Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity

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Abstract

We consider the chemotaxis-fluid system

$$\begin{cases} n_t + u\cdot\nabla n &= \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(x,n,c)\cdot\nabla c), \\ c_t + u\cdot\nabla c &= \Delta c - nf(c), \\ u_t + \nabla P &= \Delta u + n\nabla\phi, \\ \nabla\cdot u &= 0, \end{cases} \tag{0.1}$$

in a bounded convex domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, where $\phi\in W^{1,\infty}(\Omega)$ and D, f and S are given functions with values in $[0,\infty), [0,\infty)$ and $\mathbb{R}^{3\times 3}$, respectively.

In the existing literature, the derivation of results on global existence and qualitative behavior essentially relies on the use of energy-type functionals which seem to be available only in special situations, necessarily requiring the matrix-valued S to actually reduce to a scalar function of c which, along with f, in addition should satisfy certain quite restrictive structural conditions.

The present work presents a novel a priori estimation method which allows for removing any such additional hypothesis: Besides appropriate smoothness assumptions, in this paper it is only required that f is locally bounded in $[0,\infty)$, that S is bounded in $\Omega\times[0,\infty)^2$, and that $D(n)\ge k_D n^{m-1}$ for all $n\ge 0$ with some $k_D>0$ and some

$$m > \frac{7}{6}.$$

It is shown that then for all reasonably regular initial data, a corresponding initial-boundary value problem for (0.1) possesses a globally defined weak solution.

The method introduced here is efficient enough to moreover provide global boundedness of all solutions thereby obtained in that, inter alia, $n\in L^\infty(\Omega\times(0,\infty))$. Building on this boundedness property, it can finally even be proved that in the large time limit, any such solution approaches the spatially homogeneous equilibrium $(\overline{n_0},0,0)$ in an appropriate sense, where $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$, provided that merely $n_0\not\equiv 0$ and $f>0$ on $(0,\infty)$. To the best of our knowledge, these are the first results on boundedness and asymptotics of large-data solutions in a three-dimensional chemotaxis-fluid system of type (0.1).

Key words: chemotaxis, Stokes, nonlinear diffusion, global existence, boundedness, stabilization
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1 Introduction

We consider the chemotaxis-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n &= \nabla\cdot\Big(D(n)\nabla n\Big) - \nabla\cdot\Big(nS(x,n,c)\cdot\nabla c\Big), \qquad & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t &= \Delta u - \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u &= 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

in a bounded domain $\Omega\subset\mathbb{R}^N$, where the main focus of this work will be on the case $N=3$. Systems of this type arise in the modeling of populations of aerobic bacteria when suspended into sessile drops of water ([5], [31]). In this setting, $n=n(x,t)$ and $c=c(x,t)$ denote the density of the cell population and the oxygen concentration, respectively, and $u=u(x,t)$ and $P=P(x,t)$ represent the fluid velocity and the associated pressure. The essential modeling hypotheses underlying (1.2) are that cell movement is partially directed by gradients of the chemical which they consume, that convection transports both cells and oxygen, and that the presence of bacteria, which are slightly heavier than water, influences the fluid motion through buoyant forces in an external gravitational potential ϕ. The additional assumption that the fluid flow be comparatively slow is reflected in the fact that in (1.2) its evolution is described by the Stokes equations rather than the full Navier-Stokes system ([21]). Related mechanisms of chemotaxis-fluid interaction also arise in different biological contexts such as biomixing-based fertilization strategies of certain benthic invertebrates ([14], [15]).

Approaches based on a natural energy functional. According to the model specification underlying the numerical simulations in the original work [31], analytical studies in the existing mathematical literature concentrate on the particular version of (1.2) obtained on considering

$$n_t + u\cdot\nabla n = \nabla\cdot\Big(n^{m-1}\nabla n\Big) - \nabla\cdot\Big(n\chi(c)\nabla c\Big), \qquad x\in\Omega,\ t>0, \tag{1.3}$$

as the first equation therein, with $m\ge 1$ and a scalar chemotactic sensitivity function $\chi:\ [0,\infty)\to\mathbb{R}$. Here an essential step forward in the analysis was marked by the observation that under suitable structural assumptions linking χ to the oxygen consumption rate f, this class of versions of (1.2) admits for certain natural quasi-Lyapunov functionals which involve the logarithmic entropy $\int_\Omega n\ln n$. Indeed, when tracking the time evolution of the latter, the appearing crucial cross-diffusion-related integral $\int_\Omega \chi(c)\nabla c\cdot\nabla n$ can precisely be cancelled upon adding the result of a suitable testing procedure in the second equation in (1.2), where further integrals arising during the latter can be controlled conveniently under certain conditions on the relationship between χ and f. In the prototypical case $m=1, \chi\equiv 1$ and $f(c)=c$, for instance, this gives rise to an inequality of the form

$$\frac{d}{dt}\bigg\{\int_\Omega n\ln n + \frac12\int_\Omega \frac{|\nabla c|^2}{c}\bigg\} + \int_\Omega \frac{|\nabla n|^2}{n} + \int_\Omega c|D^2\ln c|^2 \le C\int_\Omega |u|^4, \qquad t>0, \tag{1.4}$$

with some $C>0$ ([35]). Appropriate a priori estimates gained from such energy-type inequalities may allow for the construction of global-in-time solutions, and thereby for going significantly beyond the outcome of the approach in [21], where local-in-time weak solutions were found for various boundary value problems associated with (1.2)-(1.3) without making use of such structural properties: In [6], energy-based arguments were applied to establish global existence of weak solutions, under certain technical conditions, in the case $\Omega = \mathbb{R}^2$ when $m = 1$. The particular requirement therein that $\|c_0\|_{L^\infty(\mathbb{R}^2)}$ be small was later removed in [20] at the cost of additional structural hypotheses on χ and f. That also the latter restriction can actually be relaxed was one of the results in [35], where a corresponding boundary value problem was considered in bounded convex domains (cf. the boundary conditions in (1.6) below) under milder assumptions on χ and f, without imposing any smallness condition on the initial data. A refined use of energy inequalities was performed there to derive statements on global existence of weak solutions in the three-dimensional setting and of smooth solutions in the case $N = 2$, even in the situation when the fluid evolution is governed by the full incompressible Navier-Stokes equations; recently, a further refinement of this analysis allowed for the construction of certain global weak solutions also in the corresponding three-dimensional semilinear chemotaxis-Navier-Stokes system ([37]). The powerfulness of this energy-based approach is further underlined by its ability to moreover yield information on the large time behavior of solutions if exploited properly: For instance, a generalized version of (1.4) was used in [36] as a starting point to show that in the latter two-dimensional chemotaxis-Navier-Stokes system all solutions stablilize to the spatially homogeneous equilibrium $(\overline{n_0}, 0, 0)$ in the large time limit, where $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0 > 0$ (see also [41] for an estimate on the rate of convergence, and [38] for a partial extension to the three-dimensional analogue).

As an apparently inherent drawback, any such type of energy-based reasoning seems to require quite inflexible properties of the parameter functions in (1.2), thereby possibly excluding even small perturbations of the latter. For instance, the mentioned results in [20] were inter alia built on the strong condition that $(\frac{f}{\chi})'' < 0$ be valid on $(0,\infty)$. This could only slightly be relaxed in [35], where merely $(\frac{f}{\chi})'' \le 0$ on $(0,\infty)$ was required; after all, this allowed for the choices $\chi \equiv const.$ and $f(c) = c$ in (1.3). Alternative structural conditions can be found in the more recent work [3], in which for the Navier-Stokes variant of (1.2) associated with (1.3) in $\Omega = \mathbb{R}^2$, a slightly modified version of the Lyapunov functional in (1.4) is analyzed to prove global existence of classical solutions for nonnegative and noncecreasing χ and f under the additional condition that $\|\chi - \mu f\|_{L^\infty((0,\infty))}$ be small for some $\mu \ge 0$. Even more drastically, for the construction of global weak solutions to the corresponding Cauchy problem in $\Omega = \mathbb{R}^3$ it is required there that χ precisely coincides with a fixed multiple of f. Only under appropriate smallness assumptions on the initial data, establishing global solutions seems to be achievable without any substantial structural restrictions on the parameter functions, by means of independent arguments essentially making use of the persiting negligibility of all nonlinear ingredients ([16]).

In the case of degenerate cell diffusion of porous medium type, that is, when $m > 1$ in (1.3), the above procedure yields an inequality quite similar to (1.4), again under essentially the same structural assumptions on χ and f. Correspondingly obtained a priori estimates can then be exploited to derive global existence of bounded weak solutions when $N = 2$ and $m > 1$ is arbitrary ([28]), of global weak but possibly unbounded solutions in the case $N = 3$ for any $m > 1$ ([7]), and of global weak solutions, locally bounded in $\bar{\Omega} \times [0,\infty)$, when $N = 3$ and $m > \frac{8}{7}$ ([29]), thereby going significantly beyond results achieved without making explicit use thereof ([4], [32]).

Analysis beyond natural energies. More recent experimental findings and corresponding modeling approaches suggest that chemotactic migration need not necessarily be directed exclusively toward increasing signal concentrations, but can rather have rotational components, especially near the physical boundary of the domain, and that accordingly the chemotactic sensitivity should actually be considered as a tensor with possibly nontrivial off-diagonal entries ([40]). In light of the observation that spontaneous emergence of structures indeed seems to occur mainly near droplet boundaries ([31]), it thus appears adequate to allow the parameter function S in (1.2) to attain values in $\mathbb{R}^{N\times N}$, and to thereby depart from the particular structure in (1.3). In this general situation, however, it seems no longer possible to derive inequalities of type (1.4) by means of any procedure which in a subtle way cancels contributions stemming from cross-diffusive interaction as described above.

Accordingly, the goal of the present work will be to develop an alternative a priori estimation method which is sufficiently robust so as to apply to (1.2) under very mild conditions on all parameter functions appearing therein. In fact, it turns out that our approach will provide integral estimates which will not only allow for the construction of global solutions to (1.2) that remain bounded for all times in a suitable sense, but which beyond this will also serve as a fundament for determining the large time behavior of these solutions.

Main results. In order to formulate our main results in this direction, let us specify the precise evolution problem addressed in the sequel by considering (1.2) along with the initial conditions

$$n(x,0)=n_0(x), \quad c(x,0)=c_0(x) \quad \text{and} \quad u(x,0)=u_0(x), \qquad x\in\Omega, \tag{1.5}$$

and under the boundary conditions

$$\Big(D(n)\nabla n - nS(x,n,c)\cdot\nabla c\Big)\cdot\nu=0, \quad \frac{\partial c}{\partial\nu}=0 \quad \text{and} \quad u=0 \qquad \text{on } \partial\Omega, \tag{1.6}$$

in a bounded convex domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, where throughout this paper we assume for convenience that

$$\begin{cases} n_0\in C^\kappa(\bar\Omega) \quad \text{for some } \kappa>0 \text{ with } n_0\ge 0 \text{ in } \Omega, \quad \text{that} \\ c_0\in W^{1,\infty}(\Omega) \quad \text{satisfies } c_0\ge 0 \text{ in } \Omega, \quad \text{and that} \\ u_0\in D(A_r^{\alpha_0}) \quad \text{for some } \alpha_0\in(\frac{3}{4},1) \text{ and all } r\in(1,\infty). \end{cases} \tag{1.7}$$

with A_r denoting the Stokes operator with domain $D(A_r):=W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega)\cap L^r_\sigma(\Omega)$, where $L^r_\sigma(\Omega):=\{\varphi\in L^r(\Omega)\ |\ \nabla\cdot\varphi=0\}$ for $r\in(1,\infty)$ (cf. also Section 3.1 below).
As for the diffusion coefficient in (1.2), we shall assume that D generalizes the porous-medium-like prototype $D(n)=mn^{m-1}$ by satisfying

$$D\in C^\theta_{loc}([0,\infty)) \qquad \text{for some } \theta>0, \tag{1.8}$$

as well as

$$D(n)\ge k_D n^{m-1} \qquad \text{for all } n>0 \tag{1.9}$$

with some $m>1$ and $k_D>0$, noting that this includes both degenerate and non-degenerate diffusion at $n=0$.
Apart from this, we shall merely suppose that

$$S\in C^2\Big(\bar\Omega\times[0,\infty)^2;\mathbb{R}^{3\times 3}\Big) \tag{1.10}$$

satisfies

$$|S(x,n,c)| \le S_0(c) \quad \text{for all } (x,n,c) \in \bar{\Omega} \times [0,\infty)^2 \qquad \text{with some nondecreasing } S_0 : [0,\infty) \to \mathbb{R}, \tag{1.11}$$

that

$$f \in C^1([0,\infty)) \quad \text{is nonnegative,} \tag{1.12}$$

and that

$$\phi \in W^{1,\infty}(\Omega), \tag{1.13}$$

underlining that unlike most previous studies, we do not require any monotonicity property of neither f nor S.

In the context of these assumptions, the first of our main results asserts global existence of a bounded solution in the following sense.

Theorem 1.1 *Let (1.10), (1.11), (1.12) and (1.13) hold, and suppose that D satisfies (1.8) and (1.9) with some*

$$m > \frac{7}{6}. \tag{1.14}$$

Then for any choice of n_0, c_0 and u_0 fulfilling (1.7), the problem (1.2), (1.5), (1.6) possesses at least one global weak solution (n,c,u,P) in the sense of Definition 4.1 below. This solution is bounded in $\Omega \times (0,\infty)$ in the sense that with some $C > 0$ we have

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{1.15}$$

Furthermore, c and u are continuous in $\bar{\Omega} \times [0,\infty)$ and

$$n \in C^0_{w-\star}([0,\infty); L^\infty(\Omega)); \tag{1.16}$$

that is, n is continuous on $[0,\infty)$ as an $L^\infty(\Omega)$-valued function with respect to the weak-$\star$ topology.

We note here that as compared to the global existence result in [29], the admissible range for m indicated by (1.14) is slightly smaller. However, besides requiring significantly less conditions on S and f, the statement in Theorem 1.1 goes considerably beyond the outcome in [29] in that, inter alia, *global* boundedness, rather than merely local boundedness, of solutions is obtained here. A further example for a lack of global boundedness in a system of type (1.2) can be found in [2], where global existence of possibly unbounded classical solutions is proved for the three-dimensional variant of (1.2) obtained on specifying $D \equiv 1$ and $f(c) = c$ for $c \ge 0$, and requiring that instead of (1.11), S decays for large values of n in the sense that $|S(x,n,c)| \le C(1+n^{-\alpha})$ for all $n \ge 0$ and some $C > 0$ and $\alpha > \frac{1}{6}$.

We can moreover show that all the above solutions approach the unique spatially homogeneous equilibrium corresponding to the bacterial mass $\int_\Omega n_0$ in the large time limit, provided that n_0 is nontrivial, and that the condition $f > 0$ on $(0,\infty)$ is satisfied, which is evidently necessary for such a behavior:

Theorem 1.2 *Let (1.10), (1.11), (1.12) and (1.13) hold, suppose that D satisfies (1.8) and (1.9) with some $m > \frac{7}{6}$, and assume that in addition*

$$f(c) > 0 \qquad \textit{for all } c > 0. \tag{1.17}$$

Then whenever (n_0, c_0, u_0) satisfies (1.7) with $n_0 \not\equiv 0$, the global weak solution constructed in Theorem 1.1 satisfies

$$n(\cdot,t) \stackrel{\star}{\rightharpoonup} \overline{n_0} \quad \text{in } L^\infty(\Omega), \qquad c(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{and} \qquad u(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \tag{1.18}$$

as $t \to \infty$, where $\overline{n_0} := \frac{1}{|\Omega|}\int_\Omega n_0$.

In particular, Theorem 1.1 and Theorem 1.2 provide some progress also in the fluid-free subcase of (1.2) obtained on letting $\phi \equiv 0$ and $u \equiv 0$. Indeed, for the correspondingly gained chemotaxis system with matrix-valued sensitivity the literature so far only contains very few results: Global classical solutions are known to exist in the case $N = 2$ but for small values of $\|c_0\|_{L^\infty(\Omega)}$ only ([18]); in the same two-dimensional setting, global bounded weak solutions for large initial data can be found whenever $m > 1$ ([1]); in the case $m = 1$, certain global generalized solutions, possibly unbounded, have recently been constructed for any $N \geq 1$ and arbitrarily large initial data ([39]). If this two-component system is further simplified by moreover choosing the sensitivity to be a constant scalar, again some energy arguments become available so as to yield much more comprehensive results on global existence of solutions and even on their large time behavior ([27]).

Our approach underlying the derivation of Theorem 1.1 consists at its core in an analysis of the functional

$$y(t) := \int_\Omega n^p(\cdot,t) + \int_\Omega |\nabla c(\cdot,t)|^{2q} + \int_\Omega |A^{\frac{1}{2}} u(\cdot,t)|^2, \qquad t \geq 0, \tag{1.19}$$

for solutions of certain regularized versions of (1.2) (see Section 2), where we eventually intend to choose $p > 1$ arbitrarily large. For this purpose, in Section 3.2 we shall first follow standard testing procedures to gain some basic information on the time evolution of each of the summands in (1.19) separately. It will turn out in Section 3.3 and in Section 3.4 that under suitable conditions on the relationship between the exponents $p > 1$ and $q > 1$ herein it is possible to estimate the respective ill-signed contributions appropriately, and thereby establish an ODI for y containing an absorptive linear term and thus implying an upper bound for y (see Lemma 3.14 and also Lemma 3.13).

Since boundedness of y even for large p and q does not directly entail sufficient regularity properties of u, our reasoning will involve a two-step bootstrap argument: In the first step thereof we shall only rely on the natural mass conservation property (2.4) and the smoothing action of the Stokes semigroup (Section 3.1) to gain a first integral bound for the Jacobian Du (cf. Lemma 3.15 and also Corollary 3.4); since such bounds allow for estimating certain integrals stemming from the signal-fluid interaction, provided that q in (1.19) is not too large (Lemma 3.11 and Lemma 3.12), this primary information can be used to derive a bound on y for certain small p and q (Lemma 3.15). The latter in turn implies higher regularity features of Du and thus enables us to treat y for arbitrarily large p and q in a second step (Lemma 3.16).

In deriving the convergence properties asserted in Theorem 1.2 in Section 5, we shall essentially make use of the boundedness statement from Theorem 1.1: In fact, the latter will enable us to exploit the finiteness of $\int_0^\infty \int_\Omega nf(c)$ and $\int_0^\infty \int_\Omega |\nabla c|^2$ (Lemma 3.20) to firstly obtain boundedness also of $\int_0^\infty \int_\Omega |\nabla n^\alpha|^2$ for some $\alpha > 1$ (Lemma 3.21), and to secondly prove that as a consequence of these three integral inequalities, all our solutions asymptotically become homogeneous in space and thus satisfy (1.18) (Lemma 5.1, Lemma 5.2 and Lemma 5.3).

2 Approximation by non-degenerate problems

Our goal is to construct solutions of (1.2) as limits of solutions to appropriately regularized problems. To achieve this, we approximate the diffusion coefficient function in (1.2) by introducing a family $(D_\varepsilon)_{\varepsilon\in(0,1)}$ of functions

$$\begin{aligned}
&D_\varepsilon \in C^2([0,\infty)) \quad \text{such that} \quad D_\varepsilon(n) \ge \varepsilon \quad \text{for all } n \ge 0 \text{ and } \varepsilon \in (0,1) \quad \text{and} \\
&D(n) \le D_\varepsilon(n) \le D(n) + 2\varepsilon \quad \text{for all } n \ge 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Next, it will be convenient to deal with homogeneous Neumann boundary conditions for both n and c rather than with the nonlinear no-flux relation in (1.6). In order to achieve this at least during our approximation procedure, following [18] we moreover fix families $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ and $(\chi_\varepsilon)_{\varepsilon\in(0,1)}$ of functions

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{with} \quad 0 \le \rho_\varepsilon \le 1 \text{ in } \Omega \quad \text{and} \quad \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0$$

and

$$\chi_\varepsilon \in C_0^\infty([0,\infty)) \quad \text{satisfying} \quad 0 \le \chi_\varepsilon \le 1 \text{ in } [0,\infty) \quad \text{and} \quad \chi_\varepsilon \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon \searrow 0,$$

and define smooth approximations $S_\varepsilon \in C^2(\bar\Omega \times [0,\infty)^2; \mathbb{R}^{3\times3})$ of S by letting

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x)\cdot\chi_\varepsilon(n)\cdot S(x,n,c), \qquad x \in \bar\Omega,\ n \ge 0,\ c \ge 0, \tag{2.1}$$

for $\varepsilon \in (0,1)$. Then for any such ε, the regularized problems

$$\begin{cases}
\partial_t n_\varepsilon + u_\varepsilon\cdot\nabla n_\varepsilon = \nabla\cdot\Big(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\Big) - \nabla\cdot\Big(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big), & x\in\Omega,\ t>0,\\
\partial_t c_\varepsilon + u_\varepsilon\cdot\nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon f(c_\varepsilon), & x\in\Omega,\ t>0,\\
\partial_t u_\varepsilon + \nabla P_\varepsilon = \Delta u_\varepsilon + n_\varepsilon\nabla\phi, & x\in\Omega,\ t>0,\\
\nabla\cdot u_\varepsilon = 0, & x\in\Omega,\ t>0,\\
\frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x\in\partial\Omega,\ t>0,\\
n_\varepsilon(x,0)=n_0(x), \quad c_\varepsilon(x,0)=c_0(x), \quad u_\varepsilon(x,0)=u_0(x), & x\in\Omega,
\end{cases} \tag{2.2}$$

are globally solvable in the classical sense:

Lemma 2.1 *Let $\varepsilon \in (0,1)$. Then there exist functions*

$$\begin{cases}
n_\varepsilon \in C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\bar\Omega\times(0,\infty)),\\
c_\varepsilon \in C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\bar\Omega\times(0,\infty)),\\
u_\varepsilon \in C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\bar\Omega\times(0,\infty)),\\
P_\varepsilon \in C^{1,0}(\bar\Omega\times(0,\infty)),
\end{cases}$$

such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.2) classically in $\Omega\times(0,\infty)$, and such that n_ε and c_ε are nonnegative in $\Omega\times(0,\infty)$.

PROOF. By an adaptation of well-established fixed point arguments, one can readily verify the existence of a local-in-time smooth solution, nonnegative in its first two components by the maximum principle, and extensible up to a maximal time $T_{max,\varepsilon} \in (0,\infty]$ which in the case $T_{max,\varepsilon} < \infty$ has the property that

$$\limsup_{t \nearrow T_{max,\varepsilon}} \Big(\|n_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|c_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|u_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} \Big) = \infty \qquad (2.3)$$

(cf. [35, Lemma 2.1] and [25, Lemma 2.1], for instance). Since (2.1) ensures that for fixed $\varepsilon \in (0,1)$ the function $S_\varepsilon(x,n,c)$ vanishes for all sufficiently large n, one may apply standard a priori estimation techniques to infer that for any such ε and each $T > 1$ there exists $C(\varepsilon,T) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|c_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} + \|u_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} \le C(\varepsilon,T) \qquad \text{for all } t \in (\tau, \tilde T_{max,\varepsilon}),$$

where $\tau := \min\{1, \frac{1}{2}T_{max,\varepsilon}\}$ and $\tilde T_{max,\varepsilon} := \min\{T, T_{max,\varepsilon}\}$ (cf. e.g. [35, Sect. 5], [13] and [26]). As a consequence of this and (2.3), we actually must have $T_{max,\varepsilon} = \infty$, as desired. $\Box$

The following basic properties of solutions to (2.2) are immediate.

Lemma 2.2 *The solution of (2.2) satisfies*

$$\|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega n_0 \qquad \textit{for all } t > 0 \qquad (2.4)$$

as well as

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t > 0. \qquad (2.5)$$

PROOF. The mass conservation property (2.4) directly follows by integrating the first equation in (2.2) over Ω. Moreover, using that both n_ε and f are nonnegative we can readily derive the inequality (2.5) by applying a parabolic comparison argument to the second equation in (2.2). $\Box$

3 A priori estimates

We proceed to derive ε-independent estimates for the approximate solutions constructed above. Throughout this section, for $\varepsilon \in (0,1)$ we let $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ denote the global solution of (2.2).

3.1 $W^{1,r}$ regularity of u implied by L^p regularity of n

Our first goal is to draw a consequence of a supposedly known bound for n_ε in $L^\infty((0,\infty);L^p(\Omega))$ on the regularity of the spatial derivative Du_ε of u_ε. Since in view of (2.4) we intend to apply this inter alia to the case $p = 1$ in a first step (cf. Lemma 3.15), applying smoothing estimates for the Stokes semigroup seems not fully straightforward in our situation.

In order to prepare our results in this direction, let us recall that for each $r \in (1,\infty)$, the Helmholtz projection acts as a bounded linear operator $\mathcal{P}_r$ from $L^r(\Omega)$ onto its subspace $L^r_\sigma(\Omega) = \{\varphi \in L^r(\Omega) \mid \nabla\cdot\varphi = 0\}$ of all solenoidal vector fields. Moreover, the realization A_r of the Stokes operator A in $L^r_\sigma(\Omega)$ with domain $D(A_r) = W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega) \cap L^r_\sigma(\Omega)$ is sectorial in $L^r_\sigma(\Omega)$ and hence possesses closed fractional powers A_r^β with dense domains $D(A_r^\beta)$ for any $\beta \in \mathbb{R}$ ([9]), and A_r generates an analytic semigroup $(e^{-tA_r})_{t\ge0}$ in $L^r_\sigma(\Omega)$. In the sequel, since $\mathcal{P}_r\psi$ and $A_r^\beta\varphi$ as well as $e^{-tA_r}\varphi$ are actually

independent of $r \in (1,\infty)$ for each $\psi \in C_0^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega) \cap L^r_\sigma(\Omega)$, $\beta \in \mathbb{R}$ and $t \geq 0$, we may suppress the subscript r in $\mathcal{P}_r, A_r^\beta$ and e^{-tA_r} whenever there is no danger of confusion.
Then among well-known embedding and regularity estimates we will especially need the following in the sequel.

Lemma 3.1 *i) Let $r > 1$. Then for all $\beta > \frac{1}{2}$ one can find $C = C(r,\beta) > 0$ such that*

$$\|D\varphi\|_{L^r(\Omega)} \leq C\|A^\beta\varphi\|_{L^r(\Omega)} \qquad \textit{for all } \varphi \in D(A_r^\beta). \tag{3.1}$$

ii) There exists $\mu > 0$ with the following property: For all $r \in (1,\infty)$ and $p \in (1,r]$ and each $\beta \geq 0$ there exists $C = C(p,r,\beta)$ such that whenever $\varphi \in L^p_\sigma(\Omega)$, we have

$$\|A^\beta e^{-tA}\varphi\|_{L^r(\Omega)} \leq Ct^{-\beta-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}e^{-\mu t}\|\varphi\|_{L^p(\Omega)} \qquad \textit{for all } t > 0. \tag{3.2}$$

Proof. i) The inequality (3.1) can be derived from [8, Lemma II.17.1] by adapting the argument in [8, Lemma II.17.2] in a straightforward manner.
ii) For (3.2), we may refer the reader to [10, p.201], noting that the exponentially decaying factor on the right-hand side can be justified by precisely following the reasoning in [23, Proposition 48.5]; cf. also [34, Lemma 1.3]). □

Evidently, the regularity properties for u_ε are linked to those of the forcing term $g_\varepsilon := \mathcal{P}[n_\varepsilon\nabla\phi]$ appearing in the version $\partial_t u_\varepsilon + Au_\varepsilon = \mathcal{P}[n_\varepsilon\nabla\phi]$ of the Stokes subsystem of (2.2) when projected to the respective spaces of divergence-free functions. Since in a first step, our basic information (2.4) only asserts a bound for g_ε with respect to the norm in $L^1(\Omega)$, and since standard results apparently do not apply directly to this non-reflexive situation, we briefly include the following two lemmata to prepare an adequate estimation of g_ε.
Firstly, a further embedding property of the domains of fractional powers of A allows us to control, upon a slight lifting through a negative fractional power of A, the norm of functions in the space $L^{p_0}(\Omega)$ by the norm of the unlifted function in $L^p(\Omega)$ with some p smaller than p_0. In Lemma 3.3, the case $p_0 = \infty$ will be of particular importance for treating the L^1 situation mentioned above.

Lemma 3.2 *Suppose that $1 < p < p_0 \leq \infty$, and that $\delta \in (0,1)$ is such that $2\delta - \frac{3}{p} > -\frac{3}{p_0}$. Then there exists $C > 0$ such that*

$$\|A^{-\delta}\psi\|_{L^{p_0}(\Omega)} \leq C\|\psi\|_{L^p(\Omega)} \qquad \textit{for all } \psi \in L^p(\Omega). \tag{3.3}$$

Proof. According to [9, Theorem 3] and [12, Theorem 1.6.1], our assumption on δ ensures that $D(A_p^\delta) \hookrightarrow L^{p_0}(\Omega)$, which means that there exists $C_1 > 0$ such that

$$\|\varphi\|_{L^{p_0}(\Omega)} \leq C_1\|A^\delta\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in D(A_p^\delta).$$

Thus, if we fix $\psi \in C_0^\infty(\Omega)$ and apply this to $\varphi := A^{-\delta}\psi$, we see that the inequality in (3.3) holds with $C := C_1$. For arbitrary $\psi \in L^p(\Omega)$, (3.3) easily follows from this by completion. □

By means of a straightforward duality argument, we can thereby indeed use a knowledge on the size of a function in $L^1(\Omega)$ to control a slightly lifted variant of its solenoidal part in a reflexive L^p space. More generally, we have the following.

Lemma 3.3 *Assume that $1 \le p < p_0 < \infty$, and that $\delta \in (0,1)$ is such that $2\delta - \frac{3}{p} > -\frac{3}{p_0}$. Then there exists $C > 0$ such that*

$$\|A^{-\delta}\mathcal{P}\psi\|_{L^{p_0}(\Omega)} \le C\|\psi\|_{L^p(\Omega)} \qquad \textit{for all } \psi \in C_0^\infty(\Omega). \tag{3.4}$$

Consequently, the operator $A^{-\delta}\mathcal{P}$ possesses a unique extension to all of $L^{p_0}(\Omega)$ with norm controlled according to (3.4).

Proof. Let $\varphi \in C_0^\infty(\Omega)$. Then since both $A^{-\delta}\mathcal{P}\psi$ and $A^{-\delta}\mathcal{P}\varphi$ are divergence free and $A^{-\delta}$ is symmetric, we have

$$\int_\Omega A^{-\delta}\mathcal{P}\psi \cdot \varphi = \int_\Omega A^{-\delta}\mathcal{P}\psi \cdot \mathcal{P}\varphi = \int_\Omega \mathcal{P}\psi \cdot A^{-\delta}\mathcal{P}\varphi = \int_\Omega \psi \cdot A^{-\delta}\mathcal{P}\varphi.$$

Now with $p' := \frac{p}{p-1} \in (1,\infty]$ and $p_0' := \frac{p_0}{p_0-1}$ we have $1 < p_0' < p' \le \infty$, and the assumption $2\delta - \frac{3}{p} > -\frac{3}{p_0}$ ensures that $2\delta - \frac{3}{p_0'} > -\frac{3}{p'}$. Therefore we may invoke Lemma 3.2 and use the boundedness of the projection $\mathcal{P}$ in $L^{p_0'}(\Omega)$ to find $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} \left|\int_\Omega A^{-\delta}\mathcal{P}\psi \cdot \varphi\right| &\le \|\psi\|_{L^p(\Omega)} \cdot \|A^{-\delta}\mathcal{P}\varphi\|_{L^{p'}(\Omega)} \\ &\le C_1\|\psi\|_{L^p(\Omega)} \cdot \|\mathcal{P}\varphi\|_{L^{p_0'}(\Omega)} \\ &\le C_2\|\psi\|_{L^p(\Omega)} \cdot \|\varphi\|_{L^{p_0'}(\Omega)} \qquad \text{for all } \varphi \in C_0^\infty(\Omega). \end{aligned}$$

By a standard duality argument, this implies (3.4). $\square$

An application of this to the Stokes equations in (2.2) now yields the following implication of some presupposed boundedness property of n_ε to the regularity features of u_ε.

Corollary 3.4 *Let $p \in [1,\infty)$ and $r \in [1,\infty]$ be such that*

$$\begin{cases} r < \frac{3p}{3-p} & \textit{if } p \le 3, \\ r \le \infty & \textit{if } p > 3. \end{cases} \tag{3.5}$$

Then for all $K > 0$ there exists $C = C(p,r,K)$ such that if for some $\varepsilon \in (0,1)$ and $T > 0$ we have

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le K \qquad \textit{for all } t \in (0,T), \tag{3.6}$$

then

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le C \qquad \textit{for all } t \in (0,T). \tag{3.7}$$

Proof. In view of (3.5), it is evidently sufficient to consider the case $r > p$ only, in which we can fix $r_0 \in (p,r)$ such that with β as in (1.7) we have

$$\frac{1}{2} + \frac{3}{2}\Big(\frac{1}{r_0} - \frac{1}{r}\Big) < \beta. \tag{3.8}$$

Since (3.5) moreover ensures that

$$\Big\{\frac{1}{2} + \frac{3}{2}\Big(\frac{1}{r_0} - \frac{1}{r}\Big)\Big\} - \Big\{1 - \frac{3}{2}\Big(\frac{1}{p} - \frac{1}{r_0}\Big)\Big\} = -\frac{1}{2} + \frac{3}{2}\Big(\frac{1}{p} - \frac{1}{r}\Big) < 0,$$

we can thus choose $\beta_0 \in (\frac{1}{2}, \beta)$ fulfilling

$$\frac{1}{2} + \frac{3}{2}\Big(\frac{1}{r_0} - \frac{1}{r}\Big) < \beta_0 < 1 - \frac{3}{2}\Big(\frac{1}{p} - \frac{1}{r_0}\Big), \tag{3.9}$$

then pick $\delta \in (0,1)$ small enough such that still

$$\beta_0 + \delta < 1 - \frac{3}{2}\Big(\frac{1}{p} - \frac{1}{r_0}\Big), \tag{3.10}$$

and finally fix some $p_0 > p$ sufficiently close to p such that

$$2\delta - \frac{3}{p} > -\frac{3}{p_0}. \tag{3.11}$$

Then in the variation-of-constants representation

$$u_\varepsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds, \qquad t \in (0,T), \tag{3.12}$$

with $A = A_{r_0}$ and $\mathcal{P} = \mathcal{P}_{r_0}$, we apply A^{β_0} on both sides and use Lemma 3.1 i) along with the left inequality in (3.9) to find $C_1 > 0$ such that

$$\begin{aligned} \|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} &\le C_1\|A^{\beta_0}u_\varepsilon(\cdot,t)\|_{L^{r_0}(\Omega)} \\ &\le C_1\|A^{\beta_0}e^{-tA}u_0\|_{L^{r_0}(\Omega)} + C_1\int_0^t \Big\|A^{\beta_0+\delta}e^{-(t-s)A}A^{-\delta}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^{r_0}(\Omega)}ds \end{aligned} \tag{3.13}$$

for all $t \in (0,T)$. Here since $u_0 \in D(A^{\beta_0})$ by (1.7) and the fact that $\beta_0 < \beta$, we have

$$\|A^{\beta_0}e^{-tA}u_0\|_{L^{r_0}(\Omega)} = \|e^{-tA}A^{\beta_0}u_0\|_{L^{r_0}(\Omega)} \le C_2 \qquad \text{for all } t \in (0,T)$$

with some $C_2 > 0$. Furthermore, since $p_0 > p \ge 1$, Lemma 3.1 ii) applies to show that there exist $C_3 > 0$ and $\mu > 0$ fulfilling

$$\Big\|A^{\beta_0+\delta}e^{-(t-s)A}A^{-\delta}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^{r_0}(\Omega)} \le C_3(t-s)^{-\beta_0-\delta-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{r_0})}e^{-\mu(t-s)}\Big\|A^{-\delta}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^{p_0}(\Omega)}$$

for all $t \in (0,T)$ and $s \in (0,t)$, where thanks to Lemma 3.3 and the boundedness of $\nabla\phi$, we can use (3.6) to see that

$$\Big\|A^{-\delta}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]\Big\|_{L^{p_0}(\Omega)} \le C_4\|n_\varepsilon(\cdot,s)\nabla\phi\|_{L^p(\Omega)} \le C_5\|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \le C_5K \qquad \text{for all } s \in (0,T)$$

with some $C_4 > 0$ and $C_5 > 0$. Therefore, from (3.13) we altogether obtain that

$$\begin{aligned} \|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} &\le C_1C_2 + C_1C_3C_5K\int_0^t (t-s)^{-\beta_0-\delta-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{r_0})}e^{-\mu(t-s)}ds \\ &\le C_1C_2 + C_1C_3C_5C_6K \qquad \text{for all } t \in (0,T), \end{aligned}$$

where

$$C_6 := \int_0^\infty \sigma^{-\beta_0-\delta-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{r_0})}e^{-\mu\sigma}d\sigma$$

is finite according to (3.10). This proves (3.7). $\square$

3.2 Standard testing procedures

We now turn to the analysis of the coupled functional in (1.19). Here we first apply standard testing procedures to gain the inequalities in the following three lemmata. Further estimating the respective right-hand sides therein will then be done separately in the sequel.
Let us begin by testing the first equation in (2.2) against powers of n_ε.

Lemma 3.5 *Let $p>1$. Then for all $\varepsilon\in(0,1)$,*

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{2(p-1)k_D}{(m+p-1)^2}\int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 \le \frac{(p-1)S_1^2}{2k_D}\int_\Omega n_\varepsilon^{p+1-m}|\nabla c_\varepsilon|^2 \qquad \textit{for all } t>0, \tag{3.14}$$

where k_D is as in (1.9) and

$$S_1 := S_0(\|c_0\|_{L^\infty(\Omega)})$$

with S_0 taken from (1.11).

Proof. We multiply the first equation in (2.2) by n_ε^{p-1} and integrate by parts over Ω. Since $S_\varepsilon(x,n_\varepsilon,c_\varepsilon)$ vanishes whenever $x\in\partial\Omega$ according to (2.1), this yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)\int_\Omega n_\varepsilon^{p-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 = (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \qquad \text{for all } t>0. \tag{3.15}$$

Here we use the definition of D_ε and (1.9) to see that

$$(p-1)\int_\Omega n_\varepsilon^{p-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 \ge (p-1)k_D\int_\Omega n_\varepsilon^{p+m-3}|\nabla n_\varepsilon|^2 \qquad \text{for all } t>0,$$

and next combine (2.1) with (1.11) and (2.5) to obtain

$$|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)| \le S_1 \qquad \text{in } \Omega\times(0,\infty),$$

so that using (1.9) and Young's inequality we can estimate

$$\begin{aligned}(p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) &\le (p-1)S_1\int_\Omega n_\varepsilon^{p-1}|\nabla n_\varepsilon|\cdot|\nabla c_\varepsilon| \\ &\le \frac{(p-1)k_D}{2}\int_\Omega n_\varepsilon^{p+m-3}|\nabla n_\varepsilon|^2 + \frac{(p-1)S_1^2}{2k_D}\int_\Omega n_\varepsilon^{p+1-m}|\nabla c_\varepsilon|^2\end{aligned}$$

for all $t>0$, whence (3.14) readily follows from (3.15). $\Box$

In order to obtain a first information on the time evolution also of $\int_\Omega|\nabla c_\varepsilon|^{2q}$, we make use of the convexity of Ω in the following lemma.

Lemma 3.6 *Let $q>1$ and $\varepsilon\in(0,1)$. Then*

$$\begin{aligned}\frac{1}{2q}\frac{d}{dt}\int_\Omega|\nabla c_\varepsilon|^{2q} + \frac{2(q-1)}{q^2}\int_\Omega\Big|\nabla|\nabla c_\varepsilon|^q\Big|^2 + \frac{1}{2}\int_\Omega|\nabla c_\varepsilon|^{2q-2}|D^2c_\varepsilon|^2 \\ \le \frac{(2q-2+\sqrt{3})^2 f_1}{2}\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} + \int_\Omega|\nabla c_\varepsilon|^{2q}\cdot|Du_\varepsilon| \qquad \textit{for all } t>0,\end{aligned} \tag{3.16}$$

where

$$f_1 := \|f\|_{L^\infty((0,\|c_0\|_{L^\infty(\Omega)}))}. \tag{3.17}$$

PROOF. Differentiating the second equation in (2.2) and using that $\Delta|\nabla c_\varepsilon|^2 = 2\nabla c_\varepsilon\cdot\nabla\Delta c_\varepsilon + 2|D^2c_\varepsilon|^2$, we obtain the pointwise identity

$$\begin{aligned}\frac{1}{2}\Big(|\nabla c_\varepsilon|^2\Big)_t &= \nabla c_\varepsilon\cdot\nabla\Big\{\Delta c_\varepsilon - n_\varepsilon f(c_\varepsilon) - u_\varepsilon\cdot\nabla c_\varepsilon\Big\}\\ &= \frac{1}{2}\Delta|\nabla c_\varepsilon|^2 - |D^2c_\varepsilon|^2 - \nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) - \nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) \quad \text{in } \Omega\times(0,\infty). \quad (3.18)\end{aligned}$$

We multiply this by $(|\nabla c_\varepsilon|^2)^{q-1}$ and integrate by parts over Ω. Since $\frac{\partial c_\varepsilon}{\partial\nu}=0$ on $\partial\Omega$ along with the convexity of Ω ensures that $\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}\le 0$ on $\partial\Omega$ ([19, Lemma I.1, p.350]), this results in the inequality

$$\begin{aligned}&\frac{1}{2q}\frac{d}{dt}\int_\Omega|\nabla c_\varepsilon|^{2q} + \frac{q-1}{2}\int_\Omega|\nabla c_\varepsilon|^{2q-4}\Big|\nabla|\nabla c_\varepsilon|^2\Big|^2 + \int_\Omega|\nabla c_\varepsilon|^{2q-2}|D^2c_\varepsilon|^2\\ &\quad\le -\int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) - \int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) \qquad \text{for all } t>0. \quad (3.19)\end{aligned}$$

Here in the first integral on the right we again integrate by parts to estimate, using that

$$|f(c_\varepsilon)|\le f_1 \qquad \text{in } \Omega\times(0,\infty)$$

by (2.5) and (3.17),

$$\begin{aligned}-\int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) &= \int_\Omega n_\varepsilon f(c_\varepsilon)|\nabla c_\varepsilon|^{2q-2}\Delta c_\varepsilon + \int_\Omega n_\varepsilon f(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^{2q-2}\\ &\le f_1\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-2}|\Delta c_\varepsilon| + f_1\int_\Omega n_\varepsilon|\nabla c_\varepsilon|\cdot\Big|\nabla|\nabla c_\varepsilon|^{2q-2}\Big|\end{aligned}$$

for all $t>0$. Since $|\Delta c_\varepsilon|\le\sqrt{3}|D^2c_\varepsilon|$ by the Cauchy-Schwarz inequality, and since

$$\nabla|\nabla c_\varepsilon|^{2q-2} = 2(q-1)|\nabla c_\varepsilon|^{2q-4}D^2c_\varepsilon\cdot\nabla c_\varepsilon \qquad \text{in } \Omega\times(0,\infty),$$

in view of Young's inequality this implies that

$$\begin{aligned}-\int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(n_\varepsilon f(c_\varepsilon)) &\le \sqrt{3}f_1\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-2}|D^2c_\varepsilon| + 2(q-1)f_1\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-3}|D^2c_\varepsilon\cdot\nabla c_\varepsilon|\\ &\le (2q-2+\sqrt{3})f_1\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-2}|D^2c_\varepsilon|\\ &\le \frac{1}{2}\int_\Omega|\nabla c_\varepsilon|^{2q-2}|D^2c_\varepsilon|^2 + \frac{(2q-2+\sqrt{3})^2f_1^2}{2}\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} \qquad (3.20)\end{aligned}$$

for all $t>0$. As for the rightmost integral in (3.19), we first differentiate $u_\varepsilon\cdot\nabla c_\varepsilon$ to gain the decomposition

$$-\int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) = -\int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(Du_\varepsilon\cdot\nabla c_\varepsilon) - \int_\Omega|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) \quad (3.21)$$

for all $t>0$, and then use the pointwise identity

$$\begin{aligned}|\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) &= u_\varepsilon\cdot\Big(|\nabla c_\varepsilon|^{2q-2}D^2c_\varepsilon\cdot\nabla c_\varepsilon\Big)\\ &= \frac{1}{2q}u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^{2q} \qquad \text{in } \Omega\times(0,\infty)\end{aligned}$$

to infer on integrating by parts that

$$
\begin{aligned}
-\int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(D^2c_\varepsilon\cdot u_\varepsilon) &= -\frac{1}{2q}\int_\Omega u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^{2q} \\
&= \frac{1}{2q}\int_\Omega (\nabla\cdot u_\varepsilon)|\nabla c_\varepsilon|^{2q} \\
&= 0 \qquad \text{for all } t>0,
\end{aligned}
$$

because u_ε is solenoidal. Therefore, (3.21) entails that

$$
\begin{aligned}
-\int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) &= -\int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(Du_\varepsilon\cdot\nabla c_\varepsilon) \\
&\le \int_\Omega |\nabla c_\varepsilon|^{2q}\cdot|Du_\varepsilon| \qquad \text{for all } t>0,
\end{aligned}
$$

which combined with (3.20) and (3.19) yields (3.16). □

Finally, the following basic inequality for $\int_\Omega |A^{\frac12}u_\varepsilon|^2$ is standard.

Lemma 3.7 *For any $\varepsilon\in(0,1)$, we have*

$$
\frac{d}{dt}\int_\Omega |A^{\frac12}u_\varepsilon|^2+\int_\Omega |Au_\varepsilon|^2 \le \|\nabla\phi\|^2_{L^\infty(\Omega)}\cdot\int_\Omega n_\varepsilon^2 \qquad \textit{for all } t>0. \tag{3.22}
$$

Proof.
We apply the Helmholtz projection $\mathcal{P}$ to the third equation in (2.2) and test the resulting identity $\partial_t u_\varepsilon+Au_\varepsilon=\mathcal{P}(n_\varepsilon\nabla\phi)$ by Au_ε. Using that $A^{\frac12}$ is self-adjoint in $L^2_\sigma(\Omega)$ and that $\mathcal{P}$ acts as an orthogonal projection in this Hilbert space, by Young's inequality we obtain that

$$
\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega |A^{\frac12}u_\varepsilon|^2+\int_\Omega |Au_\varepsilon|^2 &= \int_\Omega Au_\varepsilon\cdot\mathcal{P}(n_\varepsilon\nabla\phi) \\
&\le \frac12\int_\Omega |Au_\varepsilon|^2+\frac12\int_\Omega |\mathcal{P}(n_\varepsilon\nabla\phi)|^2 \\
&\le \frac12\int_\Omega |Au_\varepsilon|^2+\frac12\int_\Omega |n_\varepsilon\nabla\phi|^2 \\
&\le \frac12\int_\Omega |Au_\varepsilon|^2+\frac12\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \qquad \text{for all } t>0,
\end{aligned}
$$

and thereby precisely arrive at (3.22). □

3.3 Estimating the right-hand sides in (3.14), (3.16) and (3.22)

We next plan to estimate the right-hand sides in the above inequalities appropriately by using suitable interpolation arguments along with the basic a priori information provided by Lemma 2.2. Here the following auxiliary interpolation lemma, extending a similar statement known in the two-dimensional case ([18]) to the present framework, will play an important role in making efficient use of the known L^∞ bound (2.5) for c_ε.

Lemma 3.8 *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, that $q \ge 1$ and that*

$$\lambda \in [2q+2, 4q+1]. \tag{3.23}$$

Then there exists $C > 0$ such that for all $\varphi \in C^2(\bar{\Omega})$ fulfilling $\varphi \cdot \frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ we have

$$\|\nabla\varphi\|_{L^\lambda(\Omega)} \le C \Big\| |\nabla\varphi|^{q-1} D^2\varphi \Big\|_{L^2(\Omega)}^{\frac{2\lambda-6}{(2q-1)\lambda}} \|\varphi\|_{L^\infty(\Omega)}^{\frac{6q-\lambda}{(2q-1)\lambda}} + C\|\varphi\|_{L^\infty(\Omega)}. \tag{3.24}$$

Proof. We integrate by parts to rewrite

$$\|\nabla\varphi\|_{L^\lambda(\Omega)}^\lambda = -\int_\Omega \varphi|\nabla\varphi|^{\lambda-2}\Delta\varphi - (\lambda-2)\int_\Omega \varphi|\nabla\varphi|^{\lambda-4}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi), \tag{3.25}$$

where by the Cauchy-Schwarz inequality we see that

$$\Big| -(\lambda-2)\int_\Omega \varphi|\nabla\varphi|^{\lambda-4}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi)\Big| \le (\lambda-2)\|\varphi\|_{L^\infty(\Omega)}\cdot I\cdot\Big(\int_\Omega |\nabla\varphi|^{2\lambda-2q-2}\Big)^{\frac12} \tag{3.26}$$

with

$$I := \Big\| |\nabla\varphi|^{q-1}D^2\varphi\Big\|_{L^2(\Omega)}.$$

Likewise, using that $|\Delta\varphi| \le \sqrt{3}|D^2\varphi|$ we can estimate

$$\begin{aligned}\Big| -\int_\Omega \varphi|\nabla\varphi|^{\lambda-2}\Delta\varphi\Big| &\le \sqrt{3}\|\varphi\|_{L^\infty(\Omega)}\int_\Omega |\nabla\varphi|^{\lambda-2}|D^2\varphi| \\ &\le \sqrt{3}\|\varphi\|_{L^\infty(\Omega)}\cdot I\cdot\Big(\int_\Omega|\nabla\varphi|^{2\lambda-2q-2}\Big)^{\frac12}.\end{aligned} \tag{3.27}$$

Now by the Gagliardo-Nirenberg inequality, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}\Big(\int_\Omega |\nabla\varphi|^{2\lambda-2q-2}\Big)^{\frac12} &= \Big\||\nabla\varphi|^q\Big\|_{L^{\frac{2(\lambda-q-1)}{q}}(\Omega)}^{\frac{\lambda-q-1}{q}} \\ &\le C_1\Big\|\nabla|\nabla\varphi|^q\Big\|_{L^2(\Omega)}^{\frac{\lambda-q-1}{q}\cdot a}\Big\||\nabla\varphi|^q\Big\|_{L^{\frac{\lambda}{q}}(\Omega)}^{\frac{\lambda-q-1}{q}\cdot(1-a)} + C_1\Big\||\nabla\varphi|^q\Big\|_{L^{\frac{\lambda}{q}}(\Omega)}^{\frac{\lambda-q-1}{q}} \\ &\le C_2\cdot I^{\frac{\lambda-q-1}{q}\cdot a}\cdot\|\nabla\varphi\|_{L^\lambda(\Omega)}^{(\lambda-q-1)(1-a)} + C_1\|\nabla\varphi\|_{L^\lambda(\Omega)}^{\lambda-q-1}\end{aligned}$$

with

$$a = \frac{3q(\lambda-2q-2)}{(\lambda-q-1)(6q-\lambda)} \in [0,1],$$

so that combining (3.25)-(3.27) yields $C_3 > 0$ fulfilling

$$\begin{aligned}\|\nabla\varphi\|_{L^\lambda(\Omega)}^\lambda &\le C_3\|\varphi\|_{L^\infty(\Omega)}\cdot I^{1+\frac{\lambda-q-1}{q}\cdot a}\cdot\|\nabla\varphi\|_{L^\lambda(\Omega)}^{(\lambda-q-1)(1-a)} + C_3\|\varphi\|_{L^\infty(\Omega)}\cdot I\cdot\|\nabla\varphi\|_{L^\lambda(\Omega)}^{\lambda-q-1} \\ &= C_3\|\varphi\|_{L^\infty(\Omega)}\cdot I^{\frac{2\lambda-6}{6q-\lambda}}\cdot\|\nabla\varphi\|_{L^\lambda(\Omega)}^{\frac{(4q+1-\lambda)\lambda}{6q-\lambda}} + C_3\|\varphi\|_{L^\infty(\Omega)}\cdot I\cdot\|\nabla\varphi\|_{L^\lambda(\Omega)}^{\lambda-q-1}.\end{aligned} \tag{3.28}$$

Here we invoke Young's inequality to find $C_4 > 0$ and $C_5 > 0$ such that

$$C_3\|\varphi\|_{L^\infty(\Omega)} \cdot I^{\frac{2\lambda-6}{6q-\lambda}} \cdot \|\nabla\varphi\|_{L^\lambda(\Omega)}^{\frac{(4q+1-\lambda)\lambda}{6q-\lambda}} \le \frac{1}{4}\|\nabla\varphi\|_{L^\lambda(\Omega)}^{\lambda} + C_4\|\varphi\|_{L^\infty(\Omega)}^{\frac{6q-\lambda}{2q-1}} \cdot I^{\frac{2\lambda-6}{2q-1}} \tag{3.29}$$

and

$$C_3\|\varphi\|_{L^\infty(\Omega)} \cdot I \cdot \|\nabla\varphi\|_{L^\lambda(\Omega)}^{\lambda-q-1} \le \frac{1}{4}\|\nabla\varphi\|_{L^\lambda(\Omega)}^{\lambda} + C_5\|\varphi\|_{L^\infty(\Omega)}^{\frac{\lambda}{q+1}} \cdot I^{\frac{\lambda}{q+1}}, \tag{3.30}$$

where by the same token,

$$\begin{aligned} C_5\|\varphi\|_{L^\infty(\Omega)}^{\frac{\lambda}{q+1}} \cdot I^{\frac{\lambda}{q+1}} &= C_5 \cdot \Big(\|\varphi\|_{L^\infty(\Omega)}^{\frac{6q-\lambda}{2q-1}} \cdot I^{\frac{2\lambda-6}{2q-1}}\Big)^{\frac{(2q-1)\lambda}{(q+1)(2\lambda-6)}} \cdot \|\varphi\|_{L^\infty(\Omega)}^{\frac{(3\lambda-6q-6)\lambda}{(q+1)(2\lambda-6)}} \\ &\le C_5\|\varphi\|_{L^\infty(\Omega)}^{\frac{6q-\lambda}{2q-1}} \cdot I^{\frac{2\lambda-6}{2q-1}} + C_5\|\varphi\|_{L^\infty(\Omega)}^{\lambda}. \end{aligned} \tag{3.31}$$

In consequence, (3.28)-(3.31) prove (3.24). $\square$

A first application thereof will appear in the following lemma which estimates the integral on the right of (3.14) under a smallness assumption on the integrability exponent p arrearing therein.

Lemma 3.9 *Let $m \ge 1, q > 1$ and $p > \max\{1, m-1\}$ be such that*

$$p < \Big(2m - \frac{4}{3}\Big)q + m - 1. \tag{3.32}$$

Then for all $\eta > 0$ there exists $C = C(p,q,\eta) > 0$ with the property that for all $\varepsilon \in (0,1)$,

$$\int_\Omega n_\varepsilon^{p+1-m}|\nabla c_\varepsilon|^2 \le \eta\int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + \eta\int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2c_\varepsilon|^2 + C \qquad \textit{for all } t > 0. \tag{3.33}$$

Proof. We apply the Hölder inequality with exponents $\frac{q+1}{q}$ and $q+1$ to obtain

$$\begin{aligned} \int_\Omega n_\varepsilon^{p+1-m}|\nabla c_\varepsilon|^2 &\le \Big(\int_\Omega n_\varepsilon^{\frac{(p+1-m)(q+1)}{q}}\Big)^{\frac{q}{q+1}} \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^{2q+2}\Big)^{\frac{1}{q+1}} \\ &= \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+1-m)(q+1)}{(p+m-1)q}}(\Omega)}^{\frac{2(p+1-m)}{p+m-1}} \cdot \|\nabla c_\varepsilon\|_{L^{2q+2}(\Omega)}^2 \qquad \text{for all } t > 0. \end{aligned} \tag{3.34}$$

In view of the Gagliardo-Nirenberg inequality (see [33] for a version involving integrability exponents less than one) and (2.4), we can find $C_1 = C_1(p,q) > 0$ and $C_2 = C_2(p,q) > 0$ such that

$$\begin{aligned} \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+1-m)(q+1)}{(p+m-1)q}}(\Omega)}^{\frac{2(p+1-m)}{p+m-1}} &\le C_1\|\nabla n_\varepsilon^{\frac{p+1-m}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1-m)}{p+m-1}\cdot a}\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+1-m)}{p+m-1}\cdot(1-a)} + C_1\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+1-m)}{p+m-1}} \\ &\le C_2\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1-m)}{p+m-1}\cdot a} + C_2 \qquad \text{for all } t > 0, \end{aligned} \tag{3.35}$$

where $a \in (0,1)$ is determined by the relation

$$-\frac{3(p+m-1)q}{2(p+1-m)(q+1)} = \Big(1 - \frac{3}{2}\Big) \cdot a - \frac{3(p+m-1)}{2} \cdot (1-a). \tag{3.36}$$

Here we note that indeed $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2(p+1-m)(q+1)}{(p+m-1)q}}(\Omega)$, because

$$\begin{aligned}6(p+m-1)q - 2(p+1-m)(q+1) &= 6pq + 6(m-1)q - 2pq - 2p + 2(m-1)q + 2(m-1) \\ &= (4q-2)p + 8(m-1)q + 2(m-1) \\ &> 0\end{aligned}$$

and hence $\frac{2(p+1-m)(q+1)}{(p+m-1)q} < 6$. On solving (3.36) with respect to a, we see that (3.35) becomes

$$\begin{aligned}\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+1-m)(q+1)}{(p+m-1)q}}(\Omega)}^{\frac{2(p+1-m)}{p+m-1}} &\le C_2 \|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{6[(p+1-m)(q+1)-q]}{(3p+3m-4)(q+1)}} + C_2 \\ &\le C_3 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\}^{\frac{3[(p+1-m)(q+1)-q]}{(3p+3m-4)(q+1)}} \qquad \text{for all } t>0 \quad (3.37)\end{aligned}$$

with some $C_3 = C_3(p,q) > 0$.
As for the term on the right of (3.34) involving c_ε, we invoke Lemma 3.8, which in conjunction with (2.5) provides $C_4 = C_4(q) > 0$ and $C_5 = C_5(q) > 0$ satisfying

$$\begin{aligned}\|\nabla c_\varepsilon\|_{L^{2q+2}(\Omega)}^2 &\le C_4 \Big\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \Big\|_{L^2(\Omega)}^{\frac{2}{q+1}} \|c_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2}{q+1}} + C_4 \|c_\varepsilon\|_{L^\infty(\Omega)}^2 \\ &\le C_5 \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + 1 \Big\}^{\frac{1}{q+1}} \qquad \text{for all } t>0.\end{aligned}$$

Together with (3.37) and (3.34), by Young's inequality this shows that for each $\eta > 0$ we can find $C_6 = C_6(p,q,\eta) > 0$ such that

$$\begin{aligned}\int_\Omega n_\varepsilon^{p+1-m} |\nabla c_\varepsilon|^2 &\le C_3 C_5 \cdot \Big\| \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\|^{\frac{3[(p+1-m)(q+1)-q]}{(3p+3m-4)(q+1)}} \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + 1 \Big\}^{\frac{1}{q+1}} \\ &\le \eta \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + 1 \Big\} + C_6 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\}^{\frac{3[(p+1-m)(q+1)-q]}{(3p+3m-4)q}} \quad (3.38)\end{aligned}$$

for all $t > 0$. Since our hypothesis (3.32) warrants that

$$3[(p+1-m)(q+1)-q] - (3p+3m-4)q = 3p - 6mq + 4q - 3m + 3 < 0$$

and thus

$$\frac{3[(p+1-m)(q+1)-q]}{(3p+3m-4)q} < 1,$$

another application of Young's inequality shows that with some $C_7 = C_7(p,q,\eta) > 0$ we have

$$C_6 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\}^{\frac{3[(p+1-m)(q+1)-q]}{(3p+3m-4)q}} \le \eta \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\} + C_7 \qquad \text{for all } t>0.$$

The claimed inequality (3.33) thus results from (3.38). $\square$

By pursuing quite a similar strategy, under the assumption that p is suitably large as related to q we can estimate the first integral on the right of (3.16), even when enlarged by a further zero-order integral, as follows.

Lemma 3.10 *Let $m \geq 1$ and $q > 1$, and suppose that $p > 1$ satisfies*

$$p > \frac{3q - 3m + 4}{3}. \tag{3.39}$$

Then for all $\eta > 0$ there exists $C(p, q, \eta) > 0$ such that

$$\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} + \int_\Omega n_\varepsilon^2 \leq \eta \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + \eta \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + C \qquad \text{for all } t > 0. \tag{3.40}$$

Proof. Once more by the Hölder inequality,

$$\begin{aligned} \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} &\leq \Big(\int_\Omega n_\varepsilon^{q+1} \Big)^{\frac{2}{q+1}} \cdot \Big(\int_\Omega |\nabla c_\varepsilon|^{2q+2} \Big)^{\frac{q-1}{q+1}} \\ &= \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2(q+1)}{p+m-1}}(\Omega)}^{\frac{4}{p+m-1}} \cdot \|\nabla c_\varepsilon\|_{L^{2q+2}(\Omega)}^{2q-2} \qquad \text{for all } t > 0. \end{aligned} \tag{3.41}$$

Here we observe that (3.39) in particular ensures that

$$6 - \frac{2(q+1)}{p+m-1} = \frac{6p - 2q + 6m - 8}{p+m-1} > \frac{(6q - 6m + 8) - 2q + 6m - 8}{p+m-1} = \frac{4q}{p+m-1} > 0$$

and hence $\frac{2(q+1)}{p+m-1} < 6$. We thus may invoke the Gagliardo-Nirenberg inequality, which combined with (2.4) provides $C_1 = C_1(p,q) > 0$ and $C_2 = C_2(p,q) > 0$ such that

$$\begin{aligned} \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2(q+1)}{p+m-1}}(\Omega)}^{\frac{4}{p+m-1}} &\leq C_1 \|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{4}{p+m-1} \cdot a} \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{4}{p+m-1} \cdot (1-a)} + C_1 \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{4}{p+m-1}} \\ &\leq C_2 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\}^{\frac{2}{p+m-1} \cdot a} \qquad \text{for all } t > 0 \end{aligned} \tag{3.42}$$

with

$$-\frac{3(p+m-1)}{2(q+1)} = \Big(1 - \frac{3}{2}\Big) \cdot a - \frac{3(p+m-1)}{2} \cdot (1-a),$$

that is, with

$$a = \frac{q}{q+1} \cdot \frac{3(p+m-1)}{3p + 3m - 4} \in (0,1). \tag{3.43}$$

Likewise, Lemma 3.8 along with (2.5) warrants that

$$\begin{aligned} \|\nabla c_\varepsilon\|_{L^{2q+2}(\Omega)}^{2q-2} &\leq C_3 \Big\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \Big\|_{L^2(\Omega)}^{\frac{2q-2}{q+1}} \|c_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2q-2}{q+1}} + C_3 \|c_\varepsilon\|_{L^\infty(\Omega)}^{2q-2} \\ &\leq C_4 \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + 1 \Big\}^{\frac{q-1}{q+1}} \qquad \text{for all } t > 0 \end{aligned} \tag{3.44}$$

with appropriate positive constants $C_3 = C_3(q)$ and $C_4 = C_4(q)$.
In light of (3.42), (3.43) and (3.44), we may use Young's inequality in (3.41) to see that given $\eta > 0$ we can find $C_5 = C_5(p,q,\eta) > 0$ fulfilling

$$\begin{aligned}\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} &\le C_2 C_4 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\}^{\frac{6q}{(q+1)(3p+3m-4)}} \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + 1 \Big\}^{\frac{q-1}{q+1}} \\ &\le \frac{\eta}{2} \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + 1 \Big\} + C_5 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + 1 \Big\}^{\frac{3q}{3p+3m-4}} \qquad \text{for all } t > 0.\end{aligned}$$

Now by (3.39) we have

$$\frac{3q}{3p+3m-4} < \frac{3q}{3 \cdot \frac{3q-3m+4}{3} + 3m - 4} = 1,$$

so that another application of Young's inequality yields $C_6 = C_6(p,q,\eta) > 0$ such that

$$\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} \le \frac{\eta}{2} \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + \frac{\eta}{2} \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + C_6 \qquad \text{for all } t > 0.$$

Since the integral $\int_\Omega n_\varepsilon^2$ can be estimated similarly upon a straightforward simplification of the above argument, this establishes (3.40). $\Box$

As for the rightmost integral in (3.16), in order to avoid later repetitions in bootstrapping arguments we find it convenient to estimate this quantity on the basis of a supposedly known bound for Du_ε in $L^\infty((0,\infty); L^r(\Omega))$ for some $r > 1$. In the more favorable case when r is large, under a comparatively mild restriction on q we then obtain the following.

Lemma 3.11 *Let $m \ge 1$ and $r > \frac{3}{2}$, and suppose that $q \ge r-1$ is such that*

$$(4-2r)q \le r-1. \tag{3.45}$$

Then for all $\eta > 0$ and each $K > 0$ there exists $C = C(q,r,\eta,K) > 0$ such that if for some $\varepsilon \in (0,1)$ and $T > 0$ we have

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le K \qquad \textit{for all } t \in (0,T), \tag{3.46}$$

then

$$\int_\Omega |\nabla c_\varepsilon|^{2q} \cdot |Du_\varepsilon| \le \eta \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + C \qquad \textit{for all } t \in (0,T). \tag{3.47}$$

Proof. We invoke the Hölder inequality with exponents $\frac{r}{r-1}$ and r to see that

$$\begin{aligned}\int_\Omega |\nabla c_\varepsilon|^{2q} \cdot |Du_\varepsilon| &\le \Big(\int_\Omega |\nabla c_\varepsilon|^{\frac{2qr}{r-1}} \Big)^{\frac{r-1}{r}} \cdot \Big(\int_\Omega |Du_\varepsilon|^r \Big)^{\frac{1}{r}} \\ &\le K \cdot \|\nabla c_\varepsilon\|^{2q}_{L^{\frac{2qr}{r-1}}(\Omega)} \qquad \text{for all } t \in (0,T)\end{aligned} \tag{3.48}$$

due to (3.46). Since $q \ge r-1$ ensures that $\lambda := \frac{2qr}{r-1}$ satisfies

$$\lambda - (2q+2) = \frac{2qr - (2q+2)(r-1)}{r-1} = \frac{2q-2r+2}{r-1} = \frac{2(q-r+1)}{r-1} \ge 0,$$

and since (3.45) warrants that

$$(4q+1)-\lambda = \frac{(4q+1)(r-1)-2qr}{r-1} = \frac{2qr-4q+r-1}{r-1} = \frac{r-1-(4-2r)q}{r-1} \ge 0,$$

we may apply Lemma 3.8 and (2.5) to see that with some $C_1 = C_1(q,r) > 0$ and $C_2 = C_2(q,r) > 0$ we have

$$\begin{aligned} \|\nabla c_\varepsilon\|^{2q}_{L^{\frac{2qr}{r-1}}(\Omega)} &\le C_1 \Big\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \Big\|^{\frac{2q(2\lambda-6)}{(2q-1)\lambda}}_{L^2(\Omega)} \|c_\varepsilon\|^{\frac{2q(6q-\lambda)}{(2q-1)\lambda}}_{L^\infty(\Omega)} + C_1 \|c_\varepsilon\|^{2q}_{L^\infty(\Omega)} \\ &\le C_2 \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \Big\}^{\frac{2q(\lambda-3)}{(2q-1)\lambda}} + C_2 \qquad \text{for all } t \in (0,T), \end{aligned} \tag{3.49}$$

because clearly $\lambda < 6q$. As

$$\frac{2q(\lambda-3)}{(2q-1)\lambda} = \frac{2q(1-\frac{3}{\lambda})}{2q-1} = \frac{2q-\frac{3(r-1)}{r}}{2q-1} = \frac{2qr-3r+3}{(2q-1)r} = \frac{2qr-3r+3}{2qr-r} < 1$$

thanks to our assumption $r > \frac{3}{2}$, by means of Young's inequality we can easily derive (3.47) from (3.48) and (3.49). $\Box$

Also when r is small, however, we can arrive at a similar conclusion, albeit under a slightly stronger assumption on q.

Lemma 3.12 *Let $m \ge 1$, and suppose that $r \in (1, \frac{3}{2}]$ and*

$$q \in \Big(1, \frac{2r+3}{3}\Big).$$

Then for all $\eta > 0$ and $K > 0$ one can find $C = C(q,r,\eta,K) > 0$ such that if there exist $\varepsilon \in (0,1)$ and $T > 0$ fulfilling

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le K \qquad \textit{for all } t \in (0,T), \tag{3.50}$$

then

$$\int_\Omega |\nabla c_\varepsilon|^{2q} \cdot |Du_\varepsilon| \le \eta \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + \eta \int_\Omega |Au_\varepsilon|^2 + C \qquad \textit{for all } t \in (0,T). \tag{3.51}$$

Proof. By the Hölder inequality,

$$\int_\Omega |\nabla c_\varepsilon|^{2q} \cdot |Du_\varepsilon| \le \|\nabla c_\varepsilon\|^{2q}_{L^{2q+2}(\Omega)} \|Du_\varepsilon\|_{L^{q+1}(\Omega)} \qquad \text{for all } t \in (0,T), \tag{3.52}$$

where an application of Lemma 3.8 to $\lambda := 2q+2$ in combination with (2.5) yields positive constants $C_1 = C_1(q)$ and $C_2 = C_2(q)$ satisfying

$$\begin{aligned} \|\nabla c_\varepsilon\|^{2q}_{L^{2q+2}(\Omega)} &\le C_1 \Big\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \Big\|^{\frac{2q[2(2q+2)-6]}{(2q-1)(2q+2)}}_{L^2(\Omega)} \|c_\varepsilon\|^{\frac{2q[6q-(2q+2)]}{(2q-1)(2q+2)}}_{L^\infty(\Omega)} + C_1 \|c_\varepsilon\|^{2q}_{L^\infty(\Omega)} \\ &\le C_2 \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \Big\}^{\frac{q}{q+1}} + C_2 \qquad \text{for all } t \in (0,T). \end{aligned} \tag{3.53}$$

Moreover, using the Gagliardo-Nirenberg inequality and (3.50), since $q+1 \le \frac{2r+3}{3}+1 < 6$ we can find $C_3 = C_3(q,r) > 0$ and $C_4 = C_4(q,r) > 0$ such that

$$\begin{aligned} \|Du_\varepsilon\|_{L^{q+1}(\Omega)} &\le C_3 \|u_\varepsilon\|_{W^{2,2}(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \|u_\varepsilon\|_{W^{1,r}(\Omega)}^{\frac{(5-q)r}{(q+1)(6-r)}} \\ &\le C_4 \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \|Du_\varepsilon\|_{L^r(\Omega)}^{\frac{(5-q)r}{(q+1)(6-r)}} \\ &\le C_4 K^{\frac{(5-q)r}{(q+1)(6-r)}} \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \qquad \text{for all } t \in (0,T), \end{aligned} \tag{3.54}$$

because by known regularity estimates for the Stokes operator in bounded domains (see [11, p.82], [24] and the references given there) and the Poincaré inequality we know that $\|A(\cdot)\|_{L^2(\Omega)}$ and $\|D(\cdot)\|_{L^r(\Omega)}$ define norms equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ and $\|\cdot\|_{W^{1,r}(\Omega)}$, respectively, in $D(A_2)$.
Now given $\eta > 0$, we combine (3.52), (3.53) and (3.54) and invoke Young's inequality to find $C_5 = C_5(q,r,\eta,K) > 0$ such that

$$\begin{aligned} \int_\Omega |\nabla c_\varepsilon|^{2q} \cdot |Du_\varepsilon| &\le C_2 C_4 K^{\frac{(5-q)r}{(q+1)(6-r)}} \cdot \left\{ \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \right\}^{\frac{q}{q+1}} \cdot \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \\ &\quad + C_2 C_4 K^{\frac{(5-q)r}{(q+1)(6-r)}} \cdot \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \\ &\le \eta \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + C_5 \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{6-r}} \\ &\quad + C_2 C_4 K^{\frac{(5-q)r}{(q+1)(6-r)}} \cdot \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \qquad \text{for all } t \in (0,T). \end{aligned} \tag{3.55}$$

Since here our assumption $q < \frac{2r+3}{3}$ entails that

$$\frac{6(q+1-r)}{(q+1)(6-r)} < \frac{6(q+1-r)}{6-r} < 2,$$

a second application of Young's inequality yields $C_6 = C_6(q,r,\eta,K) > 0$ fulfilling

$$C_5 \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \le \frac{\eta}{2} \|Au_\varepsilon\|_{L^2(\Omega)}^2 + C_6$$

and

$$C_2 C_4 K^{\frac{(5-q)r}{(q+1)(6-r)}} \cdot \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-r)}{(q+1)(6-r)}} \le \frac{\eta}{2} \|Au_\varepsilon\|_{L^2(\Omega)}^2 + C_6$$

for all $t \in (0,T)$. Therefore, (3.55) implies (3.51). □

3.4 Combining previous estimates

Now if $m > \frac{7}{6}$, then the conditions on p in Lemma 3.9 and Lemma 3.10 can be fulfilled simultaneously for any choice of $q > 1$. Thus, resorting to such m allows for combining the above results to derive an ODI for the functional in (1.19) which contains a favorable absorptive term.

Lemma 3.13 *Let $m > \frac{7}{6}$. Let $r \geq 1$ and $q > 1$ satisfy*

$$\begin{cases} q < \frac{2r+3}{3} & \text{if } r \leq \frac{3}{2}, \\ (4-2r)q \leq r-1 & \text{if } r > \frac{3}{2}, \end{cases} \tag{3.56}$$

and assume that $p > \max\{1, m-1\}$ be such that

$$\frac{3q-3m+4}{3} < p < \Big(2m - \frac{4}{3}\Big)q + m - 1. \tag{3.57}$$

Then for all $K > 0$ one can find a constant $C = C(p,q,r,K) > 0$ such that if for some $\varepsilon \in (0,1)$ and $T > 0$ we have

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \leq K \qquad \text{for all } t \in (0,T), \tag{3.58}$$

then

$$\frac{d}{dt}\Big\{\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2q} + \int_\Omega |A^{\frac{1}{2}} u_\varepsilon|^2\Big\} \quad + \quad \frac{1}{C} \cdot \Big\{\int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + \int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 + \int_\Omega |Au_\varepsilon|^2\Big\}$$
$$\leq \quad C \qquad \text{for all } t \in (0,T). \tag{3.59}$$

Proof. We only need to add suitable multiples of the differential inequalities in Lemma 3.5, Lemma 3.6 and Lemma 3.7 and estimate the terms on the respective right-hand sides by applying Lemma 3.9, Lemma 3.10 and either Lemma 3.12 or Lemma 3.11 with appropriately small $\eta > 0$. ☐

Assuming the above boundedness property of Du_ε, upon a further analysis of (3.59) we can estimate n_ε in $L^\infty((0,\infty); L^p(\Omega))$ for certain $p \in (1,\infty)$.

Lemma 3.14 *Let $m > \frac{7}{6}$, and assume that $r \geq 1$ and $p > \max\{1, m-1\}$ are such that there exists $q > 1$ for which (3.56) and (3.57) hold. Then for all $K > 0$ there exists $C = C(p,q,r,K) > 0$ with the property that if $\varepsilon \in (0,1)$ and $T > 0$ are such that*

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \leq K \qquad \text{for all } t \in (0,T), \tag{3.60}$$

then we have

$$\int_\Omega n_\varepsilon^p(\cdot,t) \leq C \qquad \text{for all } t \in (0,T). \tag{3.61}$$

Proof. We only need to derive from (3.59) an autonomous ODI for the function $y \in C^0([0,T)) \cap C^1((0,T))$ defined by

$$y(t) := \int_\Omega n_\varepsilon^p(\cdot,t) + \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^{2q} + \int_\Omega |A^{\frac{1}{2}} u_\varepsilon(\cdot,t)|^2, \qquad t \in [0,T),$$

with an appropriate dampening term essentially dominated by

$$h(t) := \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}(\cdot,t)|^2 + \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^{2q-2}|D^2 c_\varepsilon(\cdot,t)|^2 + \int_\Omega |Au_\varepsilon(\cdot,t)|^2, \qquad t \in (0,T).$$

To this end, we first use Young's inequality, the Poincaré inequality and (2.4) to find positive constants $C_1 = C_1(p), C_2 = C_2(p)$ and $C_3 = C_3(p)$ such that

$$\begin{aligned}
\int_\Omega n_\varepsilon^p &\le C_1 \int_\Omega n_\varepsilon^{p+m-1} + C_1 \\
&\le C_2 \cdot \Big\{ \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + \|n_\varepsilon^{\frac{p+m-1}{2}}\|^2_{L^{\frac{2}{p+m-1}}(\Omega)} \Big\} + C_1 \\
&\le C_3 \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 + C_3 \qquad \text{for all } t \in (0,T). \qquad (3.62)
\end{aligned}$$

Next, from Young's inequality, Lemma 3.8 and (2.5) we obtain $C_3 = C_3(q) > 0, C_4 = C_4(q) > 0$ and $C_5 = C_5(q) > 0$ satisfying

$$\begin{aligned}
\int_\Omega |\nabla c_\varepsilon|^{2q} &\le C_3 \int_\Omega |\nabla c_\varepsilon|^{2q+2} + C_3 \\
&\le C_4 \cdot \Big\{ \Big\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \Big\|^2_{L^2(\Omega)} \|c_\varepsilon\|^2_{L^\infty(\Omega)} + \|c_\varepsilon\|^{2q+2}_{L^\infty(\Omega)} \Big\} + C_3 \\
&\le C_5 \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 + C_5 \qquad \text{for all } t \in (0,T). \qquad (3.63)
\end{aligned}$$

Finally, again by the Poincaré inequality we find $C_6 > 0$ fulfilling

$$\int_\Omega |A^{\frac12} u_\varepsilon|^2 = \int_\Omega |Du_\varepsilon|^2 \le C_6 \int_\Omega |Au_\varepsilon|^2 \qquad \text{for all } t \in (0,T), \qquad (3.64)$$

which when added to (3.62) and (3.63) shows that

$$y(t) \le C_7 \cdot h(t) + C_7 \qquad \text{for all } t \in (0,T)$$

with $C_7 := \max\{C_3, C_5, C_6\}$.
In consequence, an application of Lemma 3.13 yields $C_8 = C_8(p,q,r,K) > 0$ and $C_9 = C_9(p,q,r,K) > 0$ such that

$$y'(t) + C_8 y(t) \le C_9 \qquad \text{for all } t \in (0,T). \qquad (3.65)$$

By a comparison argument, this in particular entails that

$$y(t) \le C_{10} := \max\Big\{ y(0)\,,\, \frac{C_9}{C_8} \Big\} \qquad \text{for all } t \in (0,T), \qquad (3.66)$$

and thereby proves (3.61). □

Now in light of the mass identity (2.4), a first application of Corollary 3.4 warrants that the hypothesis (3.60) in the above lemma is satisfied for some suitably small $r > 1$. Adjusting the parameter q properly, we thereby arrive at the following result which may be viewed as an improvement of the regularity property implied by (2.4), because the number $5m - \frac{11}{3}$ appearing in (3.67) is larger than 1.

Lemma 3.15 *Let $m > \frac{7}{6}$. Then for all $p > \max\{1, m-1\}$ fulfilling*

$$p < 5m - \frac{11}{3}, \qquad (3.67)$$

one can find $C = C(p) > 0$ such that whenever $\varepsilon \in (0,1)$, we have

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{3.68}$$

PROOF. We first observe that since $m > 1$, we have $m - 1 < 5m - \frac{11}{3}$ and also $\frac{7}{3} - m < 5m - \frac{11}{3}$, whence without loss of generality we may assume that besides (3.67), p satisfies $p > m - 1$ and

$$p > \frac{7}{3} - m. \tag{3.69}$$

Now since $p < 5m - \frac{11}{3}$ by (3.67), we have $2 \cdot (6m-4) > 3p - 3m + 3$ and hence

$$\frac{3(p-m+1)}{6m-4} < 2. \tag{3.70}$$

Moreover, our assumption $m > \frac{7}{6}$ ensures that

$$-18\Big(m - \frac{7}{6}\Big)p < 18\Big(m - \frac{7}{6}\Big)\Big(m - \frac{1}{3}\Big),$$

that is,

$$(21 - 18m)p < 18m^2 - 27m + 7,$$

which is equivalent to the inequality

$$\frac{3(p-m+1)}{6m-4} < \frac{3p+3m-4}{3}. \tag{3.71}$$

According to (3.70), (3.71) and the fact that

$$\frac{3p+3m-4}{3} > 1$$

by (3.69), we can now fix $q \in (1,2)$ fulfilling

$$\frac{3(p-m+1)}{6m-4} < q < \frac{3p+3m-4}{3},$$

where the left inequality asserts that

$$p < \Big(2m - \frac{4}{3}\Big)q + m - 1,$$

and the right inequality guarantees that

$$p > \frac{3q-3m+4}{3},$$

altogether meaning that (3.57) is satisfied. Since $q < 2$, we can finally pick $r \in [1, \frac{3}{2})$ sufficiently close to $\frac{3}{2}$ such that $r > \frac{3q-3}{2}$, so that

$$q < \frac{2r+3}{3},$$

ensuring that also (3.56) is valid. Then in view of (2.4), Corollary 3.4 asserts that

$$\|Du_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le C_1 \qquad \text{for all } t>0$$

with some $C_1>0$, whence according to our choices of r, q and p we may apply Lemma 3.14 to find $C_2=C_2(p)>0$ such that

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C_2 \qquad \text{for all } t>0.$$

This proves the lemma. $\Box$

In a second step, on the basis of the knowledge just gained we may again apply Corollary 3.4 and once more combine the outcome theoreof with Lemma 3.14 to obtain bounds for n_ε in any space $L^\infty((0,\infty);L^p(\Omega))$.

Lemma 3.16 *Let $m>\frac{7}{6}$. Then for all $p>1$ there exists $C=C(p)>0$ such that for each $\varepsilon\in(0,1)$ we have*

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \textit{for all } t>0. \tag{3.72}$$

Proof. It is evidently sufficient to show that for any $p_0>\max\{1,m-1\}$ we can find some $p>p_0$ such that (3.72) holds with some $C>0$.
For this purpose, given such p_0 we first fix $q>1$ satisfying

$$q>\frac{3(p_0+1-m)}{6m-4} \tag{3.73}$$

and observe that then since $m>\frac{7}{6}$ we have

$$3q-3m+4-\Big[(6m-4)q+3m-3\Big]=(7-6m)q+7-6m<0$$

and hence

$$\frac{3q-3m+4}{3}<\frac{(6m-4)q+3m-3}{3}.$$

As (3.73) ensures that moreover

$$\frac{(6m-4)q+3m-3}{3}>\frac{3(p_0+1-m)+3m-3}{3}=p_0,$$

we can therefore pick some $p>p_0$ fulfilling

$$\frac{3q-3m+4}{3}<p<\frac{(6m-4)q+3m-3}{3}. \tag{3.74}$$

Now in order to verify (3.72) for these choices of p and q, we first use the fact that $5m-\frac{11}{3}>\frac{35}{6}-\frac{11}{3}=\frac{13}{6}$ to infer from Lemma 3.15 that there exists $C_1>0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^{\frac{13}{6}}(\Omega)} \le C_1 \qquad \text{for all } t>0.$$

Since $\frac{3\cdot\frac{13}{6}}{3-\frac{13}{6}} = \frac{39}{5} > 7$, Corollary 3.4 thereupon yields $C_2 > 0$ fulfilling

$$\|Du_\varepsilon(\cdot,t)\|_{L^7(\Omega)} \le C_2 \qquad \text{for all } t > 0.$$

As with $r := 7$, the condition $(4-2r)q \le r-1$ in (3.56) is trivially satisfied, thanks to (3.74) we may thus invoke Lemma 3.14 to establish (3.72). $\Box$

By means of a Moser-type iteration in conjunction with standard parabolic regularity arguments, we can achieve the following boundedness results.

Lemma 3.17 *Let $m > \frac{7}{6}$. Then there exists $C > 0$ such that for all $\varepsilon \in (0,1)$ the solution of (2.2) satisfies*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0 \tag{3.75}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \textit{for all } t > 0 \tag{3.76}$$

as well as

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{3.77}$$

Proof. First, the validity of estimate (3.72) in Lemma 3.16 for any $p > 3$ allows for an application of Corollary 3.4 to $r := \infty$ to infer that $(Du_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\Omega\times(0,\infty))$, and that hence (3.77) holds.
Next, using the outcome of Lemma 3.16 with suitably large p and q as a starting point, we may invoke Lemma A.1 in [26] which by means of a Moser-type iteration applied to the first equation in (2.2) establishes (3.75).
Thereupon, (3.76) finally can be derived from (3.75) and (3.77) by well-known arguments from parabolic regularity theory for the second equation in (2.2) (cf. the reasoning in [13, Lemma 4.1], for instance). $\Box$

As one further class of a priori estimates, let us finally also note straightforward consequences of Lemma 3.17 for uniform Hölder regularity properties of c_ε, ∇c_ε and u_ε.

Lemma 3.18 *Let $m > \frac{7}{6}$. Then there exists $\theta \in (0,1)$ such that for some $C > 0$ we have*

$$\|c_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t \ge 0, \tag{3.78}$$

and such that for each $\tau > 0$ we can find $C(\tau) > 0$ such that

$$\|\nabla c_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t \ge \tau. \tag{3.79}$$

Proof. Writing the second equation in the form

$$\partial_t c_\varepsilon = \Delta c_\varepsilon + g_\varepsilon(x,t), \qquad x\in\Omega,\ t>0,$$

with $g_\varepsilon(x,t) := -n_\varepsilon f(c_\varepsilon) - u_\varepsilon\cdot\nabla c_\varepsilon$, we immediately obtain both estimates (3.78) and (3.79) e.g. from standard parabolic regularity theory ([17]), because $(g_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\Omega\times(0,\infty))$ according to Lemma 3.17. $\Box$

Lemma 3.19 *Let $m > \frac{7}{6}$. Then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le C \qquad \text{for all } t \ge 0. \tag{3.80}$$

Proof. Starting from the variation-of-constants representation

$$u_\varepsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}[n_\varepsilon(\cdot,s)\nabla\phi]ds, \qquad t>0,\ \varepsilon\in(0,1), \tag{3.81}$$

we fix $\alpha \in (\frac{3}{4},\alpha_0)$ with α_0 taken from (1.7), so that the latter ensures that $u_0 \in D(A_2^{\alpha_0}) \hookrightarrow D(A_2^{\alpha}) \hookrightarrow C^\beta(\bar{\Omega})$ for any $\beta \in (0, 2\alpha - \frac{3}{2})$ ([12], [9]), and apply A^α to both sides of (3.81). Then performing standard semigroup estimation techniques ([8]), in view of the boundedness of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega\times(0,\infty))$ guaranteed by Lemma 3.17 we infer the existence of $C_1 > 0$ and $\beta' > 0$ such that for all $\varepsilon \in (0,1)$,

$$\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_1 \qquad \text{for all } t>0$$

and

$$\|A^\alpha u_\varepsilon(\cdot,t) - A^\alpha u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} \le C_1(t-t_0)^{\beta'} \qquad \text{for all } t_0\ge 0 \text{ and each } t>t_0.$$

This implies (3.80) with some appropriately small $\theta \in (0,1)$. □

3.5 Some temporally global integrability properties

Let us next derive three estimates involving integrability over the whole positive time axis. They implicitly contain some weak decay properties of the respective integrands, and these properties will constitute a starting point for our stabilization proof below.
The first two of these estimates result from (2.2) in a straightforward manner.

Lemma 3.20 *Let $m > \frac{7}{6}$. Then the inequalities*

$$\int_0^\infty\int_\Omega n_\varepsilon f(c_\varepsilon) \le \int_\Omega c_0 \tag{3.82}$$

and

$$\int_0^\infty\int_\Omega |\nabla c_\varepsilon|^2 \le \frac{1}{2}\int_\Omega c_0^2 \tag{3.83}$$

are valid for each $\varepsilon\in(0,1)$.

Proof. We test the second equation in (2.2) by 1 and c_ε to obtain that for all $\varepsilon\in(0,1)$ and $t>0$,

$$\int_\Omega c_\varepsilon(\cdot,t) + \int_0^t\int_\Omega n_\varepsilon f(c_\varepsilon) = \int_\Omega c_0$$

and

$$\frac{1}{2}\int_\Omega c_\varepsilon^2(\cdot,t) + \int_0^t\int_\Omega|\nabla c_\varepsilon|^2 + \int_0^t\int_\Omega n_\varepsilon c_\varepsilon f(c_\varepsilon) = \frac{1}{2}\int_\Omega c_0^2,$$

respectively. From these identities, (3.82) and (3.83) immediately follow. □

A corresponding spatio-temporal integrability property of ∇n_ε can be obtained from (3.14) upon using (3.83) along with the L^∞ bound for n_ε from Lemma 3.17.

Lemma 3.21 *Assume that $m > \frac{7}{6}$, and that $p > 1$ is such that $p \geq m-1$. Then there exists $C > 0$ with the property that*

$$\int_0^\infty \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 \leq C \tag{3.84}$$

for all $\varepsilon \in (0,1)$.

Proof. Due to Lemma 3.17, there exists $C_1 > 0$ such that for all $\varepsilon \in (0,1)$ we have $n_\varepsilon \leq C_1$ in $\Omega \times (0,\infty)$. Since $p \geq m-1$, we can thereby estimate the integral on the right of (3.14) according to

$$\int_\Omega n_\varepsilon^{p+1-m} |\nabla c_\varepsilon|^2 \leq C_1^{p+1-m} \int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).$$

Hence, an integration of (3.14) shows that

$$\frac{2(p-1)k_D}{(p+m-1)^2} \int_0^t \int_\Omega |\nabla n_\varepsilon^{\frac{p+m-1}{2}}|^2 \leq \frac{1}{p} \int_\Omega n_0^p + \frac{(p-1)S_1^2}{2k_D} \cdot C_1^{p+1-m} \int_0^t \int_\Omega |\nabla c_\varepsilon|^2$$

for any such t and ε, from which (3.84) results by an application of Lemma 3.20. □

3.6 Regularity properties of time derivatives

In order to pass to the limit in the first equation in (2.2), we shall need an appropriate boundedness property of the time derivatives of certain powers of n_ε. On time intervals of a fixed finite length, this can be achieved in a straightforward way by making use of the a priori bounds derived so far.

Lemma 3.22 *Suppose that $m > \frac{7}{6}$, and let $\gamma > m$ satisfy $\gamma \geq 2(m-1)$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \|\partial_t n_\varepsilon^\gamma(\cdot,t)\|_{(W_0^{3,2}(\Omega))^\star} dt \leq C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.85}$$

Proof. On differentiation and integration by parts in (2.2), we see that for each fixed $\psi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}
\frac{1}{\gamma}\int_\Omega \partial_t n_\varepsilon^\gamma(\cdot,t)\cdot\psi &= \int_\Omega n_\varepsilon^{\gamma-1}\cdot\Big\{\nabla\cdot\Big(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\Big) - \nabla\cdot\Big(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) - u_\varepsilon\cdot\nabla n_\varepsilon\Big\}\cdot\psi \\
&= -(\gamma-1)\int_\Omega n_\varepsilon^{\gamma-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\psi - \int_\Omega n_\varepsilon^{\gamma-1}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\psi \\
&\quad +(\gamma-1)\int_\Omega n_\varepsilon^{\gamma-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\psi + \int_\Omega n_\varepsilon^\gamma\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\cdot\nabla\psi \\
&\quad +\frac{1}{\gamma}\int_\Omega n_\varepsilon^\gamma u_\varepsilon\cdot\nabla\psi \qquad \text{for all } t > 0.
\end{aligned} \tag{3.86}$$

In order to estimate the integrals on the right appropriately, we first apply Lemma 3.17 to fix positive constants C_1, C_2 and C_3 such that

$$n_\varepsilon \leq C_1, \quad |\nabla c_\varepsilon| \leq C_2 \quad \text{and} \quad |u_\varepsilon| \leq C_3 \qquad \text{in } \Omega\times(0,\infty) \qquad \text{for all } \varepsilon \in (0,1), \tag{3.87}$$

whence due to the fact that $D_\varepsilon \le D + 2\varepsilon$ in $(0,\infty)$, we also have

$$D_\varepsilon(n_\varepsilon) \le C_4 := \|D_0\|_{L^\infty((0,C_1))} + 2 \qquad \text{in } \Omega\times(0,\infty) \qquad \text{for all } \varepsilon\in(0,1). \tag{3.88}$$

Moreover, since $\gamma > m$ and $\gamma \ge 2(m-1)$, the number $p := \gamma - m + 1$ satisfies $p > 1$ and $p \ge m-1$, so that Lemma 3.21 becomes applicable so as to yield $C_5 > 0$ fulfilling

$$\int_0^\infty \int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 = \int_0^\infty \int_\Omega n_\varepsilon^{p+m-3}|\nabla n_\varepsilon|^2 \le C_5 \qquad \text{for all } \varepsilon\in(0,1). \tag{3.89}$$

Now using (3.87), (3.88) and Young's inequality, in (3.86) we find that

$$\left| -(\gamma-1)\int_\Omega n_\varepsilon^{\gamma-2} D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\psi\right| \le (\gamma-1)C_4\cdot\Big(\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2\Big)\cdot\|\psi\|_{L^\infty(\Omega)} \tag{3.90}$$

as well as

$$\begin{aligned}\left| -\int_\Omega n_\varepsilon^{\gamma-1}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\psi\right| &\le C_4\cdot\Big(\int_\Omega n_\varepsilon^{\gamma-1}|\nabla n_\varepsilon|\Big)\cdot\|\nabla\psi\|_{L^\infty(\Omega)}\\ &\le C_4\cdot\Big\{\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon^\gamma\Big\}\cdot\|\nabla\psi\|_{L^\infty(\Omega)}\\ &\le \Big\{C_4\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 + C_4C_1^\gamma|\Omega|\Big\}\cdot\|\nabla\psi\|_{L^\infty(\Omega)}\end{aligned} \tag{3.91}$$

and, similarly,

$$\begin{aligned}&\left|(\gamma-1)\int_\Omega n_\varepsilon^{\gamma-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\psi\right|\\ &\qquad\le (\gamma-1)\cdot\Big(\int_\Omega n_\varepsilon^{\gamma-1}|\nabla n_\varepsilon|\Big)\cdot S_1C_2\|\psi\|_{L^\infty(\Omega)}\\ &\qquad\le (\gamma-1)S_1C_2\cdot\Big\{\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 + C_1^\gamma|\Omega|\Big\}\cdot\|\psi\|_{L^\infty(\Omega)}\end{aligned} \tag{3.92}$$

whereas by means of (3.87) and (2.5) we can estimate

$$\left|\int_\Omega n_\varepsilon^\gamma\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\cdot\nabla\psi\right| \le C_1^\gamma S_1C_2|\Omega|\|\nabla\psi\|_{L^\infty(\Omega)} \tag{3.93}$$

and

$$\left|\frac{1}{\gamma}\int_\Omega n_\varepsilon^\gamma u_\varepsilon\cdot\nabla\psi\right| \le \frac{1}{\gamma}C_1^\gamma C_3|\Omega|\|\nabla\psi\|_{L^\infty(\Omega)} \tag{3.94}$$

for all $\varepsilon\in(0,1)$.
As in the considered three-dimensional setting we have $W_0^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$, collecting (3.90)-(3.94) we infer the existence of $C_6 > 0$ such that

$$\|\partial_t n_\varepsilon^\gamma(\cdot,t)\|_{(W_0^{3,2}(\Omega))^\star} \le C_6\cdot\Big\{\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 + 1\Big\} \qquad \text{for all } t>0 \text{ and any } \varepsilon\in(0,1).$$

According to (3.89), for each $T > 0$ we therefore have

$$\int_0^T \|\partial_t n_\varepsilon^\gamma(\cdot,t)\|_{(W_0^{3,2}(\Omega))^\star} dt \le C_5 C_6 + C_6 T \qquad \text{for all } \varepsilon \in (0,1),$$

which proves (3.85). □

In proving that the limit function n gained below approaches its spatial mean not only along certain sequences of times but in fact along the entire net $t \to \infty$, we will rely on an additional regularity estimate for $\partial_t n_\varepsilon$ which, in contrast to that in Lemma 3.22, is uniform with respect to time.

Lemma 3.23 *Let $m > \frac{7}{6}$. Then there exists $C > 0$ such that*

$$\|\partial_t n_\varepsilon(\cdot,t)\|_{(W_0^{2,2}(\Omega))^\star} \le C \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.95}$$

In particular,

$$\|n_\varepsilon(\cdot,t) - n_\varepsilon(\cdot,s)\|_{(W_0^{2,2}(\Omega))^\star} \le C|t-s| \qquad \textit{for all } t \ge 0, s \ge 0 \textit{ and } \varepsilon \in (0,1). \tag{3.96}$$

Proof. We fix $\psi \in C_0^\infty(\Omega)$ and multiply the first equation in (2.2) by ψ. Integrating by parts we find that

$$\begin{aligned}
\int_\Omega \partial_t n_\varepsilon(\cdot,t)\psi &= \int_\Omega \nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon)\psi - \int_\Omega \nabla\cdot\Big(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\psi - \int_\Omega (u_\varepsilon\cdot\nabla n_\varepsilon)\psi \\
&= \int_\Omega H_\varepsilon(n_\varepsilon)\Delta\psi + \int_\Omega n_\varepsilon\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\cdot\nabla\psi + \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\psi
\end{aligned} \tag{3.97}$$

for all $t > 0$, where we have set $H_\varepsilon(s) := \int_0^s D_\varepsilon(\sigma)d\sigma$ for $s \ge 0$. Here since by Lemma 3.17 we can find $C_1 > 0$ such that $n_\varepsilon \le C_1$ in $\Omega\times(0,\infty)$ for all $\varepsilon \in (0,1)$, recalling that $D_\varepsilon \le D + 2\varepsilon$ on $(0,\infty)$ we can estimate

$$H_\varepsilon(n_\varepsilon) \le C_2 := C_1\cdot\Big(\|D\|_{L^\infty((0,C_1))} + 2\Big) \qquad \text{in } \Omega\times(0,\infty) \quad \text{for all } \varepsilon\in(0,1).$$

If moreover we invoke Lemma 3.17 once again to pick $C_3 > 0$ and $C_4 > 0$ fulfilling

$$|\nabla c_\varepsilon| \le C_3 \quad \text{and} \quad |u_\varepsilon| \le C_4 \qquad \text{in } \Omega\times(0,\infty) \quad \text{for all } \varepsilon\in(0,1),$$

then from (3.97), (1.11) and (2.5) we can derive the inequality

$$\left|\int_\Omega \partial_t n_\varepsilon(\cdot,t)\cdot\psi\right| \le C_2\int_\Omega|\Delta\psi| + C_1C_3S_1\int_\Omega|\nabla\psi| + C_1C_4\int_\Omega|\nabla\psi| \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

where $S_1 := S_0(\|c_0\|_{L^\infty(\Omega)})$. This readily establishes (3.95) and thus also (3.96). □

4 Passing to the limit. Proof of Theorem 1.1

Our generalized solution concept reads as follows.

Definition 4.1 *Suppose that (n_0, c_0, u_0) satisfies (1.7), and let $T > 0$. Then by a* weak solution *of (1.2) in $\Omega \times (0,T)$ we mean a triple of functions*

$$\begin{aligned} & n \in L^1_{loc}(\bar\Omega \times [0,T)), \\ & c \in L^\infty_{loc}(\bar\Omega \times [0,T)) \cap L^1_{loc}([0,T); W^{1,1}(\Omega)), \\ & u \in L^1_{loc}([0,T); W^{1,1}(\Omega)), \end{aligned} \tag{4.1}$$

such that $n \geq 0$ and $c \geq 0$ in $\Omega \times (0,T)$ and

$$H(n), \ n|\nabla c| \text{ and } n|u| \quad \text{belong to } L^1_{loc}(\bar\Omega \times [0,T)), \tag{4.2}$$

that $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0,T)$, and such that

$$-\int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) = \int_0^T \int_\Omega H(n)\Delta\varphi + \int_0^T \int_\Omega n\Big(S(x,n,c)\cdot\nabla c\Big)\cdot\nabla\varphi + \int_0^T \int_\Omega nu\cdot\nabla\varphi \tag{4.3}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,T))$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,T)$, that

$$-\int_0^T \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla c\cdot\nabla\varphi - \int_0^T \int_\Omega nf(c)\varphi + \int_0^T \int_\Omega cu\cdot\nabla\varphi \tag{4.4}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,T))$, and that

$$-\int_0^T \int_\Omega u\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla u\cdot\nabla\varphi + \int_0^T \int_\Omega n\nabla\phi\cdot\varphi \tag{4.5}$$

for all $\varphi \in C_0^\infty(\Omega \times [0,T); \mathbb{R}^3)$ such that $\nabla\varphi \equiv 0$ in $\Omega \times (0,T)$, where we have set

$$H(s) := \int_0^s D(\sigma)d\sigma \qquad \text{for } s \geq 0.$$

If $(n,c,u) : \Omega \times (0,\infty) \to \mathbb{R}^5$ is a weak solution of (1.2) in $\Omega \times (0,T)$ for all $T > 0$, then we call (n,c,u) a global weak solution *of (1.2).*

In this framework, (1.2) is indeed globally solvable. This can be seen by making use of the above a priori estimates and extracting suitable subsequences in a standard manner.

Lemma 4.1 *Let $m > \frac{7}{6}$. Then there exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and that*

$$n_\varepsilon \to n \qquad \text{a.e. in } \Omega \times (0,\infty), \tag{4.6}$$

$$n_\varepsilon \stackrel{\star}{\rightharpoonup} n \qquad \text{in } L^\infty(\Omega \times (0,\infty)), \tag{4.7}$$

$$n_\varepsilon \to n \qquad \text{in } C^0_{loc}\Big([0,\infty); (W_0^{2,2}(\Omega))^\star\Big), \tag{4.8}$$

$$c_\varepsilon \to c \qquad \text{in } C^0_{loc}(\bar\Omega \times [0,\infty)), \tag{4.9}$$

$$\nabla c_\varepsilon \to \nabla c \qquad \text{in } C^0_{loc}(\bar\Omega \times (0,\infty)), \tag{4.10}$$

$$\nabla c_\varepsilon \stackrel{\star}{\rightharpoonup} \nabla c \qquad \text{in } L^\infty(\Omega \times (0,\infty)), \tag{4.11}$$

$$u_\varepsilon \to u \qquad \text{in } C^0_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{and} \tag{4.12}$$

$$Du_\varepsilon \stackrel{\star}{\rightharpoonup} Du \qquad \text{in } L^\infty(\Omega \times (0,\infty)) \tag{4.13}$$

with some triple (n,c,u) which is a global weak solution of (1.2) in the sense of Definition 4.1. Moreover, n satisfies

$$n \in C^0_{w-\star}([0,\infty); L^\infty(\Omega)) \tag{4.14}$$

as well as

$$\int_\Omega n(x,t)dx = \int_\Omega n_0(x)dx \qquad \textit{for all } t>0. \tag{4.15}$$

Proof. In view of Lemma 3.18 and Lemma 3.19, the Arzelà-Ascoli theorem along with a standard extraction procedure yields a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ with $\varepsilon_j \searrow 0$ as $j\to\infty$ such that (4.9), (4.10) and (4.12) hold with some limit functions c and u belonging to the indicated spaces. Passing to a subsequence if necessary, by means of Lemma 3.17 we can achieve that for some $n \in L^\infty(\Omega\times(0,\infty))$ we moreover have (4.7), (4.11) and (4.13).
We next fix $\gamma > m$ such that $\gamma \ge 2(m-1)$ and combine Lemma 3.22 with the estimate asserted by Lemma 3.21 for $p := 2\gamma - m + 1$ to see that for each $T>0$, $(\varepsilon^\gamma)_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{1,2}(\Omega))$ with $(\partial_t n_\varepsilon^\gamma)_{\varepsilon\in(0,1)}$ being bounded in $L^1((0,T);(W_0^{3,2}(\Omega))^\star)$. Therefore, an Aubin-Lions lemma ([30]) applies to yield strong precompactness of $(n_\varepsilon^\gamma)_{\varepsilon\in(0,1)}$ in $L^2(\Omega\times(0,T))$, whence along a suitable subsequence we have $n_\varepsilon^\gamma \to z^\gamma$ and hence $n_\varepsilon \to z$ a.e. in $\Omega\times(0,\infty)$ for some nonnegative measurable $z:\Omega\times(0,\infty)\to\mathbb{R}$. By Egorov's theorem, we know that necessarily $z=n$, so that (4.6) follows.
Finally, as the embedding $L^\infty(\Omega) \hookrightarrow (W_0^{2,2}(\Omega))^\star$ is compact, the Arzelà-Ascoli once more applies to say that the equicontinuity property (3.96) together with the boundedness of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $C^0([0,\infty);L^\infty(\Omega))$ ensures that (4.8) holds after a further extraction of an adequate subsequence.

The additional regularity property (4.14) thereafter is a consequence of (4.8) and the fact that $C_1 := \|n\|_{L^\infty(\Omega\times(0,\infty))}$ is finite: First, from the latter property it follows that there exists a null set $N\subset[0,\infty)$ such that for all $t\in[0,\infty)\setminus N$ we have $n(\cdot,t)\in L^\infty(\Omega)$ with $\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C_1$. As $[0,\infty)\setminus N$ is dense in $[0,\infty)$, for an arbitrary $t_0\in[0,\infty)$ we can find $(t_j)_{j\in\mathbb{N}}\subset[0,\infty)\setminus N$ such that $t_j\to t_0$ as $j\to\infty$, and extracting a subsequence if necessary we can also achieve that $n(\cdot,t_j) \overset{\star}{\rightharpoonup} \tilde n$ in $L^\infty(\Omega)$ as $j\to\infty$ with some $\tilde n\in L^\infty(\Omega)$ satisfying $\|\tilde n\|_{L^\infty(\Omega)}\le C_1$. Since (4.8) asserts that moreover $n(\cdot,t_j)\to n(\cdot,t_0)$ in $(W_0^{2,2}(\Omega))^\star$ as $j\to\infty$, this allows us to identify $\tilde n = n(\cdot,t_0)$ and to conclude that thus actually $n(\cdot,t)\in L^\infty(\Omega)$ for all $t\in[0,\infty)$, with $\|n(\cdot,t)\|_{L^\infty(\Omega)}\le C_1$ for all $t\ge 0$. The property (4.14) can now be verified by partially repeating this argument: Given any $t_0\ge0$ and $(t_j)_{j\in\mathbb{N}}\subset[0,\infty)$ such that $t_j\to t_0$ as $j\to\infty$ we know that $(n(\cdot,t_j))_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, and that for all $\psi\in C_0^\infty(\Omega)$ we have $\int_\Omega n(\cdot,t_j)\psi \to \int_\Omega n(\cdot,t_0)\psi$ as $j\to\infty$ by (4.8). By density of $C_0^\infty(\Omega)$ in $L^1(\Omega)$, this proves that indeed $n(\cdot,t_j)\overset{\star}{\rightharpoonup} n(\cdot,t_0)$ in $L^\infty(\Omega)$ as $j\to\infty$.

Now the verification of the claimed weak solution property of (n,c,u) is straightforward: Whereas the nonnegativity of n and c and the integrability requirements in (4.1) and (4.2) are immediate from (4.6), (4.7), (4.9), (4.11) and (4.12), the integral identities (4.3), (4.4) and (4.5) can be derived by standard arguments from the corresponding weak formulations in the approximate system (2.2) upon letting $\varepsilon=\varepsilon_j\searrow0$ and using (4.6) and (4.7) as well as (4.9)-(4.13). $\square$

Proof of Theorem 1.1. We only need to combine Lemma 4.1 with Lemma 3.17. $\square$

5 Large time behavior. Proof of Theorem 1.2

In this section we shall assume that D, S and f satisfy the assumptions in Theorem 1.2, and we will establish the convergence statements therein separately for the solution components n, c and u.

Here proving stabilization of n will require a comparatively subtle reasoning, which is due to the fact that our knowledge on compactness properties of $(n(\cdot,t))_{t>0}$, and of temporal continuity features of n, is rather limited. The core of the following argument lies in an appropriate combination of the decay information implied by Lemma 3.21 with the continuity property contained in Lemma 3.23. The nonlinearity of diffusion in the first equation in (1.2), reflected in the appearance of a nontrivial function of n in the integral in (3.84), requires the use of certain powers of n in the following proof.

Lemma 5.1 *Let $m > \frac{7}{6}$. Then with (n, c, u) as given by Theorem 1.1, we have*

$$n(\cdot,t) \overset{\star}{\rightharpoonup} \overline{n_0} \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{5.1}$$

PROOF. Let us assume that the conclusion of the lemma does not hold. Then we can find a sequence $(t_j)_{j\in\mathbb{N}} \subset (0,\infty)$ such that $t_j \to \infty$ as $j \to \infty$, and such that for some $\widetilde{\psi} \in L^1(\Omega)$ we have

$$\int_\Omega n(x,t_j)\widetilde{\psi}(x)dx - \int_\Omega \overline{n_0}\widetilde{\psi}(x)dx \ge C_1 \qquad \text{for all } j \in \mathbb{N} \tag{5.2}$$

with some $C_1 > 0$. To exploit his appropriately, according to Lemma 3.17 we take $C_2 > 0$ fulfilling

$$n(x,t) \le C_2 \qquad \text{for a.e. } (x,t) \in \Omega \times (0,\infty), \tag{5.3}$$

and then use the density of $C_0^\infty(\Omega)$ in $L^1(\Omega)$ in choosing $\psi \in C_0^\infty(\Omega)$ such that $\|\psi - \widetilde{\psi}\|_{L^1(\Omega)} \le \frac{C_1}{4C_2}$, so that by (5.2),

$$\begin{aligned} \int_\Omega n(x,t_j)\psi(x)dx - \int_\Omega \overline{n_0}\psi(x)dx \ &\ge \ \int_\Omega n(x,t_j)\widetilde{\psi}(x)dx - \int_\Omega \overline{n_0}\widetilde{\psi}(x)dx \\ &\qquad -\Big\{\|n(\cdot,t_j)\|_{L^\infty(\Omega)} + \overline{n_0}\Big\} \cdot \|\psi - \widetilde{\psi}\|_{L^1(\Omega)} \\ &\ge \ \frac{C_1}{2} \qquad \text{for all } j \in \mathbb{N}. \end{aligned} \tag{5.4}$$

Now since by Lemma 3.23 there exists $C_3 > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$\|n_\varepsilon(\cdot,t) - n_\varepsilon(\cdot,s)\|_{(W_0^{2,2}(\Omega))^\star} \le C_3|t-s| \qquad \text{for all } t \ge 0 \text{ and } s \ge 0,$$

recalling the convergence property (4.8) from Lemma 4.1 we see that

$$\|n(\cdot,t) - n(\cdot,s)\|_{(W_0^{2,2}(\Omega))^\star} \le C_3|t-s| \qquad \text{for all } t \ge 0 \text{ and } s \ge 0,$$

In particular, this implies that if we let $\tau \in (0,1)$ be such that

$$\tau \le \frac{C_1}{4C_3\|\psi\|_{W_0^{2,2}(\Omega)}},$$

then for all $j \in \mathbb{N}$ and each $t \in (t_j, t_j + \tau)$ we have

$$\begin{aligned}
\left| \int_\Omega n(x,t_j)\psi(x)dx - \int_\Omega n(x,t)\psi(x)dx \right| &\le \|n(\cdot,t_j) - n(\cdot,t)\|_{(W_0^{2,2}(\Omega))^\star} \cdot \|\psi\|_{W_0^{2,2}(\Omega)} \\
&\le C_3 |t_j - t| \cdot \|\psi\|_{W_0^{2,2}(\Omega)} \\
&\le \frac{C_1}{4},
\end{aligned}$$

and hence (5.4) entails that

$$\int_\Omega n(x,t)\psi(x)dx - \int_\Omega \overline{n_0}\psi(x)dx \ge \frac{C_1}{4} \qquad \text{for all } t \in (t_j, t_j+\tau) \text{ and each } j \in \mathbb{N}. \tag{5.5}$$

To see that this contradicts the outcome of Lemma 3.21, we fix any $p > 1$ such that $p \ge m-1$ and $p > 3-m$, and abbreviate $\gamma := \frac{p+m-1}{2} > 1$. Then taking a Poincaré constant $C_4 > 0$ such that

$$\int_\Omega |\varphi(x) - \overline{\varphi}|^2 dx \le C_4 \int_\Omega |\nabla \varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

again with $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$, from Lemma 3.21 we obtain $C_5 > 0$ such that

$$\begin{aligned}
\int_0^\infty \int_\Omega |n_\varepsilon^\gamma(x,t) - a_\varepsilon^\gamma(t)|^2 dxdt &\le C_4 \int_0^\infty \int_\Omega |\nabla n_\varepsilon^\gamma|^2 \\
&\le C_5 \qquad \text{for all } \varepsilon \in (0,1),
\end{aligned} \tag{5.6}$$

where

$$a_\varepsilon(t) := \left(\overline{n_\varepsilon^\gamma(\cdot,t)}\right)^{\frac{1}{\gamma}} = \left\{ \frac{1}{|\Omega|} \int_\Omega n_\varepsilon^\gamma(x,t)dx \right\}^{\frac{1}{\gamma}} \qquad \text{for } \varepsilon \in (0,1) \text{ and } t > 0.$$

Here since from Lemma 4.1 and the Tonelli theorem we know that for a.e. $t > 0$ we have $n_\varepsilon(\cdot,t) \to n(\cdot,t)$ a.e. in Ω as $\varepsilon = \varepsilon_j \searrow 0$, in view of the uniform boundedness of $(n_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega \times (0,\infty))$ we may apply the dominated convergence theorem to infer that

$$a_\varepsilon(t) \to a(t) \qquad \text{for a.e. } t > 0 \tag{5.7}$$

as $\varepsilon = \varepsilon_j \searrow 0$, where

$$a(t) := \left(\overline{n^\gamma(\cdot,t)}\right)^{\frac{1}{\gamma}} = \left\{ \frac{1}{|\Omega|} \int_\Omega n^\gamma(x,t)dx \right\}^{\frac{1}{\gamma}} \qquad \text{for } t > 0. \tag{5.8}$$

Again using that $n_\varepsilon \to n$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, by means of Fatou's lemma we can derive from (5.6) and (5.7) that

$$\int_0^\infty \int_\Omega |n^\gamma(x,t) - a^\gamma(t)|^2 dxdt \le C_5. \tag{5.9}$$

Thanks to the fact that $\gamma > 1$ ensures the validity of the elementary inequality

$$\frac{\xi^\gamma - \eta^\gamma}{\xi - \eta} \ge \eta^{\gamma-1} \qquad \text{for all } \xi \ge 0 \text{ and } \eta \ge 0 \text{ such that } \xi \ne \eta,$$

and since by the Hölder inequality, (4.15) and (5.8) we have

$$\overline{n_0} = \frac{1}{|\Omega|}\int_\Omega n(x,t)dx \le \frac{1}{|\Omega|} \cdot \left(\int_\Omega n^\gamma(x,t)dx\right)^{\frac{1}{\gamma}} \cdot |\Omega|^{\frac{\gamma-1}{\gamma}} = a(t) \qquad \text{for a.e. } t > 0,$$

on the left of (5.9) we can estimate

$$\begin{aligned} \int_\Omega |n^\gamma(x,t) - a^\gamma(t)|^2 dx &\ge a^{2\gamma-2}(t) \cdot \int_\Omega |n(x,t) - a(t)|^2 dx \\ &\ge \overline{n_0}^{2\gamma-2} \cdot \int_\Omega |n(x,t) - a(t)|^2 dx \qquad \text{for a.e. } t > 0. \end{aligned}$$

Therefore, (5.9) yields

$$\int_0^\infty \int_\Omega |n(x,t) - a(t)|^2 dxdt \le C_6 := \frac{C_5}{\overline{n_0}^{2\gamma-2}}. \tag{5.10}$$

We now introduce

$$n_j(x,s) := n(x, t_j + s), \qquad (x,s) \in \Omega \times (0,\tau),$$

and

$$a_j(s) := a(t_j + s), \qquad s \in (0,\tau), \tag{5.11}$$

for $j \in \mathbb{N}$. Then (5.10) implies that

$$\begin{aligned} \int_0^\tau \int_\Omega |n_j(x,s) - a_j(s)|^2 dxds &= \int_{t_j}^{t_j+\tau} \int_\Omega |n(x,t) - a(t)|^2 dxdt \\ &\to 0 \qquad \text{as } j \to \infty, \end{aligned}$$

meaning that for

$$z_j(x,s) := n_j(x,s) - a_j(s), \qquad (x,s) \in \Omega \times (0,\tau), \qquad j \in \mathbb{N},$$

we have

$$z_j \to 0 \quad \text{in } L^2(\Omega \times (0,\tau)) \qquad \text{as } j \to \infty. \tag{5.12}$$

Again by (5.3), it follows from (5.8) and (5.11) that $(a_j)_{j\in\mathbb{N}}$ is bounded in $L^2((0,\tau))$, whence passing to a subsequence if necessary we may assume that for some nonnegative $a_\infty \in L^2((0,\tau))$ we have

$$a_j \rightharpoonup a_\infty \quad \text{in } L^2((0,\tau)) \qquad \text{as } j \to \infty. \tag{5.13}$$

Therefore, in view of (4.15) we see that

$$\begin{aligned}\int_0^\tau \int_\Omega z_j(x,s)dxds &= \int_0^\tau \int_\Omega \Big(n_j(x,s) - a_j(s)\Big)dxds \\ &= \tau|\Omega|\overline{n_0} - |\Omega| \cdot \int_0^\tau a_j(s)ds \\ &\to \tau|\Omega|\overline{n_0} - |\Omega| \cdot \int_0^\tau a_\infty(s)ds \qquad \text{as } j \to \infty,\end{aligned}$$

which combined with (5.12) allows us to determine the integral of the limit in (5.13) according to

$$\int_0^\tau a_\infty(s)ds = \tau\overline{n_0}. \tag{5.14}$$

On the other hand, rewriting (5.5) in terms of n_j and integrating in time we see that

$$\begin{aligned}\frac{C_1\tau}{4} &\le \int_0^\tau \int_\Omega n_j(x,s)\psi(x)dxds - \int_0^\tau \int_\Omega \overline{n_0}\psi(x)dxds \\ &= \int_0^\tau \int_\Omega n_j(x,s)\psi(x)dxds - \tau\overline{n_0} \cdot \int_\Omega \psi(x)dx \qquad \text{for all } j \in \mathbb{N},\end{aligned} \tag{5.15}$$

where as a consequence of (5.12) and (5.13),

$$\begin{aligned}\int_0^\tau \int_\Omega n_j(x,s)\psi(x)dxds &= \int_0^\tau \int_\Omega z_j(x,s)\psi(x)dxds + \int_0^\tau \int_\Omega a_j(s)\psi(x)dxds \\ &= \int_0^\tau \int_\Omega z_j(x,s)\psi(x)dxds + \Big(\int_0^\tau a_j(s)\Big) \cdot \Big(\int_\Omega \psi(x)dx\Big) \\ &\to \Big(\int_0^\tau a_\infty(s)\Big) \cdot \Big(\int_\Omega \psi(x)dx\Big) \qquad \text{as } j \to \infty.\end{aligned}$$

Taking $j \to \infty$ in (5.15) we thus arrive at the conclusion that

$$\frac{C_1\tau}{4} \le \Big(\int_0^\tau a_\infty(s)\Big) \cdot \Big(\int_\Omega \psi(x)dx\Big) - \tau\overline{n_0} \cdot \int_\Omega \psi(x)dx,$$

which in light of (5.14) is absurd and hence proves that actually (5.1) is valid. □

Remark. Let us mention here that in the case $m < 2$ one may alternatively prove Lemma 5.1 by invoking standard Hölder estimates for solutions of degenerate parabolic equations ([22]): In fact, based on our previous estimates we can in this case obtain a bound for n, independent of $t > 1$, in the space $C^{\theta,\frac{\theta}{2}}(\bar\Omega \times [t,t+1])$ for some $\theta > 0$. This can be combined with the outcome of Lemma 3.21 to see that actually $n(\cdot,t) \to \overline{n_0}$ in $L^\infty(\Omega)$ as $t \to \infty$.

We next make essential use of the fact that f does not have positive zeroes to verify that c decays in the claimed sense.

Lemma 5.2 *Let $m > \frac{7}{6}$, and assume (1.17). Then the solution of (1.2) constructed in Theorem 1.1 satisfies*

$$c(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{5.16}$$

PROOF. If the claim was false, the for some $C_1 > 0$, some $(x_j)_{j\in\mathbb{N}} \subset \Omega$ and some $(t_j)_{j\in\mathbb{N}} \subset (0,\infty)$ such that $t_j \to \infty$ as $j \to \infty$ we would have

$$c(x_j,t_j) \geq C_1 \qquad \text{for all } j \in \mathbb{N},$$

where passing to subsequences we may assume that there exists $x_0 \in \bar{\Omega}$ such that $x_j \to \infty$ as $j \to \infty$. Since c is uniformly continuous in $\bigcup_{j\in\mathbb{N}} \left(\bar{\Omega} \times [t_j, t_j+1]\right)$ by Lemma 3.18, this entails that on extracting a further subsequence if necessary, we can find $\delta > 0$ and $\tau \in (0,1)$ such that with $B := B_\delta(x_0) \cap \Omega$ we have

$$c(x,t) \geq \frac{C_1}{2} \qquad \text{for all } x \in B, t \in (t_j, t_j+\tau) \text{ and } j \in \mathbb{N},$$

so that since $f > 0$ on $(0,\infty)$ by assumption (1.17), we see that

$$f(c(x,t)) \geq C_2 \qquad \text{for all } x \in B, t \in (t_j, t_j+\tau) \text{ and } j \in \mathbb{N} \tag{5.17}$$

with some $C_2 > 0$. Now writing

$$n_j(x,s) := n(x, t_j + s) \quad \text{and} \quad c_j(x,s) := c(x, t_j+s)$$

for $x \in \Omega, s \in (0,\tau)$ and $j \in \mathbb{N}$, from Lemma 3.20 we obtain that

$$\begin{aligned} \int_0^\tau \int_B n_j(x,s) f(c_j(x,s)) dx ds &= \int_{t_j}^{t_j+\tau} \int_B n(x,t) f(c(x,t)) dx dt \\ &\leq \int_{t_j}^{\infty} \int_B n(x,t) f(c(x,t)) dx dt \\ &\to 0 \qquad \text{as } j \to \infty. \end{aligned} \tag{5.18}$$

On the other hand, if we let $\psi(x) := \chi_B(x)$ for $x \in \Omega$, then from Lemma 5.1 we obtain that

$$\begin{aligned} \left| \int_{t_j}^{t_j+\tau} \int_\Omega n(x,t)\psi(x) dx dt - \overline{n_0}\tau|B| \right| &= \left| \int_{t_j}^{t_j+\tau} \left\{ \int_\Omega n(x,t)\psi(x)dx - \int_\Omega \overline{n_0}\psi(x)dx \right\} dt \right| \\ &\leq \tau \cdot \sup_{t\in(t_j,t_j+\tau)} \left| \int_\Omega n(x,t)\psi(x)dx - \int_\Omega \overline{n_0}\psi(x)dx \right| \\ &\to 0 \qquad \text{as } j \to \infty, \end{aligned}$$

and that hence

$$\int_0^\tau \int_B n_j(x,s) dx ds \to \overline{n_0}\tau|B| \qquad \text{as } j \to \infty. \tag{5.19}$$

Therefore, (5.17) warrants that the expression on the left of (5.18) actually satisfies

$$\begin{aligned} \liminf_{j\to\infty} \int_0^\tau \int_B n_j(x,s) f(c_j(x,s)) dx ds &\geq \liminf_{j\to\infty} \left\{ C_2 \int_0^\tau \int_B n_j(x,s) dx ds \right\} \\ &= C_2 \cdot \overline{n_0}\tau|B|, \end{aligned}$$

which contradicts (5.18) and hence proves the lemma. $\square$

Finally, decay of u will be a consequence of the stabilization property of n in Lemma 5.1. Since the latter has been asserted in the weak-$\star$ sense in $L^\infty(\Omega)$ only, an argument based on the use of appropriate linear functionals involving u seems adequate to derive this here.

Lemma 5.3 *Let $m > \frac{7}{6}$. Then with (n, c, u) as given by Theorem 1.1, we have*

$$u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{5.20}$$

Proof. Since the Hilbert space realization A_2 of the Stokes operator is positive and self-adjoint in $L^2_\sigma(\Omega)$ with compact inverse, there exists a complete orthonormal basis $(\psi_k)_{k\in\mathbb{N}}$ of eigenfunctions ψ_k of A_2 corresponding to positive eigenvalues λ_k, $k \in \mathbb{N}$. By density of $\bigcup_{N\in\mathbb{N}} \mathrm{span}\{\psi_k \mid k \le N\}$ in $L^2_\sigma(\Omega)$, in view of the uniform Hölder continuity of u in $\Omega \times (0, \infty)$, as asserted by Lemma 3.19 and Lemma 4.1, to prove (5.20) it is sufficient to show that for each $k \in \mathbb{N}$ we have

$$\int_\Omega u(x, t) \cdot \psi_k(x) dx \to 0 \qquad \text{as } k \to \infty. \tag{5.21}$$

For this purpose, we fix any such k and let

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon(x, t) \cdot \psi_k(x) dx, \qquad t \ge 0,$$

for $\varepsilon \in (0, 1)$. Then from (2.2) and the eigenfunction property of ψ_k we obtain that since $\nabla \cdot \psi_k \equiv 0$ and $\int_\Omega n_\varepsilon(\cdot, t) \equiv \overline{n_0}$ by (2.4),

$$\begin{aligned}
y'_\varepsilon(t) &= -\int_\Omega Au_\varepsilon \cdot \psi_k + \int_\Omega n_\varepsilon \nabla\phi \cdot \psi_k \\
&= -\int_\Omega u_\varepsilon \cdot A\psi_k + \int_\Omega (n_\varepsilon - \overline{n_0})\nabla\phi \cdot \psi_k + \overline{n_0} \int_\Omega \nabla\phi \cdot \psi_k \\
&= -\lambda_k \int_\Omega u_\varepsilon \cdot \psi_k + \int_\Omega (n_\varepsilon - \overline{n_0})\nabla\phi \cdot \psi_k \\
&= -\lambda_k y_\varepsilon(t) + g_\varepsilon(t) \qquad \text{for all } t > 0
\end{aligned}$$

with

$$g_\varepsilon(t) := \int_\Omega \Big(n_\varepsilon(x, t) - \overline{n_0}\Big)\nabla\phi \cdot \psi_k(x) dx, \qquad t \ge 0.$$

Upon integration, this shows that for any choice of $t_0 \ge 0$,

$$y_\varepsilon(t) = y_\varepsilon(t_0)e^{-\lambda_k(t-t_0)} + \int_{t_0}^t e^{-\lambda_k(t-s)} g_\varepsilon(s) ds \qquad \text{for all } t > t_0.$$

Since as $\varepsilon = \varepsilon_j \searrow 0$ we have $u_\varepsilon \to u$ in $C^0_{loc}(\bar\Omega \times [0, \infty))$ and $n_\varepsilon \overset{\star}{\rightharpoonup} n$ in $L^\infty(\Omega \times (0, \infty))$ by Lemma 4.1, we may take $\varepsilon = \varepsilon_j \searrow 0$ here to infer that with

$$y(t) := \int_\Omega u(x, t) \cdot \psi_k(x) dx \quad \text{and} \quad g(t) := \int_\Omega \Big(n(x, t) - \overline{n_0}\Big)\nabla\phi \cdot \psi_k(x) dx, \qquad t \ge 0,$$

we have

$$y(t) = y(t_0)e^{-\lambda_k(t-t_0)} + \int_{t_0}^t e^{-\lambda_k(t-s)} g(s)ds \qquad \text{for all } t_0 \geq 0 \text{ and any } t > t_0, \tag{5.22}$$

where thanks to Lemma 5.1 we know that

$$g(t) \to 0 \qquad \text{as } t \to \infty.$$

Accordingly, if in order to prove (5.21) we let $\delta > 0$ be given, then we can pick $t_0 > 0$ large enough fulfilling

$$|g(t)| < \frac{\lambda_k \delta}{2} \qquad \text{for all } t > t_0. \tag{5.23}$$

As $C_1 := \|y\|_{L^\infty((0,\infty))}$ is finite due to the boundedness of u in $\Omega \times (0,\infty)$ guaranteed by Lemma 3.17 and Lemma 4.1, from (5.22) we thus infer that

$$\begin{aligned} |y(t)| &\leq C_1 e^{-\lambda_k(t-t_0)} + \frac{\lambda_k \delta}{2} \int_{t_0}^t e^{-\lambda_k(t-s)} ds \\ &= C_1 e^{-\lambda_k(t-t_0)} + \frac{\lambda_k \delta}{2} \cdot \frac{1 - e^{-\lambda_k(t-t_0)}}{\lambda_k} \\ &< C_1 e^{-\lambda_k(t-t_0)} + \frac{\delta}{2} \qquad \text{for all } t > t_0. \end{aligned}$$

This implies that with $t_1 := \max\{t_0, t_0 + \frac{1}{\lambda_k} \ln \frac{2C_1}{\delta}\}$ we have

$$|y(t)| < \delta \qquad \text{for all } t > t_1,$$

which establishes (5.21) and thereby completes the proof. □

Proof of Theorem 1.2. The claimed convergence properties are precisely asserted by Lemma 5.1, Lemma 5.2 and Lemma 5.3. □

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