Spatially monotone homoclinic orbits in nonlinear parabolic equations of super-fast diffusion type

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Abstract

This work deals with positive classical solutions of the degenerate parabolic equation

$$u_t = u^p u_{xx} \qquad (\star)$$

when $p > 2$, which via the substitution $v = u^{1-p}$ transforms into the super-fast diffusion equation $v_t = (v^{m-1} v_x)_x$ with $m = -\frac{1}{p-1} \in (-1, 0)$.
It is shown that $(\star)$ possesses some entire positive classical solutions, defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$, which connect the trivial equilibrium to itself in the sense that $u(x,t) \to 0$ both as $t \to -\infty$ and as $t \to +\infty$, locally uniformly with respect to $x \in \mathbb{R}$. Moreover, these solutions have quite a simple structure in that they are monotone increasing in space.
The approach is based on the construction of two types of wave-like solutions, one of them being used for $-\infty < t \le 0$ and the other one for $0 < t < +\infty$. Both types exhibit wave speeds that vary with time and tend to zero as $t \to -\infty$ and $t \to +\infty$, respectively.
The solutions thereby obtained decay as $x \to -\infty$, uniformly with respect to $t \in \mathbb{R}$, but they are unbounded as $x \to +\infty$. It is finally shown that within the class of functions having such a behavior as $x \to -\infty$, there does *not* exist any bounded homoclinic orbit.

Key words: fast diffusion, traveling wave, entire solution, connecting orbit
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Introduction

One of the characteristic features of diffusion mechanisms is the irreversibility of time. In the simplest cases, this is reflected by the dissipation of a certain magnitude which can frequently be assigned an immediate physical meaning such as energy or entropy, for instance. In these situations, the existence of such a quantity that decreases with time clearly rules out the possibility of processes evolving from one state toward the same state in a nontrivial manner: Once a state has been left, it can never be reached again in the future.

Mathematically, this finds expression in various types of statements concerning several classes of diffusion equations. The first type of results concerns the *nonexistence of oscillatory behavior*, and thus in particular of time-periodicity, the latter being the apparently most obvious way how a system could return to a state in which it has already been at some time in the past. As a well-understood example, let us consider the semilinear parabolic equation

$$u_t = \Delta u + f(u) \tag{0.1}$$

under homogoneous Dirichlet conditions on the boundary of a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, where $f \in C^2(\mathbb{R})$. Then it is known that if either $n = 1$ ([M], [Z]) or f is analytic ([J]), then all bounded solutions of (0.1) in $\Omega \times (0, \infty)$ stabilize toward some equilibrium as $t \to \infty$. Even without assuming analyticity of f, one can derive the same conclusion if either Ω is a ball and $u \geq 0$ ([HP]), or if the set of steady states of (0.1) is either discrete or totally ordered ([L]). Results of this flavor, some of which can be carried over to quasilinear and even degenerate variants of (0.1) ([Win1]), clearly exclude time oscillations which persist up to arbitrarily large times, with amplitude bounded from below by a positive constant.
Accordingly, only few results seem to be available that report exceptions from this by detecting oscillatory behavior in parabolic equations related to (0.1). Some examples of solutions with large ω-limit sets refer to the heat equation $u_t = \Delta u$ in $\Omega = \mathbb{R}^n$ ([VZ]), (0.1) with $f(u) = u^p$ in $\Omega = \mathbb{R}^n$ with $n \geq 11$ and $p \geq \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$ ([PY]), (0.1) with some $f = f(x, u)$ ([PS]), $u_t = a(x)(\Delta u + \lambda u)$ with certain $a(x)$ and $\lambda > 0$ ([Win2]), and $u_t = u^p(\Delta u + \lambda u)$ with $p \geq 3$ and some $\lambda > 0$ ([Win3]).

Another type of results addresses the question of connectibility of equilibria and especially the *nonexistence of homoclinic orbits*. We recall that a given steady state v of a diffusion equation is said to connect to a steady state w if there exists an entire solution $u = u(x, t) \not\equiv v$ of the corresponding parabolic problem, defined for all $t \in (-\infty, \infty)$, that satisfies $u(x, t) \to v$ as $t \to -\infty$ and $u(x, t) \to w$ as $t \to +\infty$. Here, if $w \neq v$ such a solution is called a heteroclinic orbit, in the case $w = v$ a homoclinic orbit.
The study of connecting orbits is essential for the understanding of the global attractor of a system, and the most exhausting results are available for the one-dimensional version of (0.1) which is accessible to strong analytic tools such as assertions on the nonincrease of zero numbers. Here, for generic choices of f, the mere knowledge on the set of steady states is essentially sufficient to decide whether or not there exist connections between two given equilibria; this is true under both Dirichlet ([BF]) or Neumann boundary conditions in bounded spatial intervals ([FR1]), even for variants of (0.1) allowing the nonlinearity to depend on u_x ([FR2]). A particular outcome is that only heteroclinic connections do exist in this situation, which can be viewed as a consequence of the presence of an energy in the sense described above. Clearly, the latter conclusion on nonexistence of homoclinic orbits carries over to any gradient-like parabolic equation or system.

Correspondingly, the question whether at all homoclinic orbits can be observed in some parabolic problem appeared to be open for a long time. For the Fujita equation

$$u_t = \Delta u + u^p \tag{0.2}$$

in the whole space $\Omega = \mathbb{R}^n$, certain positive *singular connections* from the trivial equilibrium to itself were detected for $n \geq 3$ and $p \in (\frac{n+2}{n-2}, p_L)$, where $p_L = \infty$ if $n \leq 10$ and $p_L = \frac{n-4}{n-10}$ otherwise. In fact, these connections can be chosen to be smooth classical solutions for $t < 0$, to blow up at the origin at $t = 0$, and to continue to exist as weak solutions of (0.2) for $t > 0$ ([FM], [GV]). Only recently, for (0.2) with $n \geq 3$ and $p > \frac{n+2}{n-2}$, in [FY] a bounded positive classical solution u defined in $\mathbb{R}^n \times (-\infty, \infty)$ could be found which tends to zero as $t \to -\infty$ and for $t \to +\infty$ and thus connects the trivial equilibrium to itself in a smooth but nontrivial way. Moreover, if either $n < 10$ or $p < \frac{n-4}{n-10}$ then this solution can be chosen radially symmetric and decreasing with respect to $|x|$, decaying to zero as $|x| \to \infty$ for each fixed t.

On the other hand, without the source term u^p in (0.2), a solution with the above properties cannot exist. In fact, it is easy to see that any solution u of the heat equation in $\mathbb{R}^n$ fulfilling $u(x,t) \to 0$ as $t \to -\infty$, uniformly with respect to $x \in \mathbb{R}^n$, necessarily satisfies $u \equiv 0$. This underlines that in the above results, the detection of homoclinic connections is essentially linked to the interplay between diffusion *and reaction* in (0.2). The latter is in accordance with further precedents that reveal quite a rich dynamical structure in the variant

$$u_t = \Delta u + f(u, \nabla u) \tag{0.3}$$

of (0.1) allowing for gradients in the reaction term. For instance, one of the results in [DP] says that for any $n \geq 2$ there exist some smooth bounded domain $\Omega \subset \mathbb{R}^n$ and some nonlinearity f such that (0.3) when posed in $\Omega \times \mathbb{R}$ under homogeneous Dirichlet boundary conditions, possesses an orbit connecting a non-constant, time-periodic solution to itself.

Main results

The present work addresses the question *whether smooth homoclinic orbits also exist in parabolic equations without any source.* In other words, we ask whether there are systems, purely evolving by diffusion, which return to their ancient state $\lim_{t\to-\infty} u(x,t)$ as $t \to +\infty$. We shall see that this in fact is possible, and that it is possible for quite a simple type of nonlinear diffusion, and even in space dimension $n = 1$.

To be more precise, we shall subsequently consider positive classical solutions of the nonlinear diffusion equation

$$u_t = u^p u_{xx}, \tag{0.4}$$

for $p > 2$, where x varies over the real line and the time variable t is allowed to lie in the whole range $(-\infty, \infty)$. We note that upon the substitution $v = u^{1-p}$, (0.4) is transformed into the *super-fast diffusion equation*

$$v_t = (v^{m-1} v_x)_x \tag{0.5}$$

with parameter $m = -\frac{1}{p-1}$ located in the interval $(-1, 0)$ according to our requirement $p > 2$. The equation (0.5) has been widely studied for any $m \in \mathbb{R}$, and a summary of results can be found in [V2].

Specifically, we will be interested in homoclinic connections of the trivial solution $u \equiv 0$ of

(0.4) to itself. Of course, the most desirable result in this direction would state existence of such a solution within a set of functions having some convenient decay as $|x| \to \infty$, and approaching zero with respect to $L^\infty(\mathbb{R})$ as $t \to \pm\infty$. However, we shall see below that such a solution cannot exist (cf. Proposition 0.2), and hence we admit possibly unbounded solutions, and correspondingly resort to a weaker topology in the state space, namely that of locally uniform convergence. We ignore here the question in how far this framework really may constitute a genuine dynamical system; in fact, a complete theory in this direction appears to be lacking, which may be due to far-reaching nonuniqueness results ([ERV] and [RV]). Since we do not intend to address this general issue but nevertheless wish to deal with incisive notions, let us pursue the following somewhat naive definition of a homoclinic orbit.

Definition 0.1 *We say that a positive function $u \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ is a* homoclinic orbit of (0.4) *if*

$$u(\cdot, t) \to 0 \quad \textit{locally uniformly on } \mathbb{R} \qquad \textit{as } t \to \pm\infty. \tag{0.6}$$

Our main result states that such homoclinic connections indeed exist, and that moreover they can have quite a simple structure. In fact, they may be monotone in space and decay to zero at least in one spatial spatial direction.

Theorem 0.1 *Let $p > 2$. Then there exists a homoclinic orbit of (0.4) in the sense of Definition 0.1. More precisely, (0.4) posseses a positive classical solution $u \in C^\infty(\mathbb{R} \times \mathbb{R})$ such that for all $y \in \mathbb{R}$ one can find $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty((-\infty,y))} \le C t^{-\frac{1}{p}} (\ln t)^{\frac{2}{p}} \qquad \textit{for all } t \ge 2 \tag{0.7}$$

and

$$\|u(\cdot, t)\|_{L^\infty((-\infty,y))} \le C(-t)^{-\frac{1}{p}} \ln(-t) \qquad \textit{for all } t \le -2. \tag{0.8}$$

Moreover, this solution can be constructed in such a way that $u_x > 0$ holds in $\mathbb{R} \times \mathbb{R}$, and that $u(x,t) \to 0$ as $x \to -\infty$, even uniformly with respect to $t \in \mathbb{R}$, that is,

$$\|u\|_{L^\infty((-\infty,y)\times\mathbb{R})} \to 0 \qquad \textit{as } y \to -\infty. \tag{0.9}$$

More precise statemets on the spatial asymptotics (cf. Theorems 0.3 and 0.4 below) shall reveal that our solution, though unbounded in space for $t < 0$ (and possibly also for $t > 0$), is at least linearly bounded with respect to $x > 0$ for all $t \in \mathbb{R}$. However, it will satisfy $\int_{\mathbb{R}} u_x^2(x,t)dx = +\infty$ for all $t < 0$, so that Theorem 0.1 does not contradict the formal energy identity related to (0.4),

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{u_t^2}{u^p} dxdt + \int_{\mathbb{R}} u_x^2(x,t_2)dx = \int_{\mathbb{R}} u_x^2(x,t_1), \qquad x \in \mathbb{R},\ -\infty < t_1 < t_2 < \infty.$$

One may next wonder whether one can achieve *boundedness* of some homoclinic orbit in (0.4). Indeed, this is the case for (0.2) ([FY]), and substituting $u = \rho(w)$ we see that the

solution from Theorem 0.1 corresponds to a spatially monotone and bounded homoclinic orbit of the source-free nonlinear diffusion equation

$$w_t = \frac{\rho^p(w)}{\rho'(w)} \cdot \Big(\rho'(w) w_x\Big)_x, \qquad x \in \mathbb{R},\ t \in \mathbb{R},$$

whenever ρ is a diffeomorphism from $[0,1)$ to $[0,\infty)$ with $\rho' > 0$. In the present setting, however, a similar phenomenon is impossible:

Proposition 0.2 *Let $p > 2$. Then (0.4) possesses no bounded homoclinic orbit in the sense of Definition 0.1, having the additional property of uniform decay as $x \to -\infty$ specified in (0.9).*

Another distictive feature that deserves being emphasized is the *spatial monotonicity* of the solution in Theorem 0.1. Indeed, homoclinics with such a simple structure seem hard to detect, no matter whether they are bounded or unbounded in space. For instance, it is easy to see that the heat equation $u_t = u_{xx}$ does not possess any nonnegative spatially monotone homoclinic connecting the trivial solution to itself.

In order to provide further connection to the existing literature, let us recall that some of the known facts on (0.5) indeed underline a certain criticality of the exponent $m = -1$. This may give rise to the conjecture that our restriction $p > 2$ for (0.4) might not be of purely technical nature, although we cannot prove any nonexistence result in the less degenerate range $p \le 2$. For instance, precisely for $m > -1$ there exist the celebrated explicit Barenblatt solutions for (0.5), which for $m \in (-1,1)$ take the form

$$v_a(x,t) := \Big(\frac{2(1+m)}{1-m}\Big)^{\frac{1}{1-m}} \cdot t^{\frac{1}{1-m}} \cdot \Big(x^2 + a t^{\frac{2}{1+m}}\Big)^{-\frac{1}{1-m}}, \qquad x \in \mathbb{R},\ t > 0,$$

where $a > 0$ is arbitrary ([V2]).
The most striking difference between the ranges $m \in (-\infty,-1]$ and $m \in (-1,0)$, however, seems to be linked to the well-posedness in the corresponding Cauchy problem: It is known that for $m \le -1$, (0.5) does not possess any nontrivial nonnegative local-in-time solution whenever the initial data $v_0 := v(\cdot,0)$ belong to $L^1(\mathbb{R})$ ([V1]), whereas if $m \in (-1,0)$ then for each smooth positive $v_0 \in L^\infty(\mathbb{R})$, (0.5) is classically solvable on $\mathbb{R} \times (0,\infty)$. But in the latter case, the solution is never unique, not even in the class of arbitrarily smooth solutions ([ERV], [RV]).

Slowly traveling waves as the main ingredient

Before going into details of the proof of Theorem 0.1, let us briefly describe the main ingredients that are constitutive to our approach. We first consider the evolution governed by (0.4) for positive times, and seek for *slowly traveling wave* solutions, by which we mean solutions of (0.4) having the form

$$u(x,t) = t^{-\gamma} \cdot F(x + c \ln t), \qquad x \in \mathbb{R},\ t > 0, \tag{0.10}$$

for suitable real constants γ and $c > 0$ and an appropriate positive function F defined on $\mathbb{R}$. Substituting (0.10) into (0.4) we readily see that we should take $\gamma = \frac{1}{p}$ and choose F to be a solution of the ODE

$$F^p F'' = \frac{1}{p\alpha} F' - \frac{1}{p} F \qquad \text{on } \mathbb{R}, \tag{0.11}$$

where we have substituted $c = \frac{1}{p\alpha}$ for convenience. It might be of interest of its own that such solutions indeed exist, and that their asymptotic behavior can be described quite well, because apart from the application we have in mind here, this gives examples of how mass transport occurs in the super-fast diffusion equation (0.5). We therefore separately state our following main result on slowly traveling waves for $t > 0$, the proof of which will be given in Section 1.

Theorem 0.3 *Let $p > 2$ and $\alpha > 0$. Then there exists an increasing positive function $F_\alpha \in C^\infty(\mathbb{R})$ such that the function u_α defined by*

$$u_\alpha(x,t) := t^{-\frac{1}{p}} \cdot F_\alpha\Big(x + \frac{1}{p\alpha} \ln t\Big), \qquad x \in \mathbb{R}, t > 0, \tag{0.12}$$

is a positive classical solution $u_\alpha \in C^\infty(\mathbb{R} \times (0,\infty))$ of (0.4) in $\mathbb{R} \times (0,\infty)$. Moreover, there exist positive constants c_0, c_1, d_0 and d_1 such that

$$c_0 e^{\alpha\xi} \le F_\alpha(\xi) \le c_1 e^{\alpha\xi} \qquad \text{for all } \xi \le 0 \tag{0.13}$$

and

$$d_0(\xi+1)^{\frac{2}{p}} \le F_\alpha(\xi) \le d_1(\xi+1)^{\frac{2}{p}} \qquad \text{for all } \xi \ge 0. \tag{0.14}$$

As a particular consequence of (0.13), (0.14) and a comparison argument, in Section 3.1 we shall obtain the following intermediate result: For any sufficiently smooth positive function u_0 on $\mathbb{R}$ that satisfies $u_0(x) \le ce^{\alpha x}$ for all $x \in \mathbb{R}$ and some positive c and α, the initial-value problem for (0.4) with prescribed initial data $u(\cdot,0) = u_0$ has at least one positive classical solution u on $\mathbb{R} \times (0,\infty)$ which tends to zero as $t \to +\infty$ (Theorem 3.2).

Next, in order to find an adequate *ancient* solution defined on $\mathbb{R} \times (-\infty, 0]$, we shall pursue an ansatz similar to (0.10), with t replaced by $-t$. Upon an appropriately modified analysis, in Section 2 we will end up with the following analogue of Theorem 0.3.

Theorem 0.4 *For any $p > 2$ and each $\alpha > 0$ one can find an increasing positive $G_\alpha \in C^\infty(\mathbb{R})$ such that*

$$\tilde{u}_\alpha(x,t) := (-t)^{-\frac{1}{p}} \cdot G_\alpha\Big(x + \frac{1}{p\alpha} \ln(-t)\Big), \qquad x \in \mathbb{R}, t < 0, \tag{0.15}$$

defines a positive classical solution $\tilde{u}_\alpha \in C^\infty(\mathbb{R} \times (-\infty, 0))$ of (0.4) in $\mathbb{R} \times (-\infty, 0)$. Furthermore, for suitable positive constants c_0, c_1, d_0 and d_1 we have

$$c_0 e^{\alpha\xi} \le G_\alpha(\xi) \le c_1 e^{\alpha\xi} \qquad \text{for all } \xi \le 0 \tag{0.16}$$

as well as

$$d_0(\xi+1) \le G_\alpha(\xi) \le d_1(\xi+1) \qquad \text{for all } \xi \ge 0. \tag{0.17}$$

Finally, in our construction of a homoclinic orbit we shall use any of the latter ancient solutions for, say, $t \le -1$, and use the function $u(\cdot,-1) \in C^\infty(\mathbb{R})$ thereby obtained as data in an initial-value problem for (0.4). According to the right estimates in (0.16) and (0.17), we may apply our above intermediate result and conclude (Section 4).

In order to state a problem arising here, let us note that a by-product of Theorem 0.3 is the logarithmic correction in the time decay rate of slowly traveling waves, as induced by (0.12), (0.13) and (0.14). This correction does neither appear in the explicit solutions $u(x,t) = (\frac{p}{2(p-2)})^{\frac{1}{p}} t^{-\frac{1}{p}} x^{\frac{2}{p}}$ of (0.4) defined on the half-axis $x > 0$ for $t > 0$, nor in separated solutions of the Dirichlet problem for (0.4) in bounded intervals. It would be interesting to obtain more information about how the initial data influence the time asymptotics in (0.4).

1 Slowly traveling waves for $t > 0$

In this section we shall prove Theorem 0.3, which basically reduces to analyzing the ODE (0.11) suggested by the wave ansatz. Here, writing $F(\xi) = (p\alpha^2)^{-\frac{1}{p}} \cdot f(\alpha\xi)$, we obtain the simpler equation

$$f^p f'' = f' - f \qquad \text{on } \mathbb{R} \tag{1.1}$$

(cf. (1.15) and (1.16)). Upon the substitution $f' = \varphi(f)$, this leads to the first-order equation $f^p \cdot \varphi(f)\varphi'(f) = \varphi(f) - f$ for the new unknown $\varphi = \varphi(f)$, to be considered for positive values of f. We will not solve this singular ODE directly, but rather find certain sub- and supersolutions which will provide corresponding ordered sub- and supersolutions for (1.1) and thereby enable us to finally construct appropriate solutions of (1.1) with the desired asymptotics.
To begin with, let us consider subsolutions of the first-oder equation.

Lemma 1.1 *Let $p > 2$. Then there exist $a_+ > 0$ and $s_+ > 0$ such that the function $\varphi_+ :\ [0,\infty) \to \mathbb{R}$ defined by*

$$\varphi_+(s) := \begin{cases} s + a_+ s^{p+1} & \text{if } 0 \le s \le s_+, \\ \sqrt{\frac{2}{p-2}}\, s^{-\frac{p-2}{2}} & \text{if } s > s_+, \end{cases} \tag{1.2}$$

is continuous on $[0,\infty)$ and satisfies

$$\varphi_+'(s) \le s^{-p} - \frac{s^{1-p}}{\varphi_+(s)} \qquad \text{for all } s \in (0,\infty) \setminus \{s_+\}. \tag{1.3}$$

Proof. Let us define

$$S(a) := \left(\frac{\sqrt{p^2 + 4(p+1)a} - p - 2}{2(p+1)a} \right)^{\frac{1}{p}} \qquad \text{for } a \ge 1. \tag{1.4}$$

Since $p^2 + 4(p+1)a \ge p^2 + 4(p+1) = (p+2)^2$ for $a \ge 1$, S is well-defined and nonnegative on $[1,\infty)$ with $S(1) = 0$. Using that S is positive on $(1,\infty)$ and satisfies

$$a^{\frac{1}{2p}} S(a) \to (p+1)^{-\frac{1}{2p}} =: c_0 \qquad \text{as } a \to \infty,$$

we obtain that

$$S(a) \geq \frac{c_0}{2} a^{-\frac{1}{2p}} \qquad \text{for all } a \geq a_1$$

is valid with some $a_1 \geq 1$. Accordingly,

$$\psi(a) := S^{\frac{p}{2}}(a) + a \cdot S^{\frac{3p}{2}}(a), \qquad a \geq 1,$$

defines a continuous function $\psi : [1, \infty) \to \mathbb{R}$ such that $\psi(1) = 0$ and

$$\psi(a) \geq a \cdot S^{\frac{3p}{2}}(a) \geq a \cdot \left(\frac{c_0}{2}\right)^{\frac{3p}{2}} \cdot a^{-\frac{3}{4}} = \left(\frac{c_0}{2}\right)^{\frac{3p}{2}} \cdot a^{\frac{1}{4}} \qquad \text{for all } a \geq a_1,$$

so that by a continuity argument we can pick some $a_+ > 1$ such that $\psi(a_+) = \sqrt{\frac{2}{p-2}}$, that is,

$$S^{\frac{p}{2}}(a_+) + a_+ \cdot S^{\frac{3p}{2}}(a_+) = \sqrt{\frac{2}{p-2}}. \tag{1.5}$$

Letting $s_+ := S(a_+)$ and then defining φ_+ by (1.2), we infer from (1.5) that φ_+ is continuous on $[0, \infty)$. Moreover, for large s we find

$$\begin{aligned}
\mathcal{D}\varphi_+ &:= \varphi_+'(s) - s^{-p} + \frac{s^{1-p}}{\varphi_+(s)} \\
&< \varphi_+'(s) + \frac{s^{1-p}}{\varphi_+(s)} \\
&= -\frac{p-2}{2} \cdot \sqrt{\frac{2}{p-2}} \cdot s^{-\frac{p}{2}} + \frac{s^{1-p}}{\sqrt{\frac{2}{p-2}} \cdot s^{-\frac{p-2}{2}}} \\
&= 0 \qquad \text{for all } s > s_+,
\end{aligned}$$

whereas near $s = 0$,

$$\begin{aligned}
\mathcal{D}\varphi_+ &= 1 + (p+1)a_+ s^p - s^{-p} + \frac{s^{1-p}}{s + a_+ s^{p+1}} \\
&= 1 + (p+1)a_+ s^p - \frac{a_+}{1 + a_+ s^p} \\
&= \frac{1 + (p+2)a_+ s^p + (p+1)a_+^2 s^{2p} - a_+}{1 + a_+ s^p} \qquad \text{for all } s \in (0, s_+).
\end{aligned}$$

Computing the roots of the numerator yields the factorization

$$\begin{aligned}
&1 + (p+2)a_+ s^p + (p+1)a_+^2 s^{2p} - a_+ \\
&\quad = (p+1)a_+^2 \cdot \left(s^p - \frac{\sqrt{p^2 + 4(p+1)a_+} - p - 2}{2(p+1)a_+}\right) \cdot \left(s^p + \frac{p + 2 + \sqrt{p^2 + 4(p+1)a_+}}{2(p+1)a_+}\right),
\end{aligned}$$

whence in view of (1.4) we infer that $\mathcal{D}\varphi \leq 0$ also holds for $0 < s < s_+ = S(a_+)$, as claimed. ////

We next seek for corresponding supersolutions, maintaining some degrees of freedom that shall eventually be fixed so as to guarantee the desired ordering of sub- and supersolutions for the second-order ODE (1.1).

Lemma 1.2 *Let $p > 2$. Then for any $s_0 > 0$ there exist $s_- \in (0, s_0), a_- \in (0, \frac{1}{3s_-^p})$ and $b_- \in (0, \sqrt{\frac{2}{p-2}})$ such that $\varphi_- : [0, \infty) \to \mathbb{R}$ defined by*

$$\varphi_-(s) := \begin{cases} s - a_- s^{p+1} & \text{if } 0 \le s \le s_-, \\ b_- s^{-\frac{p-2}{2}} & \text{if } s > s_-, \end{cases} \tag{1.6}$$

is continuous on $[0, \infty)$ and fulfills

$$\varphi_-'(s) \ge s^{-p} - \frac{s^{1-p}}{\varphi_-(s)} \qquad \text{for all } s \in (0, \infty) \setminus \{s_-\}. \tag{1.7}$$

Proof. Given $s_0 > 0$, we first fix $b_- > 0$ small such that $b_- < \sqrt{\frac{2}{p-2}}$ and

$$\frac{b_-}{1 - \frac{p-2}{2} b_-^2} < s_0^{\frac{p}{2}} \tag{1.8}$$

as well as

$$\frac{p-2}{2} b_-^2 < \frac{1}{p+1}, \tag{1.9}$$

and then obtain from (1.8) that

$$s_- := \left(\frac{b_-}{1 - \frac{p-2}{2} b_-^2} \right)^{\frac{2}{p}} \tag{1.10}$$

satisfies $0 < s_- < s_0$. Since this particularly implies that $b_- < s_-^{\frac{2}{p}}$, the number

$$a_- := \frac{1 - b_- s_-^{-\frac{p}{2}}}{s_-^p} \tag{1.11}$$

is positive and, by (1.9), satisfies

$$a_- s_-^p = 1 - b_- s_-^{-\frac{p}{2}} = \frac{p-2}{2} b_-^2 < \frac{1}{p+1}, \tag{1.12}$$

so that clearly $a_- \in (0, \frac{1}{3s_-^p})$. As $\sqrt{p^2 - 4(p+1)a_-} < p$, (1.12) furthermore entails that

$$s_-^p < \frac{p + 2 - \sqrt{p^2 - 4(p+1)a_-}}{2(p+1)a_-}. \tag{1.13}$$

Upon these prelimineries, let φ_- be as in (1.6). Then (1.11) precisely asserts that φ_- is continuous at $s = s_-$ and hence on $[0, \infty)$, while from (1.10) we gain that

$$\begin{aligned}
\mathcal{D}\varphi_- &\equiv \varphi_-' - s^{-p} + \frac{s^{1-p}}{b_- s^{-\frac{p-2}{2}}} \\
&= -\frac{p-2}{2} b_- s^{-\frac{p}{2}} - s^{-p} + \frac{s^{1-p}}{b_- s^{-\frac{p-2}{2}}} \\
&= \left\{ \frac{1 - \frac{p-2}{2} b_-^2}{b_-} - s^{-\frac{p}{2}} \right\} \cdot s^{-\frac{p}{2}} \\
&\geq \left\{ \frac{1 - \frac{p-2}{2} b_-^2}{b_-} - s_-^{-\frac{p}{2}} \right\} \cdot s^{-\frac{p}{2}} \\
&= 0 \qquad \text{for all } s \in (s_-, \infty).
\end{aligned}$$

Moreover, for small s we compute

$$\begin{aligned}
\mathcal{D}\varphi &= 1 - (p+1) a_- s^p - s^{-p} + \frac{s^{1-p}}{1 - a_- s^p} \\
&= \frac{1 + a_- - (p+2) a_- s^p + (p+1) a_-^2 s^{2p}}{1 - a_- s^p} \qquad \text{for all } s \in (0, s_-).
\end{aligned}$$

Hence, using (1.13) we easily see that $\mathcal{D}\varphi_-$ is positive on $(0, s_-)$, which completes the proof. ////

Now the main step in the construction of slowly traveling waves is accomplished in the following lemma.

Lemma 1.3 *Let $p > 2$ and $\alpha > 0$. Then there exists a positive solution $F_\alpha \in C^\infty(\mathbb{R})$ of*

$$F_\alpha^p F_\alpha'' = \frac{1}{p\alpha} F_\alpha' - \frac{1}{p} F_\alpha, \qquad \xi \in \mathbb{R}. \tag{1.14}$$

Moreover, we have $F_\alpha' > 0$ on $\mathbb{R}$, and there exist positive constants c_0, c_1, d_0 and d_1 such that the two-sided estimates (0.13) and (0.14) hold.

Proof. Upon the substitution

$$F_\alpha(\xi) = (p\alpha^2)^{-\frac{1}{p}} \cdot f(\alpha\xi), \qquad \xi \in \mathbb{R}, \tag{1.15}$$

(1.14) transforms into the equation

$$\mathcal{E} f := -f^p f'' + f' - f = 0, \qquad \sigma \in \mathbb{R}. \tag{1.16}$$

In order to solve (1.16), we first claim that there exist $f_\pm \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ such that

$$\mathcal{E} f_- \leq 0 \qquad \text{and} \qquad \mathcal{E} f_+ \geq 0 \qquad \text{on } \mathbb{R}, \tag{1.17}$$

that

$$f_-(\sigma) < f_+(\sigma) \qquad \text{for all } \sigma \in \mathbb{R}, \tag{1.18}$$

and that the two-sided estimates

$$k_0 e^\sigma \le f_\pm(\sigma) \le k_1 e^\sigma \qquad \text{for all } \sigma \le 0 \tag{1.19}$$

and

$$l_0(\sigma+1)^{\frac{2}{p}} \le f_\pm(\sigma) \le l_1(\sigma+1)^{\frac{2}{p}} \qquad \text{for all } \sigma \ge 0 \tag{1.20}$$

hold with positive constants k_0, k_1, l_0 and l_1.
To see this, we let φ_+, a_+ and s_+ be as given by Lemma 1.1, next choose $s_0 > 0$ small enough such that $s_0 < s_+$ and

$$s_0^{-p} > \frac{3}{2}(s_+^{-p} + a_+), \tag{1.21}$$

and then take $s_- \in (0, s_0), a_- \in (0, \frac{1}{3s_-^p}), b_- \in (0, \sqrt{\frac{2}{p-2}})$ and φ_- as provided by Lemma 1.2.
We now define $f_\pm$ to be the solutions of

$$\begin{cases} f_+'(\sigma) = \varphi_+(f_+(\sigma)), & \sigma \in \mathbb{R}, \\ f_+(0) = s_+, \end{cases} \tag{1.22}$$

and

$$\begin{cases} f_-'(\sigma) = \varphi_-(f_-(\sigma)), & \sigma \in \mathbb{R}, \\ f_-(0) = s_-. \end{cases} \tag{1.23}$$

In view of the respective choices of φ_+ and φ_-, both these problems are explicitly solvable and we obtain

$$f_+(\sigma) = \begin{cases} e^\sigma \cdot \left[s_+^{-p} + a_+ - a_+ e^{p\sigma}\right]^{-\frac{1}{p}} & \text{if } \sigma \le 0, \\ \left(s_+^{\frac{p}{2}} + \sqrt{\frac{p^2}{2(p-2)}}\sigma\right)^{\frac{2}{p}} & \text{if } \sigma > 0, \end{cases} \tag{1.24}$$

as well as

$$f_-(\sigma) = \begin{cases} e^\sigma \cdot \left[s_-^{-p} - a_- + a_- e^{p\sigma}\right]^{-\frac{1}{p}} & \text{if } \sigma \le 0, \\ \left(s_-^{\frac{p}{2}} + \frac{pb_-}{2}\sigma\right)^{\frac{2}{p}} & \text{if } \sigma > 0. \end{cases} \tag{1.25}$$

Now differentiating (1.22) and (1.23) and using (1.2) and (1.6) we immediately deduce (1.17), whereas (1.19) and (1.20) directly result from (1.24) and (1.25).
To see (1.18), we consider the case $\sigma > 0$ first, in which we use that $s_- < s_+$ and $b_- < \sqrt{\frac{2}{p-2}}$ to derive from (1.24) and (1.25) that

$$\begin{aligned} f_-(\sigma) &= \left(s_-^{\frac{p}{2}} + \frac{pb_-}{2}\sigma\right)^{\frac{2}{p}} \\ &< \left(s_+^{\frac{p}{2}} + \frac{p\sqrt{\frac{2}{p-2}}}{2}\sigma\right)^{\frac{2}{p}} \\ &= f_+(\sigma) \qquad \text{for all } \sigma > 0. \end{aligned}$$

As to $\sigma \le 0$, however, in view of the fact that $a_- < \frac{1}{3s_-^p}$ and $s_- < s_0$, from (1.24) and (1.25) we gain

$$\begin{aligned} e^{p\sigma} \cdot \Big(f_-^{-p}(\sigma) - f_+^{-p}(\sigma) \Big) &= (s_-^{-p} - a_- + a_- e^{p\sigma}) - (s_+^{-p} + a_+ - a_- e^{p\sigma}) \\ &> s_-^{-p} - s_+^{-p} - a_- - a_+ \\ &> \frac{2}{3} s_-^{-p} - s_+^{-p} - a_+ \\ &> \frac{2}{3} s_0^{-p} s_+^{-p} - a_+ \\ &> 0 \end{aligned}$$

because of (1.21).
Having thus asserted (1.17)-(1.20), from standard theory of elliptic boundary value problems ([T]) we infer that for each $R > 0$ the problem

$$\begin{cases} \mathcal{E} f_R = 0, \qquad \sigma \in (-R, R), \\ f_R(-R) = k_1 e^{-R}, \\ f_R(R) = l_1 (R+1)^{\frac{2}{p}}, \end{cases} \tag{1.26}$$

has at least one positive solution $f_R \in C^2([-R, R])$ satisfying

$$k_0 e^\sigma \le f_R(\sigma) \le k_1 e^\sigma \qquad \text{for all } \sigma \in [-R, 0]$$

and

$$l_0 (\sigma + 1)^{\frac{2}{p}} \le f_R(\sigma) \le l_1 (\sigma + 1)^{\frac{2}{p}} \qquad \text{for all } \sigma \in [0, R].$$

These two-sided bounds allow us to apply elliptic regularity estimates to (1.26) so as to obtain that along an appropriate sequence of numbers $R_j \to \infty$ we have

$$f_{R_j} \to f \qquad \text{in } C^2_{loc}(\mathbb{R})$$

for some $f \in C^2(\mathbb{R})$ that solves $\mathcal{E} f = 0$ on $\mathbb{R}$ and fulfills

$$k_0 e^\sigma \le f(\sigma) \le k_1 e^\sigma \qquad \text{for all } \sigma \le 0 \tag{1.27}$$

and

$$l_0 (\sigma + 1)^{\frac{2}{p}} \le f(\sigma) \le l_1 (\sigma + 1)^{\frac{2}{p}} \qquad \text{for all } \sigma \ge 0. \tag{1.28}$$

Transforming back to the original variables via (1.15), we thereby obtain a solution F_α of (1.14) which, by (1.27) and (1.28), satisfies (0.13) and (0.14). By elliptic regularity arguments applied to (1.14), we furthermore infer that in fact $F_\alpha \in C^\infty(\mathbb{R})$. Finally, since (1.14) rules out the occurrence of both local minima and saddle points of F_α, in view of the asymptotics described by (0.13) and (0.14) we finally gain that F_α' must be positive throughout $\mathbb{R}$. ////

Our main result on the existence of slowly traveling wave solutions of (0.4) for positive times is now an immediate consequence.

Proof of Theorem 0.3. In view of Lemma 1.3, we only need to make sure that u_α as defined by (0.12) in fact solves (0.4). To this end, we compute, omitting the argument $x + \frac{1}{p\alpha}\ln t$ of F_α,

$$u_{\alpha t} = -\frac{1}{p} t^{-\frac{1}{p}-1} \cdot F_\alpha + t^{-\frac{1}{p}} \cdot \frac{1}{p\alpha} \cdot \frac{1}{t} \cdot F'_\alpha \qquad \text{and} \qquad u_{\alpha xx} = t^{-\frac{1}{p}} \cdot F''_\alpha$$

and thus obtain

$$\begin{aligned} u_{\alpha t} - u_\alpha^p u_{\alpha xx} &= t^{-\frac{1}{p}-1} \cdot \Big\{ \frac{1}{p} F_\alpha + \frac{1}{p\alpha} F'_\alpha + F_\alpha^p F''_\alpha \Big\} \\ &= 0 \qquad \text{on } \mathbb{R} \times (0,\infty) \end{aligned}$$

according to (1.14). ////

2 Ancient slowly traveling wave solutions

In this section we seek for traveling waves defined for $t < 0$ by pursuing the ansatz

$$u(x,t) = (-t)^{-\frac{1}{p}} \cdot G\Big(x + \frac{1}{p\alpha}\ln(-t)\Big), \qquad x \in \mathbb{R},\ t < 0.$$

Then instead of (0.11) and (1.1) we should solve

$$G^p G'' = \frac{1}{p} G - \frac{1}{p\alpha} G' \qquad \text{on } \mathbb{R} \tag{2.1}$$

and

$$g^p g'' = g - g' \qquad \text{on } \mathbb{R}, \tag{2.2}$$

respectively. We this time find it more convenient to study the latter equation directly, without order reduction. In fact, we shall see that it is possible to glue together two solutions of (2.2), one defined for negative and the other one for positive values of the variable.

The former will again be obtained using sub- and supersolutions, and our construction actually applies to any choice of $p > 0$ in (0.4).

Lemma 2.1 *Let $p > 0$. Then there exists a positive solution $g \in C^2((-\infty, 0])$ of*

$$g^p g'' = g - g', \qquad \sigma < 0, \tag{2.3}$$

and this solution satisfies

$$g'(\sigma) > 0 \qquad \textit{for all } \sigma \le 0 \tag{2.4}$$

as well as

$$c_0 e^\sigma \le g(\sigma) \le c_1 e^\sigma \qquad \textit{for all } \sigma \le 0 \tag{2.5}$$

for some $c_0 > 0$ and $c_1 > 0$.

Proof. We fix arbitrary constants $a_+ \geq 1$ and $a_- \in (0,1)$ and then pick numbers $s_+ \in (0,1)$ and $s_- \in (0, s_+)$ such that

$$a_+ s_+^p < 1 \tag{2.6}$$

and

$$(p+2)a_- s_-^p \leq 1 - a_-. \tag{2.7}$$

Then $s_+^{-p} > a_+$ and $s_-^p > 1 > a_-$, and hence the functions g_+ and g_- explicitly given by

$$g_\pm(\sigma) := \Big[(s_\pm^{-p} - a_\pm)e^{-p\sigma} + a_\pm\Big]^{-\frac{1}{p}}, \qquad \sigma \leq 0, \tag{2.8}$$

are both well-defined, positive and nondecreasing on $(-\infty, 0]$. Moreover, since $a_- < a_+$ and $s_- < s_+$ imply that

$$\begin{aligned} e^{p\sigma} \cdot \Big(g_+^{-p}(\sigma) - g_-^{-p}(\sigma)\Big) &= (s_+^{-p} - a_+ + a_+ e^{-p\sigma}) - (s_-^{-p} - a_- + a_- e^{-p\sigma}) \\ &= (s_+^{-p} - s_-^{-p}) - (a_+ - a_-)(1 - e^{-p\sigma}) \\ &< 0 \qquad \text{for all } \sigma \leq 0, \end{aligned}$$

g_+ and g_- are strictly ordered in the sense that

$$g_-(\sigma) < g_+(\sigma) \qquad \text{for all } \sigma \leq 0. \tag{2.9}$$

Next, it can easily be checked using (2.8) that g_+ and g_- are the respective solutions of the initial-value problems

$$\begin{cases} g_\pm' = g_\pm - a_\pm g_\pm^{p+1}, & \sigma < 0, \\ g_\pm(0) = s_\pm. \end{cases} \tag{2.10}$$

We claim that writing

$$\mathcal{E}g := -g^p g'' - g' + g,$$

we have the inequalities

$$\mathcal{E}g_+ > 0 \qquad \text{and} \qquad \mathcal{E}g_- < 0 \qquad \text{on } (-\infty, 0]. \tag{2.11}$$

Indeed, by (2.10) we compute

$$\begin{aligned} \mathcal{E}g_\pm &= -g_\pm^p \cdot \Big(1 - (p+1)a_\pm g_\pm^p\Big) \cdot \Big(g_\pm - a_\pm g_\pm^{p+1}\Big) - \Big(g_\pm - a_\pm g_\pm^{p+1}\Big) + g_\pm \\ &= g_\pm^{p+1} \cdot \Big(a_\pm - 1 + (p+2)a_\pm g_\pm^p - (p+1)a_\pm^2 g_\pm^{2p}\Big) \qquad \text{for } \sigma < 0. \end{aligned} \tag{2.12}$$

Since $a_+ \geq 1$ and g_+ increases on $(-\infty, 0]$, from (2.6) we obtain

$$\begin{aligned} \mathcal{E}g_+ &\geq a_+ g_+^{2p+1} \cdot \Big(p + 2 - (p+1)a_+ g_+^p\Big) \\ &\geq a_+ g_+^{2p+1} \cdot \Big(p + 2 - (p+1)a_+ s_+^p\Big) \\ &\geq a_+ g_+^{2p+1} \cdot \Big(p + 2 - (p+1)\Big) \\ &> 0 \qquad \text{on } (-\infty, 0]. \end{aligned}$$

As to g_-, in (2.12) we omit the last nonnegative summand in brackets and use the fact that also g_- increases in deriving

$$\begin{aligned} \mathcal{E}g_- &< g_-^{p+1} \cdot \Big(-(1-a_-) + (p+2)a_- g_-^p \Big) \\ &\leq g_-^{p+1} \cdot \Big(-(1-a_-) + (p+2)a_- s_-^p \Big) \\ &\leq 0 \qquad \text{on } (-\infty, 0] \end{aligned}$$

because of (2.7).
Having thereby found a pair of ordered sub- and supersolutions $g_\pm$ of (2.3), we can proceed in a standard way to construct a solution g of (2.3) fulfilling $g_- \leq g \leq g_+$ (cf. the proof of Lemma 1.3): According to [T], for each $R > 0$ the problem

$$\begin{cases} \mathcal{E}g_R = 0, \qquad \sigma \in (-\infty, 0), \\ g_R(-R) = g_+(-R), \\ g_R(0) = g_+(0), \end{cases} \tag{2.13}$$

has a solution $g_R \in C^2([-R, 0])$ such that $g_- \leq g_R \leq g_+$ in $[-R, 0]$. Using this in conjunction with elliptic regularity theory, we infer that along a suitable sequence of numbers $R = R_j \nearrow \infty$ we have $g_{R_j} \to g$ in $C^2_{loc}((-\infty, 0])$, where taking limits in (2.13) shows that g in fact solves (2.3). Since clearly $g_- \leq g \leq g_+$ on $(-\infty, 0]$, (2.5) immediately results from (2.8). Finally, (2.3) implies that $g'' > 0$ at each point where $g' = 0$, which together with (2.5) entails that actually $g' > 0$ on $(-\infty, 0]$. ////

We next extend the solution obtained above to the positive half-axis by solving a corresponding initial-value problem.

Lemma 2.2 *Let $p > 2$, $a > 0$ and $b > 0$. Then the initial-value problem*

$$\begin{cases} g^p g'' = g - g', \qquad \sigma > 0, \\ g(0) = a, \\ g'(0) = b, \end{cases} \tag{2.14}$$

has a positive solution $g \in C^2([0, \infty))$ satisfying $g' > 0$ on $[0, \infty)$ and

$$d_0(\sigma + 1) \leq g(\sigma) \leq d_1(\sigma + 1) \qquad \textit{for all } \sigma \geq 0 \tag{2.15}$$

with certain positive constants d_0 and d_1.

Proof. Let g denote the local solution of (2.14), extended up to some maximal $\sigma_{max} \in (0, \infty]$. Since $b > 0$ and g cannot attain a positive local maximum in $(0, \sigma_{max})$, we must have $g' > 0$ on $(0, \sigma_{max})$ and hence $g^p \geq a^p > 0$ on $(0, \sigma_{max})$. Therefore we have the two-sided estimate

$$-a^{-p} g' \leq g'' \leq a^{-p} g \qquad \text{on } (0, \sigma_{max}),$$

which excludes the possibility that either g or g' blows up at σ_{max}. It follows that actually $\sigma_{max} = \infty$, and that $g' > 0$ on $(0, \infty)$.
To see the right inequality in (2.15), we multiply the differential equation in (2.14) by $g^{-p}g'$ and integrate over $(0, \sigma)$ to obtain

$$\frac{1}{2}g'^2(\sigma) - \frac{1}{2}b^2 = \frac{1}{p-2}a^{2-p} - \frac{1}{p-2}g^{2-p}(\sigma) - \int_0^\sigma g^{-p}(\tau)g'^2(\tau)d\tau.$$

Dropping nonnegative terms, we conclude that

$$g'(\sigma) \le \sqrt{\frac{2}{p-2}a^{2-p} + b^2} =: \tilde{d}_1 \qquad \text{for all } \sigma > 0$$

and thereby see that the second estimate in (2.15) holds for $d_1 := \max\{a, \tilde{d}_1\}$.

In order to derive the lower bound in (2.15), let us proceed to show that there exists $\sigma_0 > 0$ such that

$$g''(\sigma) \ge 0 \qquad \text{for all } \sigma \ge \sigma_0. \tag{2.16}$$

To this end, we first observe that $\psi(\sigma) := g(\sigma) - g'(\sigma)$ satisfies

$$\begin{aligned} \psi' &= g' - g'' \\ &= g' - \frac{g - g'}{g^p} \\ &= g - \psi - \frac{\psi}{g^p} \\ &\ge -\Big(1 + \frac{1}{g^p}\Big) \cdot \psi \qquad \text{on } (0, \infty). \end{aligned}$$

Upon an ODE comparison argument, this guarantees that if $\psi(\sigma_1) \ge 0$ for some $\sigma_1 \in (0, \infty)$ then $\psi \ge 0$ on (σ_1, ∞). Thus, if (2.16) was false then $\psi \equiv g^p g''$ would be nonnegative on the whole interval $(0, \infty)$, so that we would obtain $g' > g$ and therefore $g(\sigma) > ae^\sigma$ on $(0, \infty)$. Since we have already verified the right inequality in (2.15), this yields a contradiction and thereby proves (2.16). Consequently,

$$g(\sigma) \ge g(\sigma_0) + g'(\sigma_0) \cdot (\sigma - \sigma_0) \qquad \text{for all } \sigma > \sigma_0,$$

which easily leads to the left inequality in (2.15). ////

Combining the above two lemmata in a straightforward manner, we can complete the construction of solutions to (2.1).

Corollary 2.3 *Let $p > 2$ and $\alpha > 0$. Then there exist positive constants c_0, c_1, d_0 and d_1 and a positive solution $G_\alpha \in C^\infty(\mathbb{R})$ of*

$$G_\alpha^p G_\alpha'' = \frac{1}{p}G_\alpha - \frac{1}{p\alpha}G_\alpha', \qquad \xi \in \mathbb{R}, \tag{2.17}$$

that satisfies $G_\alpha' > 0$ on $\mathbb{R}$ and (0.16) as well as (0.17).

Proof. We first apply Lemma 2.1 to obtain a positive solution $\tilde{g} \in C^2((-\infty,0])$ of $\tilde{g}^p\tilde{g}'' = \tilde{g} - \tilde{g}'$ for which (2.4) and (2.5) hold. In particular, both $a := \tilde{g}(0)$ and $b := \tilde{g}'(0)$ are positive and hence Lemma 2.2 applies to extend $\tilde{g}$ to a positive solution $g \in C^2(\mathbb{R})$ of the ODE $g^p g'' = g - g'$ on $\mathbb{R}$ which satisfies (2.5) and (2.15) and moreover $g' > 0$ on $\mathbb{R}$. Letting

$$G_\alpha(\xi) := (p\alpha^2)^{-\frac{1}{p}} g(\alpha\xi), \qquad \xi \in \mathbb{R},$$

we thus obtain a positive function $G_\alpha \in C^2(\mathbb{R})$ that solves (2.17) and fulfills $G_\alpha' > 0$ on $\mathbb{R}$ and, by (2.5) and (2.15), obviously has the properties (2.15) and (2.16) with suitable positive c_0, c_1, d_0 and d_1. Finally, by elliptic regularity arguments, from the validity of (2.17) and the positivity of G_α it follows that actually $G_\alpha \in C^\infty(\mathbb{R})$. ////

The proof of Theorem 0.4 is a direct consequence.

Proof of Theorem 0.4. Taking G_α as given by Corollary 2.3, we can copy almost word by word the proof of Theorem 0.3, and so we can omit repeating details here. ////

For later use, let us also note that upon a small time shift, we immediately obtain ancient solutions of (0.4), smooth up to $t = 0$, which approach zero as $t \to -\infty$.

Corollary 2.4 *Let $p > 2$ and $\alpha > 0$. Then the problem*

$$u_t = u^p u_{xx}, \qquad x \in \mathbb{R},\ t \le 0, \tag{2.18}$$

has a smooth positive classical solution $u \in C^\infty(\mathbb{R} \times (-\infty, 0])$ which has the property that for all $y \in \mathbb{R}$ one can find $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty((-\infty,y))} \le C(-t)^{-\frac{1}{p}} \ln(-t) \qquad \text{for all } t \in (-\infty,-2). \tag{2.19}$$

Furthermore, there exist positive constants c_0, c_1, d_0 and d_1 such that

$$c_0 e^{\alpha x} \le u(x,0) \le c_1 e^{\alpha x} \qquad \text{for all } x \le 0 \tag{2.20}$$

and

$$d_0(x+1) \le u(x,0) \le d_1(x+1) \qquad \text{for all } x \ge 0, \tag{2.21}$$

and u has the uniform decay property

$$\|u\|_{L^\infty((-\infty,y)\times(-\infty,0))} \to 0 \qquad \text{as } y \to -\infty. \tag{2.22}$$

Proof. Letting $\overline{u}_\alpha$ denote the slowly traveling wave solution provided by Theorem 0.4, we fix an arbitrary $\tau > 0$ and define $u(x,t) := \overline{u}_\alpha(x, t-\tau)$ for $x \in \mathbb{R}$ and $t \in (-\infty, 0]$. Then (2.18) is obvious from the solution property of $\overline{u}_\alpha$, and (2.19) is a consequence of the formula (0.15) and the right inequality in (0.17). Moreover, the estimates (2.20) and (2.21) immediately result from (0.16) and (0.17), so that it remains to verify (2.22). To this end, according to (0.17) let us take $\tilde{d}_1 > 0$ such that

$$G_\alpha(\xi) \le \tilde{d}_1(\xi+1) \qquad \text{for all } \xi > 0. \tag{2.23}$$

Given $\varepsilon > 0$, we can pick $t_\varepsilon < 0$ such that

$$\tilde{d}_1(\tau - t)^{-\frac{1}{p}} \cdot \Big(1 + \frac{1}{p\alpha}\ln(\tau - t)\Big) < \varepsilon \qquad \text{for all } t < t_\varepsilon \tag{2.24}$$

and then take $x_\varepsilon < 0$ fulfilling

$$x_\varepsilon < -\frac{1}{p\alpha}\ln(\tau - t_\varepsilon) \tag{2.25}$$

and

$$c_1 e^{\alpha x_\varepsilon} < \varepsilon, \tag{2.26}$$

where c_1 is as in (0.16). To see that this choice guarantees that

$$u(x,t) \le \varepsilon \qquad \text{for all } x < x_\varepsilon \text{ and } t < 0, \tag{2.27}$$

we fix such x and t and let $\xi := x + \frac{1}{p\alpha}\ln(\tau - t)$. Then in the case $\xi < 0$, (0.16) tells us that

$$u(x,t) \le c_1(\tau - t)^{-\frac{1}{p}} e^{\alpha\xi} = c_1 e^{\alpha x} \le c_1 e^{\alpha x_\varepsilon} < \varepsilon.$$

If, conversely, $\xi \ge 0$, then we necessarily have $t \le t_\varepsilon$, because (2.25) asserts that if $t < t_\varepsilon$ and $x < x_\varepsilon$ then $\xi < x_\varepsilon + \frac{1}{p\alpha}\ln(\tau - t_\varepsilon) < 0$. Therefore, (2.23) and (2.24) apply to yield

$$u(x,t) \le (\tau - t)^{-\frac{1}{p}} \cdot \tilde{d}_1\Big(x_\varepsilon + 1 + \frac{1}{p\alpha}\ln(\tau - t)\Big) \le \tilde{d}_1(\tau - t)^{-\frac{1}{p}} \cdot (1 + \frac{1}{p\alpha}\ln(\tau - 1)\Big) < \varepsilon$$

for such (x,t). This establishes (2.22). ////

3 The Cauchy problem for $t > 0$

3.1 Construction of minimal solutions

In this section we consider the forward Cauchy problem associated with (0.4),

$$\begin{cases} u_t = u^p u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $u_0 > 0$ on $\mathbb{R}$ and, for simplicity, we assume that u_0 belongs to $C^3(\mathbb{R})$. A comprehensive solution theory for (3.1) with $p > 2$ was developed in [ERV] and [RV], where actually the transformed version (0.5) was addressed in the corresponding range $m \in (-1, 0)$. In particular, it was shown there that even positive classical solutions are never unique, which excludes any hope for a comparison principle without imposing further conditions on the solutions to be compared. We therefore briefly track a straightforward construction of one particular solution of (3.1) by a suitable approximation process. This stepwise procedure

will enable us to safely apply comparison arguments, because they are valid at the level of approximations.

To be more precise, we consider the problems

$$\begin{cases} u_{Mt} = u_M^p u_{Mxx}, & x \in \mathbb{R},\ t > 0, \\ u_M(x,0) = u_{0M}(x), & x \in \mathbb{R}, \end{cases} \tag{3.2}$$

with $u_{0M} \in W^{1,\infty}_{loc}(\mathbb{R})$ given by

$$u_{0M}(x) := \min\{u_0(x), M\}, \qquad x \in \mathbb{R},$$

for $M > 0$. These problems in turn can be approximated by the family of initial-boundary value problems

$$\begin{cases} u_{MRt} = u_{MR}^p u_{MRxx}, & x \in B_R,\ t > 0, \\ u_{MR}(\pm R, t) = 0, & x = \pm R,\ t > 0, \\ u_{MR}(x,0) = u_{0MR}(x), & x \in B_R, \end{cases} \tag{3.3}$$

where $R > 0$ and $B_R := (-R, R)$. Here,

$$u_{0MR}(x) := \chi\Big(\frac{|x|}{R}\Big) \cdot u_{0M}(x), \qquad x \in [-R, R],$$

with a fixed nonincreasing function $\chi \in C^\infty([0,1))$ satisfying $\chi > 0$ in $[0,1)$, $\chi(1) = 0$ and $\chi \equiv 1$ on $[0, \frac{1}{2}]$. Consequently, u_{0R} belongs to $W^{1,\infty}(\bar{B}_R)$, is positive inside B_R and vanishes on ∂B_R.

Let us first assert solvability of (3.2) through the approximation (3.3).

Lemma 3.1 *Let $p > 0$ and $u_0 \in C^3(\mathbb{R})$ be positive. Then (3.2) possesses a positive classical solution u_M which is global in time and satisfies $u_M \le M$ in $\mathbb{R} \times (0,\infty)$. This solution can be obtained as the limit in $C^0_{loc}(\mathbb{R} \times [0,\infty)) \cap C^{2,1}_{loc}(\mathbb{R} \times (0,\infty))$ of solutions u_{MR} of (3.3) as $R \to \infty$.*

Proof. According to standard parabolic theory ([LSU]), for each $\varepsilon \in (0,1)$ the problem

$$\begin{cases} u_{MR\varepsilon t} = u_{MR\varepsilon}^p u_{MR\varepsilon xx}, & x \in B_R,\ t > 0, \\ u_{MR\varepsilon}(\pm R, t) = \varepsilon, & x = \pm R,\ t > 0, \\ u_{MR\varepsilon}(x,0) = u_{0MR\varepsilon}(x) := u_{0MR}(x) + \varepsilon, & x \in B_R, \end{cases} \tag{3.4}$$

has a positive classical solution $u_{MR\varepsilon} \in C^0(\bar{B} \times [0,\infty)) \cap C^{2,1}(\bar{B} \times (0,\infty))$ which satisfies $\varepsilon \le u_{MR\varepsilon} \le M + \varepsilon$ in $B_R \times (0,\infty)$ by comparison. Moreover, using that u_0 is positive on $\mathbb{R}$, we have $c_R := \inf_{x \in B_R} u_0(x) > 0$, and thus it is easily checked that $u_{0MR}(x) \ge \tilde{c}_R \Theta_R(x)$ holds for all $x \in B_R$ with $\Theta_R(x) := \cos\frac{\pi x}{2R}$ and $\tilde{c}_R := \frac{2Rc_R}{\pi}$. Since the separated function

$$\underline{u}(x,t) := y(t) \cdot \Theta_R(x), \qquad y(t) := \left(\tilde{c}_R^{-p} + \Big(\frac{\pi}{2R}\Big)^2 pt\right)^{-\frac{1}{p}}, \qquad x \in \bar{B}_R,\ t \ge 0,$$

can easily be verified to satisfy $\underline{u}_t \le \underline{u}^p \underline{u}_{xx}$ in $B_R \times (0,\infty)$, another comparison thus yields the two-sided ε-independent estimate

$$y(t) \cdot \Theta_R(x) \le u_{MR\varepsilon}(x,t) \le M+1 \qquad \text{for all } x \in \bar{B}_R \text{ and } t \ge 0,$$

which allows for an application of parabolic regularity theory to yield uniform estimates for $(u_{MR\varepsilon})_{\varepsilon\in(0,1)}$ in $C^{\theta,\frac{\theta}{2}}_{loc}(\bar{B}_R \times [0,\infty))$ and in $C^{2+\theta,1+\frac{\theta}{2}}_{loc}(B_R \times (0,\infty))$ for some $\theta > 0$ ([LSU]). Along with the evident ordering property of $(u_{MR\varepsilon})_{\varepsilon\in(0,1)}$, this implies that as $\varepsilon \searrow 0$ we have $u_{MR\varepsilon} \to u_{MR}$ in $C^0_{loc}(\bar{B}_R \times [0,\infty)) \cap C^{2,1}_{loc}(B_R \times (0,\infty))$ for some positive u_{MR} that solves (3.3) classically.

Now from the properties of χ it follows that u_{MR} is nondecreasing with respect to R and hence approaches a limit u_M from below. Since evidently $u_{MR} \le M$ for all R, again invoking parabolic regularity theory we infer that actually the convergence $u_{MR} \to u_M$ takes place in the asserted topology. ////

Using suitable slowly traveling wave solutions as comparison functions, we now obtain that any smooth initial data that are dominated by some exponential lead to solutions of (3.1) decaying to zero as $t \to +\infty$.

Theorem 3.2 *Let $p > 2$, and assume that $u_0 \in C^3(\mathbb{R})$ is positive on $\mathbb{R}$ and fulfills*

$$u_0(x) \le ce^{\alpha x} \qquad \text{for all } x \in \mathbb{R} \tag{3.5}$$

with some positive constants α and c. Then (3.1) possesses a global positive classical solution $u \in C^{2,1}(\mathbb{R} \times [0,\infty))$ which satisfies

$$u(x,t) \le t^{-\frac{1}{p}} \cdot F_\alpha\Big(x + x_0 + \frac{1}{p\alpha}\ln t\Big) \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0 \tag{3.6}$$

with some sufficiently large $x_0 \in \mathbb{R}$, where F_α is as provided by Theorem 0.3.
In particular, for all $y \in \mathbb{R}$ one can find $C_y > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty((-\infty,y))} \le C_y \cdot t^{-\frac{1}{p}}(\ln t)^{\frac{2}{p}} \qquad \text{for all } t \ge 2, \tag{3.7}$$

and moreover u enjoys the uniform decay property

$$\|u\|_{L^\infty((-\infty,y)\times(0,\infty))} \to 0 \qquad \text{as } y \to -\infty. \tag{3.8}$$

Remark. Let us emphasize a caveat that is implicitly contained in the above statement: It shows that even strictly convex initial data do *not* enforce solutions of (3.1) to be monotone increasing in time. This effect, which is in close accordance with the results in [RV] on non-uniqueness in (0.4) for $p > 2$, clearly stems from the degeneracy in (0.4): In the corresponding problem for the heat equation $u_t = u_{xx}$, for instance, it can easily be seen from the explicit representation formula for solutions that the condition $u_{0xx} \ge 0$ ensures $u_t \ge 0$ in $\mathbb{R} \times (0,\infty)$ (provided that u lies in the commonly used solutions class of functions which for all $T > 0$ do not exceed $ce^{\alpha x^2}$ in $\mathbb{R} \times (0,T)$ for some $c > 0$ and $\alpha > 0$.)

Proof (of Theorem 3.2.) With $F_\alpha, c_0, c_1, d_0, d_1$ and u_α taken from Lemma 1.3 and Theorem 0.3, respectively, we fix $x_0 \in \mathbb{R}$ large enough such that

$$c_0 e^{\alpha x_0} > c, \tag{3.9}$$

and then, given $R > 0$, let

$$\tau_R := e^{-p\alpha(x_0+R)}. \tag{3.10}$$

Then

$$\begin{aligned} \overline{u}_R(x,t) &:= u_\alpha(x+x_0, t+\tau_R) \\ &\equiv (t+\tau_R)^{-\frac{1}{p}} \cdot F_\alpha\Big(x + x_0 + \frac{1}{p\alpha}\ln(t+\tau_R)\Big), \qquad x \in \mathbb{R},\ t \geq 0, \end{aligned}$$

defines a smooth positive function on $\mathbb{R} \times [0,\infty)$ that solves (0.4) classically according to Corollary 0.3. Moreover, if we restrict x to be contained in $[-R,R]$, we see that

$$x + x_0 + \frac{1}{p\alpha}\ln\tau_R = x - R \leq 0,$$

so that by (0.13),

$$\begin{aligned} \overline{u}_R(x,0) &= \tau_R^{-\frac{1}{p}} \cdot F_\alpha\Big(x + x_0 + \frac{1}{p\alpha}\ln t\Big) \\ &\geq \tau_R^{-\frac{1}{p}} \cdot c_0 e^{\alpha(x+x_0+\frac{1}{p\alpha}\ln\tau_R)} \\ &= c_0 e^{\alpha x_0} e^{\alpha x} \\ &> c e^{\alpha x} \qquad \text{for all } x \in [-R,R]. \end{aligned}$$

Consequently, $\overline{u}_R(\cdot,0) > u_{0MR}$ holds in $(-R,R)$ for any $M > 0$, whence the comparison principle says that $\overline{u}_R > u_{MR}$ in $(-R,R) \times (0,\infty)$. Taking $R \to \infty$ shows that for the solution $u_M = \lim_{R\to\infty} u_{MR}$ of (3.2) constructed in Lemma 3.1 we have

$$u_M(x,t) \leq t^{-\frac{1}{p}} \cdot F_\alpha\Big(x + x_0 + \frac{1}{p\alpha}\ln t\Big) \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0. \tag{3.11}$$

Thus, given $y \in \mathbb{R}$ and $T > 0$, for all $(x,t) \in (-\infty,y) \times (0,T)$ we find that $\xi := x + x_0 + \frac{1}{p\alpha}\ln t \leq \xi_{y,T} := y + x_0 + \frac{1}{p\alpha}\ln T$, and hence for small ξ we obtain from (0.13) that

$$u_M(x,t) \leq t^{-\frac{1}{p}} \cdot c_1 e^{\alpha(x+x_0+\frac{1}{p\alpha}\ln t)} = c_1 e^{\alpha(x+x_0)} \qquad \text{if } \xi \leq 0,$$

whereas if $\xi > 0$ then $t > e^{-p\alpha(y+x_0)}$, so that

$$u_M(x,t) \leq (e^{-p\alpha(y+x_0)})^{-\frac{1}{p}} \cdot F_\alpha(\xi_{y,T}) \qquad \text{if } \xi > 0,$$

because F_α is increasing. All in all, this implies that $(u_M)_{M>0}$ is locally bounded in $\mathbb{R} \times [0,\infty)$, and since u_M evidently increases with M, we may once more invoke parabolic regularity theory to infer that $(u_M)_{M>0}$ is relatively compact and hence convergent in

$C^0_{loc}(\mathbb{R}\times[0,\infty))\cap C^{2,1}_{loc}(\mathbb{R}\times(0,\infty))$ to a limit u which is a positive classical solution of (3.1) that satisfies (3.6).
Going back to (3.11), from the right inequality in (0.14) we moreover obtain $C>0$ such that

$$u(x,t)\le C\cdot t^{-\frac{1}{p}}\cdot\Big(x+\frac{1}{p\alpha}\ln t\Big)^{\frac{2}{p}}\quad\text{for all }(x,t)\in\mathbb{R}\times(0,\infty)\text{ with }x+\frac{1}{p\alpha}\ln t\ge C,$$

which clearly implies (3.7). Finally, the proof of (3.8) can be accomplished using (0.13) and (0.14) by repeating the argument from the proof of Corollary 2.4. ////

3.2 Spatial monotonicity

As we have seen in Theorem 3.2, spatial convexity of u_0 need not be inherited by solutions of (3.1). The question whether at least spatial monotonicity is preserved during evolution therefore appears not to be obvious. In order to prepare an affirmative answer, let us provide a comparison principle for sub- and supersolutions to (3.1) with sublinear growth as $|x|\to\infty$ in an appropriate sense. Our proof uses a well-established method involving smooth approximations of the function sgn_+.

Lemma 3.3 *Let $-\infty<t_0<t_1<\infty$ and $\underline{u}$ and $\overline{u}$ be two positive functions from $C^{2,1}(\mathbb{R}\times[t_0,t_1])$ satisfying*

$$\underline{u}_t-\underline{u}^p\underline{u}_{xx}\le 0\le\overline{u}_t-\overline{u}^p\overline{u}_{xx}\qquad\text{in }\mathbb{R}\times(t_0,t_1)\tag{3.12}$$

and

$$\underline{u}(x,t_0)\le\overline{u}(x,t_0)\qquad\text{for all }x\in\mathbb{R}\tag{3.13}$$

as well as

$$\frac{1}{R}\int_{t_0}^{t_1}\underline{u}(\pm R,t)dt\to 0\qquad\text{as }R\to\infty.\tag{3.14}$$

Then

$$\underline{u}\le\overline{u}\qquad\text{in }\mathbb{R}\times(t_0,t_1).$$

Proof. We fix $\varphi\in C^\infty(\mathbb{R})$ such that $\varphi\equiv 0$ in $(-\infty,0)$ and $\varphi\equiv 1$ in $(1,\infty)$ and $0\le\varphi'\le 2$ on $\mathbb{R}$, and set $\varphi_\delta(s):=\varphi(\frac{s}{\delta})$ and $\Phi_\delta(s):=\int_0^s\varphi_\delta(\sigma)d\sigma$ for $s\in\mathbb{R}$ and $\delta>0$. Abbreviating $H(s):=-\frac{1}{p-1}s^{1-p}$ for $s>0$, from (3.12) we see that

$$\partial_t(H(\underline{u})-H(\overline{u}))\le(\underline{u}-\overline{u})_{xx}\qquad\text{in }\mathbb{R}\times(t_0,t_1).\tag{3.15}$$

We now fix $t\in(t_0,t_1)$ and $R>0$ and multiply (3.15) by $\varphi_\delta(\underline{u}-\overline{u})\cdot\Theta_R(x)$, where $\Theta_R(x):=\cos\frac{\pi x}{2R}$, $x\in[-R,R]$. On integration over $(-R,R)\times(t_0,t)$ we obtain

$$\begin{aligned}I_1(R,\delta)&:=\int_{t_0}^t\int_{-R}^R\varphi_\delta(\underline{u}-\overline{u})\cdot\partial_t(H(\underline{u})-H(\overline{u}))\cdot\Theta_R\\&\le\int_{t_0}^t\int_{-R}^R\varphi_\delta(\underline{u}-\overline{u})\cdot(\underline{u}-\overline{u})_{xx}\cdot\Theta_R\\&=:I_2(R,\delta).\end{aligned}\tag{3.16}$$

Since $\Theta_R(\pm R) = 0$ and $\Theta Rx(\pm R) = \mp\frac{\pi}{2R}$, two integrations by parts with respect to x yield

$$\begin{aligned} I_2(R,\delta) &= -\int_{t_0}^{t}\int_{-R}^{R} \varphi_\delta'(\underline{u}-\overline{u})\cdot(\underline{u}-\overline{u})_x^2\cdot\Theta_R + \int_{t_0}^{t}\int_{-R}^{R}\Phi_\delta(\underline{u}-\overline{u})\cdot\Theta_{Rxx} \\ &\quad + \frac{\pi}{2R}\int_{t_0}^{t}\Big\{\Phi_\delta(\underline{u}(R,\tau)-\overline{u}(R,\tau)) + \Phi_\delta(\underline{u}(-R,\tau)-\overline{u}(-R,\tau))\Big\}d\tau. \end{aligned}$$

Here we use that $\varphi_\delta' \geq 0$ and $\Theta_R \geq 0$ and hence $\Theta_{Rxx} \equiv -(\frac{\pi}{2R})^2\Theta_R \leq 0$, and that Φ_δ is nonnegative and nondecreasing to verify that

$$\begin{aligned} I_2(R,\delta) &\leq \frac{\pi}{2R}\int_{t_0}^{t_1}\Big\{\Phi_\delta(\underline{u}(R,\tau)) + \Phi_\delta(\underline{u}(-R,\tau))\Big\}d\tau \\ &\leq \frac{\pi}{2R}\int_{t_0}^{t_1}(\underline{u}(R,\tau)+\underline{u}(-R,\tau))d\tau, \end{aligned} \tag{3.17}$$

because $\Phi_\delta(s) \leq s$ for $s \geq 0$.
As to the term on the left of (3.16), we integrate by parts with respect to time to achieve the identity

$$\begin{aligned} I_1(R,\delta) &= \int_{t_0}^{t}\int_{-R}^{R}\varphi_\delta'(\underline{u}-\overline{u})\cdot(H(\underline{u})-H(\overline{u}))\cdot(\underline{u}_t-\overline{u}_t)\cdot\Theta_R \\ &\quad + \int_{-R}^{R}\varphi_\delta(\underline{u}-\overline{u})(\cdot,t)\cdot(H(\underline{u})-H(\overline{u}))(\cdot,t)\cdot\Theta_R \\ &=: I_{11}(R,\delta) + I_{12}(R,\delta), \end{aligned} \tag{3.18}$$

for $\underline{u} \leq \overline{u}$ at time t_0 by (3.13). Now the assumed positivity of $\overline{u}$ warrants the existence of some $c > 0$ such that $\overline{u} \geq c$ in $(-R,R)\times(t_0,t)$. Observing that $\varphi_\delta'(\underline{u}-\overline{u}) = 0$ whenever $\underline{u} \leq \overline{u}$ or $\underline{u} \geq \overline{u}+\delta$, and that $0 \leq \varphi_\delta'(\underline{u}-\overline{u}) \leq \frac{2}{\delta}$, using the mean-value theorem we obtain

$$|\varphi_\delta'(\underline{u}-\overline{u})\cdot(H(\underline{u})-H(\overline{u}))| \leq \varphi_\delta'(\underline{u}-\overline{u})\cdot\overline{u}^{-p}\cdot|\underline{u}-\overline{u}| \leq \frac{2}{\delta}\cdot c^{-p}\cdot\delta \quad \text{in } (-R,R)\times(t_0,t).$$

Therefore an application of the dominated convergence theorem ensures that for each fixed $R > 0$,

$$I_{11}(R,\delta) \to 0 \qquad \text{as } \delta \to 0, \tag{3.19}$$

because according to our hypotheses, $\underline{u}_t$ and $\overline{u}_t$ are bounded in $(-R,R)\times(t_0,t)$.
Finally, since $\varphi_\delta(s) \nearrow 1$ on $(0,\infty)$ as $\delta \searrow 0$, invoking Beppo Levi's theorem we infer that

$$I_{22}(R,\delta) \to \int_{-R}^{R}(H(\underline{u})-H(\overline{u}))_+(\cdot,t)\cdot\Theta_R \qquad \text{as } \delta \to 0 \tag{3.20}$$

for each $R > 0$. Collecting (3.16)-(3.20), in the limit $\delta \to 0$ we arrive at the inequality

$$\int_{-R}^{R}(H(\underline{u})-H(\overline{u}))_+(\cdot,t)\cdot\Theta_R \leq \frac{\pi}{2R}\int_{t_0}^{t_1}(\underline{u}(R,t)+\underline{u}(-R,t))dt$$

for all $R > 0$. Taking $R \to \infty$ here, in view of our assumption (3.14) we obtain that $H(\underline{u})(\cdot,t) \le H(\overline{u})(\cdot,t)$ in $\mathbb{R}$ and hence conclude from the strict monotonicity of H that $\underline{u} \le \overline{u}$ in $\mathbb{R} \times (t_0,t_1)$, because $t \in (t_0,t_1)$ was arbitrary. ////

As a straightforward consequence, we obtain the following.

Lemma 3.4 *Suppose that u_0 meets the requirements of Theorem 3.2 and moreover fulfils $u_{0x} \ge 0$ in $\mathbb{R}$. Then the global positive solution of (3.1) constructed in Theorem 3.2 satisfies $u_x \ge 0$ in $\mathbb{R} \times (0,\infty)$.*

Proof. For fixed $h > 0$ and $t_1 > 0$, we intend to apply Lemma 3.3 to

$$\underline{u}(x,t) := u(x,t) \qquad \text{and} \qquad \overline{u}(x,t) := u(x+h,t) \qquad \text{in } \mathbb{R} \times (0,t_1). \tag{3.21}$$

For this purpose, we observe that both $\underline{u}$ and $\overline{u}$ are solutions of the PDE $v_t = v^p v_{xx}$ in $\mathbb{R} \times (0,t_1)$, and the assumption $u_{0x} \ge 0$ implies that $\underline{u} \le \overline{u}$ at $t = 0$. In order to conclude that $\underline{u} \le \overline{u}$ in $\mathbb{R} \times (0,t_1)$, it is thus sufficient to verify (3.14), which amounts to showing that

$$I_1(R) := \frac{1}{R}\int_0^{t_1} u(R,t)dt \qquad \text{and} I_2(R) := \frac{1}{R}\int_0^{t_1} u(-R,t)dt, \qquad R > 0,$$

both vanish in the limit $R \to \infty$. To this end, we recall that Theorem 3.2 provides $x_0 > 0$ such that

$$u(R,t) \le t^{-\frac{1}{p}} \cdot F_\alpha\Big(R + x_0 + \frac{1}{p\alpha}\ln t\Big) \qquad \text{for all } t > 0 \tag{3.22}$$

with F_α taken from Theorem 0.3.
Assuming without loss of generality that $t_1 > t_\star := e^{-p\alpha(R+x_0)}$, we can split $I_1(R)$ and use (3.22) to obtain

$$\begin{aligned} I_1(R) &\le \frac{1}{R}\int_0^{t_\star} t^{-\frac{1}{p}} \cdot F_\alpha\Big(R + x_0 + \frac{1}{p\alpha}\ln t\Big)dt + \frac{1}{R}\cdot\int_{t_\star}^{t_1} t^{-\frac{1}{p}} \cdot F_\alpha\Big(R + x_0 + \frac{1}{p\alpha}\ln t\Big)dt \\ &=: I_{11}(R) + I_{12}(R) \end{aligned}$$

for all $R > 0$. According to (0.13), since $R + x_0 + \frac{1}{p\alpha}\ln t \le 0$ for $t \le t_\star$ we have

$$\begin{aligned} I_{11}(R) &\le \frac{1}{R}\int_0^{t_\star} t^{-\frac{1}{p}} \cdot c_1\, e^{\alpha(R+x_0+\frac{1}{p\alpha}\ln t)}dt = \frac{c_1}{R}\cdot e^{\alpha(R+x_0)}\cdot t_\star = \frac{c_1}{R}\cdot e^{-(p-1)\alpha(R+x_0)} \\ &\to 0 \qquad \text{as } R \to \infty, \end{aligned}$$

becasue $p \ge 1$. As to $I_{12}(R)$, from (0.14) and the monotonicity of $t \mapsto \ln t$ we infer that

$$\begin{aligned} I_{12}(R) &\le \frac{1}{R}\int_{t_\star}^{t_1} t^{-\frac{1}{p}} \cdot d_1\Big(R + x_0 + 1 + \frac{1}{p\alpha}\ln t\Big)^{\frac{2}{p}}dt \\ &\le \frac{d_1}{R}\cdot\Big(R + x_0 + 1 + \frac{1}{p\alpha}\ln t\Big)^{\frac{2}{p}}\cdot\int_0^{t_1} t^{-\frac{1}{p}}dt \end{aligned}$$

for all $R > 0$. Since $p > 2$, this proves that $I_{12}(R)$ and hence also $I_1(R)$ converges to zero as $R \to \infty$, and repeating the above argument we also obtain that $I_2(R) \to 0$ as $R \to \infty$. Therefore Lemma 3.3 applies to yield $u(x+h,t) - u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \in (0,t_1)$. Upon division by h, in the limit $h \searrow 0$ this shows that $u_x \geq 0$ in $\mathbb{R} \times (0,t_1)$ and thereby completes the proof, because $t_1 > 0$ was arbitrary. ////

4 Proofs of the main results on homoclinic orbits

We are now in the position to prove our main result concerning the existence of homoclinic orbits for (0.4).

Proof of Theorem 0.1. We fix any $\alpha > 0$ and let $u \in C^\infty(\mathbb{R} \times (-\infty, 0])$ be the ancient solution provided by Corollary 2.4. Then $u_0 := u(\cdot, 0)$ is smooth and positive on $\mathbb{R}$ and satisfies $u_0(x) \leq ce^{\alpha x}$ for all $x \in \mathbb{R}$ with some $c > 0$ by (2.20) and (2.21). Accordingly, Theorem 3.2 makes sure that u can be extended for positive times so as to yield an entire positive classical solution, which in fact belongs to $C^\infty(\mathbb{R} \times \mathbb{R})$ due to parabolic regularity theory. The decay estimates (0.7) and (0.8) directly result from (3.7) and (2.19), respectively, and from Corollary 2.4 and Theorem 3.2 it is also clear that $\|u\|_{L^\infty((-\infty,y)\times\mathbb{R})} \to 0$ as $y \to -\infty$. Finally, since clearly $u_x > 0$ in $\mathbb{R} \times (-\infty, 0]$, Lemma 3.4 asserts that $u_x \geq 0$ in $\mathbb{R} \times (0, \infty)$. Therefore the strong maximum principle guarantees that we even have $u_x > 0$ in $\mathbb{R} \times (0, \infty)$, which completes the proof. ////

Remark. It is likely to be expected that along the lines presented above, certain ancient slowly traveling waves can also be constructed in the regime $p \in (0, 2]$. However, we do not know how such solutions evolve for $t > 0$, and thus in particular have to leave open the interesting question whether or not (0.4) possesses homoclinic orbits also in the less degenerate case $p \leq 2$.

Let us finally make sure that no bounded homoclinic connections from $u \equiv 0$ to itself exist in the sense of Definition 0.1.

Proof of Proposition 0.2. Assuming on the contrary that such a homoclinic orbit u exists, we let $\varepsilon > 0, x_0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$ be given and claim that

$$u(x_0, t_0) \leq \varepsilon, \tag{4.23}$$

which will evidently lead to the absurd conclusion that $u \equiv 0$.
To this end, we take $x_1 < x_0$ such that

$$u(x_1, t) < \frac{\varepsilon}{2} \qquad \text{for all } t \in \mathbb{R}, \tag{4.24}$$

which is possible according to (0.9), and then pick a number $a > 0$ fulfilling

$$a > \frac{x_0 - 2^{\frac{p}{2}} x_1}{2^{\frac{p}{2}} - 1}\,. \tag{4.25}$$

This enables us to choose a number $\tau > -t_0$ satisfying

$$\frac{1}{\varepsilon} \cdot (x_0 + a)^{\frac{2}{p}} \leq \left\{ \frac{2(p-2)}{p} \cdot (t_0 + \tau) \right\}^{\frac{1}{p}} \leq \frac{2}{\varepsilon} \cdot (x_1 + a)^{\frac{2}{p}}. \tag{4.26}$$

We finally fix $T \in (0, t_0 + \tau)$ sufficiently close to $t_0 + \tau$ and $x_2 > x_0$ large enough such that

$$\frac{2(p-2)}{p} \cdot (t_0 + \tau - T) \leq \frac{(x_1 + a)^2}{M^p} \tag{4.27}$$

and

$$(x_2 + a)^{\frac{2}{p}} \geq \left\{ \frac{2(p-2)}{p} \cdot (t_0 + \tau) \right\}^{\frac{1}{p}} \cdot M, \tag{4.28}$$

where $M := \|u\|_{L^\infty(\mathbb{R}\times\mathbb{R})}$ is finite according to our hypothesis. Now the function $\overline{u} : \ Q := [x_1, x_2] \times [t_0 - T, t_0] \to \mathbb{R}$ defined by

$$\overline{u}(x,t) := y(t) \cdot (x + a)^{\frac{2}{p}}, \qquad (x,t) \in Q,$$

with

$$y(t) := \left\{ \frac{2(p-2)}{p} \cdot (t + \tau) \right\}^{-\frac{1}{p}}, \qquad t \in [t_0, T, t_0],$$

is smooth and positive in Q and satisfies

$$\overline{u}_t - \overline{u}^p \overline{u}_{xx} = \left\{ y' + \frac{2(p-2)}{p^2} \cdot y^{p+1} \right\} \cdot (x + a)^{\frac{2}{p}} = 0 \qquad \text{in } Q,$$

Moreover, at $t = t_0 - T$ we have

$$\begin{aligned} \overline{u}(x, t_0 - T) &= \left\{ \frac{2(p-2)}{p} \cdot (t_0 + \tau - T) \right\}^{-\frac{1}{p}} \cdot (x + a)^{\frac{2}{p}} \\ &\geq \left\{ \frac{2(p-2)}{p} \cdot (t_0 + \tau - T) \right\}^{-\frac{1}{p}} \cdot (x_1 + a)^{\frac{2}{p}} \\ &\geq M \qquad \text{for all } x \in [x_1, x_2] \end{aligned}$$

due to (4.27), whereas if $x = x_1$ then

$$\begin{aligned} \overline{u}(x_1, t) &= \left\{ \frac{2(p-2)}{p} \cdot (t + \tau) \right\}^{-\frac{1}{p}} \cdot (x_1 + a)^{\frac{2}{p}} \\ &\geq \left\{ \frac{2(p-2)}{p} \cdot (t_0 + \tau) \right\}^{-\frac{1}{p}} \cdot (x_1 + a)^{\frac{2}{p}} \\ &\geq \frac{\varepsilon}{2} \\ &> u(x_1, t) \qquad \text{for all } t \in [t_0 - T, t_0] \end{aligned}$$

as a consequence of (4.26) and (4.24). Similarly, using (4.28) and the definition of M, we see that on the right lateral boundary of Q,

$$\begin{aligned} \overline{u}(x_2,t) &\geq \left\{\frac{2(p-2)}{p}\cdot(t_0+\tau)\right\}^{-\frac{1}{p}}\cdot(x_2+a)^{\frac{2}{p}} \\ &\geq M \\ &\geq u(x_2,t) \qquad \text{for all } t\in[t_0-T,t_0]. \end{aligned}$$

Therefore, the comparison principle shows that $u \leq \overline{u}$ in Q. In view of the left inequality in (4.26), this in particular implies that

$$u(x_0,t_0) \leq \overline{u}(x_0,t_0) = \left\{\frac{2(p-2)}{p}\cdot(t_0+\tau)\right\}^{-\frac{1}{p}}\cdot(x_0+a)^{\frac{2}{p}} \leq \varepsilon$$

and thereby proves (4.23). ////

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