Stabilization in a higher-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity

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Abstract

The quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with smooth boundary, where the focus is on cases when herein the diffusivity $D(s)$ decays exponentially as $s \to \infty$.

It is shown that under the subcriticality condition that

$$\frac{S(s)}{D(s)} \leq Cs^\alpha \qquad \text{for all } s \geq 0 \tag{0.1}$$

with some $C > 0$ and $\alpha < \frac{2}{n}$, for all suitably regular initial data satisfying an essentially explicit smallness assumption on the total mass $\int_\Omega u_0$, the corresponding Neumann initial-boundary value problem for $(\star)$ possesses a globally defined bounded classical solution which moreover approaches a spatially homogeneous steady state in the large time limit. Viewed as a complement of known results on the existence of small-mass blow-up solutions in cases when in (0.1) the reverse inequality holds with some $\alpha > \frac{2}{n}$, this confirms criticality of the exponent $\alpha = \frac{2}{n}$ in (0.1) with regard to the singularity formation also for arbitrary $n \geq 2$, thereby generalizing a recent result on unconditional global boundedness in the two-dimensional situation.

As a by-product of our analysis, without any restriction on the initial data, we obtain boundedness and stabilization of solutions to a so-called volume-filling chemotaxis system involving jump probability functions which decay at sufficiently large exponential rates.

Key words: chemotaxis, degenerate diffusion, global existence, boundedness
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1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the quasilinear parabolic problem

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

which is used in mathematical biology to describe the evolution of bacterial populations, at density denoted by $u = u(x,t)$, in response to a chemical signal, at concentration $v = v(x,t)$, produced by themselves. In generalization of the classical Keller-Segel chemotaxis system obtained upon the particular choices

$$D \equiv 1 \qquad \text{and} \qquad S(s) = s, \quad s \geq 0, \tag{1.3}$$

the model (1.2) may account for various types of nonlinear diffusion and cross-diffusion mechanisms, where especially saturation effects at large cell densities appear to play a predominant role in refined modeling approaches ([10], [18]); accordingly, in such contexts it will be of particular interest to determine how far choices of D and S substantially below those in (1.3) may still lead to singularity formation in (1.2), as known to occur for large classes of initial data in the Keller-Segel system (1.2)-(1.3) when either $n \geq 3$ ([16]), or $n = 2$ and the total mass of cells is suitably large ([5], [9]).

In this respect, previous results indicate that a certain dimension-dependent power-type asymptotic behavior of the ratio $\frac{S(s)}{D(s)}$ for large values of s should be critical: It is known, for instance, that if D and S are sufficiently smooth functions on $[0,\infty)$ such that $D > 0$ on $(0,\infty)$ and

$$\liminf_{s\to\infty} \frac{s\left(\frac{S}{D}\right)'(s)}{\left(\frac{S}{D}\right)(s)} > \frac{2}{n}, \tag{1.4}$$

then still some solutions to (1.2) exist which blow up either in finite or infinite time ([15], [2], [3], [7]). On the other hand, any such unboundedness phenomenon is entirely ruled out if with some $\varepsilon > 0$ and $C > 0$ we have

$$\frac{S(s)}{D(s)} \leq C s^{\frac{2}{n}-\varepsilon} \qquad \text{for all } s \geq 1, \tag{1.5}$$

and if in addition D decays at most algebraically in the sense that

$$\liminf_{s\to\infty} \left(s^p D(s)\right) > 0 \tag{1.6}$$

for some $p > 0$ ([14]; cf. also [12] and [7] for some precedents).

In cases when D decays substantially faster, however, the literature apparently provides only quite few rigorous results on (1.2) for subcritical behavior of $\frac{S}{D}$ in the sense that e.g. (1.5) holds; this may be viewed as reflecting the circumstance that straightforward adaptations of standard regularity techniques, based e.g. on iterative arguments of Moser or DeGiorgi type, seem inappropriate in such situations. Correspondingly, the only results available so far seem to concentrate on the mere questions of global solvability without asserting boundedness ([1], [2, Theorem 1.6], [17]), or are restricted to the particular spatially two-dimensional setting with exponentially decaying D, in which the Moser-Trudinger inequality can be used to firstly derive global bounds for e^u in $L^p(\Omega)$ for some $p > 0$, from which global boundedness of arbitrary classical solutions can be obtained by means of an iterative argument ([4]).

It is the goal of the present work to provide some further rigorous evidence indicating that also in higher-dimensional situations, the power-type asymptotic behavior $\frac{S(s)}{D(s)} \simeq s^{\frac{2}{n}}$ indeed is critical regarding the global existence and boundedness in (1.2) with rapidly decreasing diffusivities, at least in cases when this decay occurs at exponential rates. To this end, we shall consider (1.2) under the assumptions that

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) \quad \text{with } u_0 > 0 \text{ in } \bar{\Omega} \qquad \text{and} \\ v_0 \in W^{1,\vartheta}(\Omega) \text{ for some } \vartheta > 2 \quad \text{with } v_0 \geq 0 \text{ in } \Omega, \end{cases} \tag{1.7}$$

and that

$$\begin{cases} D \in C^2([0,\infty)) \quad \text{is positive and} \\ S \in C^2([0,\infty)) \quad \text{is nonnegative with } S(0) = 0. \end{cases} \tag{1.8}$$

Moreover, we shall suppose that there exist constants $\beta^+ > 0, \beta^- \geq \beta^+$, $K_1 > 0$ and $K_2 > 0$ such that

$$K_1 e^{-\beta^- s} \leq D(s) \leq K_2 e^{-\beta^+ s} \qquad \text{for all } s \geq 0, \tag{1.9}$$

and that

$$\frac{S(s)}{D(s)} \leq K_3 s^\alpha \qquad \text{for all } s \geq 0 \tag{1.10}$$

with some $K_3 > 0$ and $\alpha \in [0, \frac{2}{n})$.

In order to precisely state our main results, let us recall that since $\alpha < \frac{2}{n}$ and hence $W^{1,\frac{2}{\alpha+1}}(\Omega) \hookrightarrow L^2(\Omega)$, a corresponding Poincaré-Sobolev inequality provides $K_P(\alpha) > 0$ such that

$$\|\varphi - \overline{\varphi}\|_{L^2(\Omega)} \leq K_P(\alpha) \|\nabla \varphi\|_{L^{\frac{2}{\alpha+1}}(\Omega)} \qquad \text{for all } \varphi \in W^{1,\frac{2}{\alpha+1}}(\Omega), \tag{1.11}$$

where, as throughout the sequel, by $\overline{\varphi} := \frac{1}{|\Omega|} \int_\Omega \varphi$ we denote the spatial average of arbitrary functions $\varphi \in L^1(\Omega)$.

Our main result then asserts that indeed in the full subcritical range of $\frac{S}{D}$ consistent with (1.5), solutions do not only exist globally and remain bounded, but moreover even stabilize toward a spatially homogeneous equilibrium, provided that a smallness condition on the total mass $\int_\Omega u_0$ is satisfied which is essentially explicit by merely involving the constant K_P from (1.11) beyond the parameters from (1.10).

Theorem 1.1 *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let u_0 and v_0 be compatible with (1.7), and assume that D and S are such that (1.8) holds and that there exist positive constants K_1, K_2 and K_3 with the property that (1.9) and (1.10) are valid with some $\beta^+ > 0, \beta^- \geq \beta^+$ and*

$$\alpha \in \Big[0, \frac{2}{n}\Big).$$

Then if furthermore

$$\left\{ \int_\Omega u_0 \right\}^\alpha < \frac{1}{K_3 K_P^2(\alpha)}, \tag{1.12}$$

the problem (1.2) possesses a uniquely determined global classical solution (u, v) *with*

$$\begin{cases} u \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ v \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\vartheta}(\Omega)), \end{cases} \tag{1.13}$$

such that both u *and* v *are nonnegative in* $\Omega \times (0,\infty)$*. This solution is bounded in the sense that there exists* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0, \tag{1.14}$$

and moreover we have

$$u(\cdot,t) \to \overline{u_0} \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty \tag{1.15}$$

and

$$v(\cdot,t) \to \overline{u_0} \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty. \tag{1.16}$$

In order to put our present results in perspective, let us recall that the above assumption that (1.10) holds with some $\alpha < \frac{2}{n}$ cannot substantially be relaxed if global existence of bounded solutions is required for widely arbitrary choices of initial data. More precisely, inter alia the following is implied by [15, Theorem 5.1] in conjunction with e.g. Lemma 2.1 and [4, Lemma 2.3].

Theorem A *Let* $n \ge 2$ *and* $\Omega \subset \mathbb{R}^n$ *be a ball. Suppose that* D *and* S *satisfy (1.8) as well as (1.4). Then for any choice of* $m > 0$*, there exist nonnegative and radially symmetric functions* $u_0 \in C^\infty(\bar\Omega)$ *and* $v_0 \in C^\infty(\bar\Omega)$ *such that* $\int_\Omega u_0 = m$*, and such that (1.2) possesses a classical solution which blows up either in finite or in infinite time in the sense that there exists* $T_{max} \in (0,\infty]$ *with the property that both* u *and* v *belong to* $C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max}))$*, but that* $\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$*.*

In contrast to this, our present result states that also in higher dimensions subcritical growth of $\frac{S}{D}$ in the sense that (1.10) is valid with some $\alpha < \frac{2}{n}$, plays a critical role: Namely, blow-up phenomena do *not* occur at arbitrarily small levels of the total population mass, as underlined in the following.

Corollary 1.2 *Let* $n \ge 2$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary. Suppose that* D *and* S *comply with (1.8), and that there exist positive constants* K_1, K_2 *and* K_3 *with the property that (1.9) and (1.10) are valid with some* $\beta^+ > 0, \beta^- \ge \beta^+$ *and* $\alpha \in (0, \frac{2}{n})$*. Then there exists* $\varepsilon > 0$ *such that whenever* u_0 *and* v_0 *satisfy (1.7) as well as*

$$\int_\Omega u_0 < \varepsilon, \tag{1.17}$$

the problem (1.2) possesses a uniquely determined global classical solution (u, v) *fulfilling (1.13) which has the boundedness and convergence properties (1.14), (1.15) and (1.16).*

Proof of Corollary 1.2. In view of Theorem 1.1, the desired conclusion holds if we let

$$\varepsilon := \Big(K_3 K_P(\alpha)\Big)^{\frac{1}{\alpha}},$$

for instance. $\square$

Let us finally focus on the particular version of (1.2) obtained by choosing D and S in dependence on a given function Q with $Q(u)$ measuring the probability that a cell, when localized at a position (x,t) with population density $u(x,t)$, may find space in some neighboring region; in terms of the function Q, an accordingly modified random walk approach ([10]) suggests the precise functional relationships determined by

$$D(s) = Q(s) - sQ'(s) \quad \text{and} \quad S(s) = sQ(s), \qquad s \geq 0. \tag{1.18}$$

Here choosing

$$Q(s) := e^{-\beta s}, \qquad s \geq 0,$$

for $\beta > 0$, for the corresponding volume-filling chemotaxis system

$$\begin{cases} u_t = \nabla \cdot \Big((1+\beta u) e^{-\beta u} \nabla u \Big) - \nabla \cdot (u e^{-\beta u} \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.19}$$

our analysis will yield the following result on boundedness and stabilization, thus significantly going beyond previous knowledge on global existence ([2, Theorem 1.6]) and on global boundedness in the case $n = 2$ ([4, Theorem 1.2]).

Theorem 1.3 *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that with $K_P := K_P(0)$ taken from (1.11) we have*

$$\beta > K_P^2. \tag{1.20}$$

Then for any (u_0, v_0) fulfilling (1.7), (1.19) possesses a unique global classical solution (u,v) satisfying (1.13) which is bounded in $\Omega \times (0,\infty)$ in that (1.14) holds, and the large time behavior of which is determined by (1.15) and (1.16).

Plan of the paper. At its core, our analysis will be based on the observation that with $\Phi(s) := \int_1^s \int_1^\sigma \frac{1}{S(\xi)} d\xi d\sigma$, $s > 0$, the quantity

$$\mathcal{F}(t) := \int_\Omega \Phi(u(\cdot,t)) + \frac{1}{2} \int_\Omega |\nabla v(\cdot,t)|^2 + \int_\Omega \Big(v(\cdot,t) - \overline{v(\cdot,t)} \Big)^2, \qquad t \geq 0,$$

acts as a genuine Lyapunov functional for (1.2) whenever (1.12) is satisfied (Lemma 3.6). According to (1.10) and (1.9), the global upper bound for $\mathcal{F}$ thereby implied will entail an a priori bound for e^u in $L^\infty((0,\infty); L^\beta(\Omega))$ for some $\beta > 0$ (Lemma 4.1). This will turn out to be sufficient basic regularity information for the verification that each of the functionals

$$\mathcal{G}(t) := \int_\Omega \Psi(u(\cdot,t)), \qquad t \geq 0,$$

where $\Psi(s) := \int_0^s \int_0^\sigma \frac{e^{\gamma\xi}}{D(\xi)} d\xi d\sigma$, $s \ge 0$, with arbitrarily large $\gamma > 0$, enjoys a favorable quasi-energy property ensuring boundedness of e^u even in $L^\infty((0,\infty); L^\beta(\Omega))$ for any $\beta > 0$ (Lemma 4.3). Thereafter, a Moser-type iteration for e^u will show boundedness of e^u and hence of u (Lemma 4.4 and Lemma 5.1), whereupon also higher-order regularity properties become available (Lemma 4.5) which can finally be used to derive the claimed stabilization result from the relaxation process expressed in the energy inequality for $\mathcal{F}$ (see (3.15) and (3.16) and the argument in Section 4.3).

2 Preliminaries

To begin with, let us formulate a basic result from local existence theory which can be derived by well-known arguments (see for instance [7], [13] or [4] and the references therein).

Lemma 2.1 *Suppose that D and S satisfy (1.8) and that u_0 and v_0 fulfill (1.7). Then there exist $T_{max} \in (0,\infty]$ and a unique couple of nonnegative functions*

$$\begin{cases} u \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})), \\ v \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,\vartheta}(\Omega)) \end{cases}$$

such that (u,v) is a classical solution of (1.2) in $\Omega \times (0,T_{max})$, and such that we have the alternative

$$\text{either } T_{max} = \infty, \qquad \text{or} \qquad \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} \Big) = \infty. \tag{2.1}$$

This solution has the following evident mass conservation property.

Lemma 2.2 *The solution of (1.2) satisfies*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t \in (0,T_{max}). \tag{2.2}$$

PROOF. This directly follows on integrating the first equation in (1.2) over $x \in \Omega$. $\square$

3 A conditional energy inequality associated with (1.2)

The goal of this section will be to derive the inequality (3.11) which becomes a genuine energy-type inequality under the condition (1.12). The key toward this will consist in combining information on the time evolution of the functional $\int_\Omega \Phi(u)$ for

$$\Phi(s) := \int_1^s \int_1^\sigma \frac{1}{S(\xi)} d\xi d\sigma, \qquad s \ge 0, \tag{3.1}$$

with appropriate identities gained from testing procedures applied to the second equation in (1.2). In fact, this particular choice of Φ entails the following basic identity in which the integral on the right-hand side reflects the cross-diffusive interaction in (1.2) in a form which will make this term convenient to handle. Let us mention that the construction in this section partially parallels that underlying the stabilization analysis performed in [6] for the classical Keller-Segel system.

Lemma 3.1 *The solution of (1.2) satisfies*

$$\frac{d}{dt}\int_\Omega \Phi(u) + \int_\Omega \frac{D(u)}{S(u)}|\nabla u|^2 = \int_\Omega \nabla u\cdot\nabla v \qquad \textit{for all } t\in(0,T_{max}), \tag{3.2}$$

where Φ is given by (3.1).

Proof. By straightforward computation using the first equation in (1.2), we see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Phi(u) &= \int_\Omega \Phi'(u)u_t\\ &= -\int_\Omega \Phi''(u)D(u)|\nabla u|^2 + \int_\Omega \Phi''(u)S(u)\nabla u\cdot\nabla v \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

which directly yields (3.2) in view of the fact that $\Phi''=\frac{1}{S}$. □

In order to conveniently cope with the integral on the right of (3.2), let us introduce the new dependent variable z by defining

$$z(x,t) := v(x,t) - \overline{v(\cdot,t)} \qquad \text{for } x\in\bar\Omega \text{ and } t\in[0,T_{max}). \tag{3.3}$$

Then clearly $\int_\Omega z(\cdot,t)=0$ for all $t\in(0,T_{max})$, and moreover it follows from (1.2) that z satisfies the Neumann problem

$$\begin{cases} z_t = \Delta z - z + u - \overline{u_0}, & x\in\Omega,\ t\in(0,T_{max}),\\ \frac{\partial z}{\partial\nu}=0, & x\in\partial\Omega,\ t\in(0,T_{max}),\\ z(x,0)=v_0(x)-\overline{v_0}, & x\in\Omega. \end{cases} \tag{3.4}$$

Applying a standard testing procedure to this problem will then indeed involve precisely the coupled quantity appearing in (3.2):

Lemma 3.2 *The function z defined in (3.3) satisfies*

$$\frac{d}{dt}\left\{\frac12\int_\Omega|\nabla z|^2+\frac12\int_\Omega z^2\right\}+\int_\Omega z_t^2 = -\int_\Omega\nabla u\cdot\nabla v-\int_\Omega(u-\overline{u_0})z+\int_\Omega(u-\overline{u_0})^2 \qquad \textit{for all } t\in(0,T_{max}). \tag{3.5}$$

Proof. On testing (3.4) against z_t, we obtain

$$\begin{aligned}\int_\Omega z_t^2 &= \int_\Omega \Delta z\cdot z_t - \int_\Omega zz_t + \int_\Omega (u-\overline{u_0})z_t\\ &= -\frac12\frac{d}{dt}\int_\Omega|\nabla z|^2 - \frac12\frac{d}{dt}\int_\Omega z^2 + \int_\Omega(u-\overline{u_0})z_t \qquad \text{for all } t\in(0,T_{max}),\end{aligned} \tag{3.6}$$

where again by (3.4),

$$\begin{aligned}\int_\Omega (u-\overline{u_0})z_t &= \int_\Omega (u-\overline{u_0})(\Delta z - z + u - \overline{u_0})\\ &= -\int_\Omega \nabla u\cdot\nabla z - \int_\Omega(u-\overline{u_0})z + \int_\Omega(u-\overline{u_0})^2 \qquad \text{for all } t\in(0,T_{max}).\end{aligned}$$

As clearly $\nabla z=\nabla v$, (3.6) thus implies (3.5). □

Now to appropriately rewrite the second integral on the right of (3.5), we test (3.4) by another standard multiplier.

Lemma 3.3 *For z as in (3.3) we have*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2 + \int_\Omega |\nabla z|^2 + \int_\Omega z^2 = \int_\Omega (u-\overline{u_0})z \qquad \textit{for all } t\in(0,T_{max}). \tag{3.7}$$

PROOF. This immediately results from multiplying (3.4) by z and integrating by parts over Ω. $\Box$

A combination of the preceding three lemmata yields the following identity.

Lemma 3.4 *Let Φ be taken from (3.1). Then for the solution of (1.2) we have the identity*

$$\frac{d}{dt}\Big\{\int_\Omega \Phi(u) + \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega (v-\overline{v})^2\Big\} + \int_\Omega \frac{D(u)}{S(u)}|\nabla u|^2 + \int_\Omega |\nabla v|^2 + \int_\Omega (v-\overline{v})^2 + \int_\Omega (v-\overline{v})_t^2$$
$$= \int_\Omega (u-\overline{u_0})^2 \qquad \textit{for all } t\in(0,T_{max}). \tag{3.8}$$

PROOF. We only need to add the identities provided by Lemma 3.1, Lemma 3.2 and Lemma 3.3, and recall that $z=v-\overline{v}$ and hence $\nabla z = \nabla v$. $\Box$

In order to take full advantage of the dissipative action expressed in (3.8), we shall next employ the Hölder inequality to derive the following.

Lemma 3.5 *Under the assumption (1.10), the solution of (1.2) satisfies*

$$\int_\Omega \frac{D(u)}{S(u)}|\nabla u|^2 \ge \frac{1}{K_3\cdot\Big\{\int_\Omega u_0\Big\}^\alpha}\cdot \|\nabla u\|^2_{L^{\frac{2}{\alpha+1}}(\Omega)} \qquad \textit{for all } t\in(0,T_{max}). \tag{3.9}$$

PROOF. According to (1.10), we can estimate

$$\int_\Omega \frac{D(u)}{S(u)}|\nabla u|^2 \ge \frac{1}{K_3}\int_\Omega \frac{|\nabla u|^2}{u^\alpha} \qquad \text{for all } t\in(0,T_{max}), \tag{3.10}$$

whereas on the other hand the Hölder inequality along with (2.2) implies that

$$\begin{aligned}\int_\Omega |\nabla u|^{\frac{2}{\alpha+1}} &= \int_\Omega \Big(\frac{|\nabla u|^2}{u^\alpha}\Big)^{\frac{1}{\alpha+1}}\cdot u^{\frac{\alpha}{\alpha+1}} \\ &\le \Big\{\int_\Omega \frac{|\nabla u|^2}{u^\alpha}\Big\}^{\frac{1}{\alpha+1}}\cdot\Big\{\int_\Omega u\Big\}^{\frac{\alpha}{\alpha+1}} \\ &= \Big\{\int_\Omega \frac{|\nabla u|^2}{u^\alpha}\Big\}^{\frac{1}{\alpha+1}}\cdot\Big\{\int_\Omega u_0\Big\}^{\frac{\alpha}{\alpha+1}} \qquad \text{for all } t\in(0,T_{max}).\end{aligned}$$

Since thus

$$\int_\Omega \frac{|\nabla u|^2}{u^\alpha} \ge \Big\{\int_\Omega u_0\Big\}^{-\alpha}\cdot\Big\{\int_\Omega |\nabla u|^{\frac{2}{\alpha+1}}\Big\}^{\alpha+1} \qquad \text{for all } t\in(0,T_{max}),$$

the inequality (3.9) is a consequence of (3.10). $\Box$

The announced conditional energy inequality, along with some evident consequences thereof, can now be gained on combining Lemma 3.4 with Lemma 3.5.

Lemma 3.6 *Let $\alpha \in [0, \frac{2}{n})$ and $K_3 > 0$ be as in (1.10), and let $K_P(\alpha) > 0$ be taken from (1.11). Then with Φ as defined in (3.1),*

$$\frac{d}{dt}\Big\{ \int_\Omega \Phi(u) + \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega (v-\overline{v})^2 \Big\}$$
$$+\bigg\{ \frac{1}{K_3 K_P^2(\alpha)\cdot \Big\{ \int_\Omega u_0 \Big\}^\alpha} - 1 \bigg\} \cdot \int_\Omega (u-\overline{u_0})^2 + \int_\Omega (v-\overline{v})^2 \tag{3.11}$$
$$\leq \quad 0 \qquad \textit{for all } t \in (0, T_{max}). \tag{3.12}$$

In particular, if

$$\Big\{ \int_\Omega u_0 \Big\}^\alpha < \frac{1}{K_3 K_P^2(\alpha)}, \tag{3.13}$$

then there exists $C > 0$ such that

$$\int_\Omega \Phi(u(\cdot,t)) \leq C \qquad \textit{for all } t \in (0, T_{max}), \tag{3.14}$$

and moreover we have

$$\int_0^{T_{max}} \int_\Omega (u-\overline{u_0})^2 < \infty \tag{3.15}$$

as well as

$$\int_0^{T_{max}} \int_\Omega (v-\overline{v})^2 < \infty. \tag{3.16}$$

PROOF. Combining the outcome of Lemma 3.5 with (1.11) and (2.2), we see that

$$\int_\Omega \frac{D(u)}{S(u)}|\nabla u|^2 \geq \frac{1}{K_3 \cdot \Big\{ \int_\Omega u_0 \Big\}^\alpha} \cdot \frac{1}{K_P^2(\alpha)} \int_\Omega (u-\overline{u_0})^2 \qquad \text{for all } t \in (0, T_{max}),$$

so that Lemma 3.4 in particular implies (3.11). As herein

$$\frac{1}{K_3 K_P^2(\alpha) \cdot \Big\{ \int_\Omega u_0 \Big\}^\alpha} - 1 > 0$$

according to (3.13), on integration this readily yields (3.14), (3.15) and (3.16). $\square$

4 Estimating $\int_\Omega e^{\beta u}$ for arbitrary $\beta > 0$

In this section we shall see that under the assumptions of Theorem 1.1, the function e^u belongs to $L^\infty((0,T_{max}); L^\beta(\Omega))$ for arbitrary $\beta > 0$. This will be achieved in Lemma 4.3 by means of another testing procedure in the first equation of (1.2) on the basis of a corresponding estimate in $L^{\beta_0}(\Omega)$ for some suitably small $\beta_0 > 0$ (Lemma 4.1) when combined with a pointwise boundedness property of ∇v thereby implied (Lemma 4.2).

4.1 An estimate for $\int_\Omega e^{\beta u}$ with some $\beta > 0$

Let us first make sure that according to our assumptions on D and S, the boundedness property (3.14) implies that e^u lies in $L^\infty((0,T_{max});L^\beta(\Omega))$ for all sufficiently small $\beta > 0$.

Lemma 4.1 *Suppose that the assumptions of Theorem 1.1 as well as (3.13) hold. Then for all $\beta \in (0,\beta^+)$ there exists $C = C(\beta) > 0$ such that*

$$\int_\Omega e^{\beta u(\cdot,t)} \le C \qquad \textit{for all } t \in (0,T_{max}). \tag{4.1}$$

PROOF. Given $\beta \in (0,\beta^+)$, we can fix $c_1 > 0$ such that

$$s^\alpha e^{-\beta^+ s} \le c_1 e^{-\beta s} \quad \text{for all } s \ge 0,$$

so that combining (1.10) with the right inequality in (1.9) shows that

$$\begin{aligned} S(s) &\le K_3 s^\alpha D(s) \\ &\le K_2 K_3 s^\alpha e^{-\beta^+ s} \\ &\le K_2 K_3 c_1 e^{-\beta s} \qquad \text{for all } s \ge 0. \end{aligned}$$

Therefore, for Φ as in (3.1) we have

$$\begin{aligned} \Phi(s) &= \int_1^s \int_1^\sigma \frac{1}{S(\xi)} d\xi d\sigma \\ &\ge \frac{1}{K_2K_3c_1} \int_1^s \int_1^\sigma e^{\beta\xi} d\xi d\sigma \\ &= \frac{1}{K_2K_3c_1} \cdot \left\{ \frac{1}{\beta^2} \cdot e^{\beta s} - \frac{e^\beta}{\beta} \cdot s - \frac{e^\beta}{\beta^2} + \frac{e^\beta}{\beta} \right\} \qquad \text{for all } s \ge 0 \end{aligned}$$

and hence

$$\int_\Omega e^{\beta u} \le K_2K_3c_1\beta^2 \int_\Omega \Phi(u) + \beta e^\beta \int_\Omega u + e^\beta |\Omega| \qquad \text{for all } t \in (0,T_{max}),$$

so that (4.1) is a consequence of Lemma 3.6 and (2.2). □

4.2 Bounds for ∇v in L^∞ and for $\int_\Omega e^{\beta u}$ with arbitrary $\beta > 0$

As a straightforward by-product, on choosing any $\beta \in (0,\beta^+)$ and using that then $0 \le s \mapsto e^{\beta s}$ grows faster than any algebraic function, from Lemma 4.1 together with a standard argument from parabolic regularity theory we obtain the following (see the reasoning e.g. in [7, Lemma 4.1])

Lemma 4.2 *Under the assumptions of Theorem 1.1 one can find $C > 0$ such that*

$$|\nabla v(x,t)| \le C \qquad \textit{for all } x \in \Omega \textit{ and } t \in (\tau, T_{max}), \tag{4.2}$$

where $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

With this information together with that of Lemma 4.1 at hand, by tracking the time evolution of another family of exponentially growing functionals of u we can now establish the following extension of Lemma 4.1.

Lemma 4.3 *Let the assumptions of Theorem 1.1 be satisfied. Then for all $\beta > 0$ there exists $C(\beta) > 0$ with the property that*

$$\int_\Omega e^{\beta u(\cdot,t)} \le C(\beta) \qquad \text{for all } t \in (\tau, T_{max}), \tag{4.3}$$

where again $\tau := \min\{1, \frac{1}{2}T_{max}\}$.

Proof. According to Lemma 4.1 and Lemma 4.2, we can find $\beta_0 > 0, c_1 > 0$ and $c_2 > 0$ such that

$$\int_\Omega e^{\beta_0 u} \le c_1 \qquad \text{for all } t \in (0, T_{max}) \tag{4.4}$$

and

$$|\nabla v(x,t)| \le c_2 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}), \tag{4.5}$$

and for the proof of the lemma it is evidently sufficient to restrict our considerations to the case when $\beta > 0$ is large enough fulfilling

$$\beta > \max\Big\{\beta_0 + \beta^+ \,,\, \frac{(n-2)\beta^-}{2} + \beta^+\Big\}, \tag{4.6}$$

which in particular ensures that $\gamma := \beta - \beta^+$ is positive. Writing

$$\Psi(s) := \int_0^s \int_0^\sigma \frac{e^{\gamma\xi}}{D(\xi)} d\xi d\sigma, \qquad s \ge 0, \tag{4.7}$$

from the first equation in (1.2) and the fact that $\Psi''(s) = \frac{e^{\gamma s}}{D(s)}$ for all $s \ge 0$, we obtain on using Young's inequality that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Psi(u) &= \int_\Omega \Psi'(u)u_t \\
&= -\int_\Omega \Psi''(u)D(u)|\nabla u|^2 + \int_\Omega \Psi''(u)S(u)\nabla u\cdot\nabla v \\
&= -\int_\Omega e^{\gamma u}|\nabla u|^2 + \int_\Omega e^{\gamma u}\frac{S(u)}{D(u)}\nabla u\cdot\nabla v \\
&\le -\frac{1}{2}\int_\Omega e^{\gamma u}|\nabla u|^2 + \frac{1}{2}\int_\Omega e^{\gamma u}\Big(\frac{S(u)}{D(u)}\Big)^2|\nabla v|^2 \\
&= -\frac{2}{\gamma^2}\int_\Omega \Big|\nabla e^{\frac{\gamma}{2}u}\Big|^2 + \frac{1}{2}\int_\Omega e^{\gamma u}\Big(\frac{S(u)}{D(u)}\Big)^2|\nabla v|^2 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{4.8}$$

In order to estimate the rightmost summand herein, let us fix $\delta > 0$ small such that

$$\delta < \min\Big\{\frac{2\gamma}{n-2}\,,\,\frac{2\beta_0}{n}\Big\} \tag{4.9}$$

and thereafter pick $c_3 > 0$ large enough fulfilling

$$s^{2\alpha} e^{\gamma s} \le c_3 e^{(\gamma+\delta)s} \qquad \text{for all } s \ge 0, \tag{4.10}$$

so that using (1.9) and recalling (4.5) we see that

$$\begin{aligned} \frac{1}{2} \int_\Omega e^{\gamma u} \Big(\frac{S(u)}{D(u)} \Big)^2 |\nabla v|^2 &\le \frac{c_2^2 K_3^2}{2} \int_\Omega e^{\gamma u} \cdot u^{2\alpha} \\ &\le \frac{c_2^2 c_3 K_3^2}{2} \int_\Omega e^{(\gamma+\delta)u} \qquad \text{for all } t \in (\tau, T_{max}). \end{aligned} \tag{4.11}$$

We next observe that by the first restriction contained in (4.6) we have $\gamma > \beta_0$ and hence

$$\frac{2(\gamma+\delta)}{\gamma} > \frac{2\beta_0}{\gamma}, \tag{4.12}$$

and that moreover in the case $n \ge 3$ we know from (4.9) that

$$\frac{\frac{2(\gamma+\delta)}{\gamma}}{\frac{2n}{n-2}} = \frac{(n-2)(\gamma+\delta)}{n\gamma} < \frac{(n-2) \cdot \Big(\gamma + \frac{2\gamma}{n-2}\Big)}{n\gamma} = 1$$

and that thus

$$\frac{2(\gamma+\delta)}{\gamma} < \frac{2n}{n-2} \tag{4.13}$$

holds for any $n \ge 2$. Now thanks to (4.12) and (4.13) we may invoke the Gagliardo-Nirenberg inequality to find $c_4 > 0$ such that for all $t \in (0, T_{max})$

$$\begin{aligned} \frac{c_2^2 c_3 K_3^2}{2} \int_\Omega e^{(\gamma+\delta)u} &= \frac{c_2^2 c_3 K_3^2}{2} \|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2(\gamma+\delta)}{\gamma}}(\Omega)}^{\frac{2(\gamma+\delta)}{\gamma}} \\ &\le c_4 \|\nabla e^{\frac{\gamma}{2}u}\|_{L^2(\Omega)}^{\frac{2(\gamma+\delta)}{\gamma}\cdot a} \|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2\beta_0}{\gamma}}(\Omega)}^{\frac{2(\gamma+\delta)}{\gamma}\cdot(1-a)} + c_4 \|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2\beta_0}{\gamma}}(\Omega)}^{\frac{2(\gamma+\delta)}{\gamma}} \end{aligned} \tag{4.14}$$

with $a \in (0,1)$ given by

$$a = \frac{\frac{n\gamma}{2\beta_0} - \frac{n\gamma}{2(\gamma+\delta)}}{1 - \frac{n}{2} + \frac{n\gamma}{2\beta_0}},$$

where we note that since $\gamma > \beta_0$ by (4.6), we have $1 - \frac{n}{2} + \frac{n\gamma}{2\beta_0} > 1 > 0$. As moreover the second condition on δ entailed by (4.9) guarantees that $\theta := \frac{\gamma+\delta}{\gamma} \cdot a$ satisfies

$$\begin{aligned} \Big(1 - \frac{n}{2} + \frac{n\gamma}{2\beta_0}\Big) \cdot (\theta - 1) &= \Big(\frac{n(\gamma+\delta)}{2\beta_0} - \frac{n}{2}\Big) - \Big(1 - \frac{n}{2} + \frac{n\gamma}{2\beta_0}\Big) \\ &= \frac{n\delta}{2\beta_0} - 1 < 0 \end{aligned}$$

and hence $\theta < 1$, observing that

$$\|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2\beta_0}{\gamma}}(\Omega)}^{\frac{2\beta_0}{\gamma}} = \int_\Omega e^{\beta_0 u} \le c_1 \qquad \text{for all } t \in (0, T_{max}) \tag{4.15}$$

by (4.4), we may employ Young's inequality to infer from (4.14) that with some $c_5 > 0$ we have

$$\begin{aligned} \frac{c_2^2 c_3 K_3^2}{2} \int_\Omega e^{(\gamma+\delta)u} &\le c_1^{\frac{(\gamma+\delta)(1-a)}{\beta_0}} c_4 \|\nabla e^{\frac{\gamma}{2}u}\|_{L^2(\Omega)}^{2\theta} + c_1^{\frac{\gamma+\delta}{\beta_0}} c_4 \\ &\le \frac{1}{\gamma^2} \int_\Omega \left|\nabla e^{\frac{\gamma}{2}u}\right|^2 + c_5 \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{4.16}$$

We next go back to the definition (4.7) of Ψ to see that according to (1.9),

$$\begin{aligned} \Psi(s) &\le \frac{1}{K_1} \int_0^s \int_0^\sigma e^{(\gamma+\beta^-)\xi} d\xi d\sigma \\ &= \frac{1}{K_1(\gamma+\beta^-)^2} \cdot e^{(\gamma+\beta^-)s} - \frac{1}{K_1(\gamma+\beta^-)} \cdot s - \frac{1}{K_1(\gamma+\beta^-)^2} \\ &\le \frac{1}{K_1(\gamma+\beta^-)^2} \cdot e^{(\gamma+\beta^-)s} \qquad \text{for all } s \ge 0. \end{aligned}$$

Next we recall that in view of the second restriction in (4.6)

$$\beta > \frac{n}{2}\beta^- - \big(\beta^- - \beta^+\big) > \frac{n}{2}\beta^-$$

and hence by the definition of γ, $\gamma + \beta^- = \beta > \frac{n}{2}\beta^-$, so that

$$\gamma > \frac{n-2}{2}\beta^-,$$

which implies

$$\frac{\frac{2(\gamma+\beta^-)}{\gamma}}{\frac{2n}{n-2}} = \frac{(n-2)(\gamma+\beta^-)}{n\gamma} = \frac{(n-2)\Big(1+\frac{\beta^-}{\gamma}\Big)}{n} < \frac{(n-2)\Big(1+\frac{\beta^-}{\frac{(n-2)\beta^-}{2}}\Big)}{n} = 1 \qquad \text{when } n \ge 3.$$

Moreover due to $\gamma + \beta^- > \gamma > \beta_0$ we have

$$\frac{2\beta_0}{\gamma} < \frac{2(\gamma+\beta^-)}{\gamma} < \frac{2n}{n-2}$$

for any $n \ge 2$, so that we may once more employ the Gagliardo-Nirenberg inequality to find $c_6 > 0$ satisfying

$$\begin{aligned} \int_\Omega \Psi(u) &\le \frac{1}{K_1(\gamma+\beta^-)^2} \|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2(\gamma+\beta^-)}{\gamma}}(\Omega)}^{\frac{2(\gamma+\beta^-)}{\gamma}} \\ &\le c_6 \|\nabla e^{\frac{\gamma}{2}u}\|_{L^2(\Omega)}^{\frac{2(\gamma+\beta^-)}{\gamma}\cdot b} \|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2\beta_0}{\gamma}}(\Omega)}^{\frac{2(\gamma+\beta^-)}{\gamma}\cdot(1-b)} + c_6 \|e^{\frac{\gamma}{2}u}\|_{L^{\frac{2\beta_0}{\gamma}}(\Omega)}^{\frac{2(\gamma+\beta^-)}{\gamma}} \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

where

$$b := \frac{\frac{n\gamma}{2\beta_0} - \frac{n\gamma}{2(\gamma+\beta^-)}}{1 - \frac{n}{2} + \frac{n\gamma}{2\beta_0}} \in (0,1).$$

Again in view of (4.15), this yields $c_7 > 0$ such that with $\kappa := \frac{(\gamma+\beta^-)b}{\gamma}$ we have

$$\int_\Omega \Psi(u) \le c_7 \|\nabla e^{\frac{\gamma}{2}u}\|_{L^2(\Omega)}^{2\kappa} + c_7 \qquad \text{for all } t \in (0, T_{max})$$

and hence

$$\begin{aligned} \int_\Omega \left|\nabla e^{\frac{\gamma}{2}u}\right|^2 &\ge \left\{\frac{1}{c_7}\int_\Omega \Psi(u) - 1\right\}^{\frac{1}{\kappa}} \\ &\ge \frac{1}{(2c_7)^{\frac{1}{\kappa}}} \left\{\int_\Omega \Psi(u)\right\}^{\frac{1}{\kappa}} - 1 \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{4.17}$$

because $(A-b)_+^{\frac{1}{\kappa}} \ge 2^{-\frac{1}{\kappa}} A^{\frac{1}{\kappa}} - B^{\frac{1}{\kappa}}$ for all $A \ge 0$ and $B \ge 0$.

We now collect (4.11), (4.16) and (4.17) to conclude from (4.8) that if we abbreviate $c_8 := \frac{1}{(2c_7)^{\frac{1}{\kappa}}\gamma^2}$ and $c_9 := c_5 + \frac{1}{\gamma^2}$, then $y(t) := \int_\Omega \Psi(u(\cdot,t))$, $t \in [\tau, T_{max})$, satisfies

$$y'(t) + c_8 y^{\frac{1}{\kappa}}(t) \le c_9 \qquad \text{for all } t \in (\tau, T_{max}),$$

which by an ODE comparison implies that

$$y(t) \le c_{10} := \max\left\{y(\tau)\,,\, \left(\frac{c_9}{c_8}\right)^\kappa\right\} \qquad \text{for all } t \in [\tau, T_{max}). \tag{4.18}$$

As by (4.7) and (1.9) we have

$$\begin{aligned} \Psi(s) &\ge \frac{1}{K_5}\int_0^s \int_0^\sigma e^{(\gamma+\beta^+)\xi} d\xi d\sigma \\ &= \frac{1}{K_2(\gamma+\beta^+)^2} \cdot e^{(\gamma+\beta^+)s} - \frac{1}{K_2(\gamma+\beta^+)} \cdot s - \frac{1}{K_2(\gamma+\beta^+)^2} \qquad \text{for all } s \ge 0, \end{aligned}$$

from (4.18) and (2.2) we infer that by the definition of γ,

$$\begin{aligned} \int_\Omega e^{\beta u} &= \int_\Omega e^{(\gamma+\beta^+)u} \\ &\le K_2(\gamma+\beta^+)^2 \int_\Omega \Psi(u) + (\gamma+\beta^+)\int_\Omega u + |\Omega| \\ &\le K_2(\gamma+\beta^+)^2 c_{10} + (\gamma+\beta^+)\int_\Omega u_0 + |\Omega| \qquad \text{for all } t \in [\tau, T_{max}), \end{aligned}$$

which implies (4.3), for the number $\sup_{t\in(0,\tau)} e^{\beta u(\cdot,t)}$ is clearly finite due to the continuity of u. $\square$

4.3 Boundedness of u. Proof of the main results

Thanks to a general result of a Moser-type iteration, to be provided in Lemma 5.1 in the Appendix, apart from Lemma 4.3 and Lemma 4.2 no further efforts are necessary to derive the following.

Lemma 4.4 *Under the assumptions of Theorem 1.1, the solution (u, v) of (1.2) is global in time, and there exists $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0. \tag{4.19}$$

Proof. An application of Lemma 5.1 to $b(x, t) := \nabla v(x, t)$, $(x, t) \in \bar{\Omega} \times (0, T_{max})$, shows that as a consequence of Lemma 4.2 and Lemma 4.3, u belongs to $L^\infty(\Omega \times (\tau, T_{max}))$, where once more $\tau = \min\{1, \frac{1}{2}T_{max}\}$. As moreover also ∇v is bounded in $\Omega \times (\tau, T_{max})$, in view of the extensibility criterion in Lemma 2.1 this guarantees that indeed $T_{max} = \infty$ and that (4.19) holds. □

In order to prepare our arguments concerning the large time behavior of solutions, let us briefly note that the above also implies the following Hölder regularity property.

Lemma 4.5 *Suppose that the hypotheses of Theorem 1.1 are valid. Then there exist $\theta \in (0, 1)$ and $C > 0$ such that*

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \le C \qquad \text{for all } t \ge 2. \tag{4.20}$$

Proof. Since u and ∇v are bounded in $\Omega \times (1, \infty)$ according to Lemma 4.4 and Lemma 4.2, the claim readily results from two straightforward applications of well-known Hölder estimates for scalar parabolic problems ([11, Theorem 1.3]). □

A proof of Theorem 1.1 can now be achieved by means of a standard argument.

Proof of Theorem 1.1. The statement on global existence and the boundedness property (1.14) have already been asserted by Lemma 4.4. The claims concerning the large time behavior of u and v result from combining Lemma 4.5 with the preliminary decay information implicitly contained in (3.15) and (3.16) in a straightforward manner: In fact, assuming (1.15) to be false we could find $\delta > 0$, $(t_k)_{k\in\mathbb{N}} \subset (2, \infty)$ and $(x_k)_{k\in\mathbb{N}} \subset \Omega$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\left|u(x_k, t_k) - \overline{u_0}\right| \ge \delta \qquad \text{for all } k \in \mathbb{N}.$$

According to the equicontinuity property of the set of function $u_k := u_{|\bar{\Omega} \times [k, k+1]}$ expressed in (4.20), we could thus find $r > 0$ and $\eta > 0$ such that

$$\left|u(x, t) - \overline{u_0}\right| \ge \frac{\delta}{2} \qquad \text{for all } x \in B_r(x_k) \cap \Omega, \text{ each } t \in (t_k, t_k + \eta) \text{ and any } k \in \mathbb{N}.$$

Noting that by smoothness of $\partial\Omega$ the number $c_1 := \inf_{k\in\mathbb{N}} |B_r(x_k) \cap \Omega|$ must be positive, from this we would obtain that

$$\int_{t_k}^{t_k+\eta} \int_\Omega \left|u(x, t) - \overline{u_0}\right|^2 dxdt \ge \frac{\delta^2 \eta c_1}{4} \qquad \text{for all } k \in \mathbb{N},$$

which contradicts (3.15) and thereby establishes (1.15).
The corresponding statement (1.16) on v can be verified similarly. □

The statements from both Theorem 1.2 and Theorem 1.3 thereby become immediate.

PROOF of Theorem 1.3. We only need to note that for $D(s) := (1+\beta s)e^{-\beta s}$ and $S(s) := se^{-\beta s}$, $s \ge 0$, fixing any $\varepsilon \in (0,\beta)$ we can clearly find $c_1 > 0$ such that

$$e^{-\beta s} \le D(s) \le c_1 e^{-(\beta-\varepsilon)s} \qquad \text{for all } s \ge 0,$$

and that moreover

$$\frac{S(s)}{D(s)} = \frac{s}{1+\beta s} \le \frac{1}{\beta} \qquad \text{for all } s \ge 0.$$

Therefore, namely, all statements result on applying Theorem 1.1 with the particular choices $\beta^- := \beta, \beta^+ := \beta - \varepsilon, \alpha := 0, K_1 := 1, K_2 := c_1$ and $K_3 := \frac{1}{\beta}$. $\Box$

5 Appendix: A general boundedness property

This appendix identifies a rather general setting under which bounds for $\int_\Omega e^{\beta u}$ with appropriately large $\beta > 0$ imply boundedness of subsolutions u to a scalar parabolic problem associated with the first equation in (1.2).

Lemma 5.1 *Suppose that D and S satisfy (1.8), that (1.9) holds with some $\beta^+ \in \mathbb{R}, \beta^- \ge \beta^+, K_1 > 0$ and $K_2 > 0$, and that there exist $\lambda > 0$ and $K_4 > 0$ fulfilling*

$$\frac{S(s)}{D(s)} \le K_4 e^{\lambda s} \qquad \textit{for all } s \ge 0. \tag{5.1}$$

Moreover, let $T \in (0,\infty]$ and $b \in C^{1,0}(\bar\Omega \times (0,T))$ be such that

$$b \cdot \nu \le 0 \qquad \textit{on } \partial\Omega \times (0,T) \tag{5.2}$$

and that

$$b \in L^\infty(\Omega \times (0,T)). \tag{5.3}$$

Then if $u \in C^0(\bar\Omega \times [0,T)) \cap C^{2,1}(\bar\Omega \times (0,T))$ is a nonnegative function which is such that

$$\begin{cases} u_t \le \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)b(x,t)), & x \in \Omega,\ t \in (0,T), \\ \frac{\partial u}{\partial \nu} \le 0, & x \in \partial\Omega,\ t \in (0,T), \end{cases} \tag{5.4}$$

and such that furthermore

$$e^u \in L^\infty((0,T); L^\beta(\Omega)) \qquad \textit{for all } \beta > 1, \tag{5.5}$$

we also have

$$e^u \in L^\infty(\Omega \times (0,T)). \tag{5.6}$$

PROOF. We let

$$w(x,t) := e^{u(x,t)}, \qquad x \in \bar{\Omega},\ t \in [0,T),$$

and then obtain from (5.4) that

$$\frac{\partial w}{\partial \nu} \le 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{5.7}$$

and that

$$\begin{aligned} w_t &= e^u u_t \\ &\le e^u \nabla \cdot (D(u)\nabla u) - e^u \nabla \cdot (S(u)b) \qquad\qquad (5.8) \\ &= \nabla \cdot \Big(e^u D(u) \nabla u \Big) - e^u D(u) |\nabla u|^2 \\ &\quad -\nabla \cdot \Big(e^u S(u) b \Big) + e^u S(u) b \cdot \nabla u \qquad \text{in } \Omega \times (0,T). \qquad\qquad (5.9) \end{aligned}$$

Herein, employing Young's inequality and using (5.1) and (1.9) as well as (5.3) we can estimate

$$\begin{aligned} e^u S(u) b \cdot \nabla u &\le e^u D(u)|\nabla u|^2 + \frac{1}{4} e^u \frac{S^2(u)}{D(u)} |b|^2 \\ &= e^u D(u)|\nabla u|^2 + \frac{1}{4} e^u D(u) \Big(\frac{S(u)}{D(u)} \Big)^2 |b|^2 \\ &\le e^u D(u)|\nabla u|^2 + \frac{1}{4} e^u \cdot K_2 e^{-\beta^+ u} \cdot K_4^2 e^{2\lambda u} \cdot \|b\|^2_{L^\infty(\Omega \times (0,T))} \\ &= e^u D(u)|\nabla u|^2 + c_1 e^{(1-\beta^+ + 2\lambda)u} \qquad \text{in } \Omega \times (0,T) \end{aligned}$$

with $c_1 := \frac{1}{4} K_2 K_4^2 \|b\|^2_{L^\infty(\Omega\times(0,T))}$. Writing

$$\widehat{D}(s) := D(\ln s) \qquad \text{for } s \ge 1, \tag{5.10}$$

we see that furthermore

$$e^u D(u) \nabla u = \widehat{D}(w) \nabla w \qquad \text{in } \Omega \times (0,T),$$

whence (5.8) implies that

$$w_t \le \nabla \cdot \Big(\widehat{D}(w) \nabla w \Big) + \nabla \cdot f(x,t) + g(x,t) \qquad \text{in } \Omega \times (0,T) \tag{5.11}$$

with

$$f(x,t) := -e^{u(x,t)} S(u(x,t)) b(x,t), \quad g(x,t) := c_1 e^{(1-\beta^+ + 2\lambda)u} \qquad x \in \Omega,\ t \in (0,T),$$

where again by (5.1), (1.9) and (5.3) we find that

$$\begin{aligned} |f| &\le e^u \cdot K_4 e^{\lambda u} D(u) \cdot |b| \\ &\le K_2 K_4 \|b\|_{L^\infty(\Omega\times(0,T))} \cdot e^{(1-\beta^+ + \lambda)u} \qquad \text{in } \Omega \times (0,T). \end{aligned}$$

Using (5.5), we thus infer that

$$f, g \in L^\infty((0,T); L^p(\Omega)) \qquad \text{for all } p > 1, \tag{5.12}$$

and that moreover also

$$w \in L^\infty((0,T); L^p(\Omega)) \qquad \text{for all } p > 1. \tag{5.13}$$

Now since (5.10) and the left inequality in (1.9) guarantee that $\widehat{D}$ satisfies the algebraic lower estimate

$$\widehat{D}(s) \geq K_1 e^{-\beta^- \cdot \ln s} = K_1 s^{-\beta^-} \qquad \text{for all } s \geq 1,$$

in view of (5.11), (5.7), (5.12) and (5.13) we see that a well-known result of a Moser-type reasoning ([14, Lemma A.1]) directly applies to the present situation so as to assert (5.6). □

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