Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening

Michael Winkler*
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

This paper deals with nonnegative solutions of the Neumann initial-boundary value problem for the parabolic chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u - \mu u^2, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

in bounded convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary.

It is shown that if the ratio $\frac{\mu}{\chi}$ is sufficiently large, then the unique nontrivial spatially homogeneous equilibrium given by $u = v \equiv \frac{1}{\mu}$ is globally asymptotically stable in the sense that for any choice of suitably regular nonnegative initial data (u_0, v_0) such that $u_0 \not\equiv 0$, the above problem possesses a uniquely determined global classical solution (u, v) with $(u, v)|_{t=0} = (u_0, v_0)$ which satisfies

$$\left\| u(\cdot, t) - \frac{1}{\mu} \right\|_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \left\| v(\cdot, t) - \frac{1}{\mu} \right\|_{L^\infty(\Omega)} \to 0$$

as $t \to \infty$.

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*michael.winkler@math.uni-paderborn.de

1 Introduction

Chemotaxis with logistic cell kinetics. In the understanding of collective behavior in cell populations in biology, the partially oriented movement of cells in response to chemical signals, aka *chemotaxis*, is known to play a crucial role in various contexts. This importance partly stems from the fact that when combined with the ability of cells to produce the respective signal substance themselves, chemotaxis mechanisms are among the most primitive forms of intercellular communication. Typical examples include aggregation processes such as slime mold formation in *Dictyostelium Discoideum* ([15]) or pattern formation like e.g. in colonies of *Salmonella typhimurium* ([35]), but also medically relevant processes such as tumor invasion ([3], [17]) and self-organization during embryonic development ([24]). For a broad overview over various types of chemotaxis processes, we refer the reader to the survey [11] and the references therein.

In numerous cases, the time scales of chemotactic migration interfere with those of cell proliferation and death. It is known that the interplay of these mechanisms may result in quite colorful dynamical behavior, and numerical simulations suggest that some of these facets can already be described mathematically by straightforward extensions of the classical Keller-Segel chemotaxis model ([15]) such as the parabolic system

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v) + ru - \mu u^2, \qquad x\in\Omega,\ t>0, \\ \tau v_t = d_2\Delta v - \alpha v + u, \qquad x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

for the cell density $u = u(x,t)$ and the signal concentration $v = v(x,t)$ in the physical domain $\Omega\subset\mathbb{R}^n$, under appropriate choices of the parameters $d_1, d_2, \chi, r, \mu, \alpha$ and τ ([12]). Some corresponding rigorous analytical evidence for the occurrence of colorful solution behavior in such models has recently been gained in [34]: It has been shown there that in the Neumann problem for the spatially one-dimensional version of (1.1) with $\tau = 0, d_2 = \alpha = \chi = 1$, under a smallness assumption on d_1 the solution component u may exceed the respective carrying capacity $\frac{r}{\mu}$ to an arbitrary extent at some intermediate time. After all, the logistic cell kinetic term in (1.1) exerts a somewhat stabilizing influence on the system in the sense of blow-up prevention: Whereas in the case $r = \mu = 0$ corresponding to the classical Keller-Segel system, solutions may become unbounded within finite time when $n \geq 2$ ([10], [20], [33], [18]), it is known that arbitrarily small $\mu > 0$ enforce global existence and boundedness of solutions when $n \leq 2$, and that suitably large μ similarly rule out blow-up in the case $n \geq 3$ ([22], [23], [29], [32]).

Main result. It is the goal of the present paper to investigate in more detail how the destabilizing and aggregation-supporting properties of chemotactic cross-diffusion interact with growth limitations of logistic type. Having in mind this focusing, for simplicity in presentation we conveniently normalize all parameters in (1.1) except for χ and μ and thus henceforth specifically consider the prototypical parabolic initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + u - \mu u^2, \qquad x\in\Omega,\ t>0, \\ v_t = \Delta v - v + u, \qquad x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \qquad x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x\in\Omega, \end{cases} \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Here, ν denotes the outward normal vector field on $\partial\Omega$, and the initial data u_0 and v_0 are such that

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) & \text{is nonnegative with } u_0 \not\equiv 0, \quad \text{and that} \\ v_0 \in W^{1,\infty}(\Omega) & \text{is nonnegative.} \end{cases} \tag{1.3}$$

In this framework, we shall see that largeness of the coefficient μ, as related to χ, fully stabilizes the unique spatially homogeneous steady state $(u,v) \equiv (u_c, v_c) := (\frac{1}{\mu}, \frac{1}{\mu})$ in the sense that whenever $\frac{\mu}{\chi}$ is suitably large, the equilibrium (u_c, v_c) becomes globally asymptotically stable:

Theorem 1.1 *Let $n \geq 1$, and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with smooth boundary. Then there exists $M > 0$ such that whenever $\chi > 0$ and $\mu > 0$ are such that*

$$\frac{\mu}{\chi} > M, \tag{1.4}$$

for any choice of u_0 and v_0 complying with (1.3), the problem (1.2) possesses a global classical solution (u,v) which for any $q > n$ is unique in the class of functions such that

$$\begin{aligned} &u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \qquad \text{and} \\ &v \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty((0,\infty); W^{1,q}(\Omega)), \end{aligned}$$

and which satisfies

$$\left\| u(\cdot,t) - \frac{1}{\mu} \right\|_{L^\infty(\Omega)} \to 0 \tag{1.5}$$

as well as

$$\left\| v(\cdot,t) - \frac{1}{\mu} \right\|_{L^\infty(\Omega)} \to 0 \tag{1.6}$$

as $t \to \infty$.

Thus going beyond the boundedness results in [23], [22] and [32], for suitably large values of $\frac{\mu}{\chi}$ this entirely rules out any type of persisting oscillatory behavior, such as detected numerically in [12] for appropriate choices of χ and μ. As a consequence, in the context of (1.2) nontrivial dynamics can be expected only at intermediate time scales, or in cases when the death effect measured by μ is sufficiently weak as compared to the chemotactic sensitivity χ.
This is consistent both with the mentioned results on excession of carrying capacities in [34], and with corresponding asymptotic stability properties of (u_c, v_c) in the parabolic-elliptic counterpart of (1.2) obtained from (1.1) upon letting $\tau = 0$ ([29]). Similar statements on global attractivity of spatially constant equilibria have been derived for chemotaxis systems involving two species interacting through chemotaxis and Lotka-Volterra-type competition under appropriate assumptions on the dominance of cell kinetics relative to chemotaxis ([30], [26]; cf. also [36]). Even in presence of more complicated couplings such as in chemotaxis-haptotaxis models for tumor invasion, recent analysis on some prototypical systems indicates that a large relative strength of cell kinetics enhances the attractivity properties of homogeneous steady states (see [28] and also [27]).

From a mathematical point of view, all these results in the literature refer to systems in which the evolution of the chemoattractant is governed by an elliptic equation associated with the case when

$\tau = 0$ in (1.1). Such simplifications have been known to substantially ease the respective analysis by making the corresponding PDE systems in numerous situations accessible to techniques familiar from the treatment of scalar parabolic equations (see e.g. [14] and [20]). Accordingly, the literature contains only few results on large time behavior in chemotaxis systems with parabolic signal evolution, many of them concentrating either on cases when diffusion at spatial infinity enforces convergence to zero, or also on special solution classes such as self-similar solutions (see [8], [5], [2], [4], [1] and [13] for typical examples).

Viewed against this background, Theorem 1.1 goes one step further in that it provides, to the best of our knowledge, the first rigorous result on large time behavior in the fully parabolic version of the chemotaxis system (1.1) in any of the cases when the logistic proliferation term is nontrivial. Indeed, unlike the situation when $r = \mu = 0$ in (1.1), the system (1.2) in general is apparently lacking a gradient structure relevant to the asymptotics of solutions: Whereas (1.1) with $r = \mu = 0$ admits an energy inequality ([21]) which guarantees that ω-limit sets are contained in the set of corresponding equilibria, and which in the case $n = 2$ even enforces each solution to approach such a steady state in the large time limit ([5]), the loss of this structure for $(r, \mu) \neq (0, 0)$ entails that none of these conclusions seems obvious for (1.2).

Accordingly, our proof needs to be built on an alternative reasoning, at its core based on a one-sided pointwise estimate for the coupled quantity $z := U + \frac{\chi}{2}|\nabla V|^2$, where $U := u - \frac{1}{\mu}$ and $V := v - \frac{1}{\mu}$ denote the respective deviations from u_c and v_c. This estimate will result from a parabolic comparison argument on the basis of an absorptive parabolic inequality fulfilled by z (see (3.3)), and it will imply a certain eventual smallness property of u when μ is large (Lemma 3.1). As a consequence thereof, using parabolic regularization effects we will then successively obtain bounds for ∇v and for $A^\beta U$ in $L^p(\Omega)$ for arbitrary $p > 1$ and $\beta < \frac{1}{2}$, where $A := -\Delta + \lambda$ with suitably small $\lambda > 0$ (Section 4). These will imply a pointwise estimate for Δv (Section 5) and thereupon allow for another comparison argument which complements the upper bound for u, as previously obtained, by a corresponding inequality from below (Lemma 6.1). Thus knowing that U can eventually be controlled in $L^\infty(\Omega)$ by a conveniently small constant when μ is large (see Corollary 6.2), in Section 7 we shall be able to prove by a self-map-type reasoning that if $\frac{\mu}{\chi}$ is suitably large then U actually decays exponentially in time (Lemma 7.1), and that hence the conclusion of Theorem 1.1 is valid.

2 Preliminaries

To begin with, let us state a basic result on local existence, uniqueness and extensibility of classical solutions. A proof of the following lemma can be found in [32, Lemma 1.1].

Lemma 2.1 *Let $\chi > 0$ and $\mu > 0$, and suppose that u_0 and v_0 satisfy (1.3). Then there exist $T_{max} \in (0, \infty]$ and a classical solution (u, v) of (1.2) in $\Omega \times (0, T_{max})$ which is such that*

$$\text{either } T_{max} = \infty, \text{ or } \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Moreover, for any $q > n$ this solution is uniquely determined in the class of function couples such that

$$u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \qquad \text{and}$$
$$v \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \cap L^\infty((0, T_{max}); W^{1,q}(\Omega)).$$

Throughout the sequel, given u_0 and v_0 satisfying (1.3) we let T_{max} and (u,v) be as given by Lemma 2.1, and to simplify notation we shall abbreviate the deviations from the nonzero homogeneous steady state by introducing

$$U(x,t) := u(x,t) - \frac{1}{\mu} \qquad \text{and} \qquad V(x,t) := v(x,t) - \frac{1}{\mu} \tag{2.2}$$

for $x \in \bar{\Omega}$ and $t \geq 0$. Then by straightforward computation it follows that (U,V) solves

$$\begin{cases} U_t = \Delta U - \chi \nabla \cdot (u \nabla V) - U - \mu U^2, & x \in \Omega,\ t > 0, \\ V_t = \Delta V - V + U, & x \in \Omega,\ t > 0, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ U(x,0) = u_0(x) - \frac{1}{\mu}, \quad V(x,0) = v_0(x) - \frac{1}{\mu}, & x \in \Omega. \end{cases} \tag{2.3}$$

3 An explicit bound for u via comparison. Global existence

The cornerstone of our analysis will be provided by the following lemma which, under a largeness assumption on $\frac{\mu}{\chi}$, establishes an explicit pointwise upper estimate that is universal in the sense that for each individual solution it asserts the eventual validity of an appropriate bound for u. The proof is based on a comparison argument, inspired by [32, Introduction], which makes use of a favorable parabolic differential inequality satisfied by the quantity $z := U + \frac{\chi}{2}|\nabla V|^2$ coupling both components in (2.2) (cf. (3.3) below). The following reasoning is the only place in this paper where the convexity of Ω is explicitly used.

Lemma 3.1 *Suppose that $\mu > \frac{n\chi}{4}$. Then for any choice of u_0 and v_0 fulfilling (1.3), the solution of (1.2) is global in time and satisfies*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{1}{(1-\theta)\mu}, \tag{3.1}$$

where $\theta := \frac{n\chi}{4\mu}$.

Proof. With U and V as defined in (2.2), we let $z(x,t) := U(x,t) + \frac{\chi}{2}|\nabla V(x,t)|^2$ for $x \in \bar{\Omega}$ and $t \in (0, T_{max})$. Then using (2.3), we compute

$$\begin{aligned} z_t &= U_t + \chi \nabla V \cdot \nabla V_t \\ &= \Delta U - \chi \nabla u \cdot \nabla V - \chi u \Delta V - U - \mu U^2 + \chi \nabla V \cdot \nabla \Delta V - \chi |\nabla V|^2 + \chi \nabla V \cdot \nabla U \end{aligned}$$

for all $x \in \Omega$ and $t \in (0, T_{max})$. Here the equality $u = U + \frac{1}{\mu}$ implies the cancellation

$$-\chi \nabla u \cdot \nabla V + \chi \nabla V \cdot \nabla U \equiv 0,$$

so that in light of the pointwise identity $\nabla V \cdot \nabla \Delta V = \frac{1}{2}\Delta |\nabla V|^2 - |D^2 V|^2$ we obtain

$$z_t = \Delta U + \Delta\Big(\frac{\chi}{2}|\nabla V|^2\Big) - \chi |D^2 V|^2 - \chi u \Delta V - U - \mu U^2 - \chi |\nabla V|^2 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}). \tag{3.2}$$

Now since by the Cauchy-Schwarz inequality we have $|\Delta V|^2 \le n|D^2V|^2$, Young's inequality entails that

$$\begin{aligned}
-\chi u\Delta V &\le \frac{\chi}{n}|\Delta V|^2 + \frac{n\chi}{4}u^2 \\
&\le \chi|D^2V|^2 + \frac{n\chi}{4}u^2 \\
&= \chi|D^2V|^2 + \frac{n\chi}{4}\Big(U+\frac{1}{\mu}\Big)^2 \\
&= \chi|D^2V|^2 + \frac{n\chi}{4}U^2 + \frac{n\chi}{2\mu}U + \frac{n\chi}{4\mu^2} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T_{max}).
\end{aligned}$$

From (3.2), on dropping a nonpositive term we thus obtain the inequality

$$\begin{aligned}
z_t - \Delta z + z &= -\chi|D^2V|^2 - \chi u\Delta V - \mu U^2 - \frac{\chi}{2}|\nabla V|^2 \\
&\le \frac{n\chi}{4}U^2 + \frac{n\chi}{2\mu}U + \frac{n\chi}{4\mu^2} - \mu U^2 \\
&= -\Big(\mu - \frac{n\chi}{4}\Big)\cdot\Big\{U^2 - \frac{n\chi}{2\mu(\mu-\frac{n\chi}{4})}\cdot U - \frac{n\chi}{4\mu^2(\mu-\frac{n\chi}{4})}\Big\} \\
&= -\Big(\mu - \frac{n\chi}{4}\Big)\cdot\Big\{\Big(U - \frac{n\chi}{4\mu(\mu-\frac{n\chi}{4})}\Big)^2 - \Big(\frac{n\chi}{4\mu(\mu-\frac{n\chi}{4})}\Big)^2 - \frac{n\chi}{4\mu^2(\mu-\frac{n\chi}{4})}\Big\} \\
&\le \Big(\mu - \frac{n\chi}{4}\Big)\cdot\Big\{\Big(\frac{n\chi}{4\mu(\mu-\frac{n\chi}{4})}\Big)^2 + \frac{n\chi}{4\mu^2(\mu-\frac{n\chi}{4})}\Big\} \\
&= \frac{n\chi}{4\mu^2}\cdot\Big\{\frac{n\chi}{4(\mu-\frac{n\chi}{4})} + 1\Big\} \\
&= \frac{n\chi}{\mu(4\mu-n\chi)} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T_{max}),
\end{aligned} \tag{3.3}$$

where we have used our assumption $\mu > \frac{n\chi}{4}$. In order to derive an estimate for z itself from this, we note that since Ω is convex and $\frac{\partial V}{\partial\nu} = 0$ on $\partial\Omega$, according to a well-known result ([16]) we know that $\frac{\partial|\nabla V|^2}{\partial\nu} \le 0$ on $\partial\Omega$ and hence also $\frac{\partial z}{\partial\nu} \le 0$ on $\partial\Omega$. We therefore may compare z to spatially homogeneous functions having a supersolution property with regard to the parabolic operator in (3.3). Indeed, if we abbreviate $t_0 := \min\{\frac{1}{2}T_{max}, 1\}$ and $c_1 := \|U(\cdot,t_0)\|_{L^\infty(\Omega)} + \frac{\chi}{2}\|\nabla V(\cdot,t_0)\|^2_{L^\infty(\Omega)}$ and let $y \in C^1([t_0,\infty))$ denote the solution of the initial-value problem

$$\begin{cases} y'(t) + y(t) = \frac{n\chi}{\mu(4\mu-n\chi)}, & t > t_0, \\ y(t_0) = c_1, \end{cases} \tag{3.4}$$

then from the comparison principle and the initial ordering $z(x,t_0) \le y(t_0)$, valid thanks to our choice of c_1, we infer that

$$z(x,t) \le y(t) \qquad \text{for all } x\in\Omega \text{ and } t\in(t_0,T_{max}). \tag{3.5}$$

Upon explicitly solving (3.4), for instance, we see that y is bounded and moreover satisfies

$$y(t) \to \frac{n\chi}{\mu(4\mu-n\chi)} \qquad \text{as } t\to\infty. \tag{3.6}$$

Along with (3.5), this first implies that

$$u(x,t) \le \frac{1}{\mu} + y(t) \le \frac{1}{\mu} + \|y\|_{L^\infty((t_0,\infty))} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}),$$

which in view of (2.1) warrants that actually $T_{max} = \infty$. Thereupon, (3.5) and (3.6) show that

$$\begin{aligned} \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \frac{1}{\mu} + \limsup_{t\to\infty} y(t) \\ &= \frac{1}{\mu} + \frac{n\chi}{\mu(4\mu - n\chi)} = \frac{1}{\mu} \cdot \Big(1 + \frac{n\chi}{4\mu - n\chi}\Big) = \frac{4}{4\mu - n\chi} = \frac{1}{(1-\theta)\mu}, \end{aligned}$$

whereby the proof is completed. □

4 Bounds in $L^p(\Omega)$ for ∇v and $A^\beta U$ for $\beta < \frac{1}{2}$

We proceed to derive from the pointwise inequality implied by Lemma 3.1 an estimate for ∇v with respect to the norm in $L^p(\Omega)$ for arbitrary $p > 1$.

Lemma 4.1 *Let $p > 1$, and suppose that $\mu > \frac{n\chi}{4}$. Then there exists $C(p) > 0$ such that if (u,v) solves (1.2) for some (u_0, v_0) satisfying (1.3), then*

$$\limsup_{t\to\infty} \|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(p)}{(1-\theta)\mu}, \tag{4.1}$$

where $\theta = \frac{n\chi}{4\mu}$.

Proof. According to known smoothing estimates for the Neumann heat semigroup in Ω ([25], [31]), we can choose $c_1(p) > 0$ such that

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)} \le c_1(p)\tau^{-\frac{1}{2}}\|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in L^\infty(\Omega). \tag{4.2}$$

Moreover, an application of Lemma 3.1 shows that each of the considered solutions satisfies

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{1}{(1-\theta)\mu},$$

so that for any such (u,v) we can fix some suitably large $t_0 = t_0(u,v) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2}{(1-\theta)\mu} \qquad \text{for all } t \ge t_0. \tag{4.3}$$

Then by means of the variation-of constants representation for v, we can estimate

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \|\nabla e^{(t-t_0)(\Delta-1)} v(\cdot,t_0)\|_{L^p(\Omega)} + \int_{t_0}^t \|\nabla e^{(t-s)(\Delta-1)} u(\cdot,s)\|_{L^p(\Omega)} ds \qquad \text{for all } t > t_0, \tag{4.4}$$

where (4.2) implies that

$$\begin{aligned}\|\nabla e^{(t-t_0)(\Delta-1)}v(\cdot,t_0)\|_{L^p(\Omega)} &= e^{-(t-t_0)}\|\nabla e^{(t-t_0)\Delta}v(\cdot,t_0)\|_{L^p(\Omega)} \\ &\le c_1(p)(t-t_0)^{-\frac{1}{2}}e^{-(t-t_0)}\|v(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t>t_0. \end{aligned} \tag{4.5}$$

Moreover, combining (4.2) with (4.3) yields

$$\begin{aligned}\int_{t_0}^t \|\nabla e^{(t-s)(\Delta-1)}u(\cdot,s)\|_{L^p(\Omega)}ds &\le c_1(p)\cdot\int_{t_0}^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}ds \\ &\le \frac{2c_1(p)}{(1-\theta)\mu}\cdot\int_{t_0}^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}ds \\ &= \frac{2c_1(p)}{(1-\theta)\mu}\cdot\int_0^{t-t_0} \sigma^{-\frac{1}{2}}e^{-\sigma}d\sigma \\ &\le \frac{c_2(p)}{(1-\theta)\mu} \qquad \text{for all } t>t_0, \end{aligned} \tag{4.6}$$

where $c_2(p) := 2c_1(p)\cdot\int_0^\infty \sigma^{-\frac{1}{2}}e^{-\sigma}d\sigma$. From (4.4), (4.5) and (4.6) we therefore obtain that

$$\limsup_{t\to\infty}\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{c_2(p)}{(1-\theta)\mu},$$

as desired. □

For the next lemma and also for Lemma 5.1 below, we fix any number $\lambda\in(0,1)$ and, given $p>1$, let $A=A_p$ denote the realization of the operator $-\Delta+\lambda$ under homogeneous Neumann boudary conditions in $L^p(\Omega)$. Then it is known ([9], [6]) that A is sectorial and thus possesses closed fractional powers A^κ for arbitrary $\kappa>0$, and the corresponding domains $D(A^\kappa)$ are known to have the embedding property

$$D(A^\kappa)\hookrightarrow W^{2,\infty}(\Omega) \qquad \text{if } 2\kappa-\frac{n}{p}>2. \tag{4.7}$$

Moreover, if $(e^{-tA})_{t\ge0}$ denotes the corresponding analytic semigroup, then for each $\kappa>0$ there exists $K(p,\kappa)>0$ such that

$$\|A^\kappa e^{-tA}\varphi\|_{L^p(\Omega)} \le K(p,\kappa)t^{-\kappa}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0 \text{ and each } \varphi\in L^p(\Omega). \tag{4.8}$$

These properties allow us to turn the result from Lemma 4.1 into an estimate for U which entails some uniform regularity property beyond mere integrability.

Lemma 4.2 *Suppose that $\mu>\frac{n\chi}{4}$. Then for all $p>1$ and any $\beta\in(0,\frac{1}{2})$ there exists $C(p,\beta)>0$ such that if (u_0,v_0) satisfies (1.3) and u denotes the corresponding solution of (1.2), then for $U=u-\frac{1}{\mu}$ we have*

$$\limsup_{t\to\infty}\|A^\beta U(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(p,\beta)}{(1-\theta)^2\mu}, \tag{4.9}$$

where $\theta=\frac{n\chi}{4\mu}$.

Proof. Again by Lemma 3.1, we know that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{1}{(1-\theta)\mu},$$

whereas Lemma 4.1 yields $c_1(p) > 0$ such that

$$\limsup_{t\to\infty} \|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{c_1(p)}{(1-\theta)\mu}.$$

We can thus fix $t_0 = t_0(u,v) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2}{(1-\theta)\mu} \qquad \text{for all } t \ge t_0 \tag{4.10}$$

and

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{2c_1(p)}{(1-\theta)\mu} \qquad \text{for all } t \ge t_0. \tag{4.11}$$

Next, according to standard estimates for the Neumann heat semigroup, we can find $c_2(p) > 0$ such that

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le c_2(p)\cdot(1+\tau^{-\frac12})\|\varphi\|_{L^p(\Omega)} \qquad \begin{array}{l}\text{for all } \tau>0 \text{ and any } \varphi\in C^1(\bar\Omega;\mathbb{R}^n)\\ \text{such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega\end{array} \tag{4.12}$$

([31, Lemma 1.3]). Now using (2.3) and recalling that $\nabla V \equiv \nabla v$ and that $A = -\Delta + \lambda$, we represent U according to

$$\begin{aligned} U(\cdot,t) &= e^{(t-t_0)(\Delta-1)}U(\cdot,t_0) - \chi\int_{t_0}^t e^{(t-s)(\Delta-1)}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)ds - \mu\int_{t_0}^t e^{(t-s)(\Delta-1)}U^2(\cdot,s)ds \\ &= e^{-(1-\lambda)(t-t_0)}e^{-(t-t_0)A}U(\cdot,t_0) - \chi\int_{t_0}^t e^{-(1-\frac{\lambda}{2})(t-s)}e^{-\frac{t-s}{2}A}e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)ds \\ &\quad -\mu\int_{t_0}^t e^{-(1-\lambda)(t-s)}e^{-(t-s)A}U^2(\cdot,s)ds \qquad \text{for all } t>t_0, \end{aligned}$$

and thus can estimate

$$\begin{aligned} \|A^\beta U(\cdot,t)\|_{L^p(\Omega)} &\le e^{-(1-\lambda)(t-t_0)}\|A^\beta e^{-(t-t_0)A}U(\cdot,t_0)\|_{L^p(\Omega)} \\ &\quad +\chi\int_{t_0}^t e^{-(1-\frac{\lambda}{2})(t-s)}\Big\|A^\beta e^{-\frac{t-s}{2}A}e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds \\ &\quad +\mu\int_{t_0}^t e^{-(1-\lambda)(t-s)}\Big\|A^\beta e^{-(t-s)A}U^2(\cdot,s)\Big\|_{L^p(\Omega)}ds \qquad \text{for all } t>t_0. \end{aligned} \tag{4.13}$$

Here by (4.8),

$$\begin{aligned} e^{-(1-\lambda)(t-t_0)}\|A^\beta e^{-(t-t_0)A}U(\cdot,t_0)\|_{L^p(\Omega)} &\le K(p,\beta)(t-t_0)^{-\beta}e^{-(1-\lambda)(t-t_0)}\|U(\cdot,t_0)\|_{L^p(\Omega)} \\ &\to 0 \qquad \text{as } t\to\infty, \end{aligned} \tag{4.14}$$

while (4.8), (4.10) and the Hölder inequality imply that

$$\begin{aligned}
\mu\int_{t_0}^t e^{-(1-\lambda)(t-s)}\Big\|A^\beta e^{-(t-s)A}U^2(\cdot,s)\Big\|_{L^p(\Omega)}ds &\le K(p,\beta)\mu\cdot\int_{t_0}^t (t-s)^{-\beta}e^{-(1-\lambda)(t-s)}\|U(\cdot,s)\|_{L^p(\Omega)}^2 ds \\
&\le \frac{4K(p,\beta)|\Omega|^{\frac{2}{p}}}{(1-\theta)^2\mu}\int_{t_0}^t (t-s)^{-\beta}e^{-(1-\lambda)(t-s)}ds \\
&= \frac{4K(p,\beta)|\Omega|^{\frac{2}{p}}}{(1-\theta)^2\mu}\int_0^{t-t_0}\sigma^{-\beta}e^{-(1-\lambda)\sigma}d\sigma \\
&\le \frac{c_3(p,\beta)}{(1-\theta)^2\mu} \qquad \text{for all } t>t_0 \qquad (4.15)
\end{aligned}$$

with $c_3(p,\beta) := 4K(p,\beta)|\Omega|^{\frac{2}{p}}\int_0^\infty \sigma^{-\beta}e^{-(1-\lambda)\sigma}d\sigma$. Employing (4.8), (4.12), (4.10) and (4.11), we furthermore obtain that

$$\begin{aligned}
&\chi\int_{t_0}^t e^{-(1-\frac{\lambda}{2})(t-s)}\Big\|A^\beta e^{-\frac{t-s}{2}A}e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds \\
&\le K(p,\beta)\chi\cdot\int_{t_0}^t\Big(\frac{t-s}{2}\Big)^{-\beta}e^{-(1-\frac{\lambda}{2})(t-s)}\Big\|e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds \\
&\le K(p,\beta)c_2(p)\chi\cdot\int_{t_0}^t\Big(\frac{t-s}{2}\Big)^{-\beta}\Big(1+\Big(\frac{t-s}{2}\Big)^{-\frac{1}{2}}\Big)e^{-(1-\frac{\lambda}{2})(t-s)}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le K(p,\beta)c_2(p)\chi\cdot\int_{t_0}^t\Big(\frac{t-s}{2}\Big)^{-\beta}\Big(1+\Big(\frac{t-s}{2}\Big)^{-\frac{1}{2}}\Big)e^{-(1-\frac{\lambda}{2})(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla v(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le K(p,\beta)c_2(p)\chi\cdot\frac{2}{(1-\theta)\mu}\cdot\frac{2c_1(p)}{(1-\theta)\mu}\cdot\int_{t_0}^t\Big(\frac{t-s}{2}\Big)^{-\beta}\Big(1+\Big(\frac{t-s}{2}\Big)^{-\frac{1}{2}}\Big)\cdot e^{-(1-\frac{\lambda}{2})(t-s)}ds \\
&= \frac{2^{\beta+2}c_1(p)c_2(p)K(p,\beta)\chi}{(1-\theta)^2\mu^2}\cdot\int_0^{t-t_0}\sigma^{-\beta}\Big(1+\Big(\frac{\sigma}{2}\Big)^{-\frac{1}{2}}\Big)\cdot e^{-(1-\frac{\lambda}{2})\sigma}d\sigma \qquad \text{for all } t>t_0.
\end{aligned}$$

Here we eliminate χ by substituting $\frac{\chi}{\mu}=\frac{4\theta}{n}$, so that since $\theta<1$ we infer that

$$\chi\int_{t_0}^t e^{-(1-\frac{\lambda}{2})(t-s)}\Big\|A^\beta e^{-\frac{t-s}{2}A}e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds \le \frac{c_4(p,\beta)}{(1-\theta)^2\mu} \qquad \text{for all } t>t_0 \quad (4.16)$$

with

$$c_4(p,\beta) := \frac{2^{\beta+4}c_1(p)c_2(p)K(p,\beta)\chi}{n}\cdot\int_0^\infty\sigma^{-\beta}\Big(1+\Big(\frac{\sigma}{2}\Big)^{-\frac{1}{2}}\Big)\cdot e^{-(1-\frac{\lambda}{2})\sigma}d\sigma$$

being finite, because $\beta+\frac{1}{2}<1$ thanks to our assumption on β. Inserting (4.14), (4.15) and (4.16) into (4.13) finally yields

$$\limsup_{t\to\infty}\|A^\beta U(\cdot,t)\|_{L^p(\Omega)} \le \frac{c_3(p,\beta)}{(1-\theta)^2\mu}+\frac{c_4(p,\beta)}{(1-\theta)^2\mu}$$

and thereby completes the proof. □

5 A pointwise estimate for Δv

In the above lemma we now choose p suitably large and β sufficiently close to $\frac{1}{2}$ to establish, using another parabolic regularization argument, the following pointwise estimate for Δv.

Lemma 5.1 *Suppose that $\mu > \frac{n\chi}{4}$. Then there exists $C > 0$ such that for any choice of u_0 and v_0 fulfilling (1.3), the corresponding solution of (1.2) satisfies*

$$\limsup_{t\to\infty} \|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C}{(1-\theta)^2\mu} \tag{5.1}$$

with $\theta = \frac{n\chi}{4\mu}$.

Proof. We fix an arbitrary $\gamma \in (1, \frac{3}{2})$ and can then choose positive numbers β and p such that

$$\gamma - 1 < \beta < \frac{1}{2} \tag{5.2}$$

and

$$p > \frac{n}{2(\gamma-1)}. \tag{5.3}$$

In particular, we then have $2\gamma - \frac{n}{p} > 2\gamma - 2(\gamma-1) = 2$, whence (4.7) applies to yield $c_1 > 0$ such that

$$\|\varphi\|_{W^{2,\infty}(\Omega)} \le c_1 \|A^\gamma \varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in D(A^\gamma), \tag{5.4}$$

where $A \equiv A_p$. Moreover, since clearly $p > 1$, the right inequality in (5.2) allows us to infer from Lemma 4.2 that for any such solution, with (U,V) as in (2.2) we have

$$\limsup_{t\to\infty} \|A^\beta U(\cdot,t)\|_{L^p(\Omega)} \le \frac{c_2}{(1-\theta)^2\mu}$$

holds with some $c_2 > 0$ depending on p and β only, so that we can fix $t_0 = t_0(u,v)$ such that

$$\|A^\beta U(\cdot,t)\|_{L^p(\Omega)} \le \frac{2c_2}{(1-\theta)^2\mu} \qquad \text{for all } t \ge t_0. \tag{5.5}$$

Now according to a variation-of-constants formula associated with the second equation in (2.3), we can write

$$\begin{aligned} V(\cdot,t) &= e^{(t-t_0)(\Delta-1)}V(\cdot,t_0) + \int_{t_0}^t e^{(t-s)(\Delta-1)}U(\cdot,s)ds \\ &= e^{-(1-\lambda)(t-t_0)}e^{-(t-t_0)A}V(\cdot,t_0) + \int_{t_0}^t e^{-(1-\lambda)(t-s)}e^{-(t-s)A}U(\cdot,s)ds \qquad \text{for all } t > t_0 \end{aligned}$$

and hence use (5.4) to estimate

$$\begin{aligned} \|V(\cdot,t)\|_{W^{2,\infty}(\Omega)} &\le c_1\|A^\gamma V(\cdot,t)\|_{L^p(\Omega)} \\ &\le c_1 e^{-(1-\lambda)(t-t_0)}\|A^\gamma e^{-(t-t_0)A}V(\cdot,t_0)\|_{L^p(\Omega)} \\ &\quad + c_1\int_{t_0}^t e^{-(1-\lambda)(t-s)}\|A^\gamma e^{-(t-s)A}U(\cdot,s)\|_{L^p(\Omega)}ds \qquad \text{for all } t > t_0. \end{aligned} \tag{5.6}$$

By (4.8),

$$\begin{aligned} c_1 e^{-(1-\lambda)(t-t_0)} \|A^\gamma e^{-(t-t_0)A} V(\cdot,t_0)\|_{L^p(\Omega)} &\le c_1 K(p,\gamma)(t-t_0)^{-\gamma} e^{-(1-\lambda)(t-t_0)} \|V(\cdot,t_0)\|_{L^p(\Omega)} \\ &\to 0 \qquad \text{as } t\to\infty, \end{aligned} \tag{5.7}$$

whereas (4.8) combined with (5.5) shows that

$$\begin{aligned} & c_1 \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^\gamma e^{-(t-s)A} U(\cdot,s)\|_{L^p(\Omega)} ds \\ &\qquad = c_1 \int_{t_0}^t e^{-(1-\lambda)(t-s)} \|A^{\gamma-\beta} e^{-(t-s)A} A^\beta U(\cdot,s)\|_{L^p(\Omega)} ds \\ &\qquad \le c_1 K(p,\gamma-\beta) \int_{t_0}^t (t-s)^{-(\gamma-\beta)} e^{-(1-\lambda)(t-s)} \|A^\beta U(\cdot,s)\|_{L^p(\Omega)} ds \\ &\qquad \le c_1 K(p,\gamma-\beta) \cdot \frac{2c_2}{(1-\theta)^2\mu} \cdot \int_{t_0}^t (t-s)^{-(\gamma-\beta)} e^{-(1-\lambda)(t-s)} ds \\ &\qquad \le \frac{2c_1c_2K(p,\gamma-\beta)}{(1-\theta)^2\mu} \cdot \int_0^\infty \sigma^{-(\gamma-\beta)} e^{-(1-\lambda)\sigma} d\sigma \qquad \text{for all } t>t_0. \end{aligned}$$

Since here the rightmost integral is finite due to the fact that $\gamma-\beta<1$ by (5.2), together with (5.6) and (5.7) this proves (5.1), because $\Delta v \equiv \Delta V$. □

6 Refined pointwise inequalities for u

We are now in the position to show that after some suitable waiting time, u does not lie too far below $\frac{1}{\mu}$ if μ is appropriately large. In viewing the following statement from this perspective, we note that $\theta = \frac{n\chi}{4\mu} \to 0$ as $\mu\to\infty$.

Lemma 6.1 *Let $\mu > \frac{n\chi}{4}$. Then there exists $C>0$ with the property that any solution of (1.2) emanating from some initial data (u_0,v_0) complying with (1.3) satisfies*

$$\liminf_{t\to\infty} \Big(\inf_{x\in\Omega} u(x,t) \Big) \ge \frac{1}{\mu} - \frac{C\theta}{(1-\theta)^2\mu} \tag{6.1}$$

with $\theta = \frac{n\chi}{4\mu}$.

Proof. In accordance with Lemma 5.1, we fix $c_1>0$ such that for any such solution we have

$$\limsup_{t\to\infty} \|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{c_1}{(1-\theta)^2\mu},$$

whence we can pick $t_0 = t_0(u,v) > 0$ such that

$$\|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2c_1}{(1-\theta)^2\mu} \qquad \text{for all } t\ge t_0. \tag{6.2}$$

Therefore, in the first equation in (1.2) we can estimate

$$\begin{aligned} u_t &= \Delta u - \chi \nabla u \cdot \nabla v - \chi u \Delta v + u - \mu u^2 \\ &\geq \Delta u - \chi \nabla u \cdot \nabla v - \chi u \cdot \frac{2c_1}{(1-\theta)^2 \mu} + u - \mu u^2 \\ &\geq \Delta u - \chi \nabla u \cdot \nabla v + \Big(1 - \frac{2c_1 \chi}{(1-\theta)^2 \mu}\Big) \cdot u - \mu u^2 \qquad \text{for all } x \in \Omega \text{ and } t > t_0. \end{aligned}$$

Thus, if we let $y \in C^1([t_0, \infty))$ denote the solution of

$$\begin{cases} y'(t) = (1 - \frac{2c_1\chi}{(1-\theta)^2\mu}) \cdot y(t) - \mu y^2(t), \qquad t > t_0, \\ y(t_0) = c_2 := \inf\limits_{x \in \Omega} u(x, t_0), \end{cases} \tag{6.3}$$

then the comparison principle asserts that

$$u(x,t) \geq y(t) \qquad \text{for all } x \in \Omega \text{ and } t \geq t_0. \tag{6.4}$$

As u is striclty positive in $\bar{\Omega} \times (0, \infty)$ by the strong maximum principle, c_2 must be positive, so that e.g. explicitly solving the Bernoulli-type initial-value problem (6.3) shows that

$$y(t) \to \frac{(1 - \frac{2c_1\chi}{(1-\theta)^2\mu})_+}{\mu} \qquad \text{as } t \to \infty.$$

Therefore,

$$\liminf_{t \to \infty} \Big(\inf_{x \in \Omega} u(x,t) \Big) \geq \liminf_{t \to \infty} y(t) \geq \frac{1 - \frac{2c_1\chi}{(1-\theta)^2\mu}}{\mu},$$

which upon substituting $\frac{\chi}{\mu} = \frac{4\theta}{n}$ establishes (6.1) with $C := \frac{8c_1}{n}$. □

In conjunction with Lemma 3.1, this means that for large μ, the component u of any solution eventually enters a small neighborhood of the constant $\frac{1}{\mu}$.

Corollary 6.2 *Assume that $\mu > \frac{n\chi}{4}$. Then one can find $C > 0$ with the property that if u_0 and v_0 are such that (1.3) holds, for the solution of (1.2) we have*

$$\limsup_{t \to \infty} \|U(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C\theta}{(1-\theta)^2 \mu}, \tag{6.5}$$

where $U \equiv u - \frac{1}{\mu}$ and $\theta = \frac{n\chi}{4\mu}$.

Proof. Rewritten in terms of U, Lemma 6.1 states that there exists $c_1 \geq 1$ such that for any of the solutions in question we know that

$$\limsup_{t \to \infty} \|U_-(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{c_1 \theta}{(1-\theta)^2 \mu}.$$

On the other hand, Lemma 3.1 says that

$$\limsup_{t \to \infty} \|U_+(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{1}{(1-\theta)\mu} - \frac{1}{\mu} = \frac{\theta}{(1-\theta)\mu}.$$

Thus, (6.5) holds if we let $C := c_1$. □

7 Exponential decay of U. Proof of Theorem 1.1

With the latter estimate for U at hand, we can now prove that U actually must converge to zero, uniformly with respect to $x \in \Omega$, at an exponential rate.

Lemma 7.1 *Let $\alpha \in (0,1)$. Then there exists $\theta_0 = \theta_0(\alpha) \in (0,1)$ such that for any choice of $\chi > 0$ and $\mu > 0$ such that $\theta := \frac{n\chi}{4\mu}$ satisfies $\theta \le \theta_0$, for each solution of (1.2) with initial data fulfilling (1.3) one can find $C > 0$ such that $U = u - \frac{1}{\mu}$ satisfies*

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le C\, e^{-\alpha t} \qquad \text{for all } t > 0. \tag{7.1}$$

Proof. We fix an arbitrary $p > n$ and then recall known smoothing estimates for $(e^{\tau\Delta})_{\tau\ge 0}$, which in conjunction with the Poincaré inequality yield positive constants c_1, c_2 and c_3 such that

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)} \le c_1 \|\nabla\varphi\|_{L^p(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in W^{1,p}(\Omega) \tag{7.2}$$

and

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)} \le c_2(1+\tau^{-\frac12})\|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \tau > 0 \text{ and each } \varphi \in L^\infty(\Omega) \tag{7.3}$$

as well as

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_3(1+\tau^{-\frac12-\frac{n}{2p}})\|\varphi\|_{L^p(\Omega)} \qquad \begin{array}{l}\text{for all } \tau > 0 \text{ and all } \varphi \in C^1(\bar\Omega;\mathbb{R}^n)\\ \text{fulfilling } \varphi\cdot\nu = 0 \text{ on } \partial\Omega\end{array} \tag{7.4}$$

([19], [31], [7]). We furthermore note that since $\alpha < 1$ and $\frac12 + \frac{n}{2p} < 1$, the numbers

$$c_4 := \int_0^\infty (1+\sigma^{-\frac12})e^{-(1-\alpha)\sigma}d\sigma$$

and

$$c_5 := \int_0^\infty (1+\sigma^{-\frac12-\frac{n}{2p}})e^{-(1-\alpha)\sigma}d\sigma$$

are finite. Next, applying Lemma 3.1, Lemma 4.1 and Corollary 6.2 we obtain $c_6 > 0$ and $c_7 > 0$ such that whenever $\frac{n\chi}{4\mu} < 1$ and (u_0, v_0) satisfies (1.3), then with $\theta = \frac{n\chi}{4\mu}$ we have

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{1}{(1-\theta)\mu}$$

and

$$\limsup_{t\to\infty} \|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{c_6}{(1-\theta)\mu}$$

as well as

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{c_7\theta}{(1-\theta)^2\mu},$$

whence

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2}{(1-\theta)\mu} \qquad \text{for all } t \ge t_0(u,v) \tag{7.5}$$

and

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{2c_6}{(1-\theta)\mu} \qquad \text{for all } t \ge t_0(u,v) \tag{7.6}$$

as well as

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{2c_7\theta}{(1-\theta)^2\mu}, \qquad \text{for all } t \ge t_0(u,v) \tag{7.7}$$

with some suitably large $t_0(u,v) > 0$.
We now fix $\theta_0 \in (0,1)$ small enough such that

$$\frac{2c_7\theta}{(1-\alpha)(1-\theta)^2} \le \frac{1}{6} \qquad \text{for all } \theta < \theta_0 \tag{7.8}$$

and

$$\frac{8c_2c_3c_4c_5|\Omega|^{\frac{1}{p}}\theta}{n(1-\theta)} \le \frac{1}{12} \qquad \text{for all } \theta < \theta_0, \tag{7.9}$$

and henceforth suppose that $\mu > 0$ and $\chi > 0$ are fixed numbers such that $\theta = \frac{n\chi}{4\mu}$ satisfies $\theta < \theta_0$. We then choose a large number $B > 0$ fulfilling

$$\frac{2c_7\theta}{(1-\theta)^2\mu} \le \frac{B}{6} \tag{7.10}$$

and

$$\frac{16c_1c_3c_5c_6}{n(1-\theta)^2\mu} \le \frac{B}{12}, \tag{7.11}$$

and let (u,v) solve (1.2) with some (u_0,v_0) satisfying (1.3). Then with $t_0 := t_0(u,v)$ as introduced above and (U,V) as in (2.2), we consider the set

$$S := \Big\{ T_0 \ge t_0 \ \Big|\ \|U(\cdot,t)\|_{L^\infty(\Omega)} \le B\, e^{-\alpha(t-t_0)} \quad \text{for all } t \in [t_0,T_0] \Big\}$$

and note that S is not empty, because (7.7) and (7.10) imply that $\|U(\cdot,t_0)\|_{L^\infty(\Omega)} \le \frac{B}{6}$. In particular, $T := \sup S \in (t_0,\infty]$ is well-defined, and in order to prove the lemma it is sufficient to make sure that actually

$$T = \infty. \tag{7.12}$$

To verify this, we first use (2.3) to represent ∇V according to

$$\nabla V(\cdot,t) = \nabla e^{(t-t_0)(\Delta-1)}V(\cdot,t_0) + \int_{t_0}^t \nabla e^{(t-s)(\Delta-1)}U(\cdot,s)ds \qquad \text{for all } t > t_0 \tag{7.13}$$

and use (7.2), (7.6) and the fact that $\alpha < 1$ to estimate

$$\begin{aligned}
\|\nabla e^{(t-t_0)(\Delta-1)}V(\cdot,t_0)\|_{L^p(\Omega)} &= e^{-(t-t_0)}\|\nabla e^{(t-t_0)\Delta}V(\cdot,t_0)\|_{L^p(\Omega)} \\
&\le e^{-(t-t_0)} \cdot c_1\|\nabla V(\cdot,t_0)\|_{L^p(\Omega)} \\
&\le e^{-(t-t_0)} \cdot c_1 \cdot \frac{2c_6}{(1-\theta)\mu} \\
&\le \frac{2c_1c_6}{(1-\theta)\mu} \cdot e^{-\alpha(t-t_0)} \qquad \text{for all } t > t_0.
\end{aligned}$$

Furthermore, (7.3) along with the Hölder inequality and the definitions of T and c_4 entails that

$$\begin{aligned}
\left\| \int_{t_0}^t \nabla e^{(t-s)(\Delta-1)} U(\cdot,s)ds \right\|_{L^p(\Omega)} &\le c_2 \int_{t_0}^t \left(1+(t-s)^{-\frac12}\right) e^{-(t-s)} \|U(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le c_2 |\Omega|^{\frac1p} \cdot \int_{t_0}^t \left(1+(t-s)^{-\frac12}\right) e^{-(t-s)} \|U(\cdot,s)\|_{L^\infty(\Omega)} ds \\
&\le c_2 |\Omega|^{\frac1p} \cdot \int_{t_0}^t \left(1+(t-s)^{-\frac12}\right) e^{-(t-s)} \cdot B\, e^{-\alpha(s-t_0)} ds \\
&= c_2 |\Omega|^{\frac1p} B \cdot \left(\int_0^{t-t_0} \left(1+\sigma^{-\frac12}\right) e^{-(1-\alpha)\sigma} d\sigma \right) \cdot e^{-\alpha(t-t_0)} \\
&\le c_2 c_4 |\Omega|^{\frac1p} B\, e^{-\alpha(t-t_0)} \qquad \text{for all } t \in (t_0,T),
\end{aligned}$$

whence (7.13) shows that

$$\|\nabla V(\cdot,t)\|_{L^p(\Omega)} \le \left\{ \frac{2c_1c_6}{(1-\theta)\mu} + c_2c_4|\Omega|^{\frac1p} B \right\} \cdot e^{-\alpha(t-t_0)} \qquad \text{for all } t \in (t_0,T). \tag{7.14}$$

We next write

$$\begin{aligned}
U(\cdot,t) &= e^{(t-t_0)(\Delta-1)} U(\cdot,t_0) - \chi \int_{t_0}^t e^{(t-s)(\Delta-1)} \nabla \cdot \Big(u(\cdot,s) \nabla V(\cdot,s) \Big) ds \\
&\quad -\mu \int_{t_0}^t e^{(t-s)(\Delta-1)} U^2(\cdot,s) ds \qquad \text{for all } t > t_0,
\end{aligned}$$

and thus obtain that

$$\begin{aligned}
\|U(\cdot,t)\|_{L^\infty(\Omega)} &\le e^{-(t-t_0)} \|e^{(t-t_0)\Delta} U(\cdot,t_0)\|_{L^\infty(\Omega)} \\
&\quad +\chi \int_{t_0}^t e^{-(t-s)} \Big\| e^{(t-s)\Delta} \nabla \cdot \Big(u(\cdot,s) \nabla V(\cdot,s) \Big) \Big\|_{L^\infty(\Omega)} ds \\
&\quad +\mu \int_{t_0}^t e^{-(t-s)} \Big\| e^{(t-s)\Delta} U^2(\cdot,s) \Big\|_{L^\infty(\Omega)} ds \qquad \text{for all } t > t_0.
\end{aligned} \tag{7.15}$$

Here the maximum principle together with (7.7) and (7.10) ensures that

$$\begin{aligned}
e^{-(t-t_0)} \|e^{(t-t_0)\Delta} U(\cdot,t_0)\|_{L^\infty(\Omega)} &\le e^{-(t-t_0)} \|U(\cdot,t_0)\|_{L^\infty(\Omega)} \\
&\le e^{-(t-t_0)} \cdot \frac{2c_7\theta}{(1-\theta)^2\mu} \\
&\le \frac{2c_7\theta}{(1-\theta)^2\mu} \cdot e^{-\alpha(t-t_0)} \\
&\le \frac{B}{6} \cdot e^{-\alpha(t-t_0)} \qquad \text{for all } t > t_0,
\end{aligned} \tag{7.16}$$

again because $\alpha < 1$. We next recall (7.4) and (7.5) and employ the estimate (7.14) to see that

$$\chi \int_{t_0}^t e^{-(t-s)} \Big\| e^{(t-s)\Delta} \nabla \cdot \Big(u(\cdot,s) \nabla V(\cdot,s) \Big) \Big\|_{L^\infty(\Omega)} ds$$

$$
\begin{aligned}
&\le c_3\chi\int_{t_0}^t \Big(1+(t-s)^{-\frac12-\frac{n}{2p}}\Big)\,e^{-(t-s)}\|u(\cdot,s)\nabla V(\cdot,s)\|_{L^p(\Omega)}ds\\
&\le c_3\chi\int_{t_0}^t \Big(1+(t-s)^{-\frac12-\frac{n}{2p}}\Big)\,e^{-(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla V(\cdot,s)\|_{L^p(\Omega)}ds\\
&\le c_3\chi\int_{t_0}^t \Big(1+(t-s)^{-\frac12-\frac{n}{2p}}\Big)\,e^{-(t-s)}\cdot\frac{2}{(1-\theta)\mu}\cdot\left\{\frac{2c_1c_6}{(1-\theta)\mu}+c_2c_4|\Omega|^{\frac1p}B\right\}\cdot e^{-\alpha(s-t_0)}ds
\end{aligned}
$$

for all $t\in(t_0,T)$. Since $\frac{\chi}{\mu}=\frac{4\theta}{n}$, in view of the definition of c_5, the restrictions (7.11) and (7.9) thus imply that

$$
\begin{aligned}
&\chi\int_{t_0}^t e^{-(t-s)}\Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla V(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds\\
&\quad\le \frac{8c_3\theta}{n(1-\theta)}\cdot\left\{\frac{2c_1c_6}{(1-\theta)\mu}+c_2c_4|\Omega|^{\frac1p}B\right\}\cdot\left(\int_0^{t-t_0}\Big(1+\sigma^{-\frac12-\frac{n}{2p}}\Big)\,e^{-(1-\alpha)\sigma}d\sigma\right)\cdot e^{-\alpha(t-t_0)}\\
&\quad\le \left\{\frac{16c_1c_3c_5c_6\theta}{n(1-\theta)^2\mu}+\frac{8c_2c_3c_4c_5|\Omega|^{\frac1p}\theta}{n(1-\theta)}\cdot B\right\}\cdot e^{-\alpha(t-t_0)}\\
&\quad\le \left\{\frac{B}{12}+\frac{1}{12}\cdot B\right\}\cdot e^{-\alpha(t-t_0)}\\
&\quad= \frac{B}{6}\cdot e^{-\alpha(t-t_0)}\qquad\text{for all } t\in(t_0,T).
\end{aligned}
\tag{7.17}
$$

Finally, to treat the third summand on the right of (7.15) we combine (7.7) with the definition of T to find that

$$
\|U^2(\cdot,s)\|_{L^\infty(\Omega)}=\|U(\cdot,s)\|_{L^\infty(\Omega)}\cdot\|U(\cdot,s)\|_{L^\infty(\Omega)}\le\frac{2c_7\theta}{(1-\theta)^2\mu}\cdot B\,e^{-\alpha(s-t_0)}\qquad\text{for all } s\in(t_0,T).
$$

Therefore, thanks to (7.8) and once more due to the comparison priciple, the integral in question can be estimated according to

$$
\begin{aligned}
\mu\int_{t_0}^t e^{-(t-s)}\Big\|e^{(t-s)\Delta}U^2(\cdot,s)\Big\|_{L^\infty(\Omega)}ds &\le \mu\int_{t_0}^t e^{-(t-s)}\|U^2(\cdot,s)\|_{L^\infty(\Omega)}ds\\
&\le \mu\cdot\frac{2c_7\theta}{(1-\theta)^2\mu}\cdot B\cdot\int_{t_0}^t e^{-(t-s)}\cdot e^{-\alpha(s-t_0)}ds\\
&= \frac{2c_7\theta}{(1-\theta)^2}\cdot\left(\int_0^{t-t_0}e^{-(1-\alpha)\sigma}d\sigma\right)\cdot B\,e^{-\alpha(t-t_0)}\\
&\le \frac{2c_7\theta}{(1-\theta)^2}\cdot\frac{1}{1-\alpha}\cdot B\,e^{-\alpha(t-t_0)}\\
&\le \frac16 B\,e^{-\alpha(t-t_0)}\qquad\text{for all } t\in(t_0,T).
\end{aligned}
$$

In conjunction with (7.15), (7.16) and (7.17), this yields

$$
\|U(\cdot,t)\|_{L^\infty(\Omega)}\le 3\cdot\frac{B}{6}\,e^{-\alpha(t-t_0)}=\frac{B}{2}\,e^{-\alpha(t-t_0)}\qquad\text{for all } t\in(t_0,T),
$$

which by continuity of U implies that indeed T cannot be finite. This shows (7.12) and hence proves the lemma. $\square$

Now our main result can be obtained by combining Lemma 7.1 with a straightforward consequence thereof for the asymptotics of V.

Proof of Theorem 1.1. We fix any $\alpha \in (0,1)$ and let $\theta_0 = \theta_0(\alpha)$ be as thereupon provided by Lemma 7.1. We then set $M := \frac{n}{4\theta_0}$ and suppose that $\chi > 0$ and $\mu > 0$ are such that $\frac{\mu}{\chi} \geq M$. Then given u_0 and v_0 fulfilling (1.3), we apply Lemma 7.1 to find $c_1 > 0$ such that with (U,V) as in (2.2) we have

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1\, e^{-\alpha t} \qquad \text{for all } t > 0. \tag{7.18}$$

Now writing V in the form

$$V(\cdot,t) = e^{t(\Delta-1)}\Big(v_0 - \frac{1}{\mu}\Big) + \int_0^t e^{(t-s)(\Delta-1)} U(\cdot,s)ds, \qquad t > 0,$$

from the maximum principle we infer that

$$\begin{aligned} \|V(\cdot,t)\|_{L^\infty(\Omega)} &\leq e^{-t}\Big\|e^{t\Delta}\Big(v_0 - \frac{1}{\mu}\Big)\Big\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)}\|e^{(t-s)\Delta}U(\cdot,s)\|_{L^\infty(\Omega)} \\ &\leq e^{-t}\Big\|v_0 - \frac{1}{\mu}\Big\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)}\|U(\cdot,s)\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0. \end{aligned}$$

Abbreviating $c_2 : \|v_0 - \frac{1}{\mu}\|_{L^\infty(\Omega)}$, from (7.18) we therefore obtain

$$\begin{aligned} \|V(\cdot,t)\|_{L^\infty(\Omega)} &\leq c_2\, e^{-t} + c_1 \int_0^t e^{-(t-s)}\, e^{-\alpha s} ds \\ &= c_2\, e^{-t} + \frac{c_1}{1-\alpha}(e^{-\alpha t} - e^{-t}) \qquad \text{for all } t > 0. \end{aligned} \tag{7.19}$$

In light of the definitions of U and V, (7.18) and (7.19) establish (1.5) and (1.6), respectively. $\square$

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