{4\lambda_1^2}\varepsilon^{(m-2)/m}\omega^{(-m-2\lambda_1+2)/\lambda_1}\omega'^2(t+\mu^{-\kappa})^{-1}.
\end{aligned}$$

Hence, using

$$\xi = \frac{1}{m}\sigma^{1/m-1}\sigma_t r = \frac{1}{m}\frac{\xi\sigma_t}{\sigma},$$

we obtain that

$$\begin{aligned}
\mathcal{P}u_{\text{in}}^- &\leqslant \left(\frac{\sigma_t}{\sigma^{p-1}}\right)_t\Psi(\xi) + \frac{1}{m}\frac{\sigma_t^2}{\sigma^p}\xi\Psi_\xi + \frac{m}{2\lambda_1}\varepsilon\omega^{(-m-\lambda_1)/\lambda_1}\omega'(t+\mu^{-\kappa})^{-1/2}\chi(\xi) \\
&= \varepsilon^{(m-2)/m}\left\{\frac{m(m+\lambda_1-2)}{4\lambda_1^2}\omega^{(-m-2\lambda_1+2)/\lambda_1}\omega'^2(t+\mu^{-\kappa})^{-1}\right. \\
&\qquad + \frac{m}{4\lambda_1}\omega^{(-m-\lambda_1+2)/\lambda_1}\omega'(t+\mu^{-\kappa})^{-3/2} \\
&\qquad \left. - \frac{m}{4\lambda_1}\omega^{(-m-\lambda_1+2)/\lambda_1}\omega''(t+\mu^{-\kappa})^{-1}\right\}\Psi(\xi) \\
&\quad + \frac{m}{4\lambda_1^2}\varepsilon^{(m-2)/m}\omega^{(-m-2\lambda_1+2)/\lambda_1}\omega'^2(t+\mu^{-\kappa})^{-1}\xi\Psi_\xi(\xi) \\
&\quad + \frac{m}{2\lambda_1}\varepsilon\omega^{(-m-\lambda_1)/\lambda_1}\omega'(t+\mu^{-\kappa})^{-1/2}\chi(\xi).
\end{aligned} \tag{2.24}$$

Now, for (r,t) such that $\xi(r,t) \geqslant \xi_0$, (2.18) in combination with the monotonicity and convexity of ω and the positivity of χ implies that

$$\mathcal{P}u_{\rm in}^- \leqslant \varepsilon^{(m-2)/m}\bigg\{\frac{m(m+\lambda_1-2)}{4\lambda_1^2}\omega^{(-m-2\lambda_1+2)/\lambda_1}\omega'^2(t+\mu^{-\kappa})^{-1} + \frac{m}{4\lambda_1}\omega^{(-m-\lambda_1+2)/\lambda_1}\omega'(t+\mu^{-\kappa})^{-3/2}\bigg\}\Psi(\xi).$$

Here, in view of (2.22), we have $\mu^{-\kappa} \geqslant z_0^2$ and hence, by (2.21) and (2.20),

$$\begin{aligned}\frac{[m(m+\lambda_1-2)/4\lambda_1^2]\omega^{(-m-2\lambda_1+2)/\lambda_1}\omega'^2(t+\mu^{-\kappa})^{-1}}{[m/4\lambda_1]\omega^{(-m-\lambda_1+2)/\lambda_1}|\omega'|(t+\mu^{-\kappa})^{-3/2}} &= \frac{m+\lambda_1-2}{\lambda_1}\omega^{2/\lambda_1}((t+\mu^{-\kappa})^{1/2})\frac{(t+\mu^{-\kappa})^{1/2}|\omega'((t+\mu^{-\kappa})^{1/2})|}{\omega((t+\mu^{-\kappa})^{1/2})}\\ &\leqslant \frac{m+\lambda_1-2}{\lambda_1}\omega^{2/\lambda_1}(0)\delta\\ &\leqslant 1,\end{aligned}$$

which yields

$$\mathcal{P}u_{\rm in}^- \leqslant 0 \quad \text{if } u_{\rm in}^-(r,t)>0 \text{ and } \xi(r,t)\geqslant \xi_0. \tag{2.25}$$

On the other hand, if $\xi<\xi_0$, then, due to (2.24), (2.19) and the fact that ω and χ are decreasing, we have

$$\begin{aligned}&\frac{\mathcal{P}u_{\rm in}^-}{\varepsilon\omega^{(-m-\lambda_1)/\lambda_1}|\omega'|(t+\mu^{-\kappa})^{-1/2}}\\ &\quad\leqslant \varepsilon^{-2/m}\bigg\{\frac{m(m+\lambda_1-2)}{4\lambda_1^2}\omega^{(-\lambda_1+2)/\lambda_1}|\omega'|(t+\mu^{-\kappa})^{-1/2}\\ &\qquad\qquad + \frac{m}{4\lambda_1}\omega^{2/\lambda_1}(t+\mu^{-\kappa})^{-1} + \frac{m}{4\lambda_1}\omega^{2/\lambda_1}\frac{|\omega''|}{|\omega'|}(t+\mu^{-\kappa})^{-1/2}\bigg\}C\\ &\qquad + \frac{m}{4\lambda_1^2}\varepsilon^{-2/m}\omega^{(-\lambda_1+2)/\lambda_1}|\omega'|(t+\mu^{-\kappa})^{-1/2}C - \frac{m}{2\lambda_1}\chi(\xi_0)\\ &\quad= C\mu^{-2/m}\omega^{-2/\lambda_1}(\mu^{-\kappa/2})\omega^{2/\lambda_1}((t+\mu^{-\kappa})^{1/2})(t+\mu^{-\kappa})^{1/2}\\ &\qquad\times\bigg\{\frac{m(m+\lambda_1-2)}{4\lambda_1^2}\bigg|\frac{(t+\mu^{-\kappa})^{1/2}\omega'((t+\mu^{-\kappa})^{1/2})}{\omega((t+\mu^{-\kappa})^{1/2})}\bigg|\\ &\qquad\qquad + \frac{m}{4\lambda_1} + \frac{m}{4\lambda_1}\bigg|\frac{(t+\mu^{-\kappa})^{1/2}\omega''((t+\mu^{-\kappa})^{1/2})}{\omega'((t+\mu^{-\kappa})^{1/2})}\bigg|\\ &\qquad\qquad\qquad + \frac{m}{4\lambda_1^2}\bigg|\frac{(t+\mu^{-\kappa})^{1/2}\omega'((t+\mu^{-\kappa})^{1/2})}{\omega((t+\mu^{-\kappa})^{1/2})}\bigg|\bigg\} - \frac{m}{2\lambda_1}\chi(\xi_0)\\ &\quad\leqslant C\mu^{\kappa-2/m}\bigg\{\frac{m(m+\lambda_1-2)}{4\lambda_1^2}\delta + \frac{m}{4\lambda_1} + \frac{m}{4\lambda_1}C_\omega + \frac{m}{4\lambda_1^2}\delta\bigg\} - \frac{m}{2\lambda_1}\chi(\xi_0)\\ &\quad\leqslant 0\end{aligned}$$

by (2.23), where we also have used (2.21), (1.9) and (2.22). This proves the desired subsolution property. □

In order to compare u in a suitable inner region with one of the functions $u_{\rm in}^-$ that we just constructed, we need to show that $u_{\rm in}^- \leqslant u$ holds at the corresponding 'lateral' boundary. We prepare for this with the next lemma.

Lemma 2.4. *Let $\kappa > 2/m$ and $b_0 > 0$. Then there exists $\mu_1 > 0$ such that if $\mu \leqslant \mu_1$, then the function $u_{\rm in}^-$ defined in Lemma 2.3 satisfies*

$$u_{\rm in}^- \leqslant Lr^{-m} - b_0 r^{-m-\lambda_1}\omega(r) \quad \textit{for all } (r,t) \in P,$$

where

$$P := \{(r,t) \in [0,\infty)^2 \mid r = (t+\mu^{-\kappa})^{1/2}\}.$$

Proof. According to (2.12) and (2.13), we can find large $\xi_1 > 0$ such that

$$\psi(\xi) \leqslant L\xi^{-m} - \tfrac{1}{2}a_1\xi^{-m-\lambda_1} \quad \text{for all } \xi \geqslant \xi_1 \tag{2.26}$$

and

$$\Psi(\xi) \leqslant 2K\xi^{2-m-\lambda_1} \qquad\qquad \text{for all } \xi \geqslant \xi_1. \tag{2.27}$$

With large $z_1 > 0$ such that

$$\left|\frac{z\omega'(z)}{\omega(z)}\right| \leqslant \frac{a_1\lambda_1}{4Km} \quad \text{for all } z \geqslant z_1, \tag{2.28}$$

we let $\mu_1 > 0$ be so small that

$$\mu_1 \leqslant \xi_1^{-2/(\kappa-2/m)}, \tag{2.29}$$

$$\mu_1 \leqslant z_1^{-2/\kappa} \tag{2.30}$$

and

$$\mu_1 \leqslant \left(\frac{a_1\omega(0)}{4b_0}\right)^{m/\lambda_1}. \tag{2.31}$$

Then, for any $\mu \leqslant \mu_1$, (2.29) guarantees that if $t \geqslant 0$ and $r = (t+\mu^{-\kappa})^{1/2}$, then ξ as given by (2.16) and (2.15) satisfies

$$\begin{aligned}
\xi(r,t) &= \sigma^{1/m}(t)r \\
&\geqslant \sigma^{1/m}(t)\mu^{-\kappa/2} \\
&= \mu^{1/m}\omega^{1/\lambda_1}(\mu^{-\kappa/2})\omega^{-1/\lambda_1}((t+\mu^{-\kappa})^{1/2})\mu^{-\kappa/2} \\
&\geqslant \mu^{1/m}\mu^{-\kappa/2} \\
&\geqslant \xi_1.
\end{aligned}$$

Hence, from (2.26), (2.27) and (2.30) we obtain that, at $r = (t+\mu^{-\kappa})^{1/2}$,

$$\begin{aligned}
u_{\text{in}}^- &\leqslant \sigma\biggl(L\xi^{-m} - \tfrac{1}{2}a_1\xi^{-m-\lambda_1} + \frac{\sigma_t}{\sigma^p}2K\xi^{2-m-\lambda_1}\biggr) \\
&= Lr^{-m} - \tfrac{1}{2}a_1\sigma^{-\lambda_1/m}r^{-m-\lambda_1} + 2K\sigma^{(-m-\lambda_1)/m}\sigma_t r^{2-m-\lambda_1} \\
&= Lr^{-m} - \tfrac{1}{2}a_1\varepsilon^{-\lambda_1/m}\omega((t+\mu^{-\kappa})^{1/2})r^{-m-\lambda_1} \\
&\qquad - 2K\frac{m}{2\lambda_1}\varepsilon^{-\lambda_1/m}\omega'((t+\mu^{-\kappa})^{1/2})(t+\mu^{-\kappa})^{-1/2}\cdot r^{2-m-\lambda_1} \\
&= Lr^{-m} - \varepsilon^{-\lambda_1/m}\biggl\{\tfrac{1}{2}a_1 - \frac{Km}{\lambda_1}\biggl|\frac{r\omega'(r)}{\omega(r)}\biggr|\biggr\}r^{-m-\lambda_1}\omega(r) \\
&\leqslant Lr^{-m} - \tfrac{1}{4}a_1\varepsilon^{-\lambda_1/m}r^{-m-\lambda_1}\omega(r).
\end{aligned}$$

Since

$$\varepsilon^{-\lambda_1/m} = \mu^{-\lambda_1/m}\omega^{-1}(\mu^{-\kappa/2}) \geqslant \mu^{-\lambda_1/m}\omega^{-1}(0)$$

due to the fact that ω decreases on $(0,\infty)$, the restriction (2.31) on μ_1 yields the desired inequality. □

Lemma 2.5. *Suppose that $u_0 = u_0(r)$ is continuous and positive for $r \geqslant 0$ and that it satisfies*

$$u_0(r) \geqslant Lr^{-m} - b_- r^{-m-\lambda_1}\omega(r) \quad \textit{for all } r > 0$$

with some positive constant b_-. Then there exists $\mu_2 > 0$ such that, whenever $\mu \leqslant \mu_2$, the function u_{in}^- introduced in Lemma 2.3 satisfies

$$u_{\text{in}}^-(r,0) \leqslant u_0(r) \quad \textit{for all } r \in [0, \mu^{-\kappa/2}]. \tag{2.32}$$

Proof. In a similar way to the proof of Lemma 2.4, we first choose $\xi_1 \geqslant 0$ such that (2.26) and (2.27) hold. Since ψ and Ψ are continuous and $0 < \psi(\xi) < L\xi^{-m}$ for all $\xi \geqslant 0$, we can then fix $C > 0$ satisfying

$$\frac{\Psi(\xi)}{\psi(\xi)} \leqslant C \quad \text{for all } \xi \leqslant \xi_1 \tag{2.33}$$

and find that

$$\nu := \tfrac{1}{2}\min(L - \xi^m\psi(\xi)) \tag{2.34}$$

is positive. Next we let $r_0 > 0$ be large enough that

$$\frac{r^{\lambda_1}}{\omega(r)} \geqslant \frac{b_-}{\nu} \quad \text{for all } r \geqslant r_0 \tag{2.35}$$

and set

$$\delta := \min\{u_0(r) \mid r \leqslant r_0\}, \tag{2.36}$$

which is greater than zero because u_0 is positive. By (1.8), we can find $z_2 > 0$ satisfying

$$\biggl|\frac{z\omega'(z)}{\omega(z)}\biggr| \leqslant \min\biggl\{\frac{a_1\lambda_1}{4Km}, 1\biggr\} \quad \text{for all } z \geqslant z_2, \tag{2.37}$$

and, finally, we take $\mu_2 > 0$ so small that

$$\mu_2 \leqslant z_2^{-2/\kappa}, \tag{2.38}$$

$$\mu_2 \leqslant \left(\frac{2\lambda_1}{mC}\frac{\nu}{L-2\nu}\right)^{1/(\kappa-(2/m))}, \tag{2.39}$$

$$\mu_2 \leqslant \frac{L-2\nu}{L-\nu}\delta, \tag{2.40}$$

$$\mu_2 \leqslant \left(\frac{a_1}{4b_-\omega(0)}\right)^{m/\lambda_1}. \tag{2.41}$$

In deriving (2.32), we may evidently assume that $u_{\mathrm{in}}^-(r,0) > 0$ and first consider those $r \leqslant \mu^{-\kappa/2}$ for which $\xi = \sigma^{1/m}(0)r = \mu^{1/m}r \geqslant \xi_1$ holds. At such points, from (2.26), (2.27) and (2.37) we obtain

$$\begin{aligned}
u_{\mathrm{in}}^-(r,0) &\leqslant \sigma(0)\left\{L\xi^{-m} - \tfrac{1}{2}a_1\xi^{-m-\lambda_1} + \frac{\sigma_t(0)}{\sigma^p(0)}\xi^{2-m-\lambda_1}\right\}\\
&= Lr^{-m} - \tfrac{1}{2}a_1\sigma^{-\lambda_1/m}(0)r^{-m-\lambda_1} + 2K\sigma^{(-m-\lambda_1)/\lambda_1}(0)\sigma_t(0)r^{2-m-\lambda_1}\\
&= Lr^{-m} - \tfrac{1}{2}a_1\varepsilon^{-\lambda_1/m}\omega(\mu^{-\kappa/2})r^{-m-\lambda_1}\\
&\qquad - \frac{Km}{\lambda_1}\varepsilon^{-\lambda_1/m}\omega'(\mu^{-\kappa/2})r^{-m-\lambda_1}\mu^{\kappa/2}r^{2-m-\lambda_1}\\
&= Lr^{-m} - \mu^{-\lambda_1/m}\left\{\tfrac{1}{2}a_1 - \frac{Km}{\lambda_1}\left|\frac{\mu^{-\kappa/2}\omega'(\mu^{-\kappa/2})}{\omega(\mu^{-\kappa/2})}\right|\mu^{\kappa}r^2\right\}r^{-m-\lambda_1}\\
&\leqslant Lr^{-m} - \tfrac{1}{4}a_1\mu^{-\lambda_1/m}r^{-m-\lambda_1}\\
&\leqslant Lr^{-m} - \frac{a_1}{4\omega(0)}\mu^{-\lambda_1/m}r^{-m-\lambda_1}\omega(r)\\
&\leqslant Lr^{-m} - b_-r^{-m-\lambda_1}\omega(r),
\end{aligned}$$

because ω is decreasing. Hence,

$$u_{\mathrm{in}}^-(r,0) \leqslant u_0(r) \quad \text{if } \mu^{-1/m} \leqslant r \leqslant \mu^{-\kappa/2}. \tag{2.42}$$

Next, if $\xi < \xi_1$, then by (2.33), (2.37)–(2.39),

$$\begin{aligned}
\frac{[\sigma_t(0)/\sigma^p(0)]\Psi(\xi)}{\psi(\xi)} &\leqslant C\frac{\sigma_t(0)}{\sigma^p(0)}\\
&= -\frac{mC}{2\lambda_1}\varepsilon^{-2/m}\omega^{(2-\lambda_1)/\lambda_1}(\mu^{-\kappa/2})\omega'(\mu^{-\kappa/2})\\
&= \frac{mC}{2\lambda_1}\mu^{\kappa-2/m}\left|\frac{\mu^{-\kappa/2}\omega'(\mu^{-\kappa/2})}{\omega(\mu^{-\kappa/2})}\right|\\
&\leqslant \frac{mC}{2\lambda_1}\mu^{\kappa-2/m}\\
&\leqslant \frac{\nu}{L-2\nu}.
\end{aligned} \tag{2.43}$$

Since (2.34) implies that $\psi(\xi) \leqslant (L-2\nu)\xi^{-m}$ for all $\xi < \xi_1$, we thus obtain

$$\begin{aligned} u_{\text{in}}^{-}(r,0) &= \sigma(0)\psi(\xi)\left\{1 + \frac{[\sigma_t(0)/\sigma^p(0)]\Psi(\xi)}{\psi(\xi)}\right\} \\ &= \frac{L-\nu}{L-2\nu}\sigma(0)\psi(\xi) \\ &\leqslant \frac{L-\nu}{L-2\nu}\sigma(0)(L-2\nu)\xi^{-m} \\ &= (L-\nu)r^{-m} \quad \text{for all } r \leqslant \mu^{-1/m}\xi_1. \end{aligned} \tag{2.44}$$

By definition (2.35) of r_0, however, in the case where $r \geqslant r_0$, we have

$$\begin{aligned} u_0(r) &\geqslant Lr^{-m} - b_- r^{-m-\lambda_1}\omega(r) \\ &\geqslant Lr^{-m} - \nu r^{-m}, \end{aligned}$$

which, combined with (2.44), yields

$$u_{\text{in}}^{-}(r,0) \leqslant u_0(r) \quad \text{if } r_0 \leqslant r < \mu^{-1/m}\xi_1, \tag{2.45}$$

so that we are left with small r satisfying $r < r_0$. With regard to these, we recall (2.36) and use (2.43) and the trivial estimate $\psi(\xi) \leqslant 1$ to obtain

$$\begin{aligned} u_{\text{in}}^{-}(r,0) &\leqslant \sigma(0)\left(1 + \frac{\nu}{L-2\nu}\right)\psi(\xi) \\ &\leqslant \frac{L-\nu}{L-2\nu}\sigma(0) \\ &= \frac{L-\nu}{L-2\nu}\mu \\ &\leqslant \delta \\ &\leqslant u_0(r) \quad \text{for } r < r_0. \end{aligned}$$

Together with (2.42) and (2.45), this proves (2.32). □

Combining the above estimates, we can now derive a lower bound of radial solutions.

Proposition 2.6. *Assume that $u_0 = u_0(r)$ is a continuous and positive function of $r \geqslant 0$ satisfying*

$$u_0(r) \geqslant Lr^{-m} - b_- r^{-m-\lambda_1}\omega(r) \quad \textit{for all } r > 0$$

with some $b_- > 0$. Then there exists $c > 0$ such that the solution u of (2.1) satisfies

$$u(0,t) \geqslant c\omega^{-m/\lambda_1}(t^{1/2}) \quad \textit{for all } t > 0. \tag{2.46}$$

Proof. Let $b_0 > 0$ be the constant provided by Corollary 2.2, and take any $\mu > 0$ satisfying $\mu < \min\{1, \mu_0, \mu_1, \mu_2\}$ with μ_0, μ_1 and μ_2 taken from Lemmas 2.3, 2.4 and 2.5, respectively. Then the function u_{in}^{-} defined by (2.17) satisfies $u_{\text{in}}^{-} \geqslant u$ for $r = (t + \mu^{-\kappa})^{1/2}$, $t \geqslant 0$, by Corollary 2.2 and Lemma 2.4, whereas Lemma 2.5 guarantees that $u_{\text{in}}^{-} \geqslant u$ also at $t = 0$. Since $u_{\rm in}^-$ is a subsolution of (2.1) by Lemma 2.3, the comparison principle shows that $u_{\rm in}^- \geqslant 0$ holds for all $t \geqslant 0$ and $r \leqslant (t + \mu^{-\kappa})^{1/2}$. In particular,

$$\begin{aligned} u(0,t) &\geqslant u_{\rm in}^-(0,t) \\ &= \varepsilon\omega^{-m/\lambda_1}((t+\mu^{-\kappa})^{1/2}) \\ &\geqslant \varepsilon\omega^{-m/\lambda_1}(t^{1/2}) \quad \text{for all } t > 0, \end{aligned}$$

because ω is decreasing. □

3. Upper bound

In this section we give an upper bound for the solution of (2.1) by constructing a suitable super-solution of (2.1). We first consider an appropriate outer region.

Lemma 3.1. *Suppose that*

$$u_0(r) \leqslant Lr^{-m} \quad \text{for all } r > 0 \tag{3.1}$$

and

$$u_0(r) \leqslant Lr^{-m} - b_+ r^{-l}\omega(r) \quad \text{for all } r \geqslant 1 \tag{3.2}$$

hold with a positive constant b_+. Then there exists $B > 0$ such that the solution u of (2.1) satisfies

$$u(r,t) \leqslant Lr^{-m} - \tfrac{1}{2}b_+ r^{-m-\lambda_1}\omega(r) \quad \text{for all } t \geqslant 0 \text{ and } r \geqslant B(t+1)^{1/2}. \tag{3.3}$$

Proof. We let $C > 0$ satisfy

$$\left|\frac{z\omega'(z)}{\omega(z)}\right| \leqslant C \quad \text{and} \quad \left|\frac{z^2\omega''(z)}{\omega(z)}\right| \leqslant C \quad \text{for all } z \geqslant 0, \tag{3.4}$$

which is possible in view of (1.8) and (1.9). We next fix $b_2 > 0$ such that

$$b_2 \geqslant 2b_+[(m+\lambda_1)|m+\lambda_1+2-N| + |N-1-2m-2\lambda_1|C + C] \tag{3.5}$$

and, finally, we take $B > 0$ so large that

$$B \geqslant \sqrt{2}\sqrt{(m+\lambda_1+2)|m+\lambda_1+4-N| + |N-5-2m-2\lambda_1|C + C} \tag{3.6}$$

and

$$B \geqslant \sqrt{\frac{2b_2}{b_+}}. \tag{3.7}$$

Then

$$u_{\rm out}^+(r,t) := \min\{Lr^{-m}, Lr^{-m} - b_+ r^{-m-\lambda_1}\omega(r) + b_2 r^{-m-\lambda_1-2}\omega(r)(t+1)\}$$

satisfies

$$u_{\rm out}^+(r,0) \geqslant u_0(r) \quad \text{for all } r \geqslant 0 \tag{3.8}$$

by (3.1) and (3.2). Moreover, at each point (r,t) where $u_{\mathrm{out}}^{+}(r,t) < Lr^{-m}$, we have $(u_{\mathrm{out}}^{+})^p < (Lr^{-m})^p$ and, thus, repeating the computation in (2.7), we find

$$\begin{aligned}
\mathcal{P}u_{\mathrm{out}}^{+} &= b_2 r^{-m-\lambda_1-2}\omega(r) + (Lr^{-m})^p \\
&\quad + b_1\{(m+\lambda_1)(m+\lambda_1+2-N)r^{-m-\lambda_1-2}\omega(r) \\
&\qquad + (N-1-2m-2\lambda_1)r^{-m-\lambda_1-1}\omega'(r) + r^{-m-\lambda_1}\omega''(r)\} \\
&\quad - b_2\{(m+\lambda_1+2)(m+\lambda_1+4-N)r^{-m-\lambda_1-4}\omega(r) \\
&\qquad + (N-5-2m-2\lambda_1)r^{-m-\lambda_1-3}\omega'(r) + r^{-m-\lambda_1-2}\omega''(r)\}(t+1) - (u_{\mathrm{out}}^{+})^p \\
&> b_2 r^{-m-\lambda_1-2}\omega(r) \\
&\quad \times \biggl\{1 + \frac{b_1}{b_2}\biggl[(m+\lambda_1)(m+\lambda_1+2-N) \\
&\qquad + (N-1-2m-2\lambda_1)\frac{r\omega'(r)}{\omega(r)} + \frac{r^2\omega''(r)}{\omega(r)}\biggr] \\
&\qquad - \biggl[(m+\lambda_1+2)(m+\lambda_1+4-N) \\
&\qquad\qquad + (N-5-2m-2\lambda_1)\frac{r\omega'(r)}{\omega(r)} + \frac{r^2\omega''(r)}{\omega(r)}\biggr]\frac{t+1}{r^2}\biggr\}.
\end{aligned}$$

Using (3.4)–(3.6), for all (r,t) satisfying $r \geqslant B(t+1)^{1/2}$ and $u_{\mathrm{out}}^{+}(r,t) < Lr^{-m}$, we obtain

$$\begin{aligned}
\mathcal{P}u_{\mathrm{out}}^{+} &> b_2 r^{-m-\lambda_1-2}\omega(r) \\
&\quad \times \biggl\{1 - \frac{b_1}{b_2}[(m+\lambda_1)|m+\lambda_1+2-N| + |N-1-2m-2\lambda_1|C + C] \\
&\qquad - [(m+\lambda_1+2)|m+\lambda_1+4-N| + |N-5-2m-2\lambda_1|C + C]\frac{1}{B^2}\biggr\} \\
&\geqslant b_2 r^{-m-\lambda_1-2}\omega(r)\{1 - \tfrac{1}{2} - \tfrac{1}{2}\} \\
&= 0.
\end{aligned}$$

Since $(r,t) \mapsto Lr^{-m}$ is a solution of (2.1), it follows that u_{out}^{+} is a super-solution for all $r \geqslant 0$ and $t \geqslant 0$, and therefore, by (3.8), the comparison principle implies $u \leqslant u_{\mathrm{out}}^{+}$ for all $r \geqslant 0$ and $t \geqslant 0$. In particular, recalling (3.7), we have

$$\begin{aligned}
u(r,t) &\leqslant u_{\mathrm{out}}^{+}(r,t) \\
&\leqslant Lr^{-m} - b_+ r^{-m-\lambda_1}\omega(r) + \tfrac{1}{2}b_+ B^2 r^{-m-\lambda_1-2}\omega(r)(t+1) \\
&\leqslant Lr^{-m} - \tfrac{1}{2}b_+ r^{-m-\lambda_1}\omega(r)
\end{aligned}$$

for all $t \geqslant 0$ and $r \geqslant B(t+1)^{1/2}$, which proves (3.3). □

We also need the following elementary property of ω, which, along with (1.8), is a simple consequence of its positivity and monotonicity.

Lemma 3.2. *For any $\Lambda > 0$, there exists $z_\Lambda > 0$ such that*

$$\omega(\Lambda z) \geqslant \tfrac{1}{2}\omega(z) \quad \textit{for all } z \geqslant z_\Lambda.$$

Proof. We evidently may assume $\Lambda > 1$. We define z_Λ as any sufficiently large number satisfying

$$\left|\frac{z\omega'(z)}{\omega(z)}\right| \leqslant \frac{1}{2(\Lambda-1)} \qquad \text{for all } z \geqslant z_\Lambda. \tag{3.9}$$

Then

$$\omega'(z) \geqslant -\frac{1}{2(\Lambda-1)}\frac{\omega(z)}{z} \quad \text{for all } z \geqslant z_\Lambda \tag{3.10}$$

and thus

$$\begin{aligned}\omega(\Lambda z) - \omega(z) &= \int_z^{\Lambda z} \omega'(s)\,\mathrm{d}s \\ &\geqslant -\frac{1}{2(\Lambda-1)}\int_z^{\Lambda z}\frac{\omega(s)}{s}\,\mathrm{d}s \quad \text{for all } z \geqslant z_\Lambda.\end{aligned}$$

Since $s \mapsto \omega(s)/s$ decreases on $(0,\infty)$, we obtain

$$\begin{aligned}\omega(\Lambda z) - \omega(z) &\geqslant -\frac{1}{2(\Lambda-1)}(\Lambda z - z)\frac{\omega(z)}{z} \\ &= -\tfrac{1}{2}\omega(z) \quad \text{for all } z \geqslant z_\Lambda,\end{aligned}$$

which proves the lemma. □

We are now in a position to give an upper bound for radial solutions. The proof closely follows that of [**2**, Lemma 4.3], but we include a complete proof here for convenience.

Proposition 3.3. *Suppose that u_0 satisfies (3.1) and (3.2) and that, for each $\alpha > u_0(0)$, u_0 intersects φ_α exactly once in $(0,\infty)$. Then there exists $C > 0$ such that the solution u of (2.1) satisfies*

$$u(0,t) \leqslant C\omega^{-m/\lambda_1}(t^{1/2}) \quad \textit{for all } t > 0. \tag{3.11}$$

Proof. We let $\sigma(t) := u(0,t)$ and we may assume that σ is unbounded, since otherwise (3.11) is trivial. Thus, there exists $t_0 > 0$ such that $\sigma(t_0) > \sigma(0)$. Then, for each $t > t_0$, $u(\cdot,t)$ does not intersect $\varphi_{\sigma(t_0)}$ because the number of intersections of $u(\cdot,t)$ with the equilibrium $\varphi_{\sigma(t_0)}$ initially equals 1 and drops at time t_0. Since σ is unbounded, this means that $u(\cdot,t) > \varphi_{\sigma(t_0)}$ for all $t > t_0$. In particular, $\sigma(t) > \sigma(0)$ for all $t > t_0$ and, hence, we may repeat the above argument with t_0 replaced by any $t_1 \geqslant t_0$ to obtain $u(\cdot,t) > \varphi_{\sigma(t_1)}$ for all $t > t_1$. Taking $t \searrow t_1$, we infer that

$$u(r,t) \geqslant \varphi_{\sigma(t)}(r) \quad \text{for all } t \geqslant t_0 \text{ and } r \geqslant 0.$$

By (2.12) and evident scaling properties of φ_α, there exists $M > 0$ such that if $\alpha^{1/m}r \geqslant M$, then

$$\begin{aligned}\varphi_\alpha(r) &= \alpha\varphi_1(\alpha^{1/m}r) \\ &\geqslant \alpha\{L(\alpha^{1/m}r)^{-m} - 2a_1(\alpha^{1/m}r)^{-m-\lambda_1}\} \\ &= Lr^{-m} - 2a_1\alpha^{-\lambda_1/m}r^{-m-\lambda_1}.\end{aligned}$$

Thus, if

$$T := \left(\frac{M}{B\sigma^{1/m}(0)}\right)^2 - 1$$

with B as provided by Lemma 3.1, for all $t \geqslant \max\{T, t_0\}$ and $r = B(t+1)^{1/2}$ we have

$$\sigma^{1/m}(t)r \geqslant \sigma^{1/m}(0)B(T+1)^{1/2} = M$$

and therefore

$$u(r,t) \geqslant \varphi_{\sigma(t)}(r) \geqslant Lr^{-m} - 2a_1\sigma^{-\lambda_1/m}(t)r^{-m-\lambda_1} \quad \text{at } r = B(t+1)^{1/2} \tag{3.12}$$

for such t. On the other hand, from Lemma 3.1, we see that

$$u(r,t) \leqslant Lr^{-m} - \tfrac{1}{2}b_+ r^{-m-\lambda_1}\omega(r) \quad \text{at } r = B(t+1)^{1/2} \text{ for all } t \geqslant 0. \tag{3.13}$$

Combining (3.12) with (3.13) and solving with respect to $\sigma(t)$, we obtain

$$\sigma(t) \leqslant \left(\frac{4a_1}{b_+}\right)^{m/\lambda_1} \omega^{-m/\lambda_1}(B(t+1)^{1/2}) \quad \text{for all } t \geqslant \max\{T, t_0\}. \tag{3.14}$$

Now the observation that

$$B(t+1)^{1/2} \leqslant \sqrt{2}Bt^{1/2}$$

in conjunction with Lemma 3.2, applied to $\Lambda := \sqrt{2}B$, yields

$$\omega(B(t+1)^{1/2}) \geqslant \omega(\sqrt{2}Bt^{1/2}) \geqslant \tfrac{1}{2}\omega(t^{1/2}) \quad \text{for all } t \geqslant z_\Lambda^2,$$

and (3.14) thereby easily leads to (3.11). □

4. Proof of Theorem 1.1

In this section we complete a proof of Theorem 1.1 by using the upper and lower estimates of radial solutions.

Given an initial value $u_0(x)$ satisfying (1.2) and (1.6), we define radially symmetric functions by

$$\underline{u}_0(r) := \min\{u_0(x) \colon |x| \leqslant r\}, \quad r \geqslant 0,$$

and

$$\bar{u}_0(r) := \max\{u_0(x) \colon |x| \geqslant r\}, \quad r \geqslant 0.$$

Then

(i) $\underline{u}_0(r)$ and $\bar{u}_0(r)$ are continuous and decreasing in $r \geqslant 0$,

(ii) $0 \leqslant \underline{u}_0(|x|) \leqslant u_0(x) \leqslant \bar{u}_0(|x|) \leqslant \varphi_\infty(|x|)$ for all $x \in \mathbb{R}^N \setminus \{0\}$ and

(iii) $\underline{u}_0(|x|)$ and $\bar{u}_0(|x|)$ satisfy (1.6).

Let $\underline{u}(r,t)$ and $\bar{u}(r,t)$ denote the solutions of (2.1) with the initial values $\underline{u}_0(r)$ and $\bar{u}_0(r)$, respectively. Then the solutions exist globally in time and are decreasing in r for all $t>0$. Moreover, by the comparison principle, the solution of (1.1) satisfies

$$\underline{u}(|\cdot|,t)\|_{L^\infty} \leqslant \|u(\cdot,t)\|_{L^\infty} \leqslant \|\bar{u}(|\cdot|,t)\|_{L^\infty}, \quad x\in\mathbb{R}^N,$$

for all $t>0$. Since $\underline{u}(r,t)$ and $\bar{u}(r,t)$ are decreasing in r for each $t>0$, since

$$\underline{u}(0,t) \geqslant c\omega^{-m/\lambda_1}(t^{1/2}) \quad \text{for all } t>0$$

by Proposition 2.6, and since

$$\bar{u}(0,t) \leqslant C\omega^{-m/\lambda_1}(t^{1/2}) \quad \text{for all } t>0$$

by Proposition 3.3, we obtain the desired estimates of the grow-up rate of the solution of (1.1).

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References

1. M. Fila, J. R. King, M. Winkler and E. Yanagida, Optimal lower bound of the grow-up rate for a supercritical parabolic equation, *J. Diff. Eqns* **228** (2006), 339–356.
2. M. Fila, J. R. King, M. Winkler and E. Yanagida, Grow-up of solutions of a semilinear parabolic equation with a critical exponent, *Adv. Diff. Eqns* **12** (2007), 1–26.
3. M. Fila, J. R. King, M. Winkler and E. Yanagida, Linear behaviour of solutions of a super-linear heat equation, *J. Math. Analysis Applic.* **340** (2008), 401–409.
4. M. Fila and M. Winkler, Rate of convergence to a singular steady state of a supercritical parabolic equation, *J. Evol. Eqns* **8** (2008), 673–692.
5. M. Fila, M. Winkler and E. Yanagida, Grow-up rate of solutions for a supercritical semilinear diffusion equation, *J. Diff. Eqns* **205** (2004), 365–389.
6. M. Fila, M. Winkler and E. Yanagida, Convergence rate for a parabolic equation with supercritical nonlinearity, *J. Dynam. Diff. Eqns* **17** (2005), 249–269.
7. M. Fila, M. Winkler and E. Yanagida, Slow convergence to zero for a parabolic equation with supercritical nonlinearity, *Math. Annalen* **340** (2008), 477–496.
8. M. Fila, M. Winkler and E. Yanagida, Convergence to self-similar solutions for a semilinear parabolic equation, *Discrete Contin. Dynam. Syst.* **21** (2008), 703–716.
9. V. Galaktionov and J. L. Vázquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, *Commun. Pure Appl. Math.* **50** (1997), 1–67.
10. C. Gui, W.-M. Ni and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$, *Commun. Pure Appl. Math.* **45** (1992), 1153–1181.
11. C. Gui, W.-M. Ni and X. Wang, Further study on a nonlinear heat equation, *J. Diff. Eqns* **169** (2001), 588–613.
12. M. Hoshino and E. Yanagida, Sharp estimate of the convergence rate for a semilinear parabolic equation with supercritical nonlinearity, *Nonlin. Analysis TMA* **69** (2008), 3136–3152.

13. D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Ration. Mech. Analysis* **49** (1973), 241–269.
14. N. Mizoguchi, Growup of solutions for a semilinear heat equation with supercritical nonlinearity, *J. Diff. Eqns* **227** (2006), 652–669.
15. P. Poláčik and E. Yanagida, On bounded and unbounded global solutions of a supercritical semilinear heat equation, *Math. Annalen* **327** (2003), 745–771.
16. P. Poláčik and E. Yanagida, Nonstabilizing solutions and grow-up set for a supercritical semilinear diffusion equation, *Diff. Integ. Eqns* **17** (2004), 535–548.
17. P. Quittner and Ph. Souplet, *Superlinear parabolic problems: blow-up, global existence and steady states,* Birkhäuser Advanced Texts (Birkhäuser, Basel, 2007).
18. C. Stinner, Very slow convergence to zero for a supercritical semilinear parabolic equation, *Adv. Diff. Eqns* **14** (2009), 1085–1106.
19. C. Stinner, Very slow convergence rates in a semilinear parabolic equation, *NoDEA. Nonlin. Diff. Eqns Applic.* **17** (2010), 213–227.
20. X. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Am. Math. Soc.* **337** (1993), 549–590.